Lösung zu Ü8_C1

$$\boxed{\mathcal{F}(\alpha * \eta) = \sum_{\sigma \in Q^P} \prod_{p \in P} \alpha(p, \sigma p) \cdot \mathcal{F}(\underbrace{\eta \circ \sigma}_{(\eta(\sigma p))_{p \in P}})}$$

P Faktorenmenge

Q Summandenmenge (für jeden Faktor p)

$$(\alpha * \eta) p = \underbrace{\sum_{q \in Q} \alpha(p,q) \cdot \eta q}_{Q \text{ Summanden im Faktor } p}.$$

$$\mathcal{F}\Big(\big(\sum_{q \in Q} \alpha(p,q) \cdot \eta q \big)_{p \in P} \Big)$$

$$= \sum_{\sigma \in Q^P} \mathcal{F}\Big(\big(\alpha(p, \sigma p) \cdot \eta(\sigma p) \big)_{p \in P} \Big)$$

$$\boxed{\sigma = (\sigma p)_{p \in P}}$$

$\sigma = (\sigma p)_{p \in P} \in Q^P$

" σ ~~σp~~ wählt im Faktor p den Summanden σp "

Über **alle** solche Auswahlen wird summiert.

d.h. über alle $\sigma \in Q^P$.

Lösung zu Ü8_C1

$$\boxed{\mathcal{F}(\alpha * \eta) = \sum_{\sigma \in Q^P} \prod_{p \in P} \alpha(p, \sigma p) \cdot \mathcal{F}(\eta \circ \sigma)}$$

$P = [2]$ & $Q = [3]$: $(\alpha * \eta)p = \sum_{q \in Q} \alpha(p,q) \cdot \eta q = \alpha(p,1)\eta 1 + \alpha(p,2)\eta 2 + \alpha(p,3)\eta 3$

Setze $a_{pq} := \alpha(p,q)$

$x_q := \eta q$

Dann ist $(\alpha * \eta)p = \sum_{q \in Q} a_{pq} x_q = a_{p1}x_1 + a_{p2}x_2 + a_{p3}x_3$

$\mathcal{F}(\alpha * \eta) = \mathcal{F}(a_{11}x_1 + a_{12}x_2 + a_{13}x_3, a_{21}x_1 + a_{22}x_2 + a_{23}x_3)$

Setze für $\gamma = (\gamma 1, \gamma 2) \in M^P$

$\mathcal{F}\gamma = \gamma 1 \bullet \gamma 2$

Dann ist $\mathcal{F}(\alpha * \eta) = (a_{11}x_1 + a_{12}x_2 + a_{13}x_3) \bullet (a_{21}x_1 + a_{22}x_2 + a_{23}x_3)$

$$= \sum_{(i,j) \in [3]^2} (a_{1i} x_i) \bullet (a_{2j} x_j)$$

$$= \sum_{(i,j) \in [3]} a_{1i} \cdot a_{2j} \cdot x_i \bullet x_j$$

$$= \sum_{\sigma \in Q^P} a_{1,\sigma 1}\, a_{2,\sigma 2} \overbrace{x_{\sigma 1} \bullet x_{\sigma 2}}^{\mathcal{F}(x_{\sigma 1}, x_{\sigma 2})}$$

$$= \sum_{\sigma \in Q^P} \prod_{p \in P} \underbrace{a_{p,\sigma p}}_{\alpha(p,\sigma p)} \cdot \mathcal{F}\underbrace{(x_{\sigma 1}, x_{\sigma 2})}_{\eta \circ \sigma}$$

$$= \sum_{\sigma \in Q^P} \prod_{p \in P} \alpha(p, \sigma p) \cdot \mathcal{F}(\eta \circ \sigma). \quad \square$$

$$\prod_{i=1}^{m} (x_{i,1}+\dots+x_{i,n}) = \sum_{\sigma\in[n]^{[m]}} \prod_{i=1}^{n} x_{i,\sigma i}$$

D.h.

$$(x_{11}+\dots+x_{1n})(x_{21}+\dots+x_{2n})\cdot\ldots\cdot(x_{m1}+\dots+x_{mn})$$

$$= \sum_{\sigma\in\{1,\dots,n\}^m} x_{1,\sigma 1}\cdot x_{2,\sigma 2}\cdot\ldots\cdot x_{m,\sigma m}$$

Allgemein:

$$F(x_{11}+\dots+x_{1n},\ x_{21}+\dots+x_{2n},\ \dots,\ x_{m,1}+\dots+x_{m,n})$$

$$= \sum_{\sigma\in\{1..n\}^m} F(x_{1,\sigma 1}, x_{2,\sigma 2}, \dots, x_{m,\sigma m})$$

D.h.

$$F\left(\left(\sum_{j=1}^{n} x_{i,j}\right)_{i\in\{1,\dots,m\}}\right) = \sum_{\sigma}$$

$$\boxed{\prod_{p\in P}\left(\sum_{q\in Q} x_{p,q}\right) = \sum_{\sigma\in Q^P}\prod_{p\in P} x_{p,\sigma_p}}$$

$$F\left(\left(\sum_{q\in Q} x_{p,q}\right)_{p\in P}\right) = \sum_{\sigma\in Q^P} F\left((x_{p,\sigma_p})_{p\in P}\right)$$

$\sigma = (\sigma_p)_{p\in P}$

$P=[m]$

$Q=[n]$

$$F\left(\left(\sum_{q\in[n]} x_{p,q}\right)_{p\in[m]}\right) = \sum_{\sigma\in[n]^{[m]}} \dots$$

$\sigma = (\sigma_p)_{p\in[m]}$

$$F\Big(\sum_{q=1}^{n} x_{1,q},\dots,\sum_{q=1}^{n} x_{m,q}$$

Ü8 H1 "Lösen durch direktes Überprüfen [AUSRECHNEN]"

(a) $x * (x \times y) = x_1(x_2y_3 - x_3y_2) + x_2(x_3y_1 - x_1y_3) + x_3(x_1y_2 - x_2y_1)$

$= \underline{x_1x_2y_3} - x_1x_3y_2 + x_2x_3y_1 - \underline{x_2x_1y_3} + x_3x_1y_2 - x_3x_2y_1$

$= 0$, d.h. $\boxed{x \perp x \times y}$. ✓

Beachte hier: $x \times y := (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$

(b) Nach der Leibnizformel ist

$$(*)\ \det(x,y,z) = x_1y_2z_3 - x_1y_3z_2 - x_3y_2z_1 - x_2y_1z_3 + x_3y_1z_2 + x_2y_3z_1$$

da $\begin{vmatrix}1&2&3\\1&2&3\end{vmatrix}$, $\begin{vmatrix}1&2&3\\3&1&2\end{vmatrix}$, $\begin{vmatrix}1&2&3\\2&3&1\end{vmatrix}$ gerade Permutationen

und $\begin{vmatrix}1&2&3\\1&3&2\end{vmatrix}$, $\begin{vmatrix}1&2&3\\3&2&1\end{vmatrix}$, $\begin{vmatrix}1&2&3\\2&1&3\end{vmatrix}$ ungerade Permutationen sind.

Außerdem ist

$$\binom{*}{*}\ x * (y \times z) = x_1(y_2z_3 - y_3z_2) + x_2(y_3z_1 - y_1z_3) + x_3(y_1z_2 - y_2z_1)$$

$$= x_1y_2z_3 - x_1y_3z_2 + x_2y_3z_1 - x_2y_1z_3 + x_3y_1z_2 - x_3y_2z_1$$

Durch direkten Vergleich von (*) und $\binom{*}{*}$ folgt nun

$\boxed{\det(x,y,z) = x * (y \times z)}$. ✓

Lösung zu Ü8_H2 "direktes Nachrechnen"

Setze $a_{ij} := \alpha(i,j)$, $b_{ij} := \beta(i,j)$, $c_{ij} := \gamma(i,j)$, $\gamma := \alpha * \beta$.

Dann ist $|\alpha|_+ = a_{11}a_{22}$ und $|\alpha|_- = a_{12}a_{21}$ und

$$|\alpha| = a_{11}a_{22}\cdot\delta_+ + a_{12}a_{21}\cdot\delta_-$$

Also ist $|\alpha*\beta| = |\gamma| = |\gamma|_+\delta_+ + |\gamma|_-\delta_-$ und

$|\gamma|_+ = c_{11}c_{22} = (a_{11}b_{11}+a_{12}b_{21})(a_{21}b_{12}+a_{22}b_{22})$ sowie

$|\gamma|_- = c_{12}c_{21} = (a_{11}b_{12}+a_{12}b_{22})(a_{21}b_{11}+a_{22}b_{21})$ wegen

$\gamma(i,j) = \alpha(i,1)\cdot\beta(1,j)+\alpha(i,2)\cdot\beta(2,j)$, d.h. $c_{ij} = a_{i1}b_{1j}+a_{i2}b_{2j}$.

Es folgt (Ausmultiplizieren):

$$|\gamma|_+ = a_{11}b_{11}\cdot a_{21}b_{12} + a_{11}b_{11}\cdot a_{22}b_{22} + a_{12}b_{21}\cdot a_{21}b_{12} + a_{12}b_{21}\cdot a_{22}b_{22}$$
$$= \underline{a_{11}a_{21}\cdot b_{11}b_{12}} + a_{11}a_{22}\cdot b_{11}b_{22} + a_{12}a_{21}\cdot b_{12}b_{21} + \underline{a_{12}a_{22}\cdot b_{21}b_{22}}$$ und

$$|\gamma|_- = a_{11}b_{12}\cdot a_{21}b_{11} + a_{11}b_{12}\cdot a_{22}b_{21} + a_{12}b_{22}\cdot a_{21}b_{11} + a_{12}b_{22}\cdot a_{22}b_{21}$$
$$= \underline{a_{11}a_{21}\cdot b_{11}b_{12}} + a_{11}a_{22}\cdot b_{12}b_{21} + a_{12}a_{21}\cdot b_{11}b_{22} + \underline{a_{12}a_{22}\cdot b_{21}b_{22}}$$

Setze $s := \underline{a_{11}a_{21}\cdot b_{11}b_{12}} + \underline{a_{12}a_{22}\cdot b_{21}b_{22}}$. Dann ist

$$|\gamma|_+ = a_{11}a_{22}\cdot b_{11}b_{22} + a_{12}a_{21}\cdot b_{21}b_{12} + s$$
$$= |\alpha|_+\cdot|\beta|_+ + |\alpha|_-\cdot|\beta|_- + s$$ und

$$|\gamma|_- = a_{11}a_{22}\cdot b_{12}b_{21} + a_{12}a_{21}\cdot b_{11}b_{22} + s$$
$$= |\alpha|_+\cdot|\beta|_- + |\alpha|_-\cdot|\beta|_+ + s$$

Andererseits ist

$|\alpha|\cdot|\beta| = (|\alpha|_+\cdot|\beta|_+ + |\alpha|_-\cdot|\beta|_-)\delta_+ + (|\alpha|_+\cdot|\beta|_- + |\alpha|_-\cdot|\beta|_+)\delta_-$,

Ergebnis:

$|\alpha*\beta| = |\gamma| = |\gamma|_+\delta_+ + |\gamma|_-\delta_- = |\alpha|\cdot|\beta| + s\delta_+ + s\delta_- = |\alpha|\cdot|\beta| + s\cdot\nu$

für $\nu := \delta_+ + \delta_-$.