Peircean Algebraic Logic and Peirce's Reduction Thesis

JOACHIM HERETH and REINHARD PÖSCHEL

Abstract

Robert Burch describes in (Burch 1991) the *Peircean Algebraic Logic (PAL)* as a language to express Peirce's "unitary logical vision" (p. 3), which Peirce tried to formulate using different logical systems. A "correct" formulation of Peirce's vision then should allow a mathematical proof of Peirce's Reduction Thesis, that all relations can be generated from the ensemble of unary, binary and ternary relations, but that at least some ternary relations cannot be reduced to relations of lower arity.

Based on Burch's algebraization, the authors further simplified the mathematical structure of PAL and removed a restriction imposed by Burch, making the resulting system in its expressiveness more similar to Peirce's system of existential graphs. The drawback however was that the proof of the Reduction Thesis from (Burch 1991) did no longer hold. A new proof has been introduced in (Hereth Correia and Pöschel 2006) and was published in full detail in (Hereth 2008).

In this paper, we give provide the proof of Peirce's Reduction Thesis using the graph notation similar to Peirce's existential graphs.

1. Introduction

The task to algebraize a logic system such as Peirce's is not an easy one. Especially the connection to the underlying philosophy or "logical vision" makes every decision how to formalize some aspect prone to discussion and possible disagreement. However, the algebraization of Peirce's logical vision by Burch (1991) has not evoked much disagreement; Zeman (1995) calls this work "a major contribution to communication between what could be called Peircean logic and the more 'traditional' approach based on the work of Frege, Peano, and Russell." As mathematicians the authors are socialized and trained in the notations coming from this 'traditional' approach. Our approach concentrates on a formalization of the logic system presented by Burch in a way analogous to other relational systems known in mathematics, in particular to relational algebras. On the one hand this allows to directly use the fundamental operations introduced by Burch, on the other hand we have two major differences: we do not distinguish interpretations and enterpretations of graphs which simplifies the treatment but probably ignores some philosophical aspects. Also, we allow more freedom of construction, making the system thus a bit more similar to the system of existential graphs. We will highlight this second difference later in the paper.

Before we present our model, we have to clarify the notion of *relation*. Let us consider the statement "John loves Mary", which Peirce (1870) called a *relative term*. While this statement may describe a relation as understood in common language, it is here considered to be only an *instance* of the relation. That is, the relation is "love(s)" and the pair (John, Mary) is an element of this relation. The order of the objects which are put into relation is also of importance. Clearly, it does not follow from the above statement that Mary loves John, so the pair (Mary, John) is not necessarily an element of the relation.

For our more abstract considerations, we replace concrete objects (or persons) by abstracts elements which will be denoted by variables such as x or y, which are chosen from some set A. In the following we assume that A has at least two elements. An ordered sequence of elements such as (x,y) or $(x_1,\dots,x_n)$ for some natural number n will be called *pair* (or *binary tuple*) or *n-ary tuple* respectively. The relations will be denoted by greek letters such as ρ. Formaly, a relation is a subset of A^n for some natural number n, that means all tuples in a relation are of the same arity. The relation is then called an *n-ary relation* (or *unary* if $n=1$, *binary* if $n=2$ and *ternary* if $n=3$). It is therefore not possible that a pair (v,w) and a triple (x,y,z) both belong to the same relation. If a tuple belongs to a relation we designate this by $\in$, for instance $(x,y)\in\rho$. As usual in mathematics, we treat relations as sets and use the normal operations on sets. If ρ and σ are binary relations, then $(x,y)\in\rho\cup\sigma$ means that the pair (x,y) is an element of ρ or of σ (or of both relations), and $(x,y)\in\rho\cap\sigma$ means that (x,y) is element of both relations.

2. Operations of Peircean Algebraic Logic

In this section we will present the basic ingredients of the Peircean Algebraic Logic (PAL). On the one hand there is the operational aspect, that is the operations and their interpretation, on the other the diagramatical aspect, the representation of the graphs.[1]

We assume that the reader is familiar with the interpretation of existential graphs, which can be applied to the diagrams in this paper. We only want to remark a technicality introduced for the translation between the graphs and the usual mathematical notation of relations.

As Burch does with the graphical syntax in (Burch 1991, chap. 11) we add numbers to the ends of arcs to number the places of the relation. In the most simple case of a single relation vertex the standard enumeration of the arcs corresponds to the places of the relation itself (denoted by the smaller numbers directly at the vertex), see for instance the left example in Fig. 1. A pair (x_1, x_2) belongs to the interpretation of this graph if (x_1, x_2) is an element of ρ. If ρ is the relation "loves" and John loves Mary, then (John, Mary) is an interpretation of the left graph. It is also possible to renumber the places, as can be seen in the right example. Using the same example, then (Mary, John) would be an interpretation of the right graph (but (John, Mary) not necessarily), that is the graph would represent the relation "is loved by". For any graph G the set of all tuples belonging to its interpretation is denoted by G^A.

Figure 1. Simple (atomic) graph with numbered and renumbered places

To denote an arbitrary graph with n places, we will use the cloud icon as seen in Fig. 2. We will use this to show constructions of graphs without recurring on their particular structure. The cloud may be replaced by an arbitrary graph.

Now, we will present the PAL-operations in the same order as in (Hereth Correia and Pöschel 2006). Let G and H be graphs and ρ and σ their respective interpretations. Let G and consequently ρ have m places, and let H and σ have n places.

(PAL1) The *product* of two graphs is represented by putting the graphs next to each other, see Fig. 3. The places of the second graph are shifted

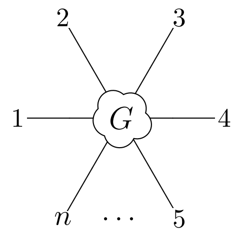

Figure 2. Example of the representation of an arbirtrary graph

by m. The corresponding term is written $G \times H$. The interpreation of

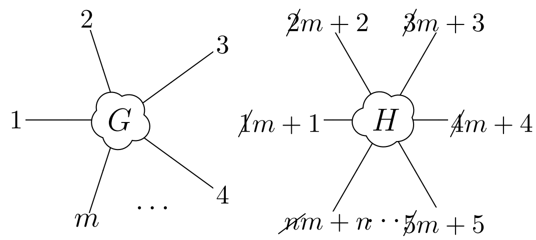

Figure 3. Product of G and H.

the product is the cross-product of the interpretations:

$$\rho \times \sigma := \{(x_1, \ldots, x_m, y_1, \ldots, y_n) \mid (x_1, \ldots, x_m) \in \rho, (y_1, \ldots, y_n) \in \sigma\}.$$

(PAL2) The *Join*-operation allows to connect two places i and j ($1 \leq i < j \leq m$) of a graph and is denoted by $\delta^{i,j}(G)$ and can be represented as in Fig. 4. The interpretation of this operation is to retain only tuples which have the same value at the places i and j and to remove these two places. Formally, this can be written as follows:

$$\begin{aligned}\delta^{i,j}(\rho) :=& \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m) \mid \\ & \exists y \in A : (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_m) \in \rho\}.\end{aligned}$$

(PAL3) The *complement* is depicted by a (slightly thicker) oval (see Fig. 5) around the graph and denoted by $\neg G$. The interpretation of this operation is the set-semantic complement, the set of all m-tuples which do *not* belong to the interpretation of G:

$$\neg\rho := \{(x_1, \ldots, x_m) \in A^m \mid (x_1, \ldots, x_m) \notin \rho\}.$$

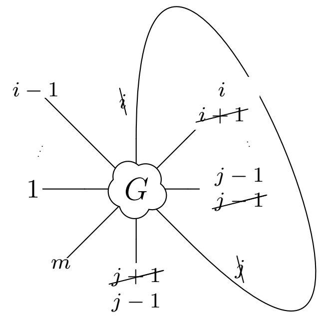

Figure 4. Join of places i and j of graph G.

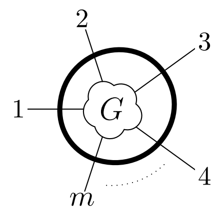

Figure 5. Complement of graph G.

(PAL4) The *permutation* operation allows to change the order of the places. Let α be a bijective mapping on $\{1,\ldots,m\}$, that is every element is mapped to some element, but no two elements to the same. Then we define

$$\pi_\alpha(\rho) := \{(x_1,\ldots,x_m) \mid (x_{\alpha(1)},\ldots,x_{\alpha(m)}) \in \rho\}.$$

The diagrammatic representation is shown in Fig. 6.

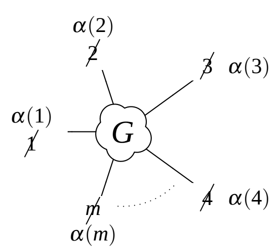

Figure 6. Applying the permutation α on graph G.

(PAL5) The last operation of PAL is not an operation in the common sense, but in the mathematical one. It is the constant *teridentity*, denoted by Id_3. Its interpretation is defined by

$$\mathsf{id}_3 := \{(x,x,x) \mid x \in A\}.$$

As the enumeration of the places is not important (id_3 is invariant under any permutation of places), it is represented without numbering the places, as shown in Fig. 7.

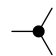

Figure 7. Representation of the teridentity.

In the following, we will investigate, what relations can be constructed by these operations or some of these operations. Let Q be a set of relations. Then $\langle Q\rangle_{\mathsf{PAL}}$ is the set of relations that can be generated applying any finite sequence of PAL-operations on these relations. We will also investigate the subset $\langle Q\rangle_{\mathsf{PAL}^-}$ of all relations that can be generated using all operations except teridentity. Formally, Peirce's Reduction Thesis can be stated as follows: let Q be the set of all unary and binary relations, then independently from the underlying set A the set $\langle Q\rangle_{\mathsf{PAL}^-}$ is a strict subset of $\langle Q\rangle_{\mathsf{PAL}}$.

Burch (1991) introduces an operation join_2, which allows the product of two relations if they are joined directly after, that is $\delta^{i,j}(\rho \times \sigma)$ with $1 \leq i \leq m < j \leq m+n$ for an m-ary relation ρ and an n-ary relation σ. He allows to generate relations using this operations in addition to join, complement, permutation and teridentity. Then, in a second step, the product of these relations may be generated. Our modified version of PAL is a simplification of this two-step process (allowing the product any time in the construction process) and allows to generate graphs which could not be generated with the original PAL. This makes the main part of Peirce's Reduction Thesis (Section 6 in this paper), that not all relations can be generated if no ternary relations are available, more difficult.

Of course, any algebraization of Peirce's logic system in general and the system of existential graphs in particular, is prone to errors due to misinterpretations. The algebraization presented here builds on the diligent work of Robert Burch, any errors introduced by further simplifications are of course our faults.

3. Generating all Relations

In this section we discuss the first part of Peirce's reduction thesis, that all relations can be generated from ternary relations. At first, ternary relations seem rather complicated as basic building blocks for arbitrary relations. But we can easily see, that unary and binary relations can be constructed from ternary relations. Let ρ be an unary relation, we define $\Delta_A := \{(x,x) \mid x \in A\}$ (this is the binary identity on A) and $\tau := \rho \times \Delta_A$. As ρ is unary and Δ_A is binary, we have τ as a ternary relation from which we can reconstruct ρ via $\rho = \delta^{2,3}(\tau)$.

Similarly, we can generate all binary relations from ternary ones. Let ρ now be an arbitrary binary relation and let us consider $A = \{x \mid x \in A\}$ as unary relation. We defin $\tau := \rho \times A$ and get $\delta^{3,4}(\delta^{5,6}(\tau \times \mathrm{id}_3)) = \rho$.

As we can generate any unary and binary relation from ternary ones, we will in the following use these as basic building blocks. Next we consider arbitrary n-ary relations with $n \geq 4$. Let us start with a simple case, we assume additionally that ρ is finite. Then we can enumerate the tuples from 1 to m and write $\rho = \{(x_{11}, \ldots, x_{1n}), \ldots, (x_{m1}, \ldots, x_{mn})\}$.

In the simplest case ($m = 1$) we have $\rho = \{x_{11}\} \times \cdots \{x_{1n}\}$. Each element can be considered as a one-element unary relation, hence we can write ρ as the product of n unary relations (see Fig. 8).

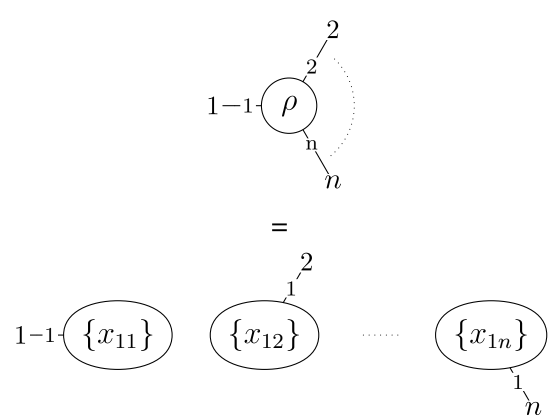

Figure 8. Generating the n-ary singleton relation ρ from unary singleton relations.

Now let ρ_k define the relation consisting of the first k tuples from ρ, that is $\rho_k := \{(x_{11}, \ldots, x_{1n}), \ldots, (x_{k1}, \ldots, x_{kn})\}$. As we have just seen, we can construct ρ_1 from unary relations. Having ρ_{k-1}, we can construct ρ_k using the PAL-operations as shown in Fig. 9. The graph may seem very complex at first, but it is composed of simple structures. In the upper inner oval we recognize a graph very similar to the lower one in Fig. 8. We can consider the graph enclosed by the upper oval as the *n*-ary singleton relation containing only the tuple $(x_{k1},\dots,x_{kn})$. The overall form of the two negation ovals enclosed by an outer negation is probably well-known to readers familar with Peirce's existential graphs. In his α-graphs, they represent disjunction. If G is the (α-)graph enclosed by the upper inner oval and H the other graph, than the logical interpretation is $\neg(\neg G \wedge \neg H) = G \vee H$. The relational interpretation of the graph in Fig. 9 is the union of the two relations (we have to connect corresponding places with teridentity-nodes, resulting in a more complicated looking diagram than the corresponding for α-graphs). The interpretation of this graph is therefore $\{(x_{k1},\dots,x_{kn})\} \cup \rho_{k-1} = \rho_k$. We assumed ρ to be finite with m elements, therefore we can construct ρ in m steps (that is $\rho_m = \rho$) using unary relations and PAL-operations (teridentity being the constant PAL-operation). Together with the considerations at the beginning of this section we have shown, that any finite relation can be constructed from ternary relations with PAL-operations.

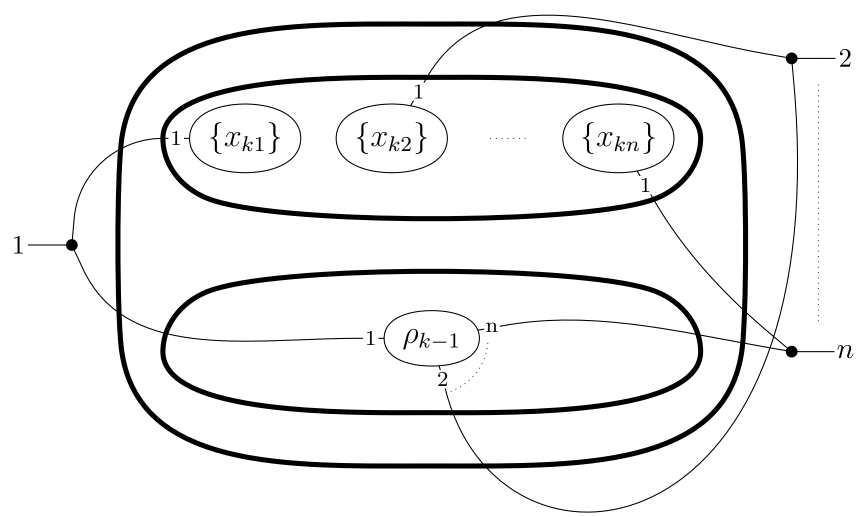

Figure 9. Union of the relation $\rho_{k-1} \subseteq \rho$ and the singleton relation $\{(x_{k1},\dots,x_{kn})\} \subseteq \rho$.

However, this construction does not work for *infinite* relations.[2] But in this case there exists a one-to-one mapping between the elements of A and the elements of $A \times A$. Let $\mu : A \times A \to A$ be such a mapping. Then we can replace the quaternary relation ρ by the ternary relation $\rho' := \{(x_1, x_2, \mu(x_3, x_4)) \mid (x_1, x_2, x_3) \in \rho\}$ and generate ρ from ρ' and μ (more precisely the graph of μ, that is the ternary relation $\{(x_1, x_2, x_3) \in A^3 \mid \mu(x_1) = (x_2, x_3)\}$, but for this article we will identify the mapping and its graph) using the injectivity of μ ($\mu(a) = \mu(b) \iff a = b$), as shown in Fig. 10).

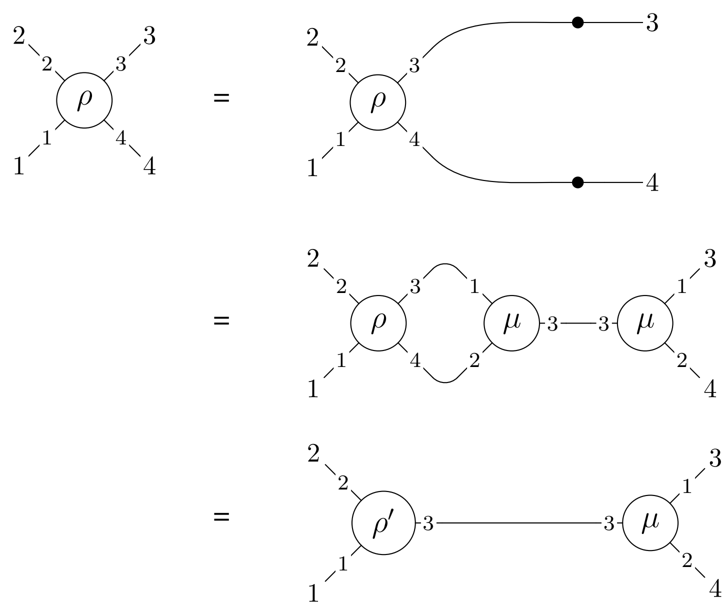

Figure 10. Constructing the quaternary relation ρ from the ternary relations ρ' and μ.

Having shown how to construct quaternary relations from ternary relations the generalization is simple. Let us assume we can construct any n-ary relation and want to construct some $n+1$-ary relation ρ. According to our assumption we can construct the n-ary relation

$$\rho' := \{(a_1, \ldots, a_{n-1}, \mu(a_n, a_{n+1})) \mid (a_1, \ldots, a_{n+1}) \in \rho\}.$$

Using the same approach as before we can construct ρ from this (constructable) n-ary relation ρ' and μ (see Fig. 11).

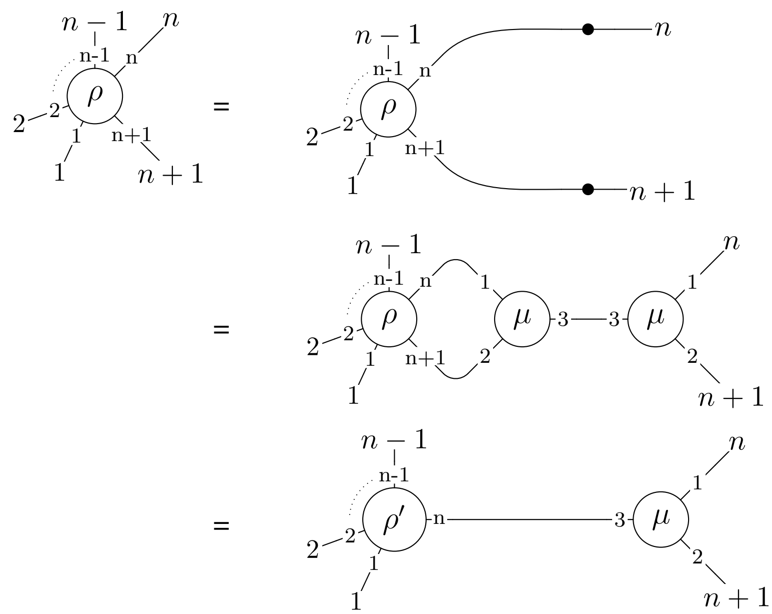

Figure 11. Constructing the $n+1$-ary relation ρ from the n-ary relation ρ' and the ternary relation μ.

4. Representations of Relations

In the last section we have shown that in PAL we can construct all relations from ternary relations. However, this is only one part of the reduction thesis. The more difficult part is to show that we actually need ternary relations, that is we cannot construct all relations from unary and binary relations alone, not using teridentity.

To show this, we introduce in this section a special representation of relations by sets of graphs. This representation is basically a normalization, analogous to the disjunctive normal form known from formulas in propositional or predicate logic. We will show, that for any set Q of graphs the interpretation of any graph G from $\langle Q\rangle_{\mathsf{PAL}}$ can be represented by the union of intersection of the interpreations of graphs constructed from $Q \cup \{\mathsf{id}_3\}$.

To make those notions more clear, we will formally introduce representations. Mathematically, a *representation* is a set of sets of graphs together with a given arity n, such that each graph has only places from 1 to n (but not necessarily all of them). To illustrate this definition, we give an advanced example in which we will construct a graph using all of the PAL-operations and explain in each step how to construct or adapt the representation.

If the graph G is atomic, that is an element of Q or if it is the teridentity, the graph is trivially a (connected) representation of itself, more precisely we have $\mathscr{R} := \{\{G\}\}$ as the representation and of course $G^A = \bigcup\{\bigcap\{H^A \mid H \in S\} \mid S \in \mathscr{R}\}$.

Building the product of two graphs G and H with the arities m and n, we have to adapt the numbering of the places for the second graph (adding m to each place) and to assure that a tuple belongs to the interpretation if and only if its projection to the first m places belongs to the interpretation of the first graph and its projection to the last n places belongs to the interpretation of the (adapted) second graph. We now proceed to formally describe the new representation. Let $\mathscr{R}_G$ be a representation of G and $\mathscr{R}_H$ a representation of H. For a graph T in the representation of H let T^{+m} denote the graph resulting from adding m to the places of T. For instance, if G is the binary relation ρ and H is the unary relation σ then σ is represented by itself (that is $\{\{\sigma\}\}$) and σ^{+2} would be the lower graph in Fig 12. In this simple case, the representation is shown as the product of the graphs. In the following, the representation of a graph will be visualized as a row of columns, each column representing one set of the representation, the graphs of the set ordered vertically in the column.

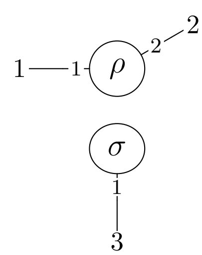

Figure 12. Product (and representation of the product) of two atomic graphs.

In the case of multiple columns this constructions becomes more complicated, we build the union of each set from the first representation with each set of the second representation (with adapted places). Formally, we define the new representation $\mathscr{R}$ by

$$\mathscr{R} := \{A \cup \{T^{+m} \mid T \in B\} \mid A \in \mathscr{R}_G, B \in \mathscr{R}_H\}.$$

A tuple t belongs to the product of the interpretations of G and H if and only if there are sets $A \in \mathscr{R}_G$ and $B \in \mathscr{R}_H$ such that the projection to the first

m places belongs to the interpretation of A (that is for each graph $T \in A$ the projection of t to the places appearing in T belongs to the interpretation of T) and the projection of t to the last n places belongs to the interpretation of B. Now it is easy to verify, that this is equivalent to the above condition, that the projection of t to the places appearing in each $T \in A$ belongs to the interpretation of T (this is exactly the same condition) *and* that the projection of t to the places appearing in T^{+m} belongs to the interpretation of T^{+m} for each $T \in B$ (this is the condition from above, just shifting the places). This shows, that the formally given representation is indeed a representation of the product of the two graphs.

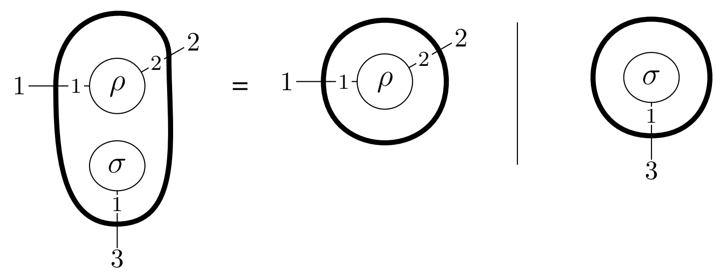

Figure 13. Complement of the graph in Fig. 12 and its representation.

The calculation of the complement of a graph is more complicated. Let again G be a graph and $\mathscr{R}$ a corresponding representation. To calculate a representation of $\neg G$ we apply the laws of DeMorgan and the distributivity of union and intersection of sets. The latter is probably well-known to the reader in the case of the intersection of two unions of two sets, but might not be apparent in the more general notation used below. Applying distributivity to the intersections of multiple unions means that we have to intersect over *all* possible choices from the unions. Mathematically, such a choice is modeled by a *choice function*, that is a function χ from the set of sets to the union of all these sets such that $\chi(A) \in A$ for each set A. To adapt this idea to our application, let us consider the expression $\bigcap\{\bigcup\{(\neg T)^A \mid T \in A\} \mid A \in \mathscr{R}\}$. Then our choice function χ is a mapping from $\mathscr{R}$ to $\bigcup\mathscr{R}$ with $\chi(A) \in A$ for each $A \in \mathscr{R}$. Let $X_{\mathscr{R}}$ be the set of all those choice functions. Distributivity then allows us to see that the above equation is (extensionally) equal to $\bigcup\{\bigcap\{(\neg T)^A \mid T \in \chi(\mathscr{R})\} \mid \chi \in X_{\mathscr{R}}\}$.

Let t be a tuple. Then we have

$$\begin{aligned} t \in (\neg G)^A &\iff t \notin \bigcup\{\bigcap\{T^A \mid T \in A\} \mid A \in \mathscr{R}\} \\ &\iff t \in \bigcap\{\bigcup\{(\neg T)^A \mid T \in A\} \mid A \in \mathscr{R}\} \\ &\iff t \in \bigcup\{\bigcap\{(\neg T)^A \mid T \in \chi(\mathscr{R})\} \mid \chi \in X_{\mathscr{R}}\} \end{aligned}$$

Consequently $\{\chi(\mathscr{R})\} \mid \chi \in X_{\mathscr{R}}\}$ is a representation of the complement of G. If $\mathscr{R}$ is as simple as in Fig. 12, then there are not many choice functions; as there is only one set to choose from, there are only as many possibilities as elements in this set, in our example only two. The result can be seen on the right side in Fig. 13.

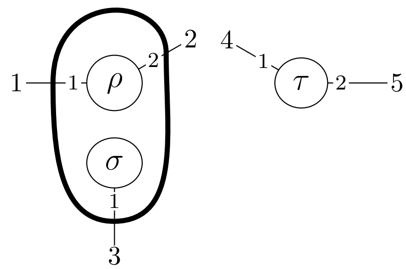

Figure 14. Product of the graph from Fig. 13 and an atomic graph.

In the next step we see a slightly more complicated case of the representation of a product. Fig. 14 shows the product of the previous graph with an atomic graph. Because the representation of the previous graphs has two sets and the atomic graph's representation contains one set the result is a representation with two sets (see Fig.15). The only graph in the only set of the atomic graphs representation (the graph itself) appears consequently in every column.

The next operation is the join between the second place of the ρ-relation and the first place of the τ-relation. Looking at the result in Fig. 17 we see that it is easy to construct the representation if both places belong to graphs in the set, the respective places are joined. In the second column there is no graph with the ρ-relation, consequently we cannot join the two places. Instead the existing place of the τ-relation is joined with the unary relation A (represented by the teridentity with two joined places). For a set of the representation where neither place belongs to any graph nothing is changed.

To describe this transformation more formally we introduce an additional notation; $i \in T$ denotes that place i appears in the graph T and consequently

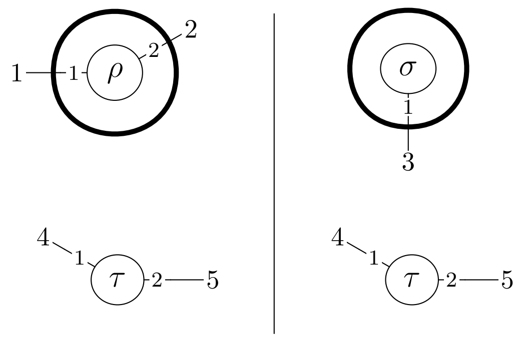

Figure 15. Representation of the product shown in Fig. 14.

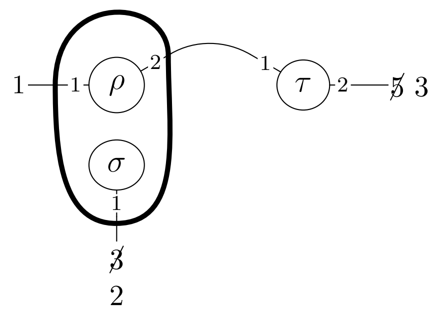

Figure 16. Join of the two places 2 and 4 of the graph from Fig. 14.

$i \in \bigcup R$ denotes that i appears in some graph in the set R. Now let $\mathscr{R}$ be the representation of the graph G with arity n for which we want to construct the representation of $\delta^{i,j}(G)$. For some set $R \in \mathscr{R}$ with $i, j \notin \bigcup R$ we define $\delta^{i,j}(R) := R$. If one of the places appears in multiple graphs of R we have first to combine them, connecting the edges belonging to the same place with teridentity graphs. Formally, we define the *meet* of two graphs T and S of the representation; to simplify the notation in the following algorithm we assume in the products the places of T numbered from 1 to n and the places of S numbered from $n+1$ to $2n$ (even if possibly not all places actually appear) and the places of the teridentity in the second product numbered from $2n+1$ to $2n+3$. Deviating from the usual operation we assume the places *not* renumbered after the join, that is we identify the places by the numbers associated after building the product, not changing the numbering.

$X \leftarrow (T \times S)$
for $p \in \{1, \dots n\}$ **where** place p appears in T and S
  do $X \leftarrow \delta^{p,2n+1}(X \times \mathsf{id}_3)$
    $X \leftarrow \delta^{n+p,2n+2}(X \times \mathsf{id}_3)$
    $X \leftarrow \alpha_{(p,2n+3)}(X)$ ▷ Renaming the remaining place of the teridentity to the original place p.
for $p \in \{1, \dots n\}$ **where** place $p+n$ appears in X
  do $X \leftarrow \alpha_{(p,n+p)}(X)$ ▷ Reshifting places originally from S
return $T \cap S := X$

Note that the result of this operation is again of arity n despite the intermediate doubling of arity. It is easy to verify that the interpretation of $T \cap S$ is indeed the intersection of the interpretations of T and S respectively. Using this derived operations we can denote the combination of all graphs where the place i appears by $T_i := \bigcap\{T \in R \mid i \in T\}$. Analogously, we create $T_j := \bigcap\{T \in R \mid j \in T\}$.

To represent the result of the join operation we have to accomodate the changed numbering of the places. Let $\alpha_{\not i}$ denote the renumbering induced by the permutation $(n, n-1, \dots, i)$, and $\alpha_{\not j}$ correspondingly induced by $(n, n-1, \dots, j)$. The combination of these two operations $\alpha_{\not j} \circ \alpha_{\not i}$ reduces the number of the places between i and j by one and the places greater than j by two; the places i and j are pushed to the end.

We will define the transformation of the graphs of a set R in the representation depending on the appearance of i and j in $\bigcup R$. If neither appears then we have only to accomodate the changed numbering of the places. We define $\delta^{i,j}(R) := \{\alpha_{\not j} \circ \alpha_{\not i}(T) \mid T \in R\}$.

If $i \in \bigcup R$ but $j \notin \bigcup R$ we have two cases; if $i \notin T$ we renumber the places as before; otherwise we have to assure that there is some $a \in A$ such that some tuple with a at place i belongs to the interpretation of R. This is done by joining place i of the graph T_i with the unary relation $A = \delta^{2,3}(\mathsf{id}_3)$. This removes place i and we only have to accomodate the shift for removing the place j. We get in this case $\delta^{i,j}(R) := \{\alpha_{\not j} \circ \alpha_{\not i}(T) \mid T \in R\ ,\ i \notin T\} \cup \{\alpha_{\not j}(\delta^{i,n+1}(T_i \times \delta^{2,3}(\mathsf{id}_3)))\}$. Analogously if $i \notin \bigcup R$ and $j \in \bigcup R$ we define $\delta^{i,j}(R) := \{\alpha_{\not j} \circ \alpha_{\not i}(T) \mid T \in R\ ,\ j \notin T\} \cup \{\alpha_{\not i}(\delta^{j,n+1}(T_j \times \delta^{2,3}(\mathsf{id}_3)))\}$.

In the fourth case, $i, j \in \bigcup R$, we join T_i and $Rgraph[T]_j$ with each other, that is $\delta^{i,j}(R) := \{\alpha_{\not j} \circ \alpha_{\not i}(T) \mid T \in R\ ,\ i, j \notin T\} \cup \{\delta^{i,j}(T_i \cap T_j)\}$.

In each of the four cases it is easy to see that

$$(x_1,\ldots,x_{n-2}) \in \bigcap\{T^A \mid T \in \delta^{i,j}(R)\}$$
$$\iff$$
$$\exists a \in A : (x_1,\ldots,x_{i-1},a,x_i,x_{i+1},\ldots,x_j,a,x_{j+1},\ldots,x_{n-2}) \in \bigcap\{T^A \mid T \in R\}.$$

Consequently, if $\mathscr{R}$ is a representation of G then $\{\delta^{i,j}(R) \mid R \in \mathscr{R}\}$ is a representation of $\delta^{i,j}(G)$, its arity being $n-2$.

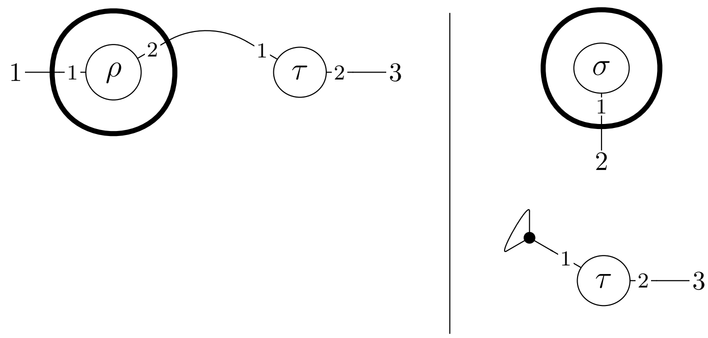

Figure 17. Representation of the graph in Fig. 16.

In our simple example only two of the four possible cases occur. In the first set (left column) both places appear. Because there is only one graph per place containing this place, the graph T_i and T_j correspond to the upper and lower graph in the first column in Fig. 16 which are then joined to provide the single graph in the first set of the new representation. In the right column we see the third case, $j = 4$ appears in the lower graph of the second column in Fig. 16, but no place $i = 2$. Therefore the places of the upper graph are simply renumbered, the place of the lower graph is connected to the unary relation A and the remaining place renumbered. The result can be seen in Fig. 17.

After this most complicated operation the renumbering of places itself is trivial. If some renumbering α is applied to G then we apply the same operation to each graph of the representation, that is if $\mathscr{R}$ is a representaiton of G then $\{\{\alpha(T) \mid T \in R\} R \in \mathscr{R}\}$ is a representation of $\alpha(G)$.

In our example we renumber the places such that the place attached to the σ-vertex is again numbered 3. The graph is shown in Fig. 18 and the corresponding representation in Fig. 19.

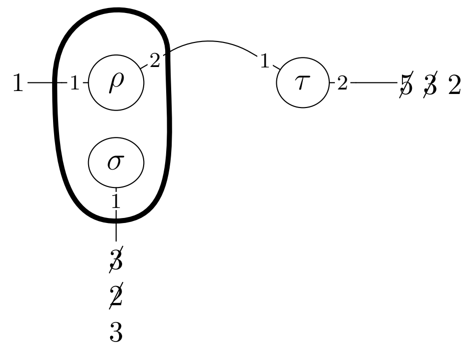

Figure 18. Renumbering the places of the graph from Fig. 16.

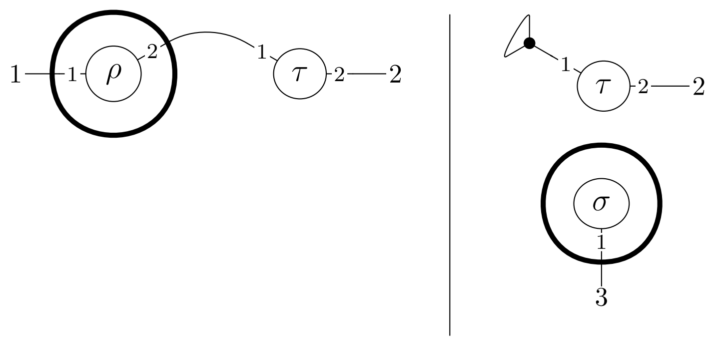

Figure 19. Representation of the graph in Fig. 18.

The last step for our example will be a second complement. As for the second time the product operation was performed, this case is slightly more complicated. Instead of one single possibility to choose from the sets of the representation there are now two (which equals the product of the cardinalities of the sets). As there is only one graph in the first set (left column) of the representation in Fig. 19, this graph appears in all columns in the representation shown in Fig. 21, where the final result can be seen.

This example also shows that it is necessary to investigate the possibility that a place appears multiple times in a set of the representation. Place 2 belongs to both graphs in the first set of the last representation, but we cannot eliminate one of these graphs. We could eliminate one if we knew that the interpretation of one were a subset of the interpretation of the other, but this will in general not be the case.

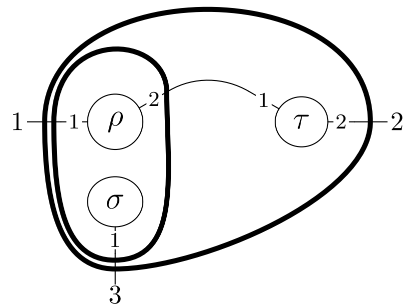

Figure 20. Complement of the graph from Fig. 18.

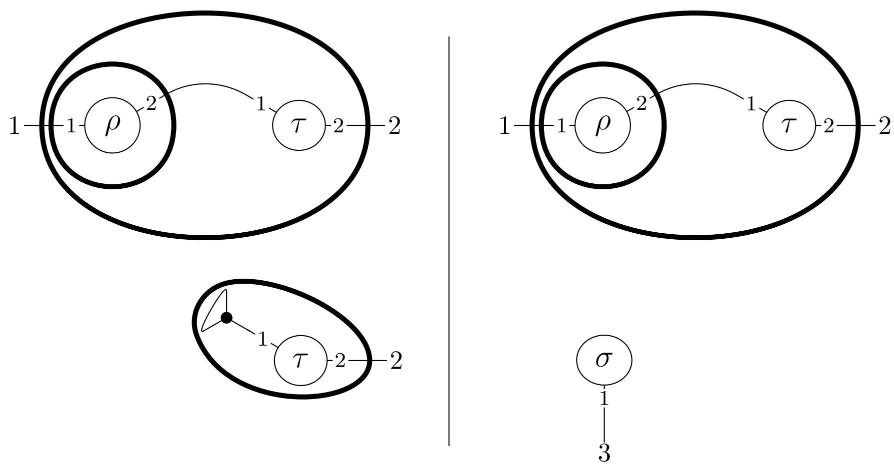

Figure 21. Representation of the graph in Fig. 20. Note that place 2 appears twice in the left column.

5. The Crux Lemma

In the last section we have shown how we can represent the interpretation of a graph by a kind of normal form, similar to the disjunctive normal form from propositional logic. This is an important step for the proof of the reduction thesis but not yet sufficient. First of all, the construction of the representation depends on the teridentity. This ternary relation is only used in constructing the representation of a graph after the join operation but by using it we cannot find a representation using unary and binary relations only (even if all atomic relations are unary or binary), which we need to prove the reduction thesis. Secondly, even if we can avoid the teridentity and have only unary and binary relations in the representation, this still does not mean that we can prove the reduction thesis: if A is finite than we can represent teridentity as the union of its tuples and each tuple can be represented as the intersection of three unary singleton relations; if we denote by $a_{(i)}$ for some element $a \in A$ the graph with a single vertex labeled by the unary relation $\{a\}$ and attached to the place i (with $i \in \{1,2,3\}$), then the teridentity on A can be represented by $\{\{a_{(1)}, a_{(2)}, a_{(3)}\} \mid a \in A\}$. No ternary relation needed. This means we have also to prove that such a representation cannot result from applying the constructions from the previous section.

The tool to solve these two issues is the *core* of binary relations. It turns out that we cannot prove that the relations attached to a certain place are not necessarily comparable to each other, but that if we use only unary and binary relations for the construction then at most one binary graph can appear attached to this place in the representation of the graph and that the core (which we will define shortly) of the interpretation of this binary graph is comparable to the interpretation of any unary graph appearing in the representation attached to the same place. Together with the property shown in the *Crux Lemma* we can then show that the interpretations of graphs constructed from unary and binary relations can never be the teridentity on A.

The core of a binary relation ρ is defined by $\mathsf{Cor}(\rho) := \{c \in A \mid \forall a,b \in A : (a,b) \in \rho \implies (a,c) \in \rho\}$. This can of course alo be represented by a graph, as shown on the left in Fig. 22. On the right of the same figure you see the graph for $\neg\mathsf{Cor}(\neg\rho)$, the complement of the core of the complement of ρ. To define when two relations are *comparable*, these two sets will be needed.

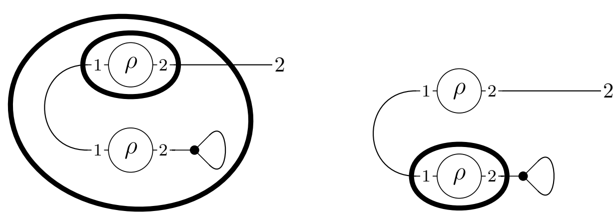

Figure 22. The core of the binary relation ρ represented as a graph on the left ($\mathsf{Cor}(\rho)$) and the complement of the core of the complement of ρ on the right ($\neg\mathsf{Cor}(\neg\rho)$)..

The notion of comparability is only defined for unary and binary relations. Two unary relations are called comparable, if the one is contained in the other (or vice-versa, obviously). Two binary relations are comparable if they are equal (that is, two different binary relations are not comparable even if one is (strictly) contained in the second). The core of a binary relation will be considered for comparing unary and binary relations. Let ρ be a binary relation and σ a unary relation. Than these two relations are called comparable if σ is contained in the core of ρ or the *complement* of the core of the *complement* of ρ is contained in σ. Investigating the last definition of comparability we see that the unary and the binary relation are compared with respect to the second place. If the comparision should be based on the first place we simply replace ρ by ρ^{-1} in the above conditions. If the unary relation σ and the binary relation ρ^{-1} are comparable we say that ρ and σ are *inverted comparable*.

The crucial instrument for the proof of Peirce's reduction thesis is the *Crux Lemma*, depicted in Fig. 23 where two configurations are shown. In the upper part we see that the relations ρ_1, ρ_2, σ_1 and σ_2 are connected, in the lower part every possible pair of these four relations is connected. It is important to note that we need a more than binary relation (a 4-identity) to represent the upper configuration but only the unary and binary relations in the lower part. The only connection between the four parts of the lower configuration is by the places. For the interpretation in the sense of a representation the same value has to be used for each place. Also it is easy to see that the upper configuration implies the lower one. When there is an element to be put at the center of the cross configuration we can use this element at the connection of each of the four parts of the lower configuration. This holds universally for any relations ρ_1, ρ_2, σ_1 and σ_2.

The converse requires that the relations stand in a certain connection – namely comparability. If ρ_1 and ρ_2 are comparable and σ_1 and σ_2 are inverted comparable, then we can deduce from the lower configuration the upper one.

The proof of this lemma has been presented in Hereth Correia and Pöschel (2006) in the usual mathematical notation. In this paper we will present the proof using the notation of existential graphs and the calculus of β-graphs.[3] We have to consider several cases because comparability has two directions resulting in four cases (two for the ρ- and σ- relations each) some of which have to be divided into subcases.

Let us look at the first case, shown in Fig. 24. There are some parts drawn with straight lines, directly derived from the premise (the lower configuration); the two graphs at the bottom with ρ_2 and σ_2 are derived from the upper two or lower two graphs in the left column of the lower configuration in Fig. 23, the part with ρ_1 and σ_1 is a copy of the corresponding graph from the

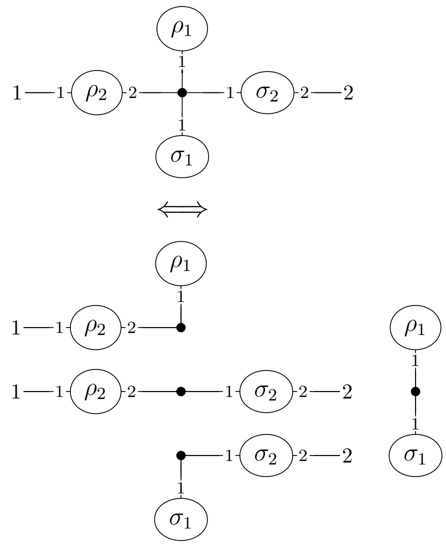

Figure 23. The Crux Lemma for comparable relations ρ_1 and ρ_2 and inverted comparable relations σ_1 and σ_2.

premise (at the right side) and the two graphs enclosed in cuts are the case we consider: that ρ_1 is a subset of the core of ρ_2 and σ_1 a subset of the core of σ_2^{-1} (compare with Fig. 22). The rules of the calculus allow us to connect graphs on the sheet of assertion with graphs enclosed by a cut (the wavy lines), then we can deiterate copies of graphs enclosed in the cuts (original

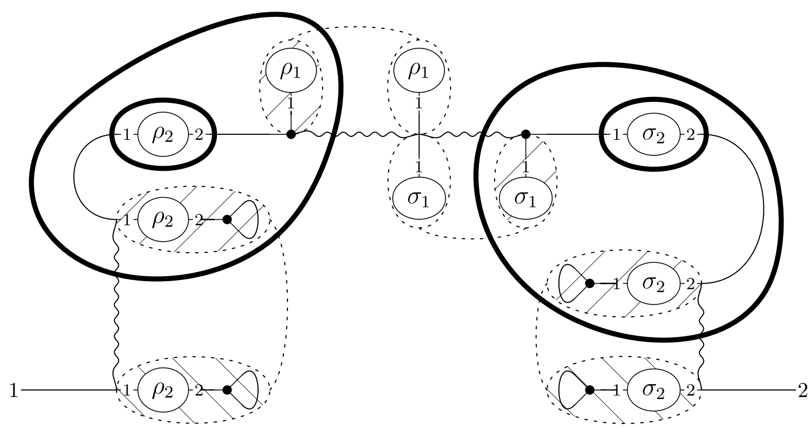

Figure 24. Proof of the Crux Lemma for the case $\rho_1 \subseteq \mathsf{Cor}(\rho_2)$ and $\sigma_1 \subseteq \mathsf{Cor}(\sigma_2)$.

and copy are surrounded and connected by a dashed line, the erased copy is marked by diagonal lines in noth-east-direction) and finally the deletion rule allows to remove some parts on the sheet of assertion (this time marked by diagonal lines in south-east-direction). Looking at the remaining parts and ignoring the double cuts we see that we arrived at the upper configuration, hence the proof for this case is complete.

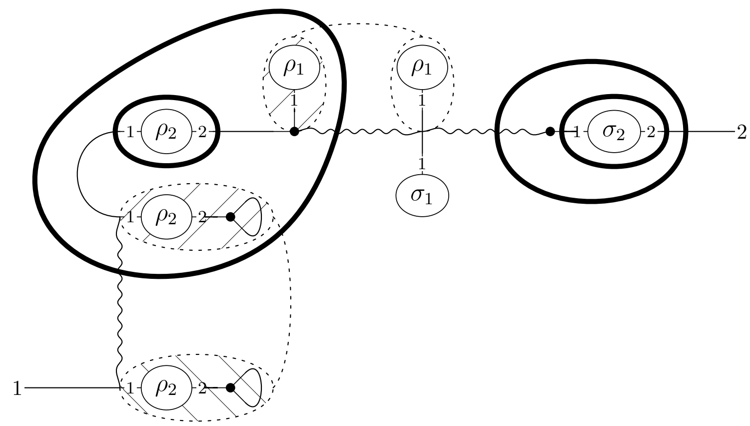

Figure 25. Proof of the Crux Lemma for the case $\rho_1 \subseteq \mathsf{Cor}(\rho_2)$ and $\forall x \in A : (x,b) \in \sigma_2$, where b is the value assigned to place 2.

The second case to consider is that ρ_1 is again a subset of the core of ρ_2 and the complement of the core of $\neg\sigma_2$ is a subset of σ_1. This case has to be divided into two subcases. The first subcase is that for any $x \in A$ the pair (x,b) is an element of σ_2 where b is the value assigned to the second place (note that by the definition of the representation the same value has to be assigned to the second place in both graphs of the lower configuration in Fig. 23 where place 2 appears and correspondingly in the upper configuration). This situation is shown in Fig. 25. We see that the core-condition for the σ-relations does not appear. Adding the wavy line from the ρ_1-σ_1-graph to the part representing the subcondition mentioned above we get on the right side (ignoring the double cut) the same graph as in the upper configuration. The left part is identical to the previous case and we can hence apply the same argumentation there. Again we arrive after applying the rules of the calculus at the upper configuration and have proven the first subcase of the second case.

For the second part of the second case (see Fig. 26) we have on the left side again the same situation as for the first case. Instead of the ρ_1-σ_1-graph from the lemma's premise we now use the ρ_1-σ_2-graph and need both the case's premise that the complement of the core of the complement of σ_2 is a subset of σ_1 (denoted by the large oval at the bottom of the figure) and the negation of the previous subcondition, that is now there is some element x such that (x,b) is *not* an element of σ_2 (the part marked by diagonal lines in south-east-direction at the right side of the figure). Given this initial situation we can again apply the rules of the calculus and arrive at the upper configuration. Now both subcases of the second case are finished.

The third case, that the complement of the core of $\neg\rho_2$ is a subset of ρ_1 and σ_1 is a subset of the core of σ_2 is simply the inversion of the second case and can be handled completely analogously.

The fourth and last case is that ρ_1 contains the complement of the core of $\neg\rho_2$ and σ_1 contains the complement of the core of $\neg\sigma_2^{-1}$. On the one hand this is the most complicated case because we have to consider four subcases, on the other hand these four proofs are just recombinations of proof parts of the previous cases. Figures 27–29 show these proofs (as for the third case the third subcase here is omitted because it is analogous to the second subcase in Fig. 28).

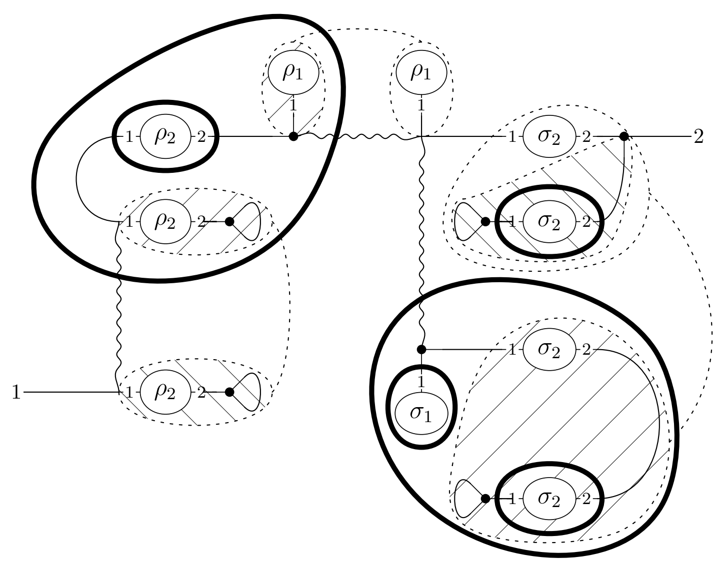

Figure 26. Proof of the Crux Lemma for the case $\rho_1 \subseteq \mathsf{Cor}(\rho_2)$, $\neg\mathsf{Cor}(\neg\sigma_2^{-1}) \subseteq \sigma_1$ and $\exists x \in A : (x,b) \notin \sigma_2$ (where b is the value assigned to place 2).

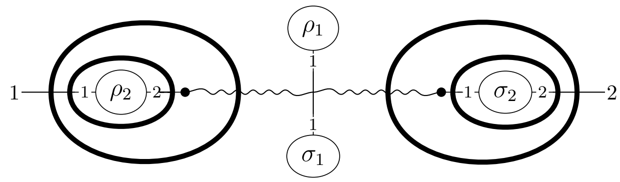

Figure 27. Proof of the Crux Lemma for the case that $\forall x \in A : (a,x) \in \rho_2$, $(x,b) \in \sigma_2$ where a is the value assigned to place 1 and b the value assigned to place 2.

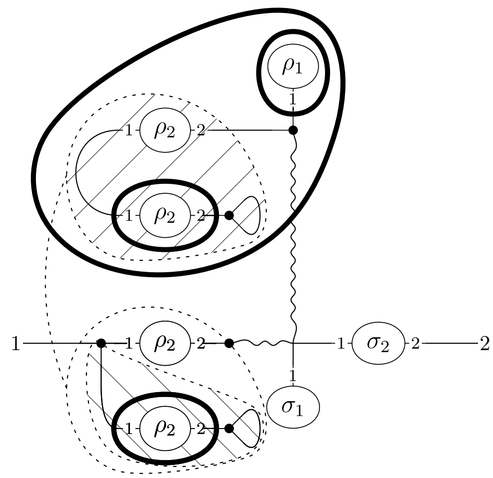

Figure 28. Proof of the Crux Lemma for the case $\neg\mathsf{Cor}(\neg\rho_2^{-1}) \subseteq \rho_1$ and$\exists x \in A$: $(a,x) \notin \rho_2$ and $\forall x \in A : (x,b) \in \sigma_2$ where a is the value assigned to place 1 and b is the value assigned to place 2.

6. Proof of Peirce's Reduction Thesis

In the previous section we have investigated the Crux Lemma which gives us (under special circumstances) an equivalence between a graph including a more than binary relation and a set of graphs constructed only from unary and binary relations. In this section we will use this lemma to show that any graph constructed from a set Q of unary and binary relations not using teridentity, that is any graph from $\langle Q \rangle_{\mathsf{PAL}^-}$, has a representation consisting of sets of unary and binary graphs from $\langle Q \cup \{A\} \rangle_{\mathsf{PAL}^-}$. These representations then allow us to show that the teridentity cannot be generated using unary and binary relations alone, thus proving Peirce's Reduction Thesis.

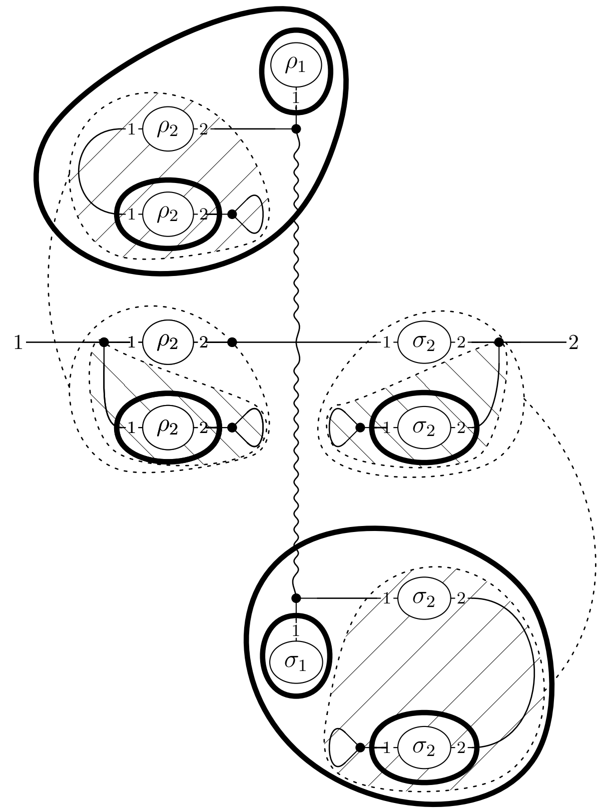

Figure 29. Proof of the Crux Lemma for the case $\neg\mathsf{Cor}(\neg\rho_2) \subseteq \rho_1$, $\neg\mathsf{Cor}(\neg\sigma_2^{-1}) \subseteq \sigma_1$ and $\exists x, y \in A : (a,x) \notin \rho_2, (x,b) \notin \sigma_2$ where a is the value assigned to place 1 and b the value assigned to place 2.

We have to make the claim about the representations more precise: Let Q be a set of unary and binary relations. Then any graph in $\langle Q \rangle_{\mathsf{PAL}^-}$ has a representation consisting of finitely many finite sets such that any graph of the representation is a unary or binary graph from $\langle Q \cup \{A\} \rangle_{\mathsf{PAL}^-}$; the interpretations of any two graphs attached to a common place are either comparable or inverted comparable (the latter if one of the relations is binary and the common place ist the "left" (lower) place of this relation). Places connected by a graph in the representation are also connected in the represented graph.

Let G be an atomic graph. Then the representation $\{\{G\}\}$ trivially fulfills these conditions. Now let G be an arbitrary graph from $\langle Q \rangle_{\mathsf{PAL}^-}$ with a representation fulfilling the above conditions. The graphs appearing in the representation of $\neg G$ constructed as described in Section 4 are the complements of the graphs in the former representation. As arity, connectedness and comparability are stable under complementation these conditions are preserved. The sets of the new representations are images of choice functions on the former representation. Because images of finite domains are finite and the only finitely many choice functions for finite families of finite sets, the finiteness condition on the former representation guarantees that there is only a finite number of finite sets in the representation of $\neg G$.

The permutation of a graph only renames the places but does not change the interpretation of the graphs in the representation (except for cases when the interpretation of a binary graph becomes inverted because the former higher place becomes the new lower place after the permutation) or the connectedness of places (relative to the represented graph). Therefore two graphs attached to a common place after the permutation have been attached to a common places before. Becaue their interpretations have been comparable or inverted comparable before by our assumption, they are comparable or inverted comparable afterwards too (comparable changes to inverted comparable and vice-versa when the permutation switches higher and lower places of a binary relation).

For the join operation we first simplify the representation. Until now we have allowed any number of relations attached to the same place in any of the sets of the representation. In the Crux Lemma we have only one unary and one binary relation for each place. By our assumpition any two graphs attached to a common place have comparable interpretations. If both graphs are binary they have the same interpretation. Also, because any place can be connected to at most one other place by unary and binary relations, the second place the graphs are attached to must be the same. Hence both graphs are identical with respect to attached places and interpretation. Therefore, if multiple binary graphs attached to the same places appear in one set of the representation one has to retain only one. Similarly, the interpretations of any two unary graphs attached to the same place are comparable, that is the interpretation of one is a subset of the interpretation of the other. Because there is only a finite number of graphs attached to the same place in any set of the representation, there is one graph with the smallest interpretation. The interpretation of the representation does not change if we eliminate all but this smallest graph.

The other problem that may arise is that there are not too many but too few graphs to apply the lemma. If there is no unary (or binary) relation attached to the corresponding places for the ρ-relations, we can supplement it with the unary relation A (or the binary relation A^2 respectively). After the application of the Crux Lemma we can remove those graphs where a supplemented relation appears because the interpretation of the first graph is then a superset of the product of A and the interpretation of the right graph of the lower configuration in Fig. 23 when the binary relation is supplemented and of the interpretation of the middle graph when the unary relation is supplemented; the interpretation of the middle graph (when the binary relation is supplemented) and of the right graph (when the unary relation is supplemented) is a superset of the interpretation of the lowest graph. As for the conditions of the Crux Lemma: Because the core of A^2 and of $\neg A^2 = \emptyset^2$ is A, any relation ρ_1 (supplemented or not) is comparable to the supplemented ρ_2. Analogously we handle the case that one or two of the σ-relations may lack.

After these preliminaries we see that the representation defined in Section 4 for the join-operation can be replaced by one fulfilling the desired condition. The finiteness condition is trivial because neither the number of sets in the representation nor the number of graphs in one of these sets can increase by applying the transformations.

Finally, let H be a second graph with a representation fulfilling the condition. Then the representation of the prodcut $G \times H$ defined in Section 4 will fulfill the condition. Any two graphs attached to the same place will either belong to the representation of G or – up to the shift of places – to the representation of H and hence fulfill the comparability condition. Connectedness is inherited from the original representations, too. The finiteness condition follows from the definition. This finishes the proof that every graph constructed from a set Q of unary and binary relations has a representation fulfilling the condition laid out at the beginning of this section.

Now let us consider the main question of Peirce's Reduction Thesis: if we apply the PAL-operations (but not the teridentity) only on unary and binary relations, can we then construct all relations? We will answer this question showing that at least one relation cannot be constructed: the teridentity. Let us consider any ternary graph G constructed in this way such that its interpretation contains the teridentity. As shown before it has then a representation with the conditions mentioned above. Let a and b be two distinct elements of A, then the triples (a,a,a) and (b,b,b) are both in the interpretation of G and therefore there have to be sets S_1 and S_2 in the representation of G such that $(a,a,a) \in S_1^A$ and $(b,b,b) \in S_2^A$. As G is constructed from unary and binary relations, there must be (at least) one place that is not connected to any of the other places. For this place there can only be unary graphs in the representation with comparable interpretations. We assume that this place is place 1 and that the interpretation of the graph attached to place 1 in the set S_1 is a subset of the interpretation of the graph attached to place 1 in the set S_2. As the element a must be in the interpretation of this graph in S_1 and its interpretation is a subset of the other graph, the element a is also an element of the latter interpretation. As explained above we can assume that there is no other graph attached to the place 1 in S_2, hence the triple (a,b,b) is an element of S_2^A and hence of G^A. Thus we have shown that any graph constructed from unary and binary relations without teridentity that contains the teridentity in its interpretation cannot be extensionally equal to the teridentity but only to a strict superset. Therefore teridentity cannot be constructed not using teridentity itself. This was the proof of Peirce's Reduction Thesis.

Notes

1. In this paper we use the term "graph" to denote the structure represented by the shown diagrams. These diagrams can be interpreted as icons of existential graphs. For the authors they are diagrams of mathematically defined graph structures. The algebraic definitions in the sense of graph-theory can be found in (Hereth 2008). The PAL-operations can be applied to these graphs, but they exist and can be interpreted independent from these operations, while the terms considered in this section are defined by these operations. Using the algebraic definitions it is shown in (Hereth 2008) that these graphs correspond to congruence classes of PAL-terms, where each congruence class has the same interpretation and the factorization into classes is compatible with PAL-operations. This mathematical and mainly technical result allows to consider the graphs instead of the PAL-terms (this made the proof in (Hereth Correia and Pöschel 2006) more difficult to read, but the inclusion of the graphs there was not possible due to space constraints). In this paper, we use the diagramatic representations of the graphs, which correspond intuitively and uniquely (up to some trivial isomorphisms) to the algebraic defined ones.
2. In Peirce's work there are fragments of a theory of infinite existential graphs. These have not been considered by Burch (1991) or in our algebraization. There is no common mathematical understanding of infinitary logic which we needed to model such graphs. Even with the tool of infinite existential graphs one could only construct relations with a well-ordering using the method for finite relations. For instance, a relation with the cardinality of the real numbers $\mathbb{R}$ could not be defined this way, if we do not assume the axiom of choice. This axiom allows the well-ordering of arbitrary sets, but using it makes any proof non-constructive.

   Admittedly, the construction presented in this paper recurs on a bijection between the underlying set A and its cartesian product $A \times A$. To prove that such a bijection exists for any infinite set A one uses the axiom of choice too. However, there are known bijections between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ and as well between $\mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$. Consequently, our construction can be applied to any domain which consists of the finite union of finite subdomains, enumerable subdomains or subdomains with the same cardinality as $\mathbb{R}$. For practical purposes this seems sufficient and therefore preferable to the version sketeched above.

3. For our proof we will need the rules of insertion and iteration to connect lines from the sheet of assertion to lines in negative contexts (cuts), the rules of deiteration and deletion to remove superfluous parts of the graph and the rule of double cut to remove (or simply ignore) remaining double cuts with no elements between the two cuts. We simply follow the usual rules given by Peirce and do not come near the problematic cases that Dau describes in (Dau 2008). Dau's work can however be adapted as a formal and precise calculus for the graphs presented in our paper.

References

Burch, Robert W. (1991). *A Peircean Reduction Thesis: the foundations of topological logic*. Lubbock: Texas Tech University Press.

Dau, Frithjof (2008). *Mathematical Logic with Diagrams - Based on the Existential Graphs of Peirce*. Habilitation thesis in preparation. URL `http://dr-dau.net/Papers/mathematical_logic_with_diagrams.pdf`

Hereth, Joachim (2008). *Relation Graphs and Contextual Logic: Towards Mathematical Foundations of Concept-Oriented Databases*. Ph.D. thesis, Technische Universität Dresden.

Hereth Correia, Joachim and Pöschel, Reinhard (2006). *The Teridentity and Peircean Algebraic Logic*. In Henrik Schärfe, Pascal Hitzler and Peter Øhrstrøm (eds.), *Conceptual Structures*, vol. 4068 of *LNAI*. Aalborg, Denmark: Springer, Berlin – Heidelberg – New York, 230–247. Proceedings of the 14th conference on Conceptual Structures.

Peirce, Charles Sanders (1870). *Description of a Notation for the Logic of Relatives, Resulting from an Amplification of the Conceptions of Boole's Calculus of Logic*. In *Memoirs of the American Academy of Arts and Sciences* 9:317–378. Reprinted in Peirce (1931–1935), 3.45–149.

— (1931–1935). *Collected Papers*. Cambridge, Massachusetts: Harvard University Press.

Pöschel, R. and Kalužnin, L.A. (1979). *Funktionen- und Relationenalgebren*. Berlin: Deutscher Verlag der Wissenschaften. Birkhäuser Verlag Basel, Math. Reihe Bd. 67, 1979.

Zeman, Jay (1995). *Existential graphs and thirdness.* In *Semiotica* 105:311–319.

Joachim Hereth (joachim.hereth@quda.de) was scientific assistant at the Institut für Algebra of the Technische Universität Dresden and wrote his PhD thesis on connections between databases, relational algebras and contextual logic, a research topic started by Rudolf Wille at Darmstadt University of Technology in the 1980s. Existential graphs (or the more formal relation graphs) are intended to be a human-oriented language for concept-oriented information systems. Today, he is SAP consultant for DMC Group GmbH in Munich.

Reinhard Pöschel is Professor at the Institut für Algebra of the Technische Universität Dresden (Germany). Among his main research interests is the duality between functional systems (clones of operations) and relational systems (Krasner algebras), as published in Pöschel and Kalužnin (1979). In particular, he is interested in the interconnections between Peircean Algebraic Logic and corresponding versions of Krasner algebras.