The Power of Peircean Algebraic Logic (PAL)

Joachim Hereth Correia[1] and Reinhard Pöschel[2]

[1] Darmstadt University of Technology
Department of Mathematics, Schlossgartenstr. 7
D-64289 Darmstadt, Germany
hereth@mathematik.tu-darmstadt.de
[2] Dresden University of Technology
Department of Mathematics, Institute for Algebra
D-01062 Dresden, Germany
poeschel@math.tu-dresden.de

Abstract. The existential graphs devised by Charles S. Peirce can be understood as an approach to represent and to work with relational structures long before the manifestation of relational algebras as known today in modern mathematics. Robert Burch proposed in [Bur91] an algebraization of the existential graphs and called it the *Peircean Algebraic Logic* (PAL). In this paper, we show that the expressive power of PAL is equivalent to the expressive power of Krasner-algebras (which extend relational algebras). Therefore, from the mathematical point of view these graphs can be considered as a two-dimensional representation language for first-order formulas. Furthermore, we investigate the special properties of the teridentity in this framework and Peirce's thesis, that to build all relations out of smaller ones we need at least a relation of arity three (for instance, the teridentity itself).

Introduction

For C. S. Peirce, the main puropse of logic as a mathematical discipline was to analyze and display the fundamental constituents of reasoning in an easily understandable fashion. For this, he preferred algebraic logic to the quantificational logic whose developement started at his time (and has even been partly influenced by him). Today, mathematical logic is often understood as quantificational logic. Peirce however developed a graphical language to express logic, the *existential graphs* which in his sense are better suited for the task of logic than the formulas of quantificational logic. In this paper, we elaborate that existential graphs can be considered as two-dimensional representations of logical statements.

In [Bur91] Robert Burch proposes a formulation of Peirce's existential graphs adapted to modern algebra. He calls this formalization Peircean Algebraic Logic (PAL) and proposes a graphical representation of the terms of PAL which resemble Peirce's existential graphs. His work also inspired the development of the *Contextual Logic of Relations* (see [Wil00], [Arn01], [Pol02]). For this paper, we consider a modified version of PAL. We adopt Burch's analysis of the fundamental operations of PAL, but while Burch allows juxtaposition (which corresponds to the cross product) only at the end of the process of term construction, we allow it at arbitrary places. Due to this modification, the proofs Burch is providing for Peirce's thesis do not hold anymore. Also, we will adapt a more traditional (in the mathematical sense) approach using only the concept of the "evaluation" (or "interpretation") of a term instead of the concepts "depiction", "representation", "expression" and "denotation" as used by Burch. We thereby concentrate on the mathematical properties of PAL, deliberately ignoring the philosophical argumentation by Burch.

Organization of this Paper

In the following section we will briefly introduce some basic definitions and results from relational algebra. In Section 2 we present an analoguous summary of the Peircean Algebraic Logic and its graphical representation as well as the main theorem on the equivalence of expressive power. Section 3 treats the special properties of the teridentity and the necessity of ternary relations for the construction of all relations (Peirce's thesis).

Then we consider an alternative system for the generation of PAL and conclude the paper with some remarks on the influence of PAL on the Contextual Logic of Relations and further work in this area.

1 Relational Algebras

In this section, we briefly recall the basic definitions for relational and Krasner algebras and their equivalence to systems of first-order logic formulas.

Notations 1 (Operations and Relations).
We introduce for operations $f : A^n \to A$ and relations $\varrho \subseteq A^m$ on a fixed base set A with $m, n \in \mathbb{N}_0$ and $m \geq 0$, $n \geq 1$ the following notation:

$$\begin{aligned}
\mathrm{Op}^{(n)}(A) &:= \{f \mid f : A^n \to A\} && (n\text{-ary operations})\\
\mathrm{Op}(A) &:= \bigcup_{n=1}^{\infty} \mathrm{Op}^{(n)}(A) && (\text{finitary operations})\\
\mathrm{Sym}(A) &:= \{f \in \mathrm{Op}^{(1)}(A) \mid f \text{ bijectiv}\} && (\text{permutations})\\
\mathrm{Rel}^{(m)}(A) &:= \{\varrho \mid \varrho \subseteq A^m\} && (m\text{-ary relations})\\
\mathrm{Rel}(A) &:= \bigcup_{m=0}^{\infty} \mathrm{Rel}^{(m)}(A) && (\text{finitary relations})\\
\mathfrak{P}(A) &:= \mathrm{Rel}^{(1)}(A) && (\text{power set})
\end{aligned}$$

The arity of an operation f or relation ϱ will be denoted by $\mathrm{ar}(f)$ or $\mathrm{ar}(\varrho)$, resp. In this article, 0-ary functions are not considered mainly for technical reasons (which appear in connection with clones of operations but play no role in this paper). Note that the elements of $\mathrm{Rel}^{(0)}(A)$ (0-ary relations) are the subsets of $A^0 = \{\emptyset\}$, i. e. the two relations $\emptyset$ and $\{\emptyset\}$.

An m-tuple $r \in A^m$ sometimes will be regarded as a mapping $r : \underline{m} \to A$ with $\underline{m} := \{1, \ldots, m\}$, and its components are given by $r = (r(1), \ldots, r(m))$.

Definitions 2 (Relational algebras and clones).
We consider the following (set-theoretical) operations on relations (ϱ and σ denote arbitrary m-ary and s-ary relations on A, resp.):

(R1) *Diagonal relation* Δ_A (nullary operation: to contain the equality relation $\Delta_A := \{(a, a) \mid a \in A\}$),

(R2a) *Cyclic shift of coordinates* ζ:

$$\zeta\varrho := \{(a_1, \ldots, a_m) \mid (a_2, \ldots, a_m, a_1) \in \varrho\},$$

(R2b) *Transposition of first two coordinates* τ:

$$\tau\varrho := \{(a_1, \ldots, a_m) \mid (a_2, a_1, a_3, \ldots, a_m) \in \varrho\},$$

(R3) *Identification of the first two coordinates* Δ:

$$\Delta\varrho := \{(a_1, \ldots, a_{m-1}) \mid (a_1, a_1, a_2, \ldots, a_{m-1}) \in \varrho\},$$

(R4) *Relational product* $\circ$:

$$\varrho \circ \sigma := \{(a_1, \ldots, a_{m+s-2}) \mid \exists\, b \in A : (a_1, \ldots, a_{m-1}, b) \in \varrho \text{ and } (b, a_m, \ldots, a_{m+s-2}) \in \sigma\},$$

(R5) *Adding a fictitious last component* ∇:

$$\nabla\varrho := \{(a_1, \ldots, a_m, b) \in A^{m+1} \mid (a_1, \ldots, a_m) \in \varrho \text{ and } b \in A\}.$$

(R6) *Union* (of relations of the same arity $m = s$): $\varrho \cup \sigma$,

(R7) *Complementation*: $\neg\varrho := A^m \setminus \varrho$

A set $Q \subseteq \mathrm{Rel}(A)$ of relations is called *relational algebra*, *weak Krasner algebra* or *Krasner algebra*, respectively, if Q is closed with respect to (R1)–(R5), (R1)–(R6) or (R1)–(R7), respectively (these definitions were used e. g. in [PösK79, 1.1.8]). The corresponding closures will be denoted by $\langle Q\rangle_{\mathsf{RA}}$, $\langle Q\rangle_{\mathsf{WKA}}$ and $\langle Q\rangle_{\mathsf{KA}}$, resp. For finite sets A these algebras are also called *clones* (relational clone, (weak) Krasner clone).

Please note that relational algebras contain relations of arbitrary (finite) arity and are much more general than Tarski's relation algebras as introduced in [Tar41].

Definitions 3 (Further operations on relations). There are many operations which can be derived from (R1)–(R7) given above (see also Def. 5 below), we mention some of them (again, ϱ and σ have arity m and s, resp.):

(R8) *Diagonal relations*: Let ε be a partition of $\{1, \ldots, m\}$. Then the m-ary relation ($m \in \mathbb{N}_+$)

$$d_\varepsilon^A := \{(a_1, \ldots, a_m) \in A^m \mid i \equiv_\varepsilon j \implies a_i = a_j\}$$

is called *diagonal* relation (where $i \equiv_\varepsilon j$ means that i and j belong to the same block of ε). Let D_A denote the set of all diagonal relations on A.
Special diagonal relations are the binary equality relation $\Delta_A = d_{\{\{1,2\}\}}$ and the ternary teridentity $\mathsf{id}_3^A = d_{\{\{1,2,3\}\}}$ (which plays a central role in PAL, see Section 2).

(R9) *Permutation of coordinates*: For a permutation α on the set $\{1, \ldots, m\}$ let

$$\pi_\alpha(\varrho) := \{(a_1, \ldots, a_m) \mid (a_{\alpha(1)}, \ldots, a_{\alpha(m)}) \in \varrho\}\,.$$

(R10) *Projection* onto a subset I of coordinates: For $I = \{i_1, \ldots, i_t\}$ with $1 \leq i_1 < i_2 \cdots < i_t \leq m$ we define

$$\mathrm{pr}_I(\varrho) := \{(a_{i_1}, \ldots, a_{i_t}) \mid \exists a_j (j \in \{1, \ldots, m\} \backslash I) : (a_1, \ldots, a_m) \in \varrho\}\,.$$

(R11) *Coupled deletion of coordinates*: For $1 \leq i < j \leq m$ let

$$\delta_{ij}(\varrho) := \{(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_m) \in A^{m-2} \mid \\ \exists\, b \in A : (a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_m) \in \varrho\}\,.$$

In case of $m = 2$ this definition is evaluated as follows:

$$\delta_{ij}(\varrho) := \begin{cases} \{\emptyset\} & \text{if } \exists\, b \in A : (b, b) \in \varrho\,, \\ \emptyset & \text{otherwise.} \end{cases}$$

(R12) *Intersection* of relations of the same arity $m = s$: $\varrho \cap \sigma$.

(R13) *Product* (cartesian or cross product):

$$\varrho \times \sigma := \{(a_1, \ldots, a_m, b_1, \ldots, b_s) \in A^{m+s} \mid (a_1, \ldots, a_m) \in \varrho, \\ (b_1, \ldots, b_s) \in \sigma\}\,.$$

Remark 4. It is straightforward to show that all these operations (R8)–(R13) can be derived from the operations Def. 2 (R1)–(R5) (for instance, a cyclic shift and a transposition generate all permutations, thus ζ and τ generate all π_α; $\varrho \times \sigma = (\nabla \varrho) \circ (\zeta \nabla \sigma)$ and so on, for more details see [PösK79, 1.1.9]). The interconnections of the operations (R1)–(R13) are discussed further in Section 4.

Definitions 5 (Logical operations and closures).

Relational algebras can be characterized also via closure with respect to first order formulas. To each first order formula $\varphi(R_1, \ldots, R_q; x_1, \ldots, x_m)$ with free variables $x_1, \ldots, x_m$ and m_i-ary relation (predicate) symbols R_i ($i \in \{1, \ldots, q\}$) one can assign an operation (called *logical operation*)

$$L_\varphi : \mathrm{Rel}^{(m_1)}(A) \times \cdots \times \mathrm{Rel}^{(m_q)}(A) \to \mathrm{Rel}^{(m)}(A) \quad \text{with}$$

$$L_\varphi(\varrho_1, \ldots, \varrho_q) := \{(a_1, \ldots, a_m) \in A^m \mid \models \varphi(\varrho_1, \ldots, \varrho_q; a_1, \ldots, a_m)\}\,.$$

Hereby, for atomic formulas e.g. $R_i(x, y)$ we interpret $\models \varrho_i(a, b)$ as $(a, b) \in \varrho_i$ (for elements $a, b \in A$). For instance, the formula

$$\varphi(R_1, R_2; x_1, x_2) :\equiv \exists z : R_1(x_1, z) \wedge R_2(z, x_2)$$

(with binary relation symbols) induces the operation relational product (see Def. 2 (R4)):

$$L_\varphi(\varrho_1, \varrho_2) = \{(a_1, a_2) \mid \exists z \in A : (a_1, z) \in \varrho_1 \wedge (z, a_2) \in \varrho_2\} = \varrho_1 \circ \varrho_2$$

for binary relations $\varrho_1, \varrho_2 \in \mathrm{Rel}^{(2)}(A)$.

Let $\Phi(\exists, \wedge, \dots)$ denote the set of all first order formulas that contain only the indicated quantifier $\exists$, the indicated connectives $\wedge, \dots$ and relation symbols and variables. Let $\mathrm{Lop}_A(\exists, \wedge, \dots) := \{L_\varphi \mid \varphi \in \Phi(\exists, \wedge, \dots)\}$ denote the corresponding logical operations.
Then we have (see [PösK79, 2.1.3]):

Theorem 6. *Let $Q \subseteq \mathrm{Rel}(A)$. Then*

$$\begin{aligned} Q \textit{ relational algebra} &\iff Q \textit{ closed w.r.t. } \mathrm{Lop}_A(\exists, \wedge, =)\,, \\ Q \textit{ weak Krasner algebra} &\iff Q \textit{ closed w.r.t. } \mathrm{Lop}_A(\exists, \wedge, \vee, =)\,, \\ Q \textit{ Krasner algebra} &\iff Q \textit{ closed w.r.t. } \mathrm{Lop}_A(\exists, \wedge, \vee, \neg, =)\,. \end{aligned}$$

2 Peircean Algebraic Logic (PAL)

The operations of the *Peircean Algebraic Logic and relation graphs* (PAL) are closely related to the existential graphs that Peirce developed in the late 1890s. Here we shall not go into details of existential graphs (which are also related to the conceptual graphs, see [Sow92]) and only describe the Peircean operations interpreted as operations on relations and show the corresponding graphs representing such operations (these operations on relations were also proposed by R. W. Burch in [Bur91] and modelled mathematically in power context families, see [Wil00], [Arn01] and [Pol02]).

Definitions 7 (Relation Graphs and Operations of PAL).
An m-ary relation $\varrho^A \in \mathrm{Rel}^{(m)}(A)$ will be represented graphically by a point (dot or small circle) with m outgoing “arms”, i. e. pending edges (called *hooks* in [Bur91]), where each pending edge has exactly one label from $\{1, \dots, m\}$, see Fig. 1 (a); the dot itself is labeled by ϱ. All these labels are called *relation labels* (they belong to the relation symbol ϱ and indicate to which “spot” (=component) of ϱ an incoming edge is connected, Fig. 1 (a’) makes this more clear; but we prefer the less complicated drawing of (a)). The order of the labels of the edges is not essential: two figures with the same labels in possibly different positions in the picture represent the same relation.

The so-called *teridentity* is the ternary relation

$$\mathsf{id}_3 := \mathsf{id}_3^A := \{(a, a, a) \mid a \in A\}$$

and will be represented as in Fig. 1 (b) (no relation label is necessary).

Let us fix some convention: the figures to be constructed shall be called *relation graphs* ([Pol02]). They have some “outgoing” pending edges, its number is called the *arity* of the relation graph. The pending edges of an m-ary relation graph are labelled by the elements $1, \dots, m$ (we shall call them *graph labels* in order to distinct them from the relation labels).

The constructions of PAL can be described on purely syntactical level by providing a set Σ (signature) of relation symbols (with arities) for labeling the points. However, if there is an *evaluation* (interpretation) of each relation symbol by a concrete relation (of the same arity) on a fixed base set A (i. e. we have an arity-preserving *evaluation function*

$\Sigma \to \mathrm{Rel}(A) : \varrho \mapsto \varrho^A$), then to each m-ary relation graph $\mathfrak{G}$ uniquely corresponds an m-ary relation $\mathfrak{G}^A$ on A.

In the following we describe constructions (allowed by PAL) for relation graphs in more detail. We shall give the construction which at the same time provides a *constructive definition of relation graphs* $\mathfrak{G}$ and the evaluation as relation (denoted by $\mathfrak{G}^A$). However we do this by representing relation graphs by figures. From the pure mathematical point of view, a relation graph is a multigraph with ordered valencies, pending edges and nested subgraphs as defined in [Pol02, Def. 2-8] (also called *structure* there), we try to avoid such technicalities and refer to [Pol02].

(PAL1) atomic relation graphs: Let $\varrho \in \Sigma$ be an m-ary relation symbol. Then the graph (also denoted by ϱ) in Fig. 2 (a) is an m-ary relation graph. It has m pending edges each of which has a relation label (drawn near the center) and a graph label (drawn at the outer end) which coincides with the relation label. Its corresponding relation (evaluation on base set A) is ϱ^A. The graph in Fig. 2 (b) is the relation graph of the teridentity. It has the graph labels $1, 2, 3$ for its pending edges. The evaluation is the relation id_3^A.

(PAL2) Permutation of pending edges: Let $\mathfrak{G}$ be an m-ary relation graph with corresponding relation $\mathfrak{G}^A$ and let $\alpha : \underline{m} \to \underline{m}$ be a permutation. Then the graph in Fig. 2 (c) obtained from $\mathfrak{G}$ by changing the label i to $\alpha(i)$ for every graph label $i \in \underline{m} := \{1, \ldots, m\}$ of a pending edge (outgoing half-edge), is an m-ary relation graph denoted by $\pi_\alpha(\mathfrak{G})$. Its evaluation is the relation

$$\pi_\alpha(\mathfrak{G})^A := \pi_\alpha(\mathfrak{G}^A) \text{ (see Def. 3 (R9))}.$$

(PAL3) Juxtaposition (cross product): Let $\mathfrak{G}_i$ be m_i-ary relation graphs ($i = 1, 2$). Then the graph in Fig. 2 (d) obtained by juxtaposition and relabeling the graph labels of the pending edges of $\mathfrak{G}_2$ (according to $i \mapsto m_1 + i$ for $i \in \{1, \ldots, m_2\}$) is an $(m_1 + m_2)$-ary relation graph denoted by $\mathfrak{G}_1 \times \mathfrak{G}_2$. The evaluation is given by

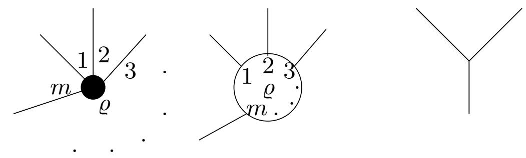
$$(\mathfrak{G}_1 \times \mathfrak{G}_2)^A := \mathfrak{G}_1^A \times \mathfrak{G}_2^A \text{ (see Def. 3 (R13))}.$$

(PAL4) Connecting pending edges: Let $\mathfrak{G}$ be an m-ary relation graph with $m \geq 2$. Let $1 \leq i < j \leq m$. Then the graph in Fig. 2 (e) obtained by connecting the pending edges with graph label i and j (deleting these graph labels and relabeling the other pending edges) is an $(m-2)$-ary relation graph denoted by $\delta_{ij}(\mathfrak{G})$. The evaluation is given by

$$(\delta_{ij}(\mathfrak{G}))^A := \delta_{ij}(\mathfrak{G}^A) \text{ (see Def. 3 (R11))}.$$

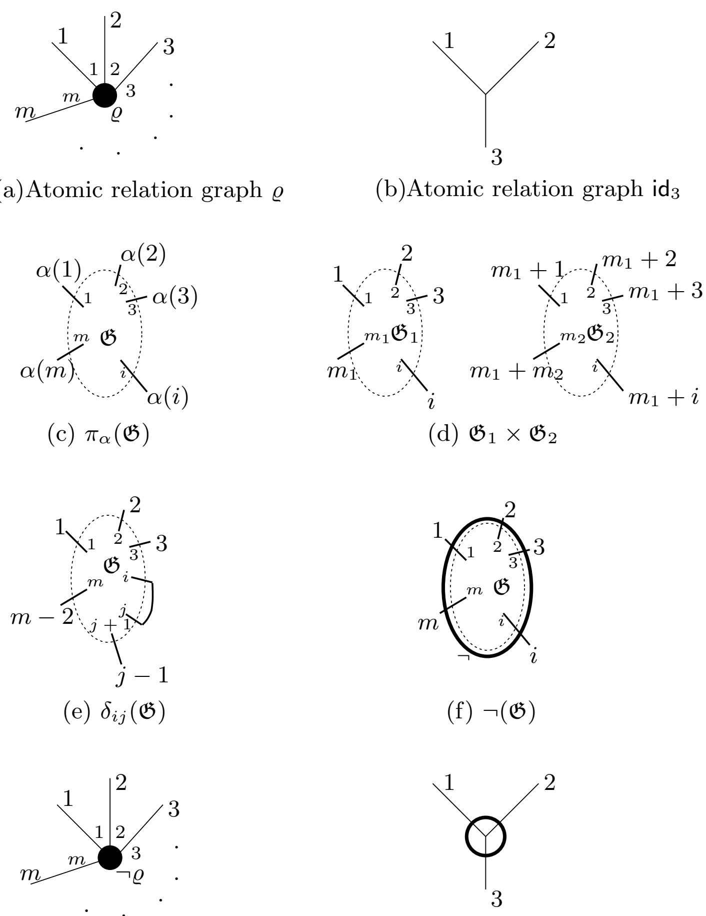
(PAL5) negation (complementation): Let $\mathfrak{G}$ be an m-ary relation graph. Then the graph in Fig. 2 (f) obtained from $\mathfrak{G}$ by drawing a simple closed curve (called also negation circle, optionally labeled by $\neg$) enclosing the whole graph and prolonging the pending edges outside the curve (while keeping the labels) is a relation graph denoted by $\neg(\mathfrak{G})$. Its evaluation is

$$(\neg(\mathfrak{G}))^A := \neg(\mathfrak{G}^A) \text{ (see Def. 2 (R7))}.$$



(a) An m-ary relation ϱ^A (a') (b)The teridentity id_3^A

Fig. 1. Graphical representation of relations in PAL



(a)Atomic relation graph ϱ (b)Atomic relation graph id_3

(c) $\pi_\alpha(\mathfrak{G})$ (d) $\mathfrak{G}_1 \times \mathfrak{G}_2$

(e) $\delta_{ij}(\mathfrak{G})$ (f) $\neg(\mathfrak{G})$

(g) local negation $\neg\varrho$ (h) $\neg\mathsf{id}_3$

Fig. 2. Construction of relation graphs
The small labels (drawn inside the dotted line) in (c)–(f) are the original graph labels of $\mathfrak{G}$ and have to be replaced by the new graph labels drawn outside the dotted line.

(PAL5') local negation: For an atomic relation graph ϱ where $\varrho \in \Sigma$, the relation graph $\neg(\varrho)$ is called locally negated atomic relation graph; it will be drawn simply also by changing the label from ϱ to $\neg\varrho$ (and not drawing the curve according to (PAL5)).
Remark: Note that local negation does not include the negation of the teridentity (which we shall represent as in Fig. 2 (h) without relation labels). This is because the negation of the teridentity allows to simulate the (unrestricted) negation, i. e., (PAL5') would be "semantically" as powerful as (PAL5) (see Proposition 11 and Remark 12).

A *relation graph* is a graph obtained from the atomic relation graphs by using (PAL1)–(PAL5) finitely often. By construction, a relation graph has graph labels for its pending edges while each other edge has a relation label on each half that is connected with an atomic relation $\varrho \in \Sigma$ (the teridentity just "splits" one edge into three "directions" and needs no relation labels).

The set of all relation graphs over a signature Σ is denoted by $\langle \Sigma \rangle_{\mathsf{PAL}}$.

The closure with respect to the rules (PAL1)–(PAL4) and (PAL5') (i. e. only local negation is allowed instead of arbitrary negation) will be denoted by $\langle \Sigma \rangle_{\mathsf{lnegPAL}}$ ("locally negated PAL"-closure). Finally, the closure with respect to (PAL1)–(PAL4) (without any negation) will be denoted by $\langle \Sigma \rangle_{\mathsf{pPAL}}$ ("positive PAL"-closure). The relation graphs in $\langle \Sigma \rangle_{\mathsf{lnegPAL}}$ and $\langle \Sigma \rangle_{\mathsf{pPAL}}$ are called *locally negated* and *positive* relation graphs, respectively.

Now the following operators on $\mathrm{Rel}(A)$ can be introduced:

Let $Q \subseteq \mathrm{Rel}(A)$ and let $\Sigma_Q := \{\underline{\varrho} \mid \varrho \in Q\}$ be the relational signature of abstract relation symbols such that $\underline{\varrho}$ has the same arity as ϱ, and the evaluation of $\underline{\varrho}$ over A is canonically given by $\underline{\varrho}^A := \varrho$. Then we define:

$$\langle Q \rangle_{\mathsf{PAL}} := \{\mathfrak{G}^A \mid \mathfrak{G} \in \langle \Sigma_Q \rangle_{\mathsf{PAL}}\}.$$

Analogously we define $\langle Q \rangle_{\mathsf{lnegPAL}}$ and $\langle Q \rangle_{\mathsf{pPAL}}$. If there is danger of confusion the base set A is added as upper index: $\langle Q \rangle^A_{\mathsf{PAL}}$.

Examples 8. Some examples of relation graphs composed by the above rules (PAL1)–(PAL5') are given in Figures 3 and 4, together with a term (in the language of relation graphs, see (PAL1)–(PAL5)) describing it.

In Fig. 3 we specify the relation graphs id_1, id_2 and ν which are all generated by the teridentity id_3 (the describing term is always given) and introduce a more convenient graphical representation for them.

The evaluation of these relations graphs is obvious:

$$\mathsf{id}_1^A = A \in \mathrm{Rel}^{(1)}(A), \quad \mathsf{id}_2^A = \Delta_A, \quad \nu^A = \{(a,b) \in A^2 \mid a \neq b\}.$$

In Fig. 4 we want to attract the attention to the generating process of relation graphs (the reader should check every step). While the graphical construction is immediate, it is easy but tedious to specify the corresponding describing terms (one carefully has to handle the labels). This demonstrates the advantage of the graphical presentation of relation graphs which is much closer to human thinking and understanding than the "linear" technical composition of terms. For instance, it im-

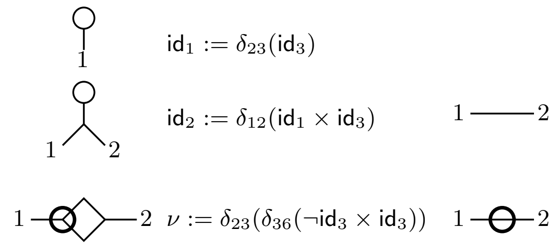


Fig. 3. Some relation graphs derivable from id_3

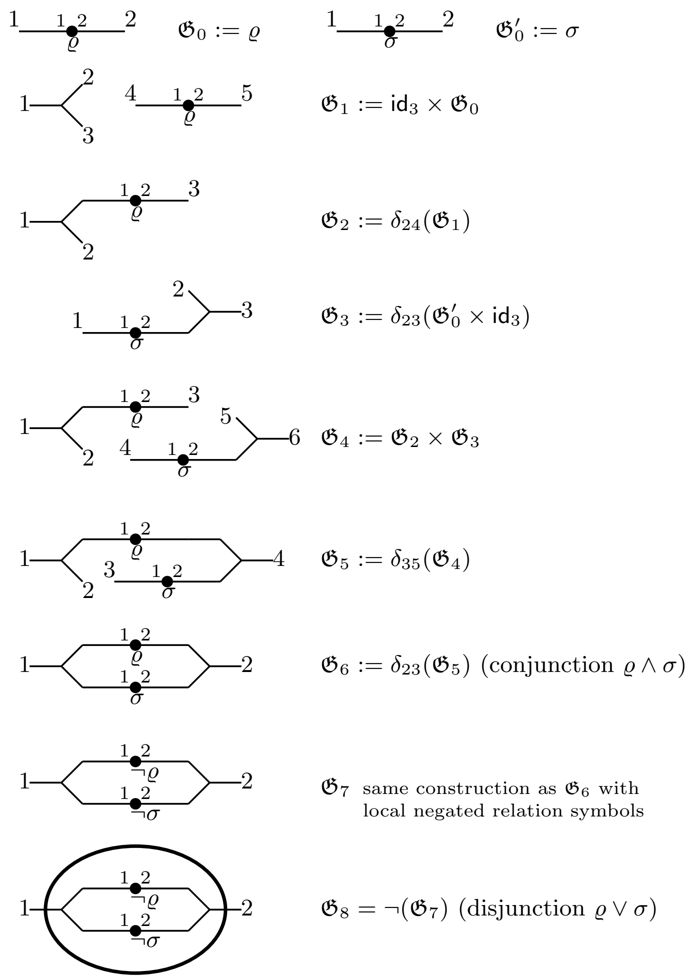


Fig. 4. Relation graphs and their description terms

mediately follows from the given drawing of the relation graphs in this figure that for a representation on any set A we have $\mathfrak{G}_6^A = \varrho^A \cap \sigma^A$ and $\mathfrak{G}_8^A = \neg(\neg\varrho^A \cap \neg\sigma^A) = \varrho^A \cup \sigma^A$. Note that there are many ways to construct other relation graphs which also represent uniformly (i. e. for all A) intersection and union of relations. e. g., one could take first the cross product $\mathsf{id}_3 \times \varrho \times \sigma \times \mathsf{id}_3$ (with 10 pending edges) and then do the necessary connections between pending edges to get $\mathfrak{G}_6$.

The following theorem shows that the operations of relational algebras and the operations of PAL are equally powerful: for a given set Q of relations one can generate the same set of derived relations. According to Def. 5 there also exists an equivalent description by first order formulas.

Theorem 9. *Let* $Q \subseteq \mathrm{Rel}(A)$ *and let* $\neg Q := \{\neg\varrho \mid \varrho \in Q\}$. *Then:*
(i) $\langle Q\rangle_{\mathsf{pPAL}} = \langle Q\rangle_{\mathsf{RA}}$,
(ii) $\langle Q\rangle_{\mathsf{lnegPAL}} = \langle Q \cup \neg Q\rangle_{\mathsf{RA}}$,
(iii) $\langle Q\rangle_{\mathsf{PAL}} = \langle Q\rangle_{\mathsf{KA}}$.

Proof. (i): From the definitions (PAL1)–(PAL5) (Def. 7) follows immediately $\langle Q\rangle_{\mathsf{pPAL}} \subseteq \langle Q\rangle_{\mathsf{RA}}$ (since $\langle Q\rangle_{\mathsf{RA}}$ is closed under (R1) (Def. 2) and (R9), (R13), (R11) (Def. 3)). Conversely, every operation from (R1)–(R5) (Def. 2) is expressible by a PAL-construction (PAL1)–(PAL5) (Def. 7): in fact, $\Delta_A = \mathsf{id}_2^A = \delta_{12}(\mathsf{id}_1 \times \mathsf{id}_3)^A$ (where $\mathsf{id}_1 := \delta_{23}(\mathsf{id}_3)$, see Fig. 3), ζ and τ are special cases of π_α, for the relational product we have $\varrho^A \circ \sigma^A = \delta_{m,m+1}(\varrho \times \sigma)^A$, and $\nabla\varrho^A = (\varrho \times \mathsf{id}_1)^A$.

(ii) is just a special case of (i) taking into account local negation.

(iii): Note that negation $\neg$ appears for Krasner algebras in (R7) (Def. 2) as well as for PAL in (PAL5) (Def. 7), and that union (R6) (Def. 2) can be expressed by negation and intersection: $\varrho\cup\sigma = \neg(\neg\varrho\cap\neg\sigma)$. Therefore (iii) immediately follows from (i). □

Remark 10. As already mentioned in Example 8 the graphical presentation of relation graphs is very convenient. Because of Theorem 9 (iii) and Theorem 6 it is as powerful as the calculus of first order logic. Thus, relations graphs can be considered as a two-dimensional representation language for first-order formulas. Note further that – in the graphical presention – one even need not worry about graph labels: a relation graph is just a collection of atomic relations (drawn in the plane like in Fig. 1) where pending edges may be connected in arbitrary way, and closed negation curves can be drawn in any appropriate way. Finally, to get a relation graph according to our definition, the remaining pending edges must be labelled by the first natural numbers $1, 2, \ldots$ (recall, the relation labels belong to the nodes and their outgoing edges, they are needed if the relation graph is to be evaluated in a relational system over A and play no role during the construction of relation graphs).

3 The Teridentity

In this section we shall deal with the special role of the teridentity id_3. From Theorem 9 (i),(iii) we get for $Q = \emptyset$ the set of all relations constructable in PAL without negation solely from the teridentity:

$$\langle\emptyset\rangle_{\mathsf{pPAL}}^A = \langle\emptyset\rangle_{\mathsf{RA}}\,, \qquad \langle\emptyset\rangle_{\mathsf{PAL}}^A = \langle\emptyset\rangle_{\mathsf{KA}}\,.$$

It is known from e. g. [PösK79, 1.1.9(R1)] that $\langle\emptyset\rangle_{\mathsf{RA}}$ equals the set D_A of all diagonal relations (see Def. 3 (R8)). Further, $\langle\emptyset\rangle_{\mathsf{KA}}$ consists of all so-called pattern relations. A *pattern relation* $\varrho \in \mathrm{Rel}^{(m)}(A)$ is a relation satisfying $\forall r, s \in A^m : r \in \varrho$ and $t(r) = t(s) \implies s \in \varrho$, where $t(r) := \{(i,j) \in \{1,\dots,m\}^2 \mid r(i) = r(j)\}$ is the *pattern* of an m-tupel in A^m.

As already mentioned the negated teridentity can simulate arbitrary negations; in fact we have:

Proposition 11. *For arbitrary finite A and arbitrary $Q \subseteq \mathrm{Rel}(A)$ we have*

$$\langle Q\rangle_{\mathsf{PAL}} = \langle Q \cup \{\neg(\mathsf{id}_3^A)\}\rangle_{\mathsf{pPAL}}.$$

Proof. It is known (e. g. [PösK79, 1.3.5]) that for $|A| \geq 3$ we have $\langle Q\rangle_{\mathsf{KA}} = \langle Q \cup \{\nu^A\}\rangle_{\mathsf{RA}}$ where $\nu^A := \{(a,b) \in A^2 \mid a \neq b\}$ is the inequality relation. As we have seen in Example 8 the inequality relation is the evaluation of the positive relation graph ν constructed with atomic relations id_3 and $\neg\mathsf{id}_3$ in Fig. 3. Thus $\langle Q\rangle_{\mathsf{PAL}} = \langle Q\rangle_{\mathsf{KA}} = \langle Q \cup \{\nu^A\}\rangle_{\mathsf{RA}} = \langle Q \cup \{\nu^A\}\rangle_{\mathsf{pPAL}} \subseteq \langle Q \cup \{\neg\mathsf{id}_3^A\}\rangle_{\mathsf{pPAL}} \subseteq \langle Q\rangle_{\mathsf{PAL}}$ and we are done.

The case $|A| = 2$ must be checked separately. We give only a sketch. It is straightforward to check that there is only one minimal clone of Boolean functions (on $A = \{0,1\}$) preserving the negated teridentity $\neg\mathsf{id}_3^A$, namely the clone $\mathrm{Sym}(2)$ generated by the negation which consists up to fictitious variables only of the identity function and negation (conjunction, disjunction, constants, any majority or minority function do not preserve $\neg\mathsf{id}_3^A$). The lattice of all Boolean clones shows that therefore $\mathrm{Sym}(2)$ is the only nontrivial clone preserving $\neg\mathsf{id}_3^A$. Using some knowlegde about the Galois closures of the Galois connections $\mathrm{Pol} - \mathrm{Inv}$ and $\mathrm{Aut} - \mathrm{Inv}$ (given by the formal context $(\mathrm{Op}(A), \mathrm{Rel}(A), \triangleright)$ where $f \triangleright \varrho :\iff f$ preserves ϱ) one can conclude $\langle Q\rangle_{\mathsf{PAL}} = \langle Q\rangle_{\mathsf{KA}} = \mathrm{Inv}\,\mathrm{Aut}\,Q = \mathrm{Inv}(\mathrm{Pol}\,Q \cap \mathrm{Sym}(2)) = \mathrm{Inv}(\mathrm{Pol}\,Q \cap \mathrm{Pol}\,\neg\mathsf{id}_3^A) = \mathrm{Inv}\,\mathrm{Pol}(Q \cup \{\neg\mathsf{id}_3^A\}) = \langle Q \cup \{\neg\mathsf{id}_3^A\}\rangle_{\mathsf{RA}} = \langle Q \cup \{\neg\mathsf{id}_3^A\}\rangle_{\mathsf{pPAL}}$. We do not go in more details here and refer to [PösK79, 1.2.3 and 1.3.5].

Remark 12. In the proof of Prop. 11 we have seen that there is a positive relation graph $\nu \in \langle\neg(\mathsf{id}_3)\rangle_{\mathsf{pPAL}}$ the evaluation of which gives the inequality relation ν^A on each set A (see Fig. 3, this even holds for infinite A). Proposition 11 implies that for every negated relation graph $\neg(\mathfrak{G}) \in \langle\Sigma\rangle_{\mathsf{PAL}}$ and given finite base set A there exists a positive relation graph $\mathfrak{G}_1 \in \langle\Sigma \cup \{\neg\mathsf{id}_3\}\rangle_{\mathsf{pPAL}}$ with the same evaluation on A: $(\neg(\mathfrak{G}))^A = \mathfrak{G}_1^A$. However, contrarily to the inequality ν, this $\mathfrak{G}_1$ usually depends on A and cannot be chosen globally for all A. Moreover, the result does not hold in general for infinite A (here modifications are necessary, see also [Pös03]).

The next results show that the teridentity is – in some sense – indispensable as atomic relation in PAL. At first we show that it cannot be (positively) generated by relations of lower arity. To make this precise let us introduce the following notation: Let $\langle\Sigma\rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$ denote the set of all relation graphs constructed by (PAL1)–(PAL5) (Def. 7) but without using the teridentity id_3 in (PAL1) (analogously $\langle\Sigma\rangle_{\mathsf{pPAL}\setminus\{\mathsf{id}_3\}}$). Thus we have $\langle\Sigma\rangle_{\mathsf{PAL}} = \langle\Sigma \cup \{\mathsf{id}_3\}\rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$ and $D_A = \langle\mathsf{id}_3^A\rangle^A_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$.

Theorem 13. *Let A be a finite set with at least two elements and let $R_2 := \mathrm{Rel}^{(1)}(A) \cup \mathrm{Rel}^{(2)}(A)$. Then*
(a) $\langle R_2\rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}} \subsetneqq \langle R_2\rangle_{\mathsf{PAL}} = \mathrm{Rel}(A)$,
(b) $\langle R_2\rangle_{\mathsf{pPAL}\setminus\{\mathsf{id}_3\}} \subsetneqq \langle R_2\rangle_{\mathsf{pPAL}} = \mathrm{Rel}(A)$.

Remarks concerning the *Proof*:
The right-hand equality of (b) implies the corresponding equality in (a) and follows from the known fact $\langle \mathrm{Rel}^{(2)}(A)\rangle_{\mathsf{RA}} = \mathrm{Rel}(A)$ (see e. g. [PösK79, 1.1.22]) using Theorem 9 (i). The inequality of (b) now follows immediately from (a).
It remains to prove the left-hand proper inclusion of (a). This can be done by showing $\mathsf{id}_3^A \notin \langle R_2\rangle^A_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$. The proof is relatively technical and we do not have enough space for the full proof, therefore it is omitted here and we refer to [HerP].

Remark 14. Since the proof of Theorem 13 (a) was not given here, we add a proof of (b) which does not use (a). For (b), it is sufficient to show $\mathsf{id}_3^A \notin \langle R_2\rangle^A_{\mathsf{pPAL}\setminus\{\mathsf{id}_3\}}$. Assume on the contrary $\mathsf{id}_3^A \in \langle R_2\rangle^A_{\mathsf{pPAL}\setminus\{\mathsf{id}_3\}}$. Then there exists a ternary positive relation graph $\mathfrak{G}$ (with three pending edges) such that every vertex has valency at most 2 and $\mathfrak{G}^A = \mathsf{id}_3^A$. We choose $\mathfrak{G}$ in such a way that it has a minimal number of vertices. But then there cannot be any non-pending edge in $\mathfrak{G}$: in fact, if an edge connects two relations ϱ_1, ϱ_2 then this can be replaced by $\sigma = \varrho_1 \circ \varrho_2$; e. g. for binary ϱ_1, ϱ_2, the graph $-\overset{1\,2}{\underset{\varrho_1}{\bullet}}-\overset{1\,2}{\underset{\varrho_2}{\bullet}}-$ would be a subgraph of $\mathfrak{G}$ and could be substituted by $-\overset{1\,2}{\underset{\sigma}{\bullet}}-$ such that the resulting graph has the same evaluation but less vertices in contradiction to vertex minimality of $\mathfrak{G}$. Consequently, $\mathfrak{G}$ has only pending edges and therefore must be of the form $\mathfrak{G}_1 := \overset{1}{\underset{\sigma_1}{\bullet}}\!-1\ \ \overset{1}{\underset{\sigma_2}{\bullet}}\!-2\ \ \overset{1}{\underset{\sigma_3}{\bullet}}\!-3$ or $\mathfrak{G}_2 := 1-\overset{1\,2}{\underset{\varrho}{\bullet}}-2\ \ \overset{1}{\underset{\sigma}{\bullet}}\!-3$ (here σ, σ_i are unary and ϱ is a binary relation symbol). But, whatever the relations $\sigma_i^A, \varrho^A, \sigma^A$ may be, neither $\mathfrak{G}_1^A = \sigma_1^A \times \sigma_2^A \times \sigma_3^A$ nor $\mathfrak{G}_2^A = \varrho^A \times \sigma^A$ equals the teridentity (provided that A has more than one element). $\square$

Remark 15. With Theorem 13 the first part of Peirce's thesis is proved, that we need (at least) ternary relations to build all relations from a set of basic relations.

The second part of the thesis is the inverse direction, that we can decompose any relation to ternary ones. According to Burch this is meant to know *which* relations (of small arity) construct a given relation. We refer for this to the theorem by Herzberger in [Her81]. This approach works if the domain A is sufficiently large, i. e. the cardinality of the domain is at least as large as the cardinality of the relation to be decomposed. This holds true for all relations in all infinite domains.

Burch proposes in [Bur91] a similar procedure for finite domains. However, to do this in general he allows to extend the underlying domain by new elements generated through the so-called *hypostatic abstraction*. While his explanations are philosophically of interest, such an extension to the underlying domain is uncommon in traditional mathematics. For this reason, we do not investigate this process in detail.

The next proposition also supports the special role of the teridentity: if a diagonal relation generates id_3 then it "contains" id_3^A (as projection), in particular, id_3^A is the only diagonal of minimal arity generating id_3 (and therefore generating all diagonal relations). The proof will be omitted here (and we refer to [HerP]).

Proposition 16. *Let A be a set with at least two elements. Then we have:*
(a) $\mathsf{id}_3^A \notin \langle \bigcup_{k=0}^{\infty} \mathrm{Rel}^{(2k)}(A)\rangle^A_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$,
(b) $\mathsf{id}_3^A \notin \langle \mathsf{id}_1^A, \mathsf{id}_2^A\rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$.
(c) *Let $\varrho^A \in D_A$ be an m-ary diagonal relation. Then $\mathsf{id}_3^A \in \langle \varrho^A\rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$ if and only if $m \geq 3$ is an odd number and there are $i, j, k \in \{1, \dots, m\}$ such that $\mathsf{id}_3^A = \mathrm{pr}_{ijk}(\varrho^A)$ (i. e. for $m{=}3$: $\varrho^A{=}\mathsf{id}_3^A$).*

4 A basic generating System for PAL

The operations $\mathsf{id}_3, \pi_\alpha, \times, \delta_{ij}, \neg$ of PAL are relatively easy to handle and intuitive. However, the operations π_α and δ_{ij} depend on the arity of the relations (and therefore must be defined for every arity m yielding formally an infinite family of operations). For theoretical investigations a finite "minimal" system of operations is desirable. This can be done as in [PösK79] or [Arn01] for relational algebras. We mention here such a basic system for the construction of relation graphs (it reflects on the syntactical level most of the operations introduced in [Arn01]). The resulting set $\langle \Sigma\rangle_{\mathsf{bPAL}}$ of relation graphs is formally a subset of $\langle \Sigma\rangle_{\mathsf{PAL}}$ but nevertheless "the same" in the sense that for each $\mathfrak{G} \in \langle \Sigma\rangle_{\mathsf{PAL}}$ there exists a $\mathfrak{B} \in \langle \Sigma\rangle_{\mathsf{bPAL}}$ such that $\mathfrak{G}^A{=}\mathfrak{B}^A$ for every semantic domain A.

Definitions 17. Let $\mathfrak{G}$ be an m-ary relation graph, $m \geq 2$. We introduce the following special PAL-operations:
(PAL7) *basic rotation*: $\zeta(\mathfrak{G}) := \pi_\alpha(\mathfrak{G})$ where α is the cyclic permutation $1 \mapsto 2, \dots, m-1 \mapsto m, m \mapsto 1$ on $\{1, \dots, m\}$.
(PAL8) *basic transposition*: $\tau(\mathfrak{G}) := \pi_\alpha(\mathfrak{G})$ where α is the transposition of the first two components: $1 \mapsto 2, 2 \mapsto 1$ and $i \mapsto i$ otherwise.
(PAL9) *Connecting the first two pending edges*: $\Delta(\mathfrak{G}) := \delta_{12}(\mathfrak{G})$.
For $m \in \{0, 1\}$, i. e. if $\mathfrak{G}$ is a unary or 0-ary relation graph, we set $\zeta(\mathfrak{G}) := \tau(\mathfrak{G}) := \Delta(\mathfrak{G}) := \mathfrak{G}$. The operations (PAL1) (atomic relations, including id_3), $\times$ (PAL3), $\neg$ (PAL5), ζ (PAL7), τ (PAL8) and Δ (PAL9) are called *basic* operations. The closure with respect to basic operation will be denoted by $\langle \Sigma\rangle_{\mathsf{bPAL}}$, and as in Def. 7 we define $\langle Q\rangle_{\mathsf{bPAL}} := \{\mathfrak{G}^A \mid \mathfrak{G} \in \langle \Sigma_Q\rangle_{\mathsf{bPAL}}\}$ for a set $Q \subseteq \mathrm{Rel}(A)$.

Formally we can state the following equality (the proof is in [HerP]):

Proposition 18. *For $Q \subseteq \mathrm{Rel}(A)$ it holds $\langle Q\rangle_{\mathsf{PAL}} = \langle Q\rangle_{\mathsf{bPAL}}$.*

5 Contextual Logic and PAL

In this paper, the authors tried to harden the links between the Peircean Algebraic Logic and the realm of relational algebras from modern mathematics, while investigating some of Peirce's claims regarding the special role of the teridentity.

However, the work on PAL has also to be considered in a broader context. On the one hand, as PAL is tightly connected with the Existential Graphs devised by Peirce, work on PAL is also of interest for the area of *Conceptual Graphs*, invented by John Sowa which are based on the Existential Graphs (see [Sow84, Sow92]).

The work on PAL has already influenced the work on the *Contextual Logic of Relations* as can be seen by the works [Wil00, Arn01, Pol02]. There a modification of PAL is used, which uses Power Context Families as semantic model of the relation graphs. Of course, the results of this paper can be easily transferred to this model. We hope that our work will contribute to the overall effort in this direction. Future research will concentrate on the investigation of deduction rules for the graphs introduced with PAL.

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