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RELATIONALLY COLLAPSING CLONES

JENS-UWE GRABOWSKI AND REINHARD PÖSCHEL

ABSTRACT. In this paper we start to investigate those sets of clones (over a finite set A) which have the same invariant relations of fixed arity m. Such sets form semi-intervals in the lattice of all clones and will be described in more detail. In particular, collapsing clones are characterized, i.e. clones which are uniquely determined by their m-ary invariant relations (or, equivalently, for which the corresponding semi-interval collapses to a single clone).

INTRODUCTION

Let $\mathcal{L}_A$ be the lattice of all clones of operations on a finite set A. Although for $|A| \geq 3$ the structure of the uncountable lattice $\mathcal{L}_A$ is very complicated, there are many attempts to investigate this lattice (for references see e.g. [Ros 77], [Pös-K 79], [Sze 86]) and to classify its elements. The classification of the clones $F \in \mathcal{L}_A$ by their n-ary operations $F^{(n)}$ or their m-ary invariant relations $\mathrm{Inv}_A^{(m)} F$, respectively, leads to equivalence classes F/n-Op and F/m-Rel, respectively. If these equivalence classes "collapse" (= consist of a single clone), i.e. if F is uniquely determined by $F^{(n)}$ or $\mathrm{Inv}_A^{(m)} F$, respectively, then F is called (operationally) n-collapsing or relationally m-collapsing, respectively.

While results on (operationally) n-collapsing clones can be found in [Ihr-P 93] in the present paper we start corresponding investigations of the lattice $\mathcal{L}_A$ from the relational point of view. In particular we describe the structure and some properties of the equivalence classes F/m-Rel. They are semi-intervals. Moreover, for each such equivalence class I there exist only finitely many clones minimal in I and, in addition these clones are finitely generated (Thm. 3.3). As a corollary we find neccessary and sufficient conditions for a clone to be relationally m-collapsing (Thm. 3.4).

For that aim it is more natural to consider the lattice $\mathcal{L}_A^*$ of all so-called relational clones which is dually isomorphic to $\mathcal{L}_A$. Some parts of the results are independent of the concrete nature of the lattice and will be formulated and

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proved for arbitrary complete lattices (Prop. 2.3).

Finally, we give several examples (minimal clones, clones of constant operations, semi-intervals of Boolean clones (i.e. clones on $A = \{0,1\}$), collapsing Boolean clones and a class F/m-Rel for $A = \{0,1,2\}$). We demonstrate in these cases how to apply the general results.

This paper is just a start and we hope that it will stimulate further research towards a relational classification of clones.

1. Basic notions and notations

1.1. Let A be a finite set. For finitary functions (operations) $f:\ A^n \to A$ and relations $\varrho \subseteq A^m$ over A we introduce the following notations:

$$\begin{aligned} O_A^{(n)} &:= \{f \mid f : A^n \to A\}, \quad O_A := \bigcup_{n=1}^{\infty} O_A^{(n)}, \\ R_A^{(m)} &:= \{\varrho \mid \varrho \subseteq A^m\}, \quad R_A := \bigcup_{m=1}^{\infty} R_A^{(m)}, \\ F^{(n)} &:= F \cap O_A^{(n)} \qquad \text{for } F \subseteq O_A, \\ Q^{(m)} &:= Q \cap R_A^{(m)} \qquad \text{for } Q \subseteq R_A. \end{aligned}$$

J_A denotes the set of all projections, i.e. the operations $e_i^n:\ A^n \to A:\ (a_1,\dots,a_n) \mapsto a_i$ (for $1 \le i \le n$, $n \in \{1,2,3,\dots\}$).

1.2. An operation $f \in O_A^{(n)}$ *preserves* a relation $\varrho \in R_A^{(m)}$ (or ϱ is *invariant* for f) if $f[r_1,\dots,r_n] \in \varrho$ for all $r_1,\dots,r_n \in \varrho$ (where $f[r_1,\dots,r_n]$ is defined by $f[r_1,\dots,r_n](i) := f(r_1(i),\dots,r_n(i))$, $i \in \{1,\dots,m\}$).

Then, for $F \subseteq O_A$, $Q \subseteq R_A$,

$$\mathrm{Pol}_A\, Q := \{f \in O_A \mid f \text{ preserves every } \varrho \in Q\}$$

is the set of so-called *polymorphisms* of Q and

$$\mathrm{Inv}_A\, F := \{\varrho \in R_A \mid \varrho \text{ is invariant for every } f \in F\}$$

denotes the set of all *invariant relations* of F.

It ist well-known that $\mathrm{Pol}_A - \mathrm{Inv}_A$ establishes a Galois connection between operations and relations and the Galois closed elements are exactly the (locally closed) clones of operations and relations, respectively. For more details we refer to e.g. [Pös-K 79], [Pös 79], [Pös 80].

Note that

$$f \in \mathrm{Pol}_A\, \varrho \iff \varrho \in \mathrm{Inv}_A\, f$$

(for $f \in O_A$ and $\varrho \in R_A$). From the algebraic point of view $f \in \mathrm{Pol}_A\, \varrho$ expresses the fact that $f:\ \langle A;\varrho\rangle^n \to \langle A;\varrho\rangle$ is a (relational) homomorphism or, equivalently, that ϱ is a subalgebra of the direct power $\langle A; f\rangle^m$ of the algebra $\langle A; f\rangle$.

1.3. We recall that a *clone* F on A (notation $F \leq O_A$) is a subset $F \subseteq O_A$ closed with respect to arbitrary compositions of functions and containing all projections. The composition $f[g_1, \dots, g_s]$ of $f \in O_A^{(s)}$ and $g_1, \dots, g_s \in O_A^{(n)}$ is defined by

$$f[g_1, \dots, g_s](a_1, \dots, a_n) := f(g_1(a_1, \dots, a_n), \dots, g_s(a_1, \dots, a_n)).$$

For $F \subseteq O_A$ let $\langle F \rangle_{O_A}$ denote the clone generated by F (i.e. the least clone containing F). For finite A every clone F can be characterized as $F = \mathrm{Pol}_A\, Q$ for a suitable set Q of relations (e.g. $Q = \mathrm{Inv}_A\, F$) and we have (cf. e.g. [Pös-K 79])

$$\langle F \rangle_{O_A} = \mathrm{Pol}_A \,\mathrm{Inv}_A\, F.$$

The clones on A form a complete algebraic lattice $\mathcal{L}_A$ with respect to inclusion (where $F_1 \wedge F_2 = F_1 \cap F_2$ and $F_1 \vee F_2 = \langle F_1 \cup F_2 \rangle_{O_A}$ are meet and join).

A relational clone Q on A (notation $Q \leq R_A$) can be defined internally as a subset $Q \subseteq R_A$ closed under some operations on relations (e.g. intersection, relational product, ...). We do not give here this definition but use equivalently the following characterization (which, however, works for finite A only). Let $Q \subseteq R_A$. Then

$$[Q]_{R_A} := \mathrm{Inv}_A \,\mathrm{Pol}_A\, Q$$

is called the *relational clone generated by* Q. In case $[Q]_{R_A} = Q$ we call Q a *relational clone*. The relational clones form a complete algebraic lattice $\mathcal{L}_A^*$ (with respect to inclusion, where meet and join are given by $Q_1 \wedge Q_2 = Q_1 \cap Q_2$, $Q_1 \vee Q_2 = [Q_1 \cup Q_2]_{R_A}$). This lattice $\mathcal{L}_A^*$ is dually isomorphic to $\mathcal{L}_A$ via the mappings

$$\mathrm{Pol}_A : \mathcal{L}_A^* \to \mathcal{L}_A : Q \mapsto \mathrm{Pol}_A\, Q$$
$$\mathrm{Inv}_A : \mathcal{L}_A \to \mathcal{L}_A^* : F \mapsto \mathrm{Inv}_A\, F.$$

The least relational clone is $D_A := \mathrm{Inv}_A\, O_A$ which consists of all (generalized) diagonal relations ([Pös-K 79]).

1.4. We collect some facts concerning the Galois connection $\mathrm{Pol}_A - \mathrm{Inv}_A$ (for details see e.g. [Pös-K 79]).

a) The mapping $\mathcal{P}(O_A) \to \mathcal{P}(O_A) : F \mapsto \mathrm{Pol}_A \,\mathrm{Inv}_A\, F$ and $\mathcal{P}(R_A) \to \mathcal{P}(R_A) : Q \mapsto \mathrm{Inv}_A \,\mathrm{Pol}_A\, Q$, resp., are closure operators on the power sets $\mathcal{P}(O_A)$ and $\mathcal{P}(R_A)$, resp. As mentioned above, the Galois closed sets are just the clones and relational clones.

b) For a clone $F \leq O_A$ and a relation $\varrho \in R_A$ with at most t elements (i.e. $|\varrho| \leq t$) we have

$$F \subseteq \mathrm{Pol}_A\, \varrho \iff F^{(t)} \subseteq \mathrm{Pol}_A\, \varrho.$$

Consequently, for $t \geq |A^m| - 1$ we have $\mathrm{Inv}_A^{(m)} F = \mathrm{Inv}_A^{(m)} F^{(t)}$ since $t \geq |\varrho|$ for any (non-trivial) $\varrho \in R_A^{(m)} \setminus \{A^m\}$.

Now we come to the crucial definitions of this paper.

1.5. Definitions. Let n-Op, m-Rel and m-Rel*, resp., be the equivalence relations on $\mathcal{L}_A$ and $\mathcal{L}_A^*$, resp., defined by their equivalence classes

$$\begin{aligned} F/n\text{-Op} &:= \{\tilde{F} \in \mathcal{L}_A \mid \tilde{F}^{(n)} = F^{(n)}\}, \\ F/m\text{-Rel} &:= \{\tilde{F} \in \mathcal{L}_A \mid \operatorname{Inv}_A^{(m)} \tilde{F} = \operatorname{Inv}_A^{(m)} F\} \quad \text{and} \\ Q/m\text{-Rel}^* &:= \{\tilde{Q} \in \mathcal{L}_A^* \mid \tilde{Q}^{(m)} = Q^{(m)}\} \quad \text{resp.,} \end{aligned}$$

for $F \in \mathcal{L}_A$ and $Q \in \mathcal{L}_A^*$.

A clone $F \leq O_A$ or $Q \leq R_A$, resp., is called *(operationally) n-collapsing* or *(relationally) m-collapsing*, resp., if $F/n\text{-Op} = \{F\}$ and $Q/m\text{-Rel}^* = \{Q\}$, resp. Moreover, a clone $F \leq O_A$ is called *relationally m-collapsing* if $F/m\text{-Rel} = \{F\}$ or, equivalently, if the clone $\operatorname{Inv}_A F$ of its invariant relations is (relationally) m-collapsing.

Note that collapsing clones are just those which are uniquely defined by their operations or relations of a fixed arity. The equivalence relations m-Rel and m-Rel* are dual to each other in the sense that

$$\tilde{F} \in F/m\text{-Rel} \iff \operatorname{Inv}_A \tilde{F} \in (\operatorname{Inv}_A F)/m\text{-Rel}^*$$

and

$$\tilde{Q} \in Q/m\text{-Rel}^* \iff \operatorname{Pol}_A \tilde{Q} \in (\operatorname{Pol}_A Q)/m\text{-Rel},$$

i.e. the operators $\operatorname{Pol}_A - \operatorname{Inv}_A$ are antiisomorphisms between F/m-Rel and Q/m-Rel*.

1.6. Remarks. The structure of F/n-Op is known to be the interval

$$F/n\text{-Op} = [\langle F^{(n)} \rangle_{O_A}, \operatorname{Sta} F^{(n)}]_{\mathcal{L}_A}$$

in the lattice $\mathcal{L}_A$ (cf. [Ihr-P 93]) where $\operatorname{Sta} F^{(n)}$ denotes the *stabilizer* of $F^{(n)}$, i.e. the set of all functions $f \in O_A^{(s)}$ $(s \in \mathbf{N})$, such that $f[f_1, \ldots, f_s] \in F^{(n)}$ for all $f_1, \ldots, f_s \in F^{(n)}$. Thus n-Op gives a partition of $\mathcal{L}_A$ into intervals. A criterion for n-collapsing clones can also be found in [Ihr-P 93].

For clones which consist of essentially unary operations only (i.e. transformation monoids), the above intervals are called *monoidal intervals*. Structural results about these monoidal intervals can be found in [Kro 95]. In [Gra 97] it is shown that binary operations suffice to test whether, for a given monoid $M = F^{(1)}$, the monoidal interval $F/1$-Op collapses or not.

2. Kernel operators

In this section we present a result on kernel operators in arbitrary complete lattices. We shall apply this to our concrete lattices $\mathcal{L}_A$ and $\mathcal{L}_A^*$ in the next section.

2.1. Definition. Let L be a complete lattice. An operator $K: \ L \to L$ is called *algebraic* if

$$K(x) = \sup\{K(x') \mid x' \leq x, \ x' \text{ is compact in } L\}^1$$

for all $x \in L$. K is called *kernel operator* if for all $x, x_1, x_2 \in L$ we have

- $K(x) \leq x$,
- $K(K(x)) = K(x)$,
- $x_1 \leq x_2 \Rightarrow K(x_1) \leq K(x_2)$.

To every operator K we associate the equivalence relation $\sim_K$ on L defined by

$$x_1 \sim_K x_2 :\Longleftrightarrow \ K(x_1) = K(x_2).$$

2.2. Examples. In connection with clones and relational clones the following operators are of interest

$$K_n: \ \mathcal{L}_A \to \mathcal{L}_A: \ F \mapsto \langle F^{(n)} \rangle_{O_A},$$
$$K_m^*: \ \mathcal{L}_A^* \to \mathcal{L}_A^*: \ Q \mapsto [Q^{(m)}]_{R_A}.$$

Both, K_n ($n \in \mathbb{N}$) and K_m^* ($m \in \mathbb{N}$), are kernel operators on $\mathcal{L}_A$ and $\mathcal{L}_A^*$, respectively. The corresponding equivalence relations are just (cf. 1.5)

$$\sim_{K_n} = n\text{-Op} \quad \text{and} \quad \sim_{K_m^*} = m\text{-Rel}^*.$$

Moreover, both they are algebraic. In fact, each $\langle F^{(n)} \rangle_{O_A}$ as well as $[Q^{(m)}]_{R_A}$ is compact.

Remark: Instead of the kernel operator K_n in $\mathcal{L}_A$ one can dually consider the closure operator C_n^* on $\mathcal{L}_A^*$ defined by

$$\begin{aligned} C_n^*(\mathrm{Inv}_A F) &:= \mathrm{Inv}_A K_n(F), \text{ i.e.} \\ C_n^*(Q) &:= \mathrm{Inv}_A K_n(\mathrm{Pol}_A Q) = \mathrm{Inv}_A \mathrm{Pol}_A^{(n)} Q \qquad \text{for } Q \in \mathcal{L}_A^*. \end{aligned}$$

Analogously, to K_m^* corresponds the closure operator

$$\begin{aligned} C_m(\mathrm{Pol}_A Q) &:= \mathrm{Pol}_A K_m^*(Q), \text{ i.e.} \\ C_m(F) &:= \mathrm{Pol}_A K_m^*(\mathrm{Inv}_A F) = \mathrm{Pol}_A \mathrm{Inv}_A^{(m)} F \qquad \text{for } F \in \mathcal{L}_A. \end{aligned}$$

In particular we have

$$\sim_{C_m} = m\text{-Rel}.$$

Thus every result on kernel operators easily can be transformed to a result on the corresponding closure operator on the dual lattice.

2.3. Proposition. *Let K be an algebraic kernel operator on a complete lattice L and let I be an equivalence class of $\sim_K$. Then I is a* (meet) semi-interval, *i.e. the following three conditions are satisfied:*

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[1] An element x' is *compact* if $x' \leq \sup T$ ($T \subseteq L$) implies $x' \leq \sup T'$ for some finite subset $T' \subseteq T$.

(a) *I has a least element o_I,*
(b) *I is* convex, *i.e. $x_1, x_2 \in I$ and $x_1 \leq x \leq x_2$ imply $x \in I$,*
(c) *for each $x \in I$ there exists a maximal (in I) element*[2] *$\hat{x} \in I$ such that $x \leq \hat{x}$.*

Proof. (a): Because of $x \sim_K K(x) \leq x$, the least element of I is $o_I = K(x)$ for any $x \in I$.

(b): Obviously, we have $K(x_1) \leq K(x) \leq K(x_2) = K(x_1)$, thus $x \in I$.

(c): We are going to apply Zorn's Lemma and therefore we consider a chain $C \subseteq I$. Obviously, $o_I \leq K(\sup C)$. On the other hand, by algebraicity of K we have

$$K(\sup C) = \sup\{K(x') \mid x' \leq \sup C,\ x' \text{ compact in } L\}.$$

From $x' \leq \sup C$ with compact x' we conclude that there exists a finite subset C' of C with $x' \leq \sup C'$ and therefore (C is a chain) a single element $x \in C$ with $x' \leq x$. Consequently $K(x') \leq K(x) = o_I$, i.e. $K(\sup C) \leq o_I$.Thus we get $K(\sup C) = o_I$, hence $\sup C \in I$. Now Zorn's Lemma implies (c). □

3. Relational and collapsing clones

3.1. Proposition 2.3 can be applied to the kernel operators introduced in 2.2. However it turns out that in these concrete cases there can be said more about the semi-intervals (see 3.3 below). We recall, a *meet semi-interval* is a union of intervals with common least element (cf. 2.3). Analogously, a *join semi-interval* is a union of intervals with common largest element. Because collapsing clones are already treated in [Ihr-P 93] (cf. 1.6), we shall deal in the following only with the relational case, i.e. with the kernel operator K_m^* and its dual, the closure operator C_m (cf. 2.2), although some results of [Ihr-P 93] are also covered by 2.3. Note that by 2.3, the equivalence classes $Q/m\text{-Rel}^*$ of relational clones with equal m-ary part $Q^{(m)}$ form a meet semi-interval while (via the dual isomorphisms Inv_A, Pol_A, cf. 1.3, 1.5) the corresponding class $F/m\text{-Rel}$ (for $F = \mathrm{Pol}_A Q$) of all clones with the same m-ary invariant relations $Q^{(m)} = \mathrm{Inv}_A^{(m)} F$ forms a join semi-interval (see Fig. 1).

3.2. In order to formulate the next results, we introduce the following notions. A clone $F \in \mathcal{L}_A$ (relational clone $Q \in \mathcal{L}_A^*$, resp.) is called *finitely relationally* or *m-relationally characterizable*(*finitely operationally characterizable*, resp.) if there exists a finite set $Q_0 \subseteq R_A$ or $Q_0 \subseteq R_A^{(m)}$ ($F_0 \subseteq O_A$, resp.) of relations (operations, resp.) such that $F = \mathrm{Pol}_A Q_0$ ($Q = \mathrm{Inv}_A F_0$, resp.).

As usual, a (relational) clone is called finitely generated if it is generated by a finite subset. There exists a purely lattice theoretic characterization of

[2] not uniquely defined: in I there may exist several maximal elements above x

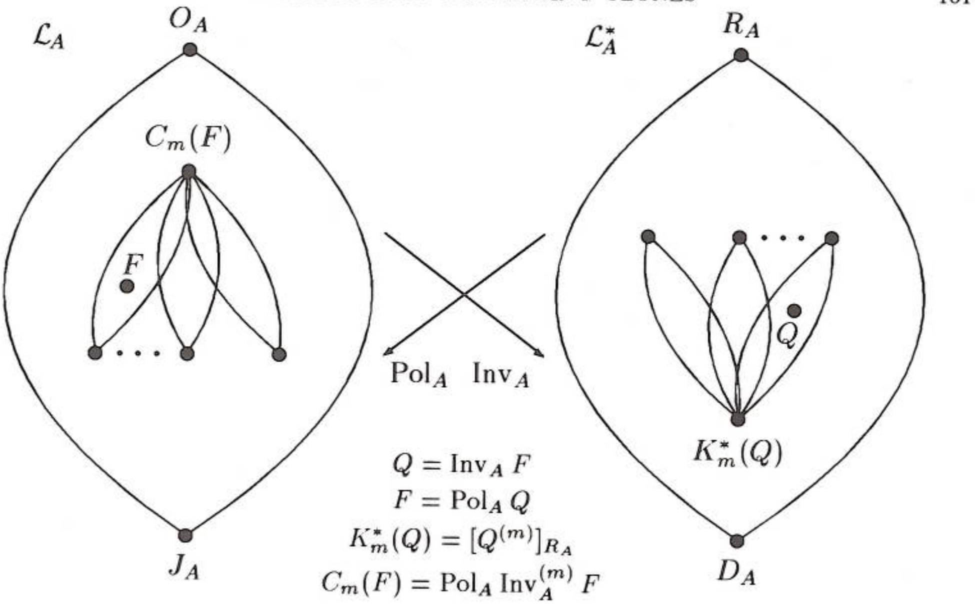

FIGURE 1. The Galois connection Pol_A – Inv_A and semi-intervals

finitely generated clones: The following conditions (a)-(d) as well as (a′)-(d′) are equivalent for $F \in \mathcal{L}_A$, $Q \in \mathcal{L}_A^*$ with $Q = \mathrm{Inv}_A F$ and $F = \mathrm{Pol}_A Q$:

(a) $F = \mathrm{Pol}_A Q$ is finitely generated,
(b) $Q = \mathrm{Inv}_A F$ is finitely operationally characterizable,
(c) the interval $[J_A, F]_{\mathcal{L}_A}$ is dually atomic,
(d) the interval $[Q, R_A]_{\mathcal{L}_A^*}$ is atomic (i.e. every clone properly containing Q contains an upper neighbour of Q).

(a′) $Q = \mathrm{Inv}_A F$ is finitely generated,
(b′) $F = \mathrm{Pol}_A Q$ is finitely relationally characterizable,
(c′) the interval $[D_A, Q]_{\mathcal{L}_A^*}$ is dually atomic,
(d′) the interval $[F, O_A]_{\mathcal{L}_A}$ is atomic.

In the following two theorems part (B) is just the translation of part (A) from the lattice $\mathcal{L}_A^*$ of relational clones to the lattice $\mathcal{L}_A$ of clones (via Pol_A – Inv_A) and needs no extra proof.

3.3. Theorem (Properties of the semi-intervals). *Let $m \in \mathbf{N}$, $F \in \mathcal{L}_A$ and $Q \in \mathcal{L}_A^*$. Then*

(A) *$I^* = Q/m\text{-}Rel^*$ is a meet semi-interval which is the union of finitely many intervals with the common least (finitely generated) element $[Q^{(m)}]_{R_A}$ where every relational clone maximal in I^* is finitely operationally characterizable.*

(B) *$I = F/m$-Rel is a join semi-interval which is the union of finitely many intervals with the common largest (finitely relationally characterizable) element* $\mathrm{Pol}_A \mathrm{Inv}_A^{(m)} F$ *where every clone minimal in I is finitely generated.*

Proof. (A) Obviously $[Q^{(m)}]_{R_A}$ is the least element in Q/m-Rel* (cf. 3.1). Because of 2.3 it remains to prove that there are finitely many maximal elements in I^* each of which is finitely operationally characterizable.
Let Q_1 be a maximal element in I^*. Let $F_1 := \mathrm{Pol}_A Q_1$ and $t := |A|^m - 1$. By 1.4(b) we have

$$\mathrm{Inv}_A^{(m)} F_1^{(t)} = \mathrm{Inv}_A^{(m)} F_1 = \mathrm{Inv}_A^{(m)} \mathrm{Pol}_A Q_1 = Q_1^{(m)},$$

i.e., $\mathrm{Inv}_A^{(m)} F_1^{(t)} \in I^*$. But $Q_1 = \mathrm{Inv}_A F_1 \subseteq \mathrm{Inv}_A F_1^{(t)}$ hence, by maximality of Q_1, we get $Q_1 = \mathrm{Inv}_A F_1^{(t)}$.
This shows that Q_1 is finitely operationally characterizable. Moreover $|O_A^{(t)}|$ is finite, hence there are only finitely many choices $F_1^{(t)} \subseteq O_A^{(t)}$. □

3.4. Theorem (Criteria for collapsing).

(A) *A relational clone $Q \in \mathcal{L}_A^*$ is (relationally) m-collapsing if and only if the following conditions are satisfied:*
 (i) *Q is finitely generated by $Q^{(m)}$,*
 (ii) *Q is finitely operationally characterizable (cf. 3.2),*
 (iii) *each upper neighbour Q' of Q (in the lattice $\mathcal{L}_A^*$) is generated by its m-ary part $Q'^{(m)}$.*

(B) *A clone $F \in \mathcal{L}_A$ is relationally m-collapsing if and only if the following conditions are satisfied:*
 (i) *F is m-relationally characterizable (i.e. $F = \mathrm{Inv}_A Q_0$ for $Q_0 \subseteq R_A^{(m)}$),*
 (ii) *F is finitely generated,*
 (iii) *each lower neighbour F' of F (in the lattice $\mathcal{L}_A$) is m-relationally characterizable.*

Proof. (A) Let Q be m-collapsing. Then $I^* = Q/m$-Rel$^* = \{Q\}$ and (i) and (ii) follow directly from 3.3(A). To prove (iii) we observe that $Q'^{(m)} \neq Q^{(m)}$ holds for any upper neighbour Q' of Q, consequently $Q = [Q^{(m)}]_{R_A} < [Q'^{(m)}]_{R_A} \leq Q'$, hence $[Q'^{(m)}]_{R_A} = Q'$.
Conversely, let (i)-(iii) be satisfied. From (i) it follows that Q is the least element of $I^* := Q/m$-Rel*. Assume there exists another relational clone in I^*. By (ii) (cf. 3.2 (b)$\Longleftrightarrow$(d)) there exists in I^* also an upper neighbour Q' of Q. By (iii) we get $Q \subset Q' = [Q'^{(m)}]_{R_A} = [Q^{(m)}]_{R_A} = Q$, a contradiction. Thus Q is m-collapsing. □

4. Examples

The following examples show in some relatively easy cases how to use the results of the preceeding section.

4.1. Minimal clones. Let $|A| \geq 3$ (for $|A| = 2$ see 4.3 below). Then the trivial clone J_A of all projections is characterizable by binary relations (see e.g. [Pös-K 79, 4.1.14]). Thus it immediately follows from 3.4 that a minimal clone (upper neighbour of J_A in $\mathcal{L}_A$) is relationally m-collapsing ($m \geq 2$) if and only if it is m-relationally characterizable (note that 3.4(ii) is trivially satisfied for minimal clones).

4.2. Clones of constant functions. Let $|A| \geq 3$ and $B \subseteq A$. Every clone

$$C_B := \langle \{c_a \mid a \in B\} \rangle_{O_A}$$

generated by a set of constant functions $c_a : A \to A : x \mapsto a$ ($a \in B$) is 3-relationally characterizable. In fact,

$$C_B = \mathrm{Pol}_A \left(\{\varrho_{i,j} \mid i, j \in B\} \cup \{\pi_3\} \right),$$

where $\varrho_{i,j} := \{(i,j)\} \cup \{(a,a) \mid a \in B\}$ and $\pi_3 := \{(x,y,z) \in A^3 \mid x = y \text{ or } y = z\}$. To see this we remark that every unary operation preserving all $\varrho_{i,j}$ must be constant, and π_3 forces an operation which preserves it to be unary. Note that every subclone of C_B is of the form $C_{B'}$ for some $B' \subseteq B$. Thus C_A and every subclone is 3-relationally characterizable. Applying 3.4 we get: all clones C_B are 3-relationally collapsing ($B \subseteq A$).

4.3. Collapsing Boolean clones ($A = \{0,1\}$). For $A = \{0,1\}$ the lattice $\mathcal{L}_A$ (Boolean clones) is well-known (it was determined by E.L. Post [Pos 41]). Figure 2 shows this lattice and we use here Post's original notations (e.g. $O_A = \mathsf{C}_1$).[3]

There are four equivalence classes w.r.t. 1-Rel (i.e. we consider the case $m = 1$). Three of them turn out to be intervals, namely $[\mathsf{O}_1, \mathsf{C}_4]_{\mathcal{L}_2}$, $[\mathsf{O}_5, \mathsf{C}_2]_{\mathcal{L}_2}$ and $[\mathsf{O}_6, \mathsf{C}_3]_{\mathcal{L}_2}$, while the equivalence class $\mathsf{C}_1/1\text{-Rel}$ (see Figure 3) is the union of the two intervals $[\mathsf{O}_4, \mathsf{C}_1]_{\mathcal{L}_2}$ and $[\mathsf{O}_8, \mathsf{C}_1]_{\mathcal{L}_2}$. The largest elements are 1-relationally characterizable. The minimal elements are finitely generated, more precisely, they are generated by unary functions, as expected from the proof of Theorem 3.3.

For $m = 2$ we find two relationally 2-collapsing clones: M_4 and C_4. The other clones belong to 17 nontrivial intervals.

For $m = 3$ there are 10 nontrivial intervals:
$[\mathsf{L}_1, \mathsf{C}_1]_{\mathcal{L}_2}$, $[\mathsf{L}_5, \mathsf{D}_3]_{\mathcal{L}_2}$ and $[\mathsf{F}_i^{\infty}, \mathsf{F}_i^{3}]_{\mathcal{L}_2}$ ($i \in \{1, \ldots, 8\}$). All other 34 clones are relationally 3-collapsing:

[3] All what now follows easily can be checked starting from few well-known facts about the clones and taking into account that the intersection of m-relationally characterizable clones is again m-relationally characterizable.

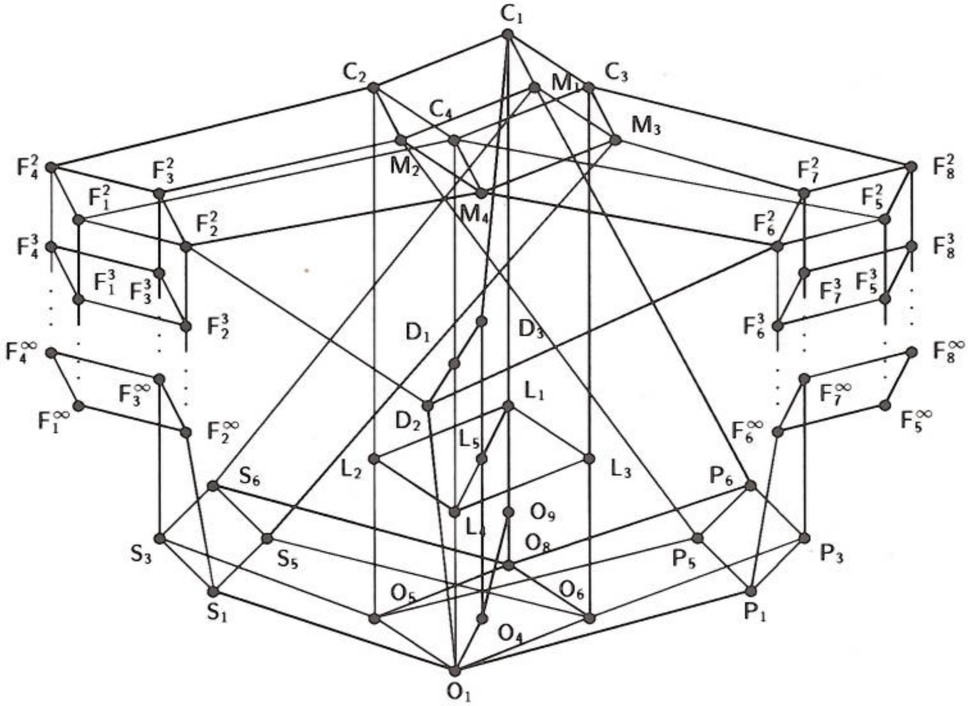

FIGURE 2. The POST-lattice $\mathcal{L}_2$

$C_2, C_3, C_4, M_1, M_2, M_3, M_4, D_1, D_2, L_2, L_3, L_4, S_1, S_3, S_5, S_6, P_1, P_3, P_5, P_6,$
$O_1, O_4, O_5, O_6, O_8, O_9, F_i^2$ $(i \in \{1, \ldots, 8\})$.

In the case $m \geq 4$, only the infinite chains $[F_i^\infty, F_i^m]_{\mathcal{L}_2}$, $i \in \{1, \ldots, 8\}$, form nontrivial equivalence classes, all other clones are relationally m-collapsing. Therefore, with exception of $m = 1$, all semi-intervals of Boolean clonesare intervals.

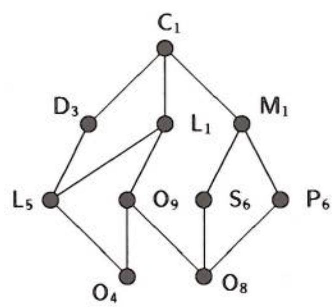

FIGURE 3. The semi-interval $C_1/1$-Rel

4.4. Example ($A = \{0,1,2\}$). Let $A = \{0,1,2\}$ and let Q be the set of the following five unary relations (= subsets of A):

$$\{0\}, \quad \{1\}, \quad \{2\}, \quad \{0,1\}, \quad \{0,2\}.$$

Each minimal clone F in the join semi-interval $(\mathrm{Pol}_A\, Q)/1\text{-Rel}$ is generated by binary functions (see proof of 3.3). Moreover, a single binary function suffices. In fact, a basis of F has to contain a function f not preserving the relation $\{1,2\}$ (the only non-trivial one missing in the above list). Then, by minimality, $F = \langle f \rangle_{O_A}$.

It turns out that there are 15 minimal clones in $(\mathrm{Pol}_A\, Q)/1\text{-Rel}$. The generating functions may be chosen, for instance, according to the following tables:

0	0	0
0	1	0
0	0	2

0	0	2
0	1	2
0	0	2

0	0	2
0	1	2
2	0	2

0	1	0
0	1	0
0	1	2

0	1	0
1	1	1
0	0	2

0	1	2
0	1	0
0	0	2

0	1	2
0	1	0
0	1	2

0	1	2
0	1	0
2	1	2

0	1	2
0	1	2
0	0	2

0	1	2
0	1	2
2	0	2

0	1	2
1	1	0
0	1	2

0	1	2
1	1	0
2	0	2

0	1	2
1	1	1
2	0	2

0	1	2
1	1	2
0	0	2

0	1	2
1	1	2
2	0	2

4.5. Problems. Many interesting clones are finitely relationally characterizable (e.g. minimal and all maximal clones). If, in addition, a clone F is finitely (operationally) generated, then every coatom in the interval $[J_A, F]_{\mathcal{L}_A}$ is also finitely relationally characterizable. Due to 3.4 there exists an m (e.g. choose the maximal arity of relations characterizing F and the coatoms) such that F is m-relationally collapsing. Let $\gamma(F)$ be the least $m \in \mathbf{N}$ such that F is m-relationally collapsing (and let $\gamma(F) = \infty$ if m does not exist). In connection with this we mention here the following problems:

(1) *Let F be a finitely relationally characterizable and finitely generated clone. Determine $\gamma(F)$.*

(2) *For fixed m, characterize clones F for which the semi-interval F/m-*Rel *becomes an interval.*

(3) *For fixed F, determine the least m such that F/m-*Rel *is an interval. Does m always exist?*

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JENS-UWE GRABOWSKI, REINHARD PÖSCHEL
TECHNISCHE UNIVERSITÄT DRESDEN
INSTITUT FÜR ALGEBRA
D – 01062 DRESDEN
GERMANY
TEL.: +49-351-463 7515
FAX: +49-351-463 4235
E-MAIL: [GRABO,POESCHEL]@MATH.TU-DRESDEN.DE