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Closure Operators on Complete Lattices

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Introduction

Galois connections play an important role in several parts of mathematics. The classical example is the Galois connection between extensions of a commutative field and the group of all field automorphisms which fix the elements of a certain subfield. This has been generalized in various directions.

To every Galois connection there is associated a pair of closure operators, and the fixed points of these closure operators form two complete lattices which are dually isomorphic.

In this paper we study these lattices via special complete sublattices arising from additional closure or kernel operators (in particular conjugate pairs of additive closure operators).

This is motivated by two examples: (1) The Galois connection *Pol* – *Inv* (polymorphisms - invariants) between operations and relations and (2) the Galois connection *Mod* – *Id* (models - identities) between algebras and identities.

In case of (1) the corresponding complete lattices are the lattice \mathbf{L}_A of all clones of functions on a (finite) set A and the dual lattice of so-called relational clones. For Boolean functions, i.e. if $|A| = 2$, the lattice \mathbf{L}_A is completely known by a result of E. L. POST obtained in 1921 ([Pos;21], [Pos;41]). In this case it is countably infinite. However, for $|A| \geq 3$, \mathbf{L}_A is uncountable ([Jan-M;59]) and widely unknown. There is not much hope to get an effective description of the lattice of all clones of operations defined on an at least three-element set although many efforts were made to understand better

the structure of \mathbf{L}_A . One of the general strategies to study this lattice could be to study complete sublattices which can be less complex than the large lattice \mathbf{L}_A .

In the second example (2) we obtain the lattice $\mathcal{L}(\tau)$ of all varieties of algebras of the same type τ and – dually – the lattice of all equational theories of this type. For arbitrary types τ not so much is known about the lattices $\mathcal{L}(\tau)$, even if one is only interested in the lattice of all subvarieties of a given variety like e.g. the variety of all semigroups. Again one could try to start with a description of complete sublattices of $\mathcal{L}(\tau)$.

In this paper we will give several characterizations of complete sublattices of a complete lattice corresponding to a Galois connection and apply this to the examples (1) and (2).

1 Operations and relations, models and identities

1.1. Operations and clones. Let $O_A = \bigcup_{n=1}^{\infty} O_A^{(n)}$ be the set of all operations defined on a set A where $O_A^{(n)}$ is the set of all n -ary operations $f : A^n \rightarrow A$ on A . The *composition* $h[f_1, \dots, f_k]$ of n -ary operations f_1, \dots, f_k and a k -ary operation h is defined by

$$h[f_1, \dots, f_k](\underline{a}) := h(f_1(\underline{a}), \dots, f_k(\underline{a})) \quad \text{for all } \underline{a} \in A^n .$$

The *projections* $e_i^{n,A}$, for short e_i^n , on the set A are n -ary operations defined by

$$e_i^{n,A}(a_1, \dots, a_n) = a_i , \quad 1 \leq i \leq n .$$

A *clone* on a set A is any set of operations on A which is closed under composition and contains all projections e_i^n for all $n \in \mathbf{N}$ and $1 \leq i \leq n$. Let \mathbf{L}_A be the lattice of all clones on A ordered by inclusion. The set O_A is the largest clone and the set J_A of all projections is the least one.

From now on let A be a finite set (if not otherwise stated). The lattice \mathbf{L}_A is atomic and dually atomic with finitely many atoms (minimal clones) and dual atoms (maximal clones). The maximal

clones were fully described by I.G. ROSENBERG in [Ros;70] for arbitrary finite sets A . A complete description of all minimal clones still is an open problem. Clones arise in many fields of mathematics, e.g. for every algebra $\underline{A} = (A; (f_i^A)_{i \in I})$ we have the clone $T(\underline{A})$ of term operations, i.e. the subclone of O_A generated by the fundamental operations $(f_i^A)_{i \in I}$.

1.2. Invariant relations. It is well-known that clones also can be characterized by relations. For a set A let $Rel_A^{(h)} = \{\varrho \mid \varrho \subseteq A^h\}$ be the set of all h -ary relations on A and let $Rel_A = \bigcup_{h=1}^{\infty} Rel_A^{(h)}$ (all finitary relations on A). An operation $f \in O_A^{(n)}$ preserves the relation $\varrho \subseteq A^h$ (or ϱ is *invariant* for f) if

$$(f(a_{11}, a_{21}, \dots, a_{n1}), \dots, f(a_{1h}, a_{2h}, \dots, a_{nh})) \in \varrho$$

whenever $(a_{11}, \dots, a_{1h}) \in \varrho, \dots, (a_{n1}, \dots, a_{nh}) \in \varrho$. For any set $F \subseteq O_A$ of operations and any set $Q \subseteq Rel_A$ of relations we define

$$\begin{aligned} Pol_A Q &:= \{f \mid f \in O_A \text{ and } \forall \varrho \in Q (f \text{ preserves } \varrho)\} \text{ and} \\ Inv_A F &:= \{\varrho \mid \varrho \in Rel_A \text{ and } \forall f \in F (f \text{ preserves } \varrho)\} \end{aligned}$$

(and we write $Pol_A \varrho, Inv_A f$ for $Pol_A \{\varrho\}, Inv_A \{f\}$).

It is easy to check that $Pol_A Q$ is a clone; moreover, every clone F satisfies $F = Pol_A Inv_A F$ ([Pös-K;79]).

1.3. Galois connections. The operators

$$\begin{aligned} Inv &: \mathcal{P}(O_A) \rightarrow \mathcal{P}(Rel_A) : F \mapsto Inv_A F \quad \text{and} \\ Pol &: \mathcal{P}(Rel_A) \rightarrow \mathcal{P}(O_A) : Q \mapsto Pol_A Q \end{aligned}$$

induced (as above in 1.2) by the relation “preserve”

$$R := \{(f, \varrho) \mid f \in O_A, \varrho \in Rel_A \text{ and } f \text{ preserves } \varrho\}.$$

are a typical example for a Galois connection (μ, ι) (here (Inv, Pol)) between two sets A (here O_A) and B (here Rel_A) induced by a

relation $R \subseteq A \times B$. In such a general setting, for given R one defines the mappings $\mu : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ and $\iota : \mathcal{P}(B) \rightarrow \mathcal{P}(A)$ by

$$\begin{aligned}\mu(T) &:= \{s \in B \mid \forall t \in T((t, s) \in R)\} \quad \text{and} \\ \iota(S) &:= \{t \in A \mid \forall s \in S((t, s) \in R)\}\end{aligned}$$

for $T \subseteq A$, $S \subseteq B$. The pair (μ, ι) is called *Galois connection* induced by R . Galois connections between A and B can be defined independently of the relation R as pairs (μ, ι) of mappings satisfying the following properties for all $T, T' \subseteq A$ and $S, S' \subseteq B$:

- (i) $T \subseteq T' \Rightarrow \mu(T) \supseteq \mu(T')$,
- (ii) $S \subseteq S' \Rightarrow \iota(S) \supseteq \iota(S')$,
- (iii) $T \subseteq \iota\mu(T)$,
- (iv) $S \subseteq \mu\iota(S)$.

Then $R = \{(t, s) \in A \times B \mid s \in \mu(\{t\})\}$ is the uniquely defined relation which induces (μ, ι) , see e.g. [Ihr;93]. For every Galois connection (μ, ι) we have further useful properties, e.g.

- (v) $\mu\iota\mu(T) = \mu(T)$,
- (vi) $\iota\mu\iota(S) = \iota(S)$,
- (vii) $\iota\mu : T \mapsto \iota\mu(T)$ is a closure operator with corresponding closure system $\mathcal{H}_{\iota\mu} = \{\iota(S) \mid S \subseteq B\}$,
- (viii) $\mu\iota : S \mapsto \mu\iota(S)$ is a closure operator with corresponding closure system $\mathcal{H}_{\mu\iota} = \{\mu(T) \mid T \subseteq A\}$,
- (ix) These closure systems are complete lattices which are dually isomorphic (their elements are called *Galois closed sets* of (μ, ι)).

Remark: Galois connections can be characterized equivalently also by the following single property:

$$(x) \quad T \subseteq \iota(S) \iff S \subseteq \mu(T) .$$

For finite A the Galois closed sets of (Inv, Pol) are just the clones (F is a clone iff $F = Pol\ Inv\ F$) and the so-called relational clones ($Q = Inv\ Pol\ Q$).

Another well-known example is the Galois connection (Id, Mod) between algebras and identities (see 1.5 below). We recall some notions.

1.4. Terms and term operations. Let $(f_i)_{i \in I}$ be an indexed set of operation symbols of type $\tau = (n_i)_{i \in I}$ where f_i is n_i -ary. Let $X = \{x_1, \dots, x_n, \dots\}$ be a countably infinite alphabet of variable symbols and let $X_n = \{x_1, \dots, x_n\}$. An n -ary *term* of type τ is defined inductively by:

- (i) the variables x_1, \dots, x_n are n -ary terms,
- (ii) if t_1, \dots, t_{n_i} are n_i -ary terms then $f_i(t_1, \dots, t_{n_i})$ is an n -ary term ($i \in I$).

Let $W_\tau(X_n)$ be the set of all n -ary terms, i.e., the smallest set which contains x_1, \dots, x_n and is closed under finite application of (ii). Let $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ denote the set of all terms of type τ . Note that $W_\tau(X_m) \subseteq W_\tau(X_n)$ for $m \leq n$, i.e. every m -ary term also can be considered as n -ary. For convenience variable symbols can also be denoted by other symbols like x, y, z .

Let $Alg(\tau)$ be the class of all algebras of type τ and let $\underline{A} = (A; (f_i^A)_{i \in I}) \in Alg(\tau)$. To each n -ary term $t \in W_\tau(X_n)$ one can consider the n -ary *term operation* $t^A : A^n \rightarrow A$ of \underline{A} defined inductively as follows:

- (1) $x_i^A := e_i^{n,A}$ ($1 \leq i \leq n$),
- (2) $f_i(t_1, \dots, t_{n_i})^A := f_i^A[t_1^A, \dots, t_{n_i}^A]$ (composition, cf. 1.1).

1.5. Identities and models. A pair (s, t) of terms is called equation or *identity* and we use the more convenient notation $s \approx t$ for identities. An algebra $\underline{A} \in Alg(\tau)$ *satisfies* an identity $s \approx t$ (or $s \approx t$ is *valid* in \underline{A} , or simply: is an identity in \underline{A}) if $s^A = t^A$ (this

makes sense since, due to 1.4, we can assume that both terms are n -ary for some n), notation: $\underline{A} \models s \approx t$. Then \underline{A} is called *model* of $s \approx t$.

1.6. The Galois connection $Mod - Id$. According to 1.3 the relation

$$\models \subseteq Alg(\tau) \times (W_\tau(X) \times W_\tau(X))$$

induces a Galois connection, namely (Id, Mod) , defined for classes $K \subseteq Alg(\tau)$ of algebras and classes Σ of identities as follows:

$$\begin{aligned} Id K &:= \{s \approx t \mid s, t \in W_\tau(X) \text{ and } K \models s \approx t\}, \\ Mod \Sigma &:= \{\underline{A} \in Alg(\tau) \mid \underline{A} \models \Sigma\}. \end{aligned}$$

The Galois closed sets are well-known by Birkhoff's theorems, namely the *varieties*

$$V(K) = Mod Id K$$

generated by K on the side of algebras, and the *equational theories*

$$Id Mod \Sigma$$

generated by Σ on the other side of identities. The closure system $\mathcal{L}(\tau) := \mathcal{H}_{Mod Id}$ (cf. 1.3(vii)) of all varieties is a complete lattice with $Alg(\tau)$ as greatest and the variety I of all one-element algebras as least element. Note that $Alg(\tau) = Mod\{x \approx x\}$ and $I = Mod\{x \approx y\}$. The dual lattice is the complete lattice $\mathcal{E} := \mathcal{H}_{Id Mod}$ of all equational theories of type τ .

2 Extensive and additive operators

2.1. Definitions. An operator $\gamma : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ is *extensive* if $T \subseteq \gamma(T)$ for all $T \subseteq A$. It is called *additive* if $\gamma(T) = \bigcup_{a \in T} \gamma(a)$ for all $T \subseteq A$ (we write $\gamma(a)$ instead of $\gamma(\{a\})$). We note that such operators For two operators $\gamma_1 : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$, $\gamma_2 : \mathcal{P}(B) \rightarrow \mathcal{P}(B)$ and a relation $R \subseteq A \times B$ we define the relations

$$\begin{aligned} R_{\gamma_1} &:= \{(t, s) \in A \times B \mid \gamma_1(t) \times \{s\} \subseteq R\}, \\ R_{\gamma_2} &:= \{(t, s) \in A \times B \mid \{t\} \times \gamma_2(s) \subseteq R\}. \end{aligned}$$

According to 1.3 these relations R , R_{γ_1} and R_{γ_2} induce Galois connections between A and B which we shall denote by (μ, ι) , $(\mu_{\gamma_1}, \iota_{\gamma_1})$ and $(\mu_{\gamma_2}, \iota_{\gamma_2})$, respectively.

In the following we assume that γ_1, γ_2 are extensive and additive.

From [Arw-D;97], [Den;97₂] we collect the following properties:

2.2. Lemma. *With the notations from 2.1 we have for extensive and additive operators γ_1, γ_2 and for all $T \subseteq A, S \subseteq B$:*

- (i) $\mu_{\gamma_1}(T) = \mu\gamma_1(T)$, (i') $\iota_{\gamma_2}(S) = \iota\gamma_2(S)$,
- (ii) $\mu_{\gamma_1}(T) \subseteq \mu(T)$, (ii') $\iota_{\gamma_2}(S) \subseteq \iota(S)$,
- (iii) $\iota_{\gamma_1}(S) \subseteq \iota(S)$, (iii') $\mu_{\gamma_2}(T) \subseteq \mu(T)$,
- (iv) $\gamma_1 \iota_{\gamma_1}(S) \subseteq \iota(S)$. (iv') $\gamma_2 \mu_{\gamma_2}(T) \subseteq \mu(T)$. \square

Additive operators $\gamma : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ are monotone (isotone) (i.e. $T_1 \subseteq T_2$ implies $\gamma(T_1) \subseteq \gamma(T_2)$ for $T \subseteq A$). It has been known for a long time that every isotone mapping of a complete lattice into itself has a fixed point (see e.g. [Kna;27]). The following theorem characterizes the fixed points of γ_1 and of γ_2 .

2.3. Theorem. *Let γ_1, γ_2 be extensive, additive operators on A and on B , respectively. Let $R \subseteq A \times B$. With the notations from 2.1 we have: For every $T \subseteq A$ with $\iota\mu(T) = T$ and for every $S \subseteq B$ with $\mu\iota(S) = S$ the following conditions (i) – (iii) and (i') – (iii'), respectively, are equivalent:*

- (i) $T = \iota\mu_{\gamma_1}(T)$, (i') $S = \mu\iota_{\gamma_2}(S)$,
- (ii) $\gamma_1(T) = T$, (ii') $\gamma_2(S) = S$,
- (iii) $\mu_{\gamma_1}(T) = \mu(T)$. (iii') $\iota_{\gamma_2}(S) = \iota(S)$. \square

PROOF. We will only show the equivalence of (i), (ii), and (iii). The other equivalences can be shown analogously.

(i) \Rightarrow (ii): $T \subseteq \gamma_1(T) \subseteq \iota\mu\gamma_1(T) = \iota\mu_{\gamma_1}(T) = T$ because of the extensivity of γ_1 , 1.3(iii) and 2.2(i).

(ii) \Rightarrow (iii): $\gamma_1(T) = T \Rightarrow \mu_{\gamma_1}(T) = \mu\gamma_1(T) = \mu(T)$ because of 2.2(i).

(iii) \Rightarrow (i): $\mu_{\gamma_1}(T) = \mu(T) \Rightarrow \iota\mu_{\gamma_1}(T) = \iota\mu(T) = T$. \square

2.4. Remarks. Let γ_1, γ_2 be additive closure operators. Then we have:

- (a) For all $T \subseteq A$ with $\iota\mu(T) = T$ and $S \subseteq B$ with $\mu\iota(S) = S$ there holds:

$$\gamma_1(T) = T \iff \iota_{\gamma_1}\mu_{\gamma_1}(T) = T ,$$

$$\gamma_2(S) = S \iff \mu_{\gamma_2}\iota_{\gamma_2}(S) = S .$$

If γ_1 is not idempotent then only one direction of these equivalences is satisfied (namely from left to right as proved in the next proposition).

- (b) For arbitrary $T \subseteq A$ and $S \subseteq B$ we have:

$$\gamma_2\mu_{\gamma_2}(T) = \mu_{\gamma_2}(T) , \quad \gamma_1\iota_{\gamma_1}(S) = \iota_{\gamma_1}(S) .$$

(a) was shown in [Den-R;95]. In order to see (b), we note that $s \in \mu_{\gamma_2}(T)$ implies $\{t\} \times \gamma_2(s) \subseteq R$ (by definition) for every $t \in T$. Consequently $\{t\} \times \gamma_2(s') \subseteq R$ for every $s' \in \gamma_2(s)$ (since γ_2 is a closure operator), and thus $\gamma_2(s) \subseteq \mu_{\gamma_2}(T)$. Finally this gives $\gamma_2\mu_{\gamma_2}(T) \subseteq \bigcup\{\gamma_2(s) \mid s \in \mu_{\gamma_2}(T)\} \subseteq \mu_{\gamma_2}(T)$. \square

2.5. Proposition. Let γ_1 be an extensive, additive operator on the set A , and let $R, R_{\gamma_1} \subseteq A \times B$, (μ, ι) , $(\mu_{\gamma_1}, \iota_{\gamma_1})$ as defined in 2.1. Then for all $T \subseteq A$ with $\iota\mu(T) = T$ we have

$$\gamma_1(T) = T \implies \iota_{\gamma_1}\mu_{\gamma_1}(T) = T .$$

PROOF. Using 2.3((i) \Leftrightarrow (ii)) we have $\gamma_1(T) = T \implies \iota\mu_{\gamma_1}(T) = T$. By 2.2(iii) we have $\iota_{\gamma_1}\mu_{\gamma_1}(T) \subseteq \iota\mu_{\gamma_1}(T) = T$. Together with $T \subseteq \iota_{\gamma_1}\mu_{\gamma_1}(T)$ this gives $\iota_{\gamma_1}\mu_{\gamma_1}(T) = T$. \square

An analogous proposition holds for $S \subseteq B$ and γ_2 .

2.6. Proposition. With the notations from 2.1 we have for all $T \subseteq A$ and $S \subseteq B$:

- (i) $\gamma_1(T) \subseteq \iota\mu(T) \iff \iota\mu_{\gamma_1}(T) \subseteq \iota\mu(T),$
- (ii) $\gamma_1 \iota\mu(T) = \iota\mu(T) \implies \gamma_1(T) \subseteq \iota\mu(T),$
- (iii) $\gamma_1 \iota_{\gamma_1}(S) = \iota_{\gamma_1}(S) \implies \mu \iota(S) \subseteq \mu_{\gamma_1} \iota_{\gamma_1}(S),$
- (iv) $\gamma_2(S) \subseteq \mu \iota(S) \iff \mu \iota_{\gamma_2}(S) \subseteq \mu \iota(S),$
- (v) $\gamma_2 \mu \iota(S) = \mu \iota(S) \implies \gamma_2(S) \subseteq \mu \iota(S)$
- (vi) $\gamma_2 \mu_{\gamma_2}(T) = \mu_{\gamma_2}(T) \implies \iota\mu(T) \subseteq \iota_{\gamma_2} \mu_{\gamma_2}(T),$
- (vii) $\iota_{\gamma_2} \mu_{\gamma_2}(T) = T \implies \iota\mu(T) = T,$
- (viii) $\mu_{\gamma_1} \iota_{\gamma_1}(S) = S \implies \mu \iota(S) = S.$

PROOF. The proof easily follows from the definitions. We only show (vii): Indeed, since $\gamma_2 \mu_{\gamma_2}(T) \subseteq \mu(T)$ implies $T = \iota_{\gamma_2} \mu_{\gamma_2}(T) = \iota_{\gamma_2} \mu_{\gamma_2}(T) \supseteq \iota\mu(T) \supseteq T$ (cf. 2.2(iv') and (i')), we immediately get (vii). \square

2.7. The three Galois connections (μ, ι) , $(\mu_{\gamma_1}, \iota_{\gamma_1})$ and $(\mu_{\gamma_2}, \iota_{\gamma_2})$ provide six complete lattices: $\mathcal{H}_{\mu\iota}$, $\mathcal{H}_{\iota\mu}$, $\mathcal{H}_{\mu_{\gamma_1}\iota_{\gamma_1}}$, $\mathcal{H}_{\iota_{\gamma_1}\mu_{\gamma_1}}$, $\mathcal{H}_{\mu_{\gamma_2}\iota_{\gamma_2}}$ and $\mathcal{H}_{\iota_{\gamma_2}\mu_{\gamma_2}}$ (cf. 1.3).

By a result of TARSKI ([Tar;53]), the set of all fixed points of an isotone mapping which maps a complete lattice into itself is also a complete lattice with respect to the same order as in the given complete lattice (for closure operators this result can be found also in [War;42]). Since additive operators are isotone we get two more complete lattices:

$$\begin{aligned} \mathcal{S}_{\gamma_1} &:= \{T \subseteq A \mid \gamma_1(T) = T \text{ and } \iota\mu(T) = T\} \quad \text{and} \\ \mathcal{S}_{\gamma_2} &:= \{S \subseteq B \mid \gamma_2(S) = S \text{ and } \mu\iota(S) = S\}. \end{aligned}$$

By definition, $\mathcal{S}_{\gamma_1} \subseteq \mathcal{H}_{\iota\mu}$ and $\mathcal{S}_{\gamma_2} \subseteq \mathcal{H}_{\mu\iota}$; by Proposition 2.5 we have $\mathcal{S}_{\gamma_1} \subseteq \mathcal{H}_{\iota_{\gamma_1}\mu_{\gamma_1}}$ and $\mathcal{S}_{\gamma_2} \subseteq \mathcal{H}_{\iota_{\gamma_2}\mu_{\gamma_2}}$. Moreover we have the following inclusions: $\mathcal{H}_{\iota_{\gamma_2}\mu_{\gamma_2}} \subseteq \mathcal{H}_{\iota\mu}$ and $\mathcal{H}_{\mu_{\gamma_2}\iota_{\gamma_2}} \subseteq \mathcal{H}_{\mu\iota}$.

Because in all these complete lattices the meet-operation is the set-theoretical intersection we obtain (see Figure 1):

- $\mathcal{H}_{\iota_{\gamma_2}\mu_{\gamma_2}}$ and \mathcal{S}_{γ_1} are complete meet-subsemilattices of $\mathcal{H}_{\iota\mu}$,
- $\mathcal{H}_{\mu_{\gamma_1}\iota_{\gamma_1}}$ and \mathcal{S}_{γ_2} are complete meet-subsemilattices of $\mathcal{H}_{\mu\iota}$,
- \mathcal{S}_{γ_1} is a complete meet-subsemilattice of $\mathcal{H}_{\iota_{\gamma_1}\mu_{\gamma_1}}$ and
- \mathcal{S}_{γ_2} is a complete meet-subsemilattice of $\mathcal{H}_{\mu_{\gamma_2}\iota_{\gamma_2}}$.

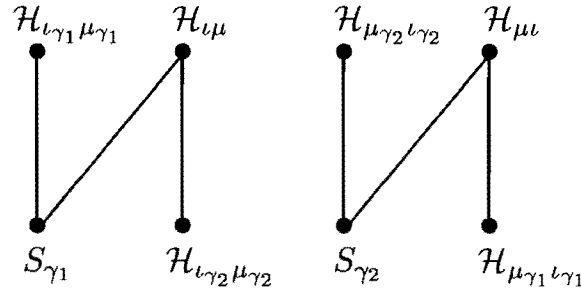


Figure 1: Diagram of inclusions

In the next section we shall study under which conditions the lattices $\mathcal{H}_{\iota_{\gamma_2}\mu_{\gamma_2}}$ and $\mathcal{H}_{\iota_{\gamma_1}\mu_{\gamma_1}}$ are complete sublattices of $\mathcal{H}_{\iota\mu}$, and when $\mathcal{H}_{\mu_{\gamma_1}\iota_{\gamma_1}}$ and $\mathcal{H}_{\mu_{\gamma_2}\iota_{\gamma_2}}$ are complete sublattices of $\mathcal{H}_{\mu\iota}$.

3 Closed subrelations

The relations $R_{\gamma_1}, R_{\gamma_2}$ (cf. 2.1) are subrelations of R . By a result in [Gan-W;96, p.112, Satz 13], (cf. [Den;97₁]) the corresponding lattices of Galois closed sets are complete sublattices of $\mathcal{H}_{\mu\iota}$ whenever these relations are closed subrelations in the following sense (cf.[Gan-W;96, p.112, Def. 50]):

3.1. Definition. Let $R, R' \subseteq A \times B$ be relations and let $(\mu, \iota), (\mu', \iota')$ be the corresponding induced Galois relations.

Then R' is called a *closed subrelation* of R if

- (i) $R' \subseteq R$,
- (ii) For all $T \subseteq A$ and $S \subseteq B$ there holds: if $\mu'(T) = S$ and $\iota'(S) = T$ then $\mu(T) = S$ and $\iota(S) = T$.

Directly from the definitions we get the following characterizations of closed subrelations ([Gan-W;96], [Den;97₂]).

3.2. Remark. *The following are equivalent:*

- (i) R' is a closed subrelation of $R \subseteq A \times B$,
- (ii) $\iota' \mu'(T) = T \implies \mu'(T) = \mu(T)$ and $\mu' \iota'(S) = S \implies \iota'(S) = \iota(S)$,

- (iii) $\mu' \iota'(S) = \mu \iota(S)$ and $\iota' \mu'(T) = \iota \mu(T)$ for all $T \subseteq A$ and for all $S \subseteq B$. \square

Examples of closed subrelations are provided by so-called conjugate pairs of closure operators:

3.3. Definition. Let $R \subseteq A \times B$ be a relation. A pair $\gamma = (\gamma_1, \gamma_2)$ of operators $\gamma_1 : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ and $\gamma_2 : \mathcal{P}(B) \rightarrow \mathcal{P}(B)$ is called *conjugate pair with respect to R* if for all $t \in A$ and $s \in B$ we have:

$$\gamma_1(t) \times \{s\} \subseteq R \iff \{t\} \times \gamma_2(s) \subseteq R .$$

According to 2.1, γ is a conjugate pair w.r.t. R iff $R_{\gamma_1} = R_{\gamma_2}$ iff $(\mu_{\gamma_1}, \iota_{\gamma_1}) = (\mu_{\gamma_2}, \iota_{\gamma_2})$. Thus we can define $R_\gamma := R_{\gamma_1} = R_{\gamma_2}$ and $(\mu_\gamma, \iota_\gamma) := (\mu_{\gamma_1}, \iota_{\gamma_1})$. Now we have:

3.4. Theorem. Let $R \subseteq A \times B$ be a relation between sets A and B and let $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_1 : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$, $\gamma_2 : \mathcal{P}(B) \rightarrow \mathcal{P}(B)$ be a conjugate pair of additive closure operators. Then R_γ is a closed subrelation of R .

PROOF. We shall prove 3.2(ii) with $R' := R_\gamma = R_{\gamma_1} = R_{\gamma_2}$. Let (μ, ι) and (μ', ι') be the Galois connections induced by R and R' , respectively. Assume $\iota' \mu'(T) = T$. Then, by 2.6(vii), we also have $\iota \mu(T) = T$ (using $\iota' \mu' = \iota_{\gamma_2} \mu_{\gamma_2}$). Therefore we can apply 2.4(a) (using $\iota' \mu' = \iota_{\gamma_1} \mu_{\gamma_1}$) and get $\gamma_1(T) = T$. From 2.3(ii) \Rightarrow (iii) we finally conclude $\mu'(T) = \mu(T)$. This shows the first condition in 3.2(ii). The second one follows analogously. \square

Further properties are collected in the following lemma.

3.5. Lemma. Let $\gamma = (\gamma_1, \gamma_2)$ be a conjugate pair of additive closure operators with respect to the relation $R \subseteq A \times B$. Then for all $T \subseteq A$, $S \subseteq B$ we have:

- | | | | |
|-------|--|--------|--|
| (i) | $\gamma_2 \mu_\gamma(T) = \mu_\gamma(T)$, | (i') | $\gamma_1 \iota_\gamma(S) = \iota_\gamma(S)$, |
| (ii) | $\gamma_1 \iota \mu_\gamma(T) = \iota \mu_\gamma(T)$, | (ii') | $\gamma_2 \mu \iota_\gamma(S) = \mu \iota_\gamma(S)$, |
| (iii) | $\mu_\gamma \iota_\gamma(S) = \mu \iota_\gamma(S)$, | (iii') | $\iota_\gamma \mu_\gamma(T) = \iota \mu_\gamma(T)$. |
- \square

The above mentioned one-to-one-correspondence between closed subrelations and complete sublattices is specified by the following theorem:

3.6. Theorem([Gan-W;96], [Den;97₁]). *Let $R', R \subseteq A \times B$ be relations and let (μ', ι') , (μ, ι) be the corresponding induced Galois connections. If R' is a closed subrelation of R , then $\underline{\mathcal{U}}(R') := \mathcal{H}_{\iota'\mu'}$ is a complete sublattice of $\mathcal{H}_{\iota\mu}$.*

Conversely, for every complete sublattice \mathcal{U} of $\mathcal{H}_{\iota\mu}$ the relation

$$\underline{R}(\mathcal{U}) := \bigcup \{T \times S \mid T \in \mathcal{U} \text{ and } S = \mu(T)\}$$

is a closed subrelation of R . Moreover, we have $\underline{\mathcal{U}}(\underline{R}(\mathcal{U})) = \mathcal{U}$, $\underline{R}(\underline{\mathcal{U}}(R')) = R'$. \square

Theorem 3.6 together with Theorem 3.4 answers the question from the end of Section 2 for a conjugate pair $\gamma = (\gamma_1, \gamma_2)$ of additive closure operators, namely:

3.7. Theorem. *If $\gamma = (\gamma_1, \gamma_2)$ is a conjugate pair of additive closure operators with respect to the relation R then $\mathcal{S}_{\gamma_1} = \mathcal{H}_{\iota_\gamma \mu_\gamma}$ ($= \mathcal{H}_{\iota_{\gamma_1} \mu_{\gamma_1}} = \mathcal{H}_{\iota_{\gamma_2} \mu_{\gamma_2}}$) and this is a complete sublattice of $\mathcal{H}_{\iota\mu}$. Dually, $\mathcal{S}_{\gamma_2} = \mathcal{H}_{\mu_\gamma \iota_\gamma}$ is a complete sublattice of $\mathcal{H}_{\mu\iota}$.*

We want to add here a direct proof without using Theorem 3.6. The results of the next section will give additional evidence of this Theorem.

PROOF. From the definitions in 2.7 and 2.4, 2.6(vii),(viii) we get $\mathcal{S}_{\gamma_1} = \mathcal{H}_{\iota_\gamma \mu_\gamma}$ and $\mathcal{S}_{\gamma_2} = \mathcal{H}_{\mu_\gamma \iota_\gamma}$. From 2.7 we also know that these lattices are complete meet-subsemilattices of $\mathcal{H}_{\iota\mu}$ and $\mathcal{H}_{\mu\iota}$, respectively. Let $\{T_j \mid j \in J\} \subseteq \mathcal{H}_{\iota_\gamma \mu_\gamma} = \mathcal{S}_{\gamma_1}$. Then, by Theorem 2.3, we have $\mu_\gamma(T_j) = \mu(T_j)$ for all $j \in J$. Now, using the Galois connection properties for μ and μ_γ , we conclude

$$\mu\left(\bigcup_{j \in J} T_j\right) = \bigcap_{j \in J} \mu(T_j) = \bigcap_{j \in J} \mu_\gamma(T_j) = \mu_\gamma\left(\bigcup_{j \in J} T_j\right). \quad (*)$$

Further we get:

$$\bigvee_{j \in J} T_j = \iota_\gamma \mu_\gamma\left(\bigcup_{j \in J} T_j\right) \stackrel{3.2(\text{iii})}{=} \iota_\mu \mu_\gamma\left(\bigcup_{j \in J} T_j\right) \stackrel{(*)}{=} \iota_\mu \mu\left(\bigcup_{j \in J} T_j\right) = \bigvee_{j \in J} T_j$$

(where $\bigvee^{\mathcal{H}}$ denotes the join (supremum) in a complete lattice \mathcal{H}). \square
 Note that $\mathcal{H}_{\iota\gamma\mu\gamma} = \mathcal{S}_{\gamma_1} = \mathcal{H}_{\gamma_1} \cap \mathcal{H}_{\iota\mu}$, where $\mathcal{H}_{\gamma_1} := \{T \subseteq A \mid \gamma_1(T) = T\}$ denotes the closure system of γ_1 .

4 Closure and kernel operators on complete lattices

From now on we assume that $\gamma_1 : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ and $\gamma_2 : \mathcal{P}(B) \rightarrow \mathcal{P}(B)$ form a conjugate pair of additive closure operators (with respect to a relation $R \subseteq A \times B$). We ask how the lattice \mathcal{S}_{γ_1} is situated in the lattice $\mathcal{H}_{\iota\mu}$: for this aim we ask for the least γ_1 -closed class ${}^{\iota}T$ containing a set $T \in \mathcal{H}_{\iota\mu}$ and for the greatest γ_1 -closed class ${}_{\gamma_1}T$ contained in T (cf. 2.7). The corresponding operators $\Gamma_1 : T \mapsto {}^{\iota}T$ and $\Gamma_2 : T \mapsto {}_{\gamma_1}T$ will be studied in more detail.

We remark that all results in this section can be dualized for the closure operator γ_2 just interchanging μ and ι .

4.1. Definition. Let T be an arbitrary subset of A . Then we define (cf. 2.7):

$$\begin{aligned} {}^{\iota}T &:= \bigcap \{T' \in \mathcal{S}_{\gamma_1} \mid T' \supseteq T\} \\ {}_{\gamma_1}T &:= \bigvee_{\mathcal{S}_{\gamma_1}} \{T' \in \mathcal{S}_{\gamma_1} \mid T' \subseteq T\} = \bigvee_{\mathcal{H}_{\iota\mu}} \{T' \in \mathcal{S}_{\gamma_1} \mid T' \subseteq T\}. \end{aligned}$$

(The last equation holds since \mathcal{S}_{γ_1} is a complete sublattice of $\mathcal{H}_{\iota\mu}$ by 3.7.)

These sets have the following properties:

4.2. Proposition. *Let $T \subseteq A$. Then:*

(a) ${}^{\iota}T = \iota\gamma\mu\gamma(T) = \iota\mu\gamma(T) = \iota\mu\gamma_1(T)$, in particular, ${}^{\iota}T$ is the $\iota\gamma\mu\gamma$ -closed class generated by T .

(b) If $T = \iota\mu(T)$ then we have

$$(i) \quad {}_{\gamma_1}T = T \iff {}^{\iota}T = T \iff \gamma_1(T) = T,$$

- (ii) $\gamma_1 T$ is the greatest $\iota_\gamma \mu_\gamma$ -closed class contained in T ,
- (iii) $\gamma_1 T = \iota_\gamma \mu(T) = \iota_{\gamma_2} \mu(T)$.

PROOF. (a): Recall $\mathcal{S}_{\gamma_1} = \mathcal{H}_{\iota_\gamma \mu_\gamma}$. Thus the closure operators $T \mapsto \gamma_1 T$ and $T \mapsto \iota_\gamma \mu_\gamma(T)$ corresponding to these closure systems must coincide. Further, $\iota_\gamma \mu_\gamma(T) \in \mathcal{S}_{\gamma_1} = \mathcal{H}_{\gamma_1} \cap \mathcal{H}_{\iota_\mu}$ implies (cf 2.3(ii) \Rightarrow (i)) $\iota_\gamma \mu_\gamma(T) = \iota_{\mu_\gamma} \iota_\gamma \mu_\gamma(T) = \iota_\mu(T)$. The last equality follows from 2.2(i).

(b): (i): The first equivalence follows from the definitions, the second one follows directly from 2.4(a) and $\mathcal{S}_{\gamma_1} = \mathcal{H}_{\iota_\gamma \mu_\gamma}$.

(ii): From Definition 4.1 we directly conclude that $\gamma_1 T \in \mathcal{S}_{\gamma_1} = \mathcal{H}_{\iota_\gamma \mu_\gamma} \subseteq \mathcal{H}_{\iota_\mu}$ is $\iota_\gamma \mu_\gamma$ -closed and contained in T (the latter because T is assumed to be ι_μ -closed). It is the largest such set since every $\iota_\gamma \mu_\gamma$ -closed T' is also in \mathcal{S}_{γ_1} and therefore contained in $\gamma_1 T$ by definition.

(iii) From the definitions and the properties of a Galois connection we conclude for any $T, T' \in \mathcal{H}_{\iota_\mu}$ and $S' \in \mathcal{H}_{\mu}$:

$$\begin{aligned} T' \subseteq T &\iff \mu(T') \supseteq \mu(T) \\ S' = \mu(T') \in \mathcal{S}_{\gamma_2} &\iff T' = \iota(S') \in \mathcal{S}_{\gamma_1}. \end{aligned}$$

Thus we have:

$$\begin{aligned} \gamma_1 T &= \bigvee^{\mathcal{H}_{\iota_\mu}} \{T' \in \mathcal{S}_{\gamma_1} \mid T' \subseteq T\} = \iota_\mu \cup \{T' \in \mathcal{S}_{\gamma_1} \mid T' \subseteq T\} \\ &= \iota \cap \{\mu(T') \in \mathcal{S}_{\gamma_2} \mid T' \subseteq T\} \\ &= \iota \cap \{\mu(T') \in \mathcal{S}_{\gamma_2} \mid \mu(T') \supseteq \mu(T)\} \\ &= \iota \cap \{S' \in \mathcal{S}_{\gamma_2} \mid S' \supseteq \mu(T)\} = \iota_{\gamma_2} \mu(T) \stackrel{2.2(i')}{=} \iota_\gamma \mu(T). \quad \square \end{aligned}$$

4.3. γ_1 -intervals. For $T \in \mathcal{H}_{\iota_\mu}$ let $[\gamma_1 T, \gamma_1 T]$ denote the interval between $\gamma_1 T$ and $\gamma_1 T$ in the lattice \mathcal{H}_{ι_μ} . Such intervals will be called γ_1 -intervals in \mathcal{H}_{ι_μ} .

Different sets T may have the same γ_1 -interval. Therefore one can define on \mathcal{H}_{ι_μ} the equivalence relation $T_1 \sim T_2 : \iff [\gamma_1 T_1, \gamma_1 T_1] = [\gamma_1 T_2, \gamma_1 T_2]$.

Sets $T \in \mathcal{H}_{\iota_\mu}$ are called $\iota_\gamma \mu_\gamma$ -collapsing if $[\gamma_1 T, \gamma_1 T] = \{T\}$, i.e. if the interval ‘‘collapses’’. Thus collapsing sets are uniquely characterized by their $\iota_\gamma \mu_\gamma$ -closure (cf. 4.2).

4.4. Proposition. *Let T, T_1, T_2 be $\iota\mu$ -closed subsets of A . Then we have:*

- (i) $T \in \mathcal{S}_{\gamma_1}$ iff $[\gamma_1 T, \gamma_1 T] = \{T\}$,
- (ii) $T_1 \subseteq T_2$ implies $\gamma_1 T_1 \subseteq \gamma_1 T_2$ and $\gamma_1 T_1 \subseteq \gamma_1 T_2$,
- (iii) $\gamma_1 T_1 \overset{\mathcal{H}_{\iota\mu}}{\wedge} \gamma_1 T_2 = \gamma_1 (T_1 \overset{\mathcal{H}_{\iota\mu}}{\wedge} T_2)$,
- (iv) $\gamma_1 T_1 \overset{\mathcal{H}_{\iota\mu}}{\vee} \gamma_1 T_2 = \gamma_1 (T_1 \overset{\mathcal{H}_{\iota\mu}}{\vee} T_2)$.

PROOF. (i) and (ii) directly follow from Definition 4.1.

(iii): Since \mathcal{S}_{γ_1} is a complete sublattice of $\mathcal{H}_{\iota\mu}$ (cf. 3.7), the left hand side is an element of \mathcal{S}_{γ_1} which is contained in both, T_1 and T_2 , consequently it is also contained in the greatest set from \mathcal{S}_{γ_1} containing T_1 and T_2 , i.e. in the right side of (iii).

Conversely, the right hand side of (iii) is (by definition) an element of \mathcal{S}_{γ_1} and it is contained (by (ii)) in both, $\gamma_1 T_1$ and $\gamma_1 T_2$, and thus also in the left hand side.

(iv) is dual to (iii) (change “ \wedge ”, “is contained in”, “ $\gamma_1 T$ ” into “ \vee ”, “contains”, “ $\gamma_1 T$ ”). \square

4.5. Proposition.

- (i) *The mapping $\Gamma_1 : \mathcal{H}_{\iota\mu} \rightarrow \mathcal{H}_{\iota\mu} : T \mapsto \gamma_1 T$ is a closure operator on $\mathcal{H}_{\iota\mu}$ satisfying*

$$\gamma_1 \left(\overset{\mathcal{H}_{\iota\mu}}{\bigvee} \{T_i \mid i \in I\} \right) = \overset{\mathcal{H}_{\iota\mu}}{\bigvee} \{\gamma_1 T_i \mid i \in I\}.$$

- (ii) *The mapping $\Gamma_2 : \mathcal{H}_{\iota\mu} \rightarrow \mathcal{H}_{\iota\mu} : T \mapsto \gamma_1 T$ is a kernel operator on $\mathcal{H}_{\iota\mu}$ satisfying*

$$\gamma_1 \left(\bigcap \{T_i \mid i \in I\} \right) = \bigcap \{\gamma_1 T_i \mid i \in I\}.$$

PROOF. The equations are generalizations of 4.4(iv) and (iii) and can be proved in the same manner (note that \bigcap equals $\overset{\mathcal{H}_{\iota\mu}}{\wedge}$ in the lattice $\mathcal{H}_{\iota\mu}$).

Γ_1 is a closure operator since it coincides with the closure operator $\iota_\gamma \mu_\gamma$ (cf. 4.2(a)).

What concerns Γ_2 , it is isotone by 4.4(ii), intensive, i.e. $\Gamma_2(T) \subseteq T$ for $T \in \mathcal{H}_{\iota\mu}$, by 4.2(b(ii)), and idempotent since $\gamma_1 T \in \mathcal{S}_{\gamma_1}$ is γ_1 -closed and therefore $\gamma_1(\gamma_1 T) = \gamma_1 T$ by 4.2(b(i)). Thus Γ_2 is a kernel operator. \square

4.6. Remarks. From the definitions and 4.4(iii),(iv) we conclude the following properties of Γ_1 and Γ_2 :

- a) The mapping Γ_1 is a join-retraction from $\mathcal{H}_{\iota\mu}$ onto $\mathcal{S}_{\gamma_1} \subseteq \mathcal{H}_{\iota\mu}$, i.e. an idempotent join homomorphism which is the identity map on \mathcal{S}_{γ_1} .
- b) Analogously, the mapping Γ_2 is a meet-retraction from $\mathcal{H}_{\iota\mu}$ onto \mathcal{S}_{γ_1} , i.e. an idempotent meet homomorphism which is the identity map on \mathcal{S}_{γ_1} .
- c) Note that, in general, Γ_1 does not preserve meets and Γ_2 does not preserve joins.

4.7. There is a well-known 1-1-correspondence between closure operators $\varphi : \mathcal{L} \rightarrow \mathcal{L}$ on a complete lattice \mathcal{L} and closure systems \mathcal{S} on \mathcal{L} (i.e. subsets of \mathcal{L} closed under arbitrary meets), namely via:

$$\mathcal{S} := \text{Fix}(\varphi) := \mathcal{H}_\varphi = \{T \in \mathcal{L} \mid \varphi(T) = T\}, \text{ i.e., assigning (as closure system) the set of all fixed points to a given closure operator } \varphi : \mathcal{L} \rightarrow \mathcal{L};$$

$$\varphi(T) := \varphi_{\mathcal{S}}(T) := \bigwedge^{\mathcal{L}} \{T' \in \mathcal{S} \mid T \leq T'\} \text{ for } T \in \mathcal{L} \\ \text{(where } \leq \text{ denotes the ordering in the lattice } \mathcal{L}\text{).}$$

We have $\mathcal{H}_{\varphi_{\mathcal{S}}} = \mathcal{S}$ and $\varphi_{\mathcal{H}_\varphi} = \varphi$ for closure systems \mathcal{S} and closure operators φ .

Dually one can consider the 1-1-correspondence between kernel operators $\psi : \mathcal{L} \rightarrow \mathcal{L}$ on a complete lattice \mathcal{L} and kernel systems \mathcal{S} on \mathcal{L} (i.e. subsets of \mathcal{L} closed under arbitrary joins), given as follows:

$$\mathcal{S} := \text{Fix}(\psi) := \{T \in \mathcal{L} \mid \psi(T) = T\};$$

$$\psi(T) := \psi_{\mathcal{S}}(T) := \bigvee^{\mathcal{L}} \{T' \in \mathcal{S} \mid T \leq T'\} \text{ for } T \in \mathcal{L}.$$

We have $\text{Fix}(\psi_{\mathcal{S}}) = \mathcal{S}$ and $\psi_{\text{Fix}(\psi)} = \psi$ for kernel systems \mathcal{S} and kernel operators ψ .

The result of Tarski (mentioned in 2.7) shows that \mathcal{H}_φ (for any closure operator φ) is always a complete lattice with respect to \leq . However it is not necessarily a sublattice of \mathcal{L} and there arises the question under which additional condition we get a complete sublattice. The answer (and its dual for $Fix(\psi)$) is the following

Proposition ([Den;97₁]). (a) $Fix(\varphi) = \mathcal{H}_\varphi$ is a complete sublattice of \mathcal{L} if and only if φ satisfies

$$\varphi(\bigvee^{\mathcal{L}}\{T_i \mid i \in I\}) = \bigvee^{\mathcal{L}}\{\varphi(T_i) \mid i \in I\}. \quad (*)$$

(b) $Fix(\psi)$ is a complete sublattice of \mathcal{L} if and only if ψ satisfies

$$\psi(\bigwedge^{\mathcal{L}}\{T_i \mid i \in I\}) = \bigwedge^{\mathcal{L}}\{\psi(T_i) \mid i \in I\}. \quad (**)$$

Because of 4.5(i) we can apply this Proposition to the situation $\varphi = \Gamma_1$ ($= \iota_\gamma \mu_\gamma$, cf. 4.2(a)), $\psi = \Gamma_2$ ($= \iota_\gamma \mu$, cf. 4.2(b)) and $\mathcal{L} = \mathcal{H}_{\iota\mu}$, thus getting further proofs of 3.7, that $\mathcal{S}_{\gamma_1} = \mathcal{H}_{\iota_\gamma \mu_\gamma} = Fix(\varphi) = Fix(\psi)$ is a complete sublattice of $\mathcal{H}_{\iota\mu}$.

5 Clones and varieties

In this section we shall apply the theory developed in the previous sections to the Galois connections $(\mu, \iota) = (Inv, Pol)$ and $(\mu, \iota) = (Id, Mod)$ (cf. 1.3 and 1.6).

A. The Galois connection (Inv, Pol) and G-clones

5.1. Definitions. Let S_A be the full symmetric group of all permutations on a set A . For $f \in O_A^{(n)}$ (cf. 1.1), $\varrho \in Rel_A^{(h)}$ (cf. 1.2) and $s \in S_A$ we define $f^s \in O_A^{(n)}$ and $\varrho^s \in Rel_A^{(h)}$ by ([Gor-P;98])

$$\begin{aligned} f^s(a_1, \dots, a_n) &:= s(f(s^{-1}(a_1), s^{-1}(a_2), \dots, s^{-1}(a_n))) \\ &\quad \text{for } a_1, \dots, a_n \in A, \\ \varrho^s &:= \{(s(a_1), \dots, s(a_h)) \mid (a_1, \dots, a_h) \in \varrho\} \end{aligned}$$

Let $G \subseteq S_A$ be a subgroup of S_A , $F \subseteq O_A$ and $Q \subseteq Rel_A$. Then

we put

$$\begin{aligned}\gamma_1(F) &:= F^G := \{f^s \mid f \in F, s \in G\}, \\ \gamma_2(Q) &:= Q^G := \{\varrho^s \mid \varrho \in Q, s \in G\}.\end{aligned}$$

We adopt the convention $\gamma_1(f) := \gamma_1(\{f\})$ for $f \in O_A$ (analogously for $\gamma_2(\varrho)$).

A clone F ($= \text{Pol Inv } F$, cf. 1.1, 1.3) is called *G-clone* if $\gamma_1(F) = F$. Likewise a relational clone $Q = \text{Inv Pol } Q$ is called *relational G-clone*, if $\gamma_2(Q) = Q$.

G-clones were considered e.g. in [Gor-P;95], [Gor-P;98], [Hoa;92]. The special case where $f^s = f$ for each element f of the clone and some fixed permutation $s \in S_A$ was considered in [Dem-H;79], [Dem-H-M;79], [Dem-H-M;80], [Mar;79], and [Csá-G;80].

5.2. Lemma. *With the notations from 5.1 we have:*

- (i) $\gamma_1 : \mathcal{P}(O_A) \rightarrow \mathcal{P}(O_A)$ is an additive closure operator on O_A ;
- (ii) $\gamma_1 : \mathcal{P}(\text{Rel}_A) \rightarrow \mathcal{P}(\text{Rel}_A)$ is an additive closure operator on Rel_A ;
- (iii) For any $f \in O_A$ and $\varrho \in \text{Rel}_A$ it holds:

$$\gamma_1(f) = f^G \text{ preserves } \varrho \iff f \text{ preserves } \gamma_2(\varrho) = \varrho^G.$$

PROOF. The statements easily follow from the definitions, but can be found also in [Den;97₂], (iii) was proved in [Gor-P;95]. \square

From Lemma 5.2, the Definitions 1.1, 3.3 and Theorem 3.4 we obtain (with the notations from 5.1):

5.3. Theorem. $\gamma = (\gamma_1, \gamma_2)$ is a conjugate pair of additive closure operators with respect to the relation

$$R := \{(f, \varrho) \in O_A \times \text{Rel}_A \mid f \text{ preserves } \varrho\}.$$

The relation (cf. 2.1, 3.3)

$$R_\gamma = R_{\gamma_1} = \{(f, \varrho) \in O_A \times \text{Rel}_A \mid \gamma_1(f) \times \{\varrho\} \subseteq R\}$$

is a closed subrelation of R . \square

We recall (cf. 3.3) $R_\gamma = R_{\gamma_2} = \{(f, \varrho) \mid \{f\} \times \gamma_2(\varrho) \subseteq R\}$. The Galois connection $(\mu_\gamma, \iota_\gamma)$ induced by R_γ will be denoted by $(G\text{-Inv}, G\text{-Pol})$. With the definitions and results from the previous section we see, that G -clones are exactly the Galois closed sets with respect to the Galois connection $(G\text{-Inv}, G\text{-Pol})$; they form a complete lattice $\mathcal{S}_{\gamma_1} = \mathcal{H}_{G\text{-Pol} G\text{-Inv}}$ which we shall denote by $\mathbf{L}_{A,G}$. Thus we get:

5.4. Theorem. *Let G be a subgroup of S_A . Then the lattice $\mathbf{L}_{A,G}$ of all G -clones is a complete sublattice of the lattice \mathbf{L}_A of all clones. Moreover, if H is a subgroup of G , then $\mathbf{L}_{A,G}$ is a complete sublattice of $\mathbf{L}_{A,H}$.*

Analogously, the relational G -clones form a complete sublattice of the lattice of all relational clones. \square

5.5. Example. We consider the case $A = \{0, 1\}$ and $G = S_2 := S_A$. The lattice \mathbf{L}_2 of all clones on $A = \{0, 1\}$ is well-known. Since S_2 contains a single non-trivial function (negation $\neg : x \mapsto \neg x$), S_2 -clones F are those which are self-dual as sets (i.e. $F^s = F$ for negation $s = \neg$).

As shown e.g. in [Gor-P;98] there are exactly 14 S_2 -clones on $A = \{0, 1\}$:

$$O_1, O_4, O_8, O_9, L_1, L_4, L_5, D_2, D_1, D_3, A_4, A_1, C_4, C_1.$$

Here we use the notations for the clones given in [Jab-G-K;70] and describe them by a generating set:

$$\begin{aligned} O_1 &= J_A \text{ (projections),} \\ O_4 &= \langle \neg \rangle \text{ (projections and their negations),} \\ O_8 &= \langle c_0, c_1 \rangle \text{ (constant operations),} \\ O_9 &= \langle c_0, c_1, \neg \rangle \text{ (essentially unary operations),} \\ L_4 &= \langle g \rangle \text{ (linear idempotent operations, } g(x, y, z) := x + y + z), \\ L_5 &= \langle g, \neg \rangle \text{ (linear self-dual operations),} \\ L_1 &= \langle c_1, + \rangle \text{ (all linear operations, } + : (x, y) \mapsto x + y), \\ D_2 &= \langle h \rangle \text{ (self-dual monotone operations,} \\ &\quad h(x, y, z) := (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)), \\ D_1 &= \langle g, h \rangle \text{ (self-dual idempotent operations),} \end{aligned}$$

- $D_3 = \langle h, \neg \rangle$ (self-dual operations),
 $A_4 = \langle \wedge, \vee \rangle$ (monotone idempotent operations),
 $A_1 = \langle c_0, c_1, \wedge, \vee \rangle$ (monotone operations),
 $C_4 = \langle \wedge, \vee, g \rangle$ (idempotent operations),
 $C_1 = O_A$ (all operations).

The lattice L_{2,S_2} of S_2 -clones is given in Fig. 2.

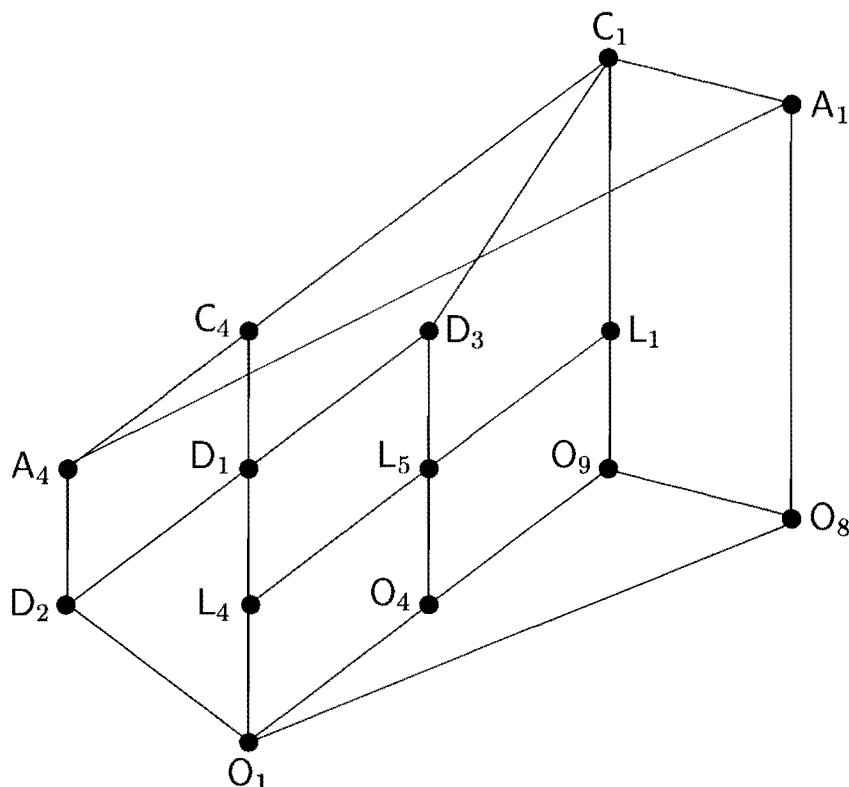


Figure 2: The lattice L_{2,S_2} of S_2 -clones

In case of a three-element set A there are already 48 S_3 -clones ([Gor-P;98]).

B. The Galois connection (Id, Mod) and solid varieties

According to 1.6 the Galois closed sets of the Galois connection $(\mu, \iota) = (Id, Mod)$ are varieties and equational theories (all of fixed type τ , cf. 1.4). We introduce a pair $(\gamma_1, \gamma_2) = (\chi_M^A, \chi_M^E)$ of closure

operators for every monoid M of hypersubstitutions as follows (for notations see also 1.4 - 1.6):

5.6. Definitions. A mapping

$$\sigma : \{f_i \mid i \in I\} \rightarrow W_\tau(X),$$

which assigns to every n_i -ary operation symbol f_i an n_i -ary term is called *hypersubstitution* (of type τ). There exists a unique extension

$$\hat{\sigma} : W_\tau(X) \rightarrow W_\tau(X) : t \mapsto \hat{\sigma}[t]$$

of σ to the set $W_\tau(X)$ of all terms, inductively defined by

- (i) $\hat{\sigma}[x] := x$ for any $x \in X$, and
- (ii) $\hat{\sigma}[f_i(t_1, \dots, t_{n_i})] := \sigma(f_i)(\hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}])$.

Here $\sigma(f_i)$ on the right hand side is to be considered as the term function $t^{W_\tau(X)}$ of the term $t = \sigma(f_i)$ in the algebra $\underline{W}_\tau(X)$ of all terms. The set $Hyp(\tau)$ of all hypersubstitutions of type τ forms a monoid $(Hyp(\tau); \circ_h, \sigma_{id})$ with respect to multiplication \circ_h given by $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ (where \circ is the usual composition of functions), i.e., $(\sigma_1 \circ_h \sigma_2)(f_i) := \hat{\sigma}_1[\sigma_2(f_i)]$. The identity element of $Hyp(\tau)$ is the identity hypersubstitution σ_{id} defined by $\sigma_{id}(f_i) = f_i(x_1, \dots, x_{n_i})$ for all $i \in I$.

For $(s \approx t) \in \Sigma \subseteq W_\tau(X) \times W_\tau(X)$ and a submonoid $M \subseteq Hyp(\tau)$ we define

$$\begin{aligned} \chi_M^E[s \approx t] &:= \{\hat{\sigma}[s] \approx \hat{\sigma}[t] \mid \sigma \in M\} \quad \text{and} \\ \chi_M^E[\Sigma] &:= \bigcup_{(s \approx t) \in \Sigma} \chi_M^E[s \approx t]. \end{aligned}$$

Let $\underline{A} \in Alg(\tau)$ be an algebra of type τ and let $\sigma \in Hyp(\tau)$ be a hypersubstitution. Then $\sigma[\underline{A}] := (A; (\sigma(f_i)^{\underline{A}})_{i \in I})$ is also an algebra of type τ called *derived algebra*.

For $\underline{A} \in \mathcal{K} \subseteq Alg(\tau)$ and a monoid $M \subseteq Hyp(\tau)$ of hypersubstitutions of type τ we define

$$\begin{aligned} \chi_M^A[\underline{A}] &:= \{\sigma[\underline{A}] \mid \sigma \in M\} \quad \text{and} \\ \chi_M^A[\mathcal{K}] &:= \bigcup_{\underline{A} \in \mathcal{K}} \chi_M^A[\underline{A}]. \end{aligned}$$

One easily checks that $(\gamma_1, \gamma_2) := (\chi_M^A, \chi_M^E)$ is a pair of closure operators satisfying

$$\chi_M^A[\underline{A}] \models s \approx t \iff \underline{A} \models \chi_M^E[s \approx t].$$

From this together with the Definitions 1.6, 3.3 and Theorem 3.4 we obtain (with the notations from 5.6):

5.7. Proposition. $\gamma = (\gamma_1, \gamma_2) := (\chi_M^A, \chi_M^E)$ is a conjugate pair of additive closure operators with respect to the relation

$$R := \{(\underline{A}, s \approx t) \in \text{Alg}(\tau) \times W_\tau^2 \mid \underline{A} \models s \approx t\}.$$

The relation (cf. 2.1, 3.3)

$$R_\gamma = R_{\gamma_1} = \{(\underline{A}, s \approx t) \in \text{Alg}(\tau) \times W_\tau^2 \mid \gamma_1(\underline{A}) \times \{s \approx t\} \subseteq R\}$$

is a closed subrelation of R . □

5.8. We recall (cf. 3.3)

$$R_\gamma = R_{\gamma_2} = \{(\underline{A}, s \approx t) \mid \{\underline{A}\} \times \gamma_2(s \approx t) \subseteq R\}.$$

The Galois connection $(\mu_\gamma, \iota_\gamma)$ induced by R_γ will be denoted by $(H_M \text{Id}, H_M \text{Mod})$ and can be characterized via M -hypersatisfaction as follows.

An algebra \underline{A} *M-hypersatisfies* an identity $s \approx t$ (and then we call $s \approx t$ a *hyperidentity* of \underline{A}), notation $\underline{A} \models_{\overline{M}\text{-hyp}} s \approx t$, if $\underline{A} \models \chi_M^E[s \approx t]$ (i.e., if $(\underline{A}, s \approx t) \in R_\gamma$, consequently $\models_{\overline{M}\text{-hyp}} = R_\gamma$).

For a class $\mathcal{V} \subseteq \text{Alg}(\tau)$ we write $\mathcal{V} \models_{\overline{M}\text{-hyp}} s \approx t$ if $\underline{A} \models_{\overline{M}\text{-hyp}} s \approx t$ for every $\underline{A} \in \mathcal{V}$.

A variety \mathcal{V} (of type τ) is called *M-solid* if $\chi_M^A[\mathcal{V}] = \mathcal{V}$. The class of all M -solid varieties will be denoted by \mathcal{S}_M . An equational theory Σ is called *M-hyperequational* if $\chi_M^E[\Sigma] = \Sigma$.

With the definitions and results from the previous sections we get:

5.9. Proposition. Let M be a monoid of hypersubstitutions of type τ .

- (i) *The Galois closed sets with respect to the Galois connection $(H_M Id, H_M Mod)$ are exactly the M -solid varieties and the M -hyperequational theories. In particular we have $\mathcal{S}_M = \mathcal{S}_{\gamma_1} = \mathcal{H}_{H_M Mod \ H_M Id}$.*
- (ii) *\mathcal{S}_M is a complete sublattice of the lattice $\mathcal{L}(\tau)$ of all varieties of type τ .*
- (iii) *If M' is a submonoid of M then \mathcal{S}_M is a complete sublattice of $\mathcal{S}_{M'}$. \square*

In the case of $M = Hyp(\tau)$ one speaks of *hyperidentities* and *solid varieties* instead of M -hyperidentities and M -solid varieties, respectively. The lattice of all solid varieties of type τ is also denoted by $\mathcal{S}(\tau)$.

The definition of a hyperidentity goes back to W. TAYLOR ([Tay;81]) and is connected with the concept of a *variety of varieties* (= *hypervariety*) introduced by W.D. NEUMANN ([Neu;70]).

Another concept of a hyperidentity was introduced by BELOUSOV ([Bel;65]) and ACZÉL ([Acz;61]), here $\sigma(f_i)$ (cf. 5.6) must be operation symbols of the same arity. Hyperidentities in the sense of Belousov were studied in the books of Yu.M. MOVSIJAN ([Mov;86], [Mov;90]).

We close with an example for a monoid M and the corresponding M -solid varieties.

5.10. Example. Let $\tau = (2)$, i.e., we consider a single binary operation symbol, say f . A hypersubstitution is called *regular* if it maps f to a binary term $t \in W_\tau(X_2)$ containing both variable symbols, x and y (cf. 1.4, here we use $X_2 = \{x, y\}$ instead of $\{x_1, x_2\}$). Clearly, the product of two regular hypersubstitutions is again regular and the identity hypersubstitution $\sigma_{id} : f \mapsto f(x, y)$ is also regular. Thus all regular hypersubstitutions of type (2) form a monoid, which we shall denote by *Reg*. The trivial variety is *Reg*-solid and thus the least element in the complete lattice \mathcal{S}_{Reg} of all *Reg*-solid varieties of type (2). DENECKE and JAMPACHON determined in [Den-J;97] the complete lattice of all *Reg*-solid varieties of commutative and idempotent algebras of type 2 (i.e. the intersection of

the complete lattice of all commutative and idempotent algebras of type (2) with the complete lattice of all *Reg*-solid varieties of type (2)), namely:

Theorem. *Let \mathcal{V} be a non-trivial variety of commutative and idempotent algebras of type (2) and let*

$$\begin{aligned}\mathcal{V}_{1,2} &:= \text{Mod}\{xy \approx yx, x^2 \approx x, (xy)y \approx xy\} \quad \text{and} \\ \mathcal{V}_{2,2} &:= \text{Mod}\{xy \approx yx, x^2 \approx x, xy^2 \approx yx^2\}.\end{aligned}$$

Then \mathcal{V} is Reg-solid if and only if $\mathcal{V}_{1,2} \subseteq \mathcal{V} \subseteq \mathcal{V}_{2,2}$.

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