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Power algebras: clones and relations

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Abstract: Every operation or relation on a base set A can be "lifted" to the power set $\mathcal{P}(A)$. In this paper clones of operations and corresponding invariant relations are studied under this process of lifting (e.g. the clone generated by all lifted operations of a clone is characterized internally by δ–closure (Thm. 2.2) and externally by special invariant relations (Thm. 2.4, 3.3)). Generalizations to multifunctions are mentioned.

1. Introduction

1.1. For any operation f as well as for a relation ϱ on a base set A one can form a "lifted" operation $f^{\#}$ or relation $\varrho^{\#}$, resp., on the power set $\mathcal{P}(A)$. Thus every algebraic structure $\underline{A}$ on A gives rise for a corresponding *power structure* $\underline{A}^{\#}$ on $\mathcal{P}(A)$ (called *power algebra*, *complex algebra* or *global*). This idea goes back to FROBENIUS in the context of group theory.

An excellent overview of known results and a universal–algebraic treatment of power structures can be found in [Bri 91]. In that paper the reader may also find many references and open problems.

A leading question is the following: How are algebraic properties of elementwise defined structures and their corresponding power structures related?

The present paper is an initial step to study the following aspects of the lifting process from structures to power structures:

1. Clones (of term operations of algebras)
2. Characterization of operations by invariant relations
3. Set–valued operations (multioperations)

The notion of a clone, i.e. a set of operations closed under composition and containing all projections, is essential in many parts of universal algebra. While the structure of all Boolean clones (i.e. clones on a two–element set) is more or less known, many problems remain open for clones on sets with more than two elements. In fact, it seems to be a hopeless task to describe completely the lattice of all clones on a finite set. Therefore it makes sense to select special classes of clones. Power algebras and their clones show that clones may be extended from "small" (finite) sets A to "large" sets $\mathcal{P}(A)$ and, vice versa, that under certain circumstances a clone on a "large" set may be reduced to a clone on a "small" set.

In *Section 2* we shall see that for a clone $\mathcal{C}$ on A, the set of all lifted operations $\mathcal{C}^\# = \{f^\# \mid f \in \mathcal{C}\}$ surprisingly does not form a clone on the power set $\mathcal{P}(A)$ (e.g. since lifting cannot produce fictitious variables). However, $\mathcal{C}^\#$ is not far from being a clone: the closure $\delta(\mathcal{C}^\#)$ with respect to adjoining fictitious variables and identification of variables gives the clone $\mathsf{clone}_{\mathcal{P}(A)}(\mathcal{C}^\#)$ generated by $\mathcal{C}^\#$ (Theorem 2.2). It is well known (cf. e.g. [Pös 79],[Pös–K 79; 1.2.1]) that every (locally closed) clone ("*Funktionenalgebra*") $\mathcal{C}$ on A can be characterized externally as the set $Pol_A Q$ of operations preserving a set Q of relations on A. For finite A, Theorem 2.4 provides such an external characterization of the clone generated by $O_A^\#$ (where O_A denotes the set of all finitary operations on A) via four special relations (two unary, a binary and a ternary one).

In *Section 3* we study in more detail the interplay between operations and relations (more precisely, the GALOIS–connection $Pol - Inv$) under the lifting process $\mathcal{C} \longmapsto \mathcal{C}^\#$. Proposition 3.2 shows that an operation f preserves a relation ϱ iff $f^\#$ preserves $\varrho^\#$; therefore the operator Pol (restricted to $O_A^\#$) commutes with lifting (Corollary 3.3).

Power structures suggest the investigation of operations with values in $\mathcal{P}(A)$. Such *multioperations*, *polyoperations* or *set–valued operations* $f : A^n \to \mathcal{P}(A)$ are a natural generalization of the operations in O_A to which the lifting process $f \longmapsto f^\#$ can be applied, too. Multioperations were first studied in the group case (i.e. for multigroups, see [Bru 58; Ch.II.7] for references). Moreover, set–valued operations have become useful in computer science for the specification of partial or non-deterministic data structures. Therefore, in *Section 4* we treat multioperations in full analogy to Sections 2 and 3 and mention some results.

considerably improved the presentation

Let us now introduce in detail all needed notions, notations and known facts.

1.2. Given a set A, let O_A and R_A, resp., denote the set of all finitary *operations* and *relations*, respectively (for technical reasons we exclude zero-ary operations which can be treated as unary constant functions), and let $\mathbf{N} = \{1, 2, 3, \ldots\}$:

$$O_A := \bigcup_{n\in\mathbf{N}} O_A^{(n)}, \quad O_A^{(n)} := \{f \mid f : A^n \longrightarrow A\},$$
$$R_A := \bigcup_{m\in\mathbf{N}} R_A^{(m)}, \quad R_A^{(m)} := \{\varrho \mid \varrho \subseteq A^m\} = \mathcal{P}(A^m).$$

$\mathcal{P}(A) = \{B \mid B \subseteq A\}$ denotes the *power set*. Every $f \in O_A^{(n)}$ can be lifted to $\mathcal{P}(A)$ by

$$f^\# : \mathcal{P}(A)^n \longrightarrow \mathcal{P}(A) : (B_1, \ldots, B_n) \longmapsto \{f(b_1, \ldots, b_n) \mid b_i \in B_i, 1 \le i \le n\}.$$

The *projections* $e_{i,A}^n$, for short e_i^n, are the operations defined by $e_i^n(x_1, \ldots, x_n) = x_i$ $(1 \le i \le n \in \mathbf{N})$, $e = e_1^1$ is the identity function.

1.3. For relations $\varrho \subseteq A^m$ the lifted relation $\varrho^\# \subseteq \mathcal{P}(A)^m$ cannot be defined as canonically as for operations. The following definition agrees with that of [Bri 91] and will suffice for our purposes, too:

$(B_1, \ldots, B_m) \in \varrho^\# :\Longleftrightarrow$

$$\forall i \in \{1, \ldots, m\} \, \forall b \in B_i \, \exists b_1 \in B_1, \ldots, b_m \in B_m : b = b_i \wedge (b_1, \ldots, b_m) \in \varrho.$$

That is, $(B_1,\dots,B_m)$ belongs to $\varrho^\#$ iff every $b\in B_i$ can be completed to an m-tupel from ϱ, i.e. $B_i\subseteq\{b\mid \varrho\cap(B_1\times\dots\times B_{i-1}\times\{b\}\times B_{i+1}\times\dots\times B_m)\neq\emptyset\}$ $(B_1,\dots,B_m\in\mathcal{P}(A))$. Equivalently, the relation $\varrho^\#$ arises from ϱ, if one takes some m-tupels from ϱ, collects the elements in each component and takes the obtained m-tupel of "component–sets" as an element of $\varrho^\#$, i.e. $\varrho^\# = \{(pr_1(\sigma),\dots,pr_m(\sigma))\mid \sigma\subseteq\varrho\}$, where $pr_i(\sigma):=\{a\mid \exists\,(a_1,\dots,a_m)\in\sigma : a=a_i\}$.

For $F\subseteq O_A$, $Q\subseteq R_A$, let

$$F^{(n)} := F\cap O_A^{(n)},\quad Q^{(m)}:=Q\cap R_A^{(m)},$$
$$F^\# := \{f^\#\mid f\in F\},\quad Q^\#:=\{\varrho^\#\mid\varrho\in Q\}.$$

1.4. An operation $f\in O_A^{(n)}$ *preserves* a relation $\varrho\in R_A^{(m)}$, or ϱ is an *invariant relation* for f, if for all $(a_{11},\dots,a_{m1})\in\varrho,\dots,(a_{1n},\dots,a_{mn})\in\varrho$ we have

$$(f(a_{11},\dots,a_{1n}),\dots,f(a_{m1},\dots,a_{mn}))\in\varrho$$

. For $F\subseteq O_A$, $Q\subseteq R_A$ define

$$Inv_AF := \{\varrho\in R_A\mid\forall f\in F:\ f \text{ preserves } \varrho\}\quad \text{(invariant relations)},$$
$$Pol_AQ := \{f\in O_A\mid\forall\varrho\in Q:\ f \text{ preserves } \varrho\}\quad (\text{"}\textit{polymorphisms}\text{"}).$$

In algebraic terminology we have $\varrho\in Inv_AF$ (ϱ m-ary) iff ϱ is a subalgebra of the m-th power of the algebra $\langle A;\ F\rangle$.

1.5. A *clone* (of operations) on A is a subset C of O_A satisfying the following conditions:

(i) C contains all projections e_i^n $(1\le i\le n\in\mathbf{N})$.
(ii) C is closed w.r.t. *superposition (composition)*, i.e.
$f\in C^{(m)}$, $g_1,\dots,g_m\in C^{(n)}$ implies $f[g_1,\dots,g_m]\in C^{(n)}$, where
$f[g_1,\dots,g_m](x_1,\dots,x_n):=f(g_1(x_1,\dots,x_n),\dots,g_m(x_1,\dots,x_n))$ for $x_1,\dots,x_n\in A$.

1.6. There are several equivalent definitions of a clone (cf. e.g. [Pös–K 79; 1.1.2–1.1.3]. The crucial closure condition is that with respect to the superposition (composition) 1.5(ii). Manipulation with projections shows that a clone is also closed with respect to so–called *place transformations* δ_π (i.e. identification of variables, adjoining fictitious variables or permutation of variables):

(iii) If C is a clone, $f\in C^{(n)}$ and $\pi:\{1,\dots,n\}\longrightarrow\{1,\dots,m\}$ is an arbitrary mapping $(m,n\in\mathbf{N})$, then $\delta_\pi(f)\in C^{(m)}$, where
$\delta_\pi(f)(x_1,\dots,x_m):=f(x_{\pi(1)},\dots,x_{\pi(n)})$ for $x_1,\dots,x_m\in A$.

Because it is needed below, we introduce the *δ–closure* of a set $F\subseteq O_A$ as follows:

$$\delta(F):=\{\delta_\pi(f)\mid f\in F^{(n)},\ \pi:\{1,\dots,n\}\to\{1,\dots,m\},\ n,m\in\mathbf{N}\}.$$

For example, $\delta(\{e\})$ is the set of all projections since $e_i^n = \delta_\pi(e)$ for $\pi:\{1\}\to\{1,\dots,n\}:1\longmapsto i$. By (iii), $\delta(C)=C$ for every clone C.

Given $F\subseteq O_A$, the *clone generated by* F (i.e., the smallest clone containing F) is denoted by $\mathsf{clone}_A(F)$.

1.7. The operators *Pol* – *Inv* (cf. 1.4) form a GALOIS–connection between sets of operations and sets of relations. For finite A the GALOIS–closed sets of operations

$Pol_A\, Inv_A\, F$ (for some $F \subseteq O_A$) are exactly the clones (for infinite A one gets so–called locally closed clones) [Pös–K 79; 1.2.1], i.e. $\mathsf{clone}_A F = Pol_A\, Inv_A\, F$. In particular, $Pol_A Q$ (for $Q \subseteq R_A$) is a clone and every clone C can be externally characterized as $C = Pol_A\, Q$.

2. Lifted clones

In this section we give an internal characterization of $\mathsf{clone}_{\mathcal{P}(A)}(C^{\#})$ for a clone $C \subseteq O_A$ by describing its elements (Thm. 2.2) via δ-closure and, for finite A, characterize $\mathsf{clone}_{\mathcal{P}(A)}(O_A^{\#})$ externally via invariant relations (Thm. 2.4).

2.1. Let f and $g_1, \ldots, g_m$ be operations on A with arities m and $n_1, \ldots, n_m$, resp., and let

$$f\langle g_1, \ldots, g_m\rangle$$

denote the operation h of arity $n_1 + \cdots + n_m$ defined by

$$h(x_{11}, \ldots, x_{1n_1}, \ldots, x_{m1}, \ldots, x_{mn_m}) := f(g_1(x_{11}, \ldots, x_{1n_1}), \ldots, g_m(x_{m1}, \ldots, x_{mn_m})),$$

(this might be called *linearized composition*).

The following two properties can easily be checked:

(i) Linearized composition is compatible with lifting (cf. [Gau 57] or the *Linearity lemma* in [Grä–W 84]), i.e.

$$h^{\#} = f^{\#}\langle g_1^{\#}, \ldots, g_m^{\#}\rangle$$

(this does not hold for the usual composition as defined in 1.5).

(ii) Every clone C is closed w.r.t. linearized composition, i.e.

$$f, g_1, \ldots, g_m \in C \Longrightarrow f\langle g_1, \ldots, g_m\rangle \in C$$

Theorem 2.2 *Let C be a clone on A. Then* (cf. 1.6)

$$\mathsf{clone}_{\mathcal{P}(A)}(C^{\#}) = \delta(C^{\#})$$

Proof. From $C^{\#} \subseteq \mathsf{clone}(C^{\#})$ and 1.6 we conclude $\delta(C^{\#}) \subseteq \delta(\mathsf{clone}(C^{\#})) = \mathsf{clone}(C^{\#})$. Thus it remains to prove that $\delta(C^{\#})$ is a clone on $\mathcal{P}(A)$:

Claim 1: *$\delta(C^{\#})$ contains all projections.*

In fact, since $e^{\#} \in C^{\#}$ is the identity on $\mathcal{P}(A)$, $\delta(C^{\#})$ contains $\delta(\{e^{\#}\})$ which is the set of all projections (cf. 1.6).

Claim 2: *$\delta(C^{\#})$ is closed w.r.t. superposition* (cf. 1.5).

To show this, let $F \in \delta(C^{\#})^{(m)}$ and $G_1, \ldots, G_m \in \delta(C^{\#})^{(n)}$. We will show that $F[G_1, \ldots, G_m]$ is the lift of a linearized composition of certain operations in C, followed by a suitable place transformation. By 1.6, there are operations f and $g_1, \ldots, g_m$ in C with arities M and $N_1, \ldots, N_m$, respectively, and place transformations $\alpha : \{1, \ldots, M\} \longrightarrow \{1, \ldots, m\}$ and $\beta_i : \{1, \ldots, N_i\} \longrightarrow \{1, \ldots, n\}$ such that $F = \delta_\alpha(f^{\#})$ and $G_i = \delta_{\beta_i}(g_i^{\#})$for $i = 1, \ldots, m$. Thus $h := f\langle g_{\alpha(1)}, \ldots, g_{\alpha(M)}\rangle$ belongs to C (cf. 2.1(ii)). Let $H := \delta_\gamma(h^{\#})$ where $\gamma :\ \{(j,k) \mid 1 \le j \le M, 1 \le k \le N_{\alpha(j)}\} \longrightarrow \{1, \ldots, n\}$ is given by $\gamma(j,k) := \beta_{\alpha(j)}(k)$. We are going to show $H = F[G_1,\dots,G_m]$ which proves claim 2 since $H \in \delta(\mathcal{C}^{\#})$. In fact, for $X_1,\dots,X_n \in \mathcal{P}(A)$ we have

$$\begin{aligned}
&F[G_1,\dots,G_m]\,(X_1,\dots,X_n)\\
&= F(G_1(X_1,\dots,X_n),\dots,G_m(X_1,\dots,X_n))\\
&= F(\delta_{\beta_1}(g_1^{\#})(X_1,\dots,X_n),\dots,\delta_{\beta_m}(g_m^{\#})(X_1,\dots,X_n))\\
&= F(g_1^{\#}(X_{\beta_1(1)},\dots,X_{\beta_1(N_1)}),\dots,g_m^{\#}(X_{\beta_m(1)},\dots,X_{\beta_m(N_m)}))\\
&= \delta_\alpha(f^{\#})(g_1^{\#}(X_{\beta_1(1)},\dots,X_{\beta_1(N_1)}),\dots,g_m^{\#}(X_{\beta_m(1)},\dots,X_{\beta_m(N_m)}))\\
&= f^{\#}(g^{\#}_{\alpha(1)}(X_{\beta_{\alpha(1)}(1)},\dots,X_{\beta_{\alpha(1)}(N_{\alpha(1)})}),\dots,g^{\#}_{\alpha(M)}(X_{\beta_{\alpha(M)}(1)},\dots,X_{\beta_{\alpha(M)}(N_{\alpha(M)})}))\\
&\overset{2.1(i)}{=} h^{\#}(X_{\beta_{\alpha(1)}(1)},\dots,X_{\beta_{\alpha(1)}(N_{\alpha(1)})},\dots,X_{\beta_{\alpha(M)}(1)},\dots,X_{\beta_{\alpha(M)}(N_{\alpha(M)})})\\
&= h^{\#}(X_{\gamma(1,1)},\dots,X_{\gamma(1,N_{\alpha(1)})},\dots,X_{\gamma(M,1)},\dots,X_{\gamma(M,N_{\alpha(M)})})\\
&= \delta_\gamma(h^{\#})(X_1,\dots,X_n)\\
&= H(X_1,\dots,X_n).
\end{aligned}$$

□

2.3. In preparation for the next theorem we introduce four special relations on the power set $\mathcal{P}(A)$:

$$\begin{aligned}
\subseteq &:= \{(B,C) \mid B \subseteq C\} \qquad \text{(inclusion relation)},\\
\varrho_\emptyset &:= \{\emptyset\},\\
\varrho_A &:= \{\{a\} \mid a \in A\} \qquad \text{(singleton relation)},\\
\varrho_0 &:= \{(\emptyset,C,D) \mid C,D \in \mathcal{P}(A)\} \cup \{(B,C,C) \mid B,C \in \mathcal{P}(A),\ B \neq \emptyset\},
\end{aligned}$$

or equivalently

$$\varrho_0 := \{(B,C,D) \mid B,C,D \in \mathcal{P}(A),\ C \neq D \Longrightarrow B = \emptyset\},$$

i.e., ϱ_0 is a "$\emptyset$–check relation".

Without difficulties one shows (cf. 1.4)

$$O_A^{\#} \subseteq Pol_{\mathcal{P}(A)}\{\subseteq, \varrho_\emptyset, \varrho_A, \varrho_0\},$$

consequently

$$\mathsf{clone}_{\mathcal{P}(A)}(O_A^{\#}) \subseteq Pol_{\mathcal{P}(A)}\{\subseteq, \varrho_\emptyset, \varrho_A, \varrho_0\}.$$

Remark. The clones $Pol_{\mathcal{P}(A)}\{\subseteq\}$ and $Pol_{\mathcal{P}(A)}\{\varrho_A\}$ are known to be maximal clones (coatoms) in the lattice of all clones on $\mathcal{P}(A)$ (cf. e.g. [Pös-K 79; 4.3.5, 4.3.7]).

Theorem 2.4 *For finite A,*

$$\mathsf{clone}_{\mathcal{P}(A)}(O_A^{\#}) = Pol_{\mathcal{P}(A)}\{\subseteq, \varrho_\emptyset, \varrho_A, \varrho_0\}.$$

Proof. Because of 2.3 and 2.2 it remains to show $Pol\{\subseteq, \varrho_\emptyset, \varrho_A, \varrho_0\} \subseteq \delta(O_A^{\#})$. Let $F \in Pol_{\mathcal{P}(A)}\{\subseteq, \varrho_\emptyset, \varrho_A, \varrho_0\}$ be an arbitrarily chosen n-ary operation.
W.l.o.g. we assume that F has no fictitious variables, which can always be produced by place transformations if F has at least one essential variable (note that all variables cannot be fictitious since in that case F must be a constant preserving $\varrho_\emptyset$, i.e. F would be constant $\emptyset$ in contradiction to $F \in Pol\,\varrho_A$).
Claim 1: *F satisfies $F(B_1,\dots,B_n) \neq \emptyset \Longleftrightarrow B_i \neq \emptyset$ for all $i \in \{1,\dots,n\}$.*

In fact, if $b_1 \in B_1, \ldots, b_n \in B_n$ then $F(B_1, \ldots, B_n) \supseteq F(\{b_1\}, \ldots, \{b_n\}) \in \varrho_A$ since F preserves $\subseteq$ and ϱ_A, consequently $F(B_1, \ldots, B_n) \neq \emptyset$. Conversely, let $B_j = \emptyset$ for some $j \in \{1, \ldots, n\}$. Since F depends essentially on all its variables, there exist $C_i \in \mathcal{P}(A)$ for $1 \leq i \leq n$ and a $C'_j \in \mathcal{P}(A)$ such that $C_j \neq C'_j$ and $D \neq D'$, where $D := F(C_1, \ldots, C_{j-1}, C_j, C_{j+1}, \ldots, C_n)$ and $D' := F(C_1, \ldots, C_{j-1}, C'_j, C_{j+1}, \ldots, C_n)$. Since F preserves ϱ_0 and $(B_1, C_1, C_1), \ldots, (\emptyset, C_j, C'_j), \ldots, (B_n, C_n, C_n) \in \varrho_0$ we get $(F(B_1, \ldots, B_n), D, D') \in \varrho_0$. Thus $F(B_1, \ldots, B_n) = \emptyset$ because $D \neq D'$ and by the definition of ϱ_0.

Claim 2: $F \in \delta(O_A^{\#})$.
We shall construct suitable α and $f \in O_A$ such that $F = \delta_\alpha(f^{\#})$. Let $A = \{a_1, \ldots a_k\}$ and $k \geq 2$ (the case $k = 1$ is trivial). For each $B \subseteq A$ with $|B| \geq 2$, choose k pairwise different k-tupel $b_1^B, \ldots, b_k^B \in B^k$ such that

$$b_i^B = (b_{i1}^B, \ldots, b_{ik}^B) \Longrightarrow \{b_{i1}^B, \ldots, b_{ik}^B\} = B$$

(e.g., fix $b_0 \in B$ and let $b_{ij}^B := b_0 \Longleftrightarrow i = j$; the other components are taken arbitrarily but according to the stated condition).
Thus, for $B, C \subseteq A$, we have

$$(*) \qquad b_i^B = b_j^C \Longrightarrow B = C \text{ and } i = j.$$

In case $|B| = 1$, i.e. $B = \{b\}$, we put $b_i^B := (b, \ldots, b)$, the unique element of B^k $(i = 1, \ldots, k)$. Moreover, for nonempty $B_1, \ldots, B_n \subseteq A$ we fix an element $a_{(B_1, \ldots, B_n)} \in F(B_1, \ldots, B_n)$ (this is possible because of claim 1).

Now we are ready to define an nk-ary operation $f : A^{nk} \longrightarrow A$ as follows:
For each k-tuple $c_j = (c_{j1}, \ldots, c_{jk}) \in A^k$ $(j = 1, \ldots, n)$ let $C_j = \{c_{j1}, \ldots, c_{jk}\}$ denote the set of its components.
Put

$$f(c_{11}, \ldots, c_{1k}, \ldots, c_{n1}, \ldots, c_{nk}) := \begin{cases} a_i & \text{if } c_1 = b_i^{C_1}, \ldots, c_n = b_i^{C_n} \\ & \text{and } a_i \in F(C_1, \ldots, C_n), \\ & i \in \{1, \ldots, k\}, \\ a_{(C_1, \ldots, C_n)} & \text{otherwise.} \end{cases}$$

f is well-defined because of $(*)$ whenever at least one C_i contains more than one element. In case $|C_1| = \cdots = |C_n| = 1$ we have $|F(C_1, \ldots, C_n)| = 1$ (since F preserves ϱ_A) and therefore f is well-defined, too.
Finally, we will show

$$(**) \qquad f^{\#}(\underbrace{B_1, \ldots, B_1}_{k}, \ldots, \underbrace{B_n, \ldots B_n}_{k}) = F(B_1, \ldots, B_n)$$

for arbitrary $B_1, \ldots, B_n \in \mathcal{P}(A)$, i.e. $F = \delta_\alpha(f^{\#})$ for a suitable place transformation α, and we are done.

In order to prove $(**)$, note that $(**)$ holds if at least one B_i equals $\emptyset$ (then the left-hand side is $\emptyset$ by definition of $f^{\#}$ and the right-hand side equals $\emptyset$ by claim 1). Thus let $B_1, \ldots, B_n$ be nonempty and $c \in f^{\#}(B_1, \ldots, B_1, \ldots, B_n, \ldots, B_n)$. By definition of f and $f^{\#}$, either $c = a_i \in F(B_1, \ldots, B_n)$ or $c = a_{(B'_1, \ldots, B'_n)}$ for suitable $B'_1 \subseteq B_1, \ldots, B'_n \subseteq B_n$, i.e. $c \in F(B'_1, \ldots, B'_n) \subseteq F(B_1, \ldots, B_n)$, in any case $c \in$

$F(B_1,\ldots,B_n)$. Conversely, if $c \in F(B_1,\ldots,B_n)$ then $c = a_i$ for some $i \in \{1,\ldots,k\}$ and $c = f(b_i^{B_1},\ldots,b_i^{B_n}) = f(b_{i1}^{B_1},\ldots,b_{ik}^{B_1},\ldots,b_{i1}^{B_n},\ldots,b_{ik}^{B_n})$ by definition of f, i.e. $c \in f^{\#}(B_1,\ldots,B_1,\ldots,B_n,\ldots,B_n)$. This proves $(**)$. □

Theorem 2.4 does not hold for infinite A since every clone of the form $Pol(\ldots)$ must be "locally closed"(cf. e.g. [Pös 79]), however, it remains as open question whether the "local closure" of the left-hand side of Theorem 2.4 will do the job.

3. Lifted relations

In this section we study in more detail the behaviour of operations and corresponding invariant relations under the process of lifting.

Remarks 3.1 a) For $\varrho \subseteq A^m$, the lifted relation $\varrho^{\#} \subseteq \mathcal{P}(A)^m$ was defined in 1.3. We give here a further equivalent characterization in order to show that this notion is more canonical than it might look at first glance.

Any family F of functions $f : A \longrightarrow B$ gives rise to a function $F^{\dagger} : A \longrightarrow \mathcal{P}(B) : a \longmapsto \{f(a) \mid f \in F\}$.

Considering $r \in A^m$ as a mapping $r : \{1,\ldots,m\} \longrightarrow A$, every $\sigma \subseteq A^m$ is a set of mappings $r : \{1,\ldots,m\} \longrightarrow A$ and the corresponding function $\sigma^{\dagger}$ is a mapping

$$\sigma^{\dagger} : \{1,\ldots,m\} \longrightarrow \mathcal{P}(A),$$

i.e. $\sigma^{\dagger} \in \mathcal{P}(A)^m$ is an element of an m-ary relation on $\mathcal{P}(A)$. With these notations we have

$$\varrho^{\#} = \mathcal{P}(\varrho)^{\dagger} := \{\sigma^{\dagger} \mid \sigma \subseteq \varrho\},$$

which is, in fact, a power construction.

b) As it is well-known, any m-ary function f can be considered as an $(m+1)$-ary relation $f^{\bullet}$. Hence there are two possibilities for lifting functions. But in general we have the following inequality:

$$(f^{\#})^{\bullet} \neq (f^{\bullet})^{\#}.$$

To show this we look at a simple example: Let A be the set $\{0,1\}$ and f the (logical) disjunction. Then $(\emptyset,\{0\},\emptyset)$ belongs to $(f^{\#})^{\bullet}$ but not to $(f^{\bullet})^{\#}$ and on the other hand $(A,A,\{1\})$ belongs to $(f^{\bullet})^{\#}$ but not to $(f^{\#})^{\bullet}$.

The following proposition is an important tool for further investigations. It shows that lifting is compatible with the preservation property.

Proposition 3.2 *For $f \in O_A$ and $\varrho \in R_A$ we have: f preserves ϱ iff $f^{\#}$ preserves $\varrho^{\#}$.*

Proof. One direction is obvious: if $f^{\#} \in Pol_{\mathcal{P}(A)}\,\varrho^{\#}$ then $f \in Pol_A\,\varrho$ (consider the one-element subsets of A only and note $(a_1,\ldots,a_m) \in \varrho \iff (\{a_1\},\ldots,\{a_m\}) \in \varrho^{\#}$ and $f(\{a_1\},\ldots\ldots,\{a_n\}) = \{f(a_1,\ldots,a_n)\}$).

Thus, let $f \in Pol_A\,\varrho$, $f \in O_A^{(n)}$, $\varrho \in R_A^{(m)}$. Consider n elements of $\varrho^{\#}$:

$$\{\,(A_{11},\ldots,A_{m1}),\ldots,(A_{1n},\ldots,A_{mn})\} \subseteq \varrho^{\#}$$

and define $B_i := f^{\#}(A_{i1},\ldots,A_{in})$ for $i = 1,\ldots,m$. In order to prove $f^{\#} \in Pol\,\varrho^{\#}$ we have to show $(B_1,\ldots,B_m) \in \varrho^{\#}$, which we shall do with the condition in 1.3(i).
For $b \in B_i$, consider $a_1 \in A_{i1},\ldots,a_n \in A_{in}$ with $b = f(a_1,\ldots,a_n)$. Because $(A_{1j},\ldots$ $\ldots,A_{mj}) \in \varrho^{\#}$, for every $j \in \{1,\ldots,n\}$ there exists $(a_{1j},\ldots,a_{mj}) \in \varrho \cap A_{1j}\times\cdots\times A_{mj}$ with $a_{ij} = a_j$. Since f preserves ϱ we get $(f(a_{11},\ldots,a_{1n}),\ldots,f(a_{m1},\ldots,a_{mn})) \in \varrho \cap B_1\times\cdots\times B_m$, consequently $b = f(a_{i1},\ldots,a_{in}) \in \{a \mid B_1\times\cdots\times B_{i-1}\times\{a\}\times B_{i+1}\times \cdots\times B_m \cap \varrho \neq \emptyset\}$ and we are done by 1.3(i). □

As an immediate consequence of 3.2 and 2.2 we get:

Corollary 3.3 *Let $Q \subseteq R_A$. Then*

(a) $(Pol_A\,Q)^{\#} = O_A^{\#} \cap Pol_{\mathcal{P}(A)}(Q^{\#})$,

(b) $\mathsf{clone}_{\mathcal{P}(A)}((Pol_A\,Q)^{\#}) = \delta(O_A^{\#} \cap Pol_{\mathcal{P}(A)}\,Q^{\#})$. □

Problem 3.4 The characterization in 3.3(b) of the clone generated by $(Pol_A\,Q)^{\#}$ has a flaw. If a clone C is given by relations, i.e. $C = Pol\,Q$, then one would like to characterize the lifted clone also by relations only. Obviously

$$(*)\qquad \delta(O_A^{\#} \cap Pol_{\mathcal{P}(A)}\,Q^{\#}) \subseteq \delta(O_A^{\#}) \cap Pol_{\mathcal{P}(A)}\,Q^{\#}$$

where the right-hand side is equal to

$$Pol\{\subseteq, \varrho_\emptyset, \varrho_A, \varrho_0\} \cap Pol\,Q^{\#} = Pol_{\mathcal{P}(A)}(Q^{\#} \cup \{\subseteq, \varrho_\emptyset, \varrho_A, \varrho_0\})$$

due to Theorem 2.4.
The problem arises whether equality holds in $(*)$, i.e. whether the relations in $Q^{\#}$ and $\{\subseteq, \varrho_\emptyset, \varrho_A, \varrho_0\}$ suffice to characterize the clone $\delta((Pol_A\,Q)^{\#})$. A counterexample due to F. Börner (personal communication) shows that equality cannot hold in general; however, we conjecture an affirmative answer with some modification using additional relations directly constructed from Q. Note that every clone $\tilde{C}$ in $O_{\mathcal{P}(A)}$ can be characterized as $Pol_{\mathcal{P}(A)}\,\tilde{Q}$ by some set $\tilde{Q} \subseteq R_{\mathcal{P}(A)}$ of relations. However, it is a serious problem for $\tilde{C} = \delta((Pol\,Q)^{\#})$ how $\tilde{Q}$ can be constructed from Q without knowing $\tilde{C}$. The above conjecture and 3.3 are attempts to solve this problem.

4. Multifunctions

The lifting process $f \longmapsto f^{\#}$ can be applied to operations $f \in O_A$ as well as to several generalizations, e.g. to partial functions or, more generally, to multifunctions.

Definition 4.1 For any set A, a function

$$f : A^n \longrightarrow \mathcal{P}(A)$$

is called an *n-ary multifunction (or multioperation)* on A.

Remark. Multifunctions and multialgebras (i.e. algebras with multioperations instead of usual operations) are models for nondeterministic processes. They include partial functions (if every image $f(a_1,\ldots,a_n)$ is either a one–element set or empty).

Definition 4.2 For a multioperation $f : A^n \longrightarrow \mathcal{P}(A)$ define

$$f^{\#} : \mathcal{P}(A)^n \longrightarrow \mathcal{P}(A)$$

by

$$f^{\#}(B_1,\dots,B_n) := \bigcup \{f(b_1,\dots,b_n) \mid b_1 \in B_1,\dots,b_n \in B_n\}.$$

Note that $f^{\#}$ – the *lifted operation* – is an ordinary (total) n-ary function on the power set. Therefore multialgebras can be considered as ordinary algebras on the power set $\mathcal{P}(A)$.

Many results for clones and invariant relations can be generalized to multioperations. For more details we refer to [Dre 93] where among other things clones of multioperations are studied. Here we mention only those results which are in parallel to the results in Sections 2 and 3.

4.3. For $\varrho \in R_A$, a multioperation $f : A^n \longrightarrow \mathcal{P}(A)$ *preserves* ϱ (or ϱ is *invariant* for f) if

$$\{(a_{11},\dots,a_{m1}),\dots,(a_{1n},\dots,a_{mn})\} \subseteq \varrho$$

implies

$$(f(a_{11},\dots,a_{1n}),\dots,f(a_{m1},\dots,a_{mn})) \in \varrho^{\#}.$$

Let M_A denote the set of all multifunctions (of arbitrary finite arity) on A. Put $Polm_A\,\varrho := \{f \in M_A \mid f \text{ preserves } \varrho\}$. Then we have:

Proposition 4.4 $\delta(M_A^{\#}) = Pol_{\mathcal{P}(A)}\{\subseteq, \varrho_0\}$, *where* ϱ_0 *is the ternary relation*

$$\varrho_0 = \{(B,C,D) \mid C \neq D \Longrightarrow B = \emptyset\}$$

as defined in 2.3.

Proposition 4.5 *A multifunction* $f \in M_A$ *preserves* $\varrho \in R_A$ *iff* $f^{\#}$ *preserves* $\varrho^{\#}$ *(i.e.* $(Polm_A\,\varrho)^{\#} = M_A^{\#} \cap Pol_{\mathcal{P}(A)}\varrho^{\#}$*).*

We omit the proofs and refer to [Dre 93].

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Zusammenfassung

Jede Operation bzw. Relation auf einer Grundmenge A kann auf die Potenzmenge $\mathcal{P}(A)$ "geliftet" werden. In dieser Arbeit wird das Verhalten von Operationenklonen und zugehörigen invarianten Relationen bei diesem Liftprozeß untersucht (z.B. wird der von den gelifteten Operationen eines Klons erzeugte Klon intern durch δ–Abgeschlossenheit (Thm. 2.2) und extern durch spezielle invariante Relationen (Thm. 2.4, 3.3) charakterisiert). Einige Verallgemeinerungen auf Multifunktionen werden aufgezeigt.

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