On Proofs of Synchronization - the Outweighing Approach

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Abstract

We develop understanding for the existence of a Lyapunov function and demonstrate why proofs for synchronization of chaotic systems based on a *naive* choice of a Lyapunov function as e.g. in [1] are not so efficient and why others as. e.g. in [2] can even lead to wrong results. We show how to apply the circle criterium in a not so common way for a more efficient proof of synchronization. Furthermore we illustrate the outweighing approach which allows the outweighing of temporary divergence by appropriate convergence with respect to a Lyapunov function which then generalizes to be a Curduneanu function.

1 Introduction

Synchronization of dynamic systems generally means that one system somehow traces the motion of the other. It has achieved much attention in terms of chaotic systems due to its possible application for communication purposes. There usually a *Master-Slave*-Configuration is adopted, i.e. an autonomous chaotic system drives another system.

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad \mathbf{x} \in \mathrm{R}^m \tag{1}$$

$$\dot{\mathbf{y}} = \mathbf{G}(\mathbf{x}, \mathbf{y}), \quad \mathbf{y} \in \mathrm{R}^n \tag{2}$$

where generally $m \geq n$ and synchronization is perceived to imply that $\mathbf{y}$ asymptotically copies a subset $\mathbf{x}_S \subseteq \mathbf{x}$. However, for the sake of easy readability we will stick to the case $m = n$ and $\mathbf{x}_S = \mathbf{x}$. The basic ideas are not changed by that. Thus a certain difference $\mathbf{z} = \mathbf{x} - \mathbf{y}$ [1] is expected to vanish as time goes on:

$$|\mathbf{x} - \mathbf{y}| \longrightarrow 0 \text{ as } t \rightarrow \infty \tag{3}$$

In other words, a certain *Synchronization Manifold* of the composition state space $\mathrm{R}^n \times \mathrm{R}^n$ is asymptotically stable [2].

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[1] One could think about a more general kind of copy: $\mathbf{z} = \mathbf{x} - \mathbf{h}(\mathbf{y})$ which corresponds to *Generalized Synchronization* (see e.g. [3]). Here we will deal exclusively with *Identical Synchronization*, i.e. $\mathbf{h}(.)$ is the identity map.

[2] Throughout the paper we mean global asymptotic stability unless we say explicitely local asymptotic stability

The natural approach to establish synchronization, i.e. the asymptotic stability of the synchronization manifold, is therefore to prove that the origin of the *Difference System*

$$\dot{\mathbf{z}} = \mathbf{F}(\mathbf{x}) - \mathbf{G}(\mathbf{x}, \mathbf{y}) = \mathbf{f}(\mathbf{x}, \mathbf{z}) \tag{4}$$

is asymptotically stable. In other words, the motion *transversal* to the synchronization manifold is supposed to die out.

It is reasonable to assume that the systems (1) and (2) are to a certain degree identical (as in all communication applications so far). That is $\mathbf{G}(\mathbf{y}, \mathbf{y}) = \mathbf{F}(\mathbf{y})$ which results in that for every initial state of (1) there exists at least an initial state of (2) for which the systems are immediately identically synchronized, namely

$$\mathbf{y}(0) = \mathbf{x}(0) \tag{5}$$

In other words the identical synchronization manifold is invariant.

Obviously, synchronization occurs when all solutions of (2) converge to one, namely the *right* one with (5). This feature has a nice old name: *Unique Asymptotic Behaviour* of the driven system (2), (see e.g. [4] or [5]). In [6] the term: *uniform-asymptotically stable with respect to a set of driving functions* was developed for the same thing.[3]

Thus *In order to establish synchronization one has to prove unique asymptotic behaviour or the asymptotic stability of the origin of a difference system.*

A common way is to apply Lyapunov's Direct Method [7] on the difference system[4]. That is one chooses a scalar, energy-like function, the Lyapunov function, on the difference state space with a minimum at the origin and establishes a strictly decreasing energy along the system flow. See figure 1 for an illustration. This has the inherent problem of how to choose the Lyapunov function.

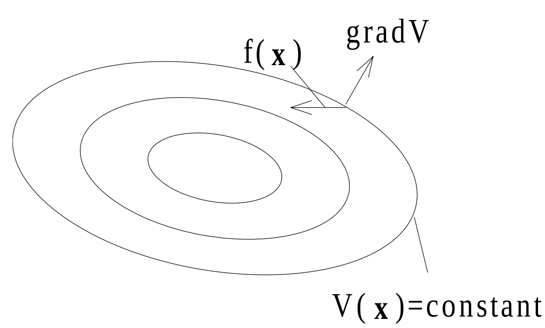


Figure 1: Illustration of decreasing energy along a system flow

One purpose of this paper is to recall and illustrate criteria which provide the existence of a suitable Lyapunov function. We will explain, why a suitable Lyapunov function is somehow a *Common Lyapunov function*. This way we work out why the approach in [2] can lead to wrong results and why a *naive* choice of a Lyapunov function as in [1] may not lead to best results.

[3]Though there all this is put into a frame which admits bidirectional interaction between both systems too

[4]It does not matter whether one considers the difference between $\mathbf{x}$ and $\mathbf{y}$ or between any two solutions of (2), since we already know, that there is one *right* solution with (5).

Another drawback associated with Lyapunov's direct method is that it requires decreasing energy always and everywhere. This way one thus cannot establish synchronization of systems which outweigh temporary divergence by appropriate convergence with respect to an energy-like function.

Therefore the second purpose of this paper is to describe and illustrate the outweighing approach for proves of synchronization. In stability theory this is known as the *Comparison Principle* [8] and the energy-like function is then called Curduneanu function [9]. In [10] a similar approach was tempted. However we think, we can give a clearer frame for this and point out its weakness.

Often synchronization depends on parameters, which describe the interaction between the systems be it coupling or feedback parameters (see e.g. [11] or [5] for a description of synchronization principles). Since synchronization proofs provide rather sufficient conditions for synchronization the analytically determined border of synchronization e.g. in terms of a coupling parameter is often far away from the parameter value of the actual onset of synchronization. The idea throughout this paper is to achieve analytical results as close as possible to the synchronization border, i.e. to improve or sharpen the proofs.

2 Theory

2.1 Common Lyapunov Function

Recall we have to establish the asymptotic stability of the zero solution of the difference system (4). For easy feasability it is often put in a time-variant, linear form:

$$\dot{\mathbf{z}} = \mathbf{A}(t) \cdot \mathbf{z},\ \mathbf{A} \in \mathrm{R}^{n \times n} \tag{6}$$

where the Matrix $\mathbf{A}$ depends on lets say the coupling parameter k and $\mathbf{x}, \mathbf{z}$ and this way on the time.

Sometimes it is chosen to consider the variational equation of (4), i.e. for infinitesimal $\mathbf{z}$ only. In this case the Matrix $\mathbf{A}$ is the Jacobian of the difference system $D_{\mathbf{z}}\mathbf{f}(\mathbf{x}, \mathbf{0})$. This corresponds to Lyapunov's Indirect Method (the linearization method) [7].

This way one only proves local asymptotic stability of the zero solution, i.e. a non-empty basin of attraction. In other words synchronization for initial states in a non-empty neighbourhood of the synchronization manifold. However, often people have been satisfied with this weaker result too, e.g. [2].

The idea is now to find a Lyapunov function $V(t, z)$ such that its time derivative is negative always and everywhere. For linear systems as in (6) quadratic forms have been proven to be handy as **Lyapunov functions**:

$$V = \mathbf{z}'\mathbf{P}\mathbf{z},\ \mathbf{P} \in \mathrm{R}^{n \times n} \tag{7}$$

$'$ stands for the tranpose of a vector or a matrix. In order to serve as a Lyapunov function $\mathbf{P}$ has to be positive definite:

Definition 1 (Positive Definite (pdf.)) $\mathbf{P}$ *is pdf., if with (7)* $V > 0,\ \forall\, \mathbf{z} \neq \mathbf{0}$.

Next we state a well known result:

Theorem 1 *For every Hurwitz matrix* $\mathbf{A}$ *and for each pdf. matrix* $\mathbf{Q}$ *exists a pdf. matrix* $\mathbf{P}$ *such that with (6) and (7)*

$$\dot{V} = \mathbf{z}'(\mathbf{A}'\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{z} = -\mathbf{z}'\mathbf{Q}\mathbf{z} < 0, \ \forall \ \mathbf{z} \neq \mathbf{0} \tag{8}$$

We have adapted a result which is actually known as the Lyapunov Matrix Equation [7].

One is tempted to assume that, if $\mathbf{A}(t)$ is Hurwitz for all t then asymptotic stability of (6) is assured, however this is wrong. Either one has to impose additional conditions on the time-variation (as e.g. in theorems for Slowly Varying Systems, [7] where V depends on t too) or one has to assure the existence of a common Lyapunov function. The fact that at each moment **a** Lyapunov function exists does not imply that there is *a common* Lyapunov function, i.e. one and the same for all $\mathbf{A}(t)$.

Thus it is not sufficient to require $\mathbf{A}(t)$ to be Hurwitz for all t in order to assure synchronization as done in [2]. An example, where a common Lyapunov function of the form (7) does not exist although all matrices are Hurwitz is illustrated in section 3.

2.2 Usage of the Kalman-Yacubovitch Lemma (KYL)

There are criteria which provide the existence of a common Lyapunov function without constructing it explicitely. The famous KYL does this in terms of Positive Realness of a transfer matrix $\mathbf{H}(s)$. Although we do not cite the KYL itself it is the basis for the theorem below.

This goes back to a typical control theory setup: An n-dimensional, linear system with an m-dimensional input and output and transfer matrix $\mathbf{H}(s)$ has a nonlinear static feedback. This feedback can be time-varying but has merely to obey a sector condition, see figure 2 for an illustration. For our purpose we state the following theorem in a slightly different form than in [7] [5]. The proof follows straight the one given there.

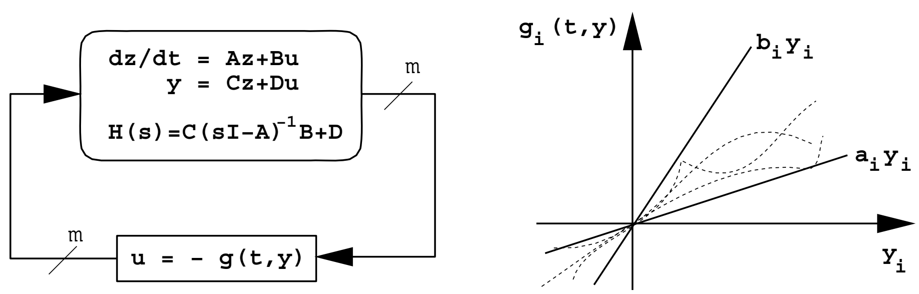


Figure 2: Feedback scheme consisteing of a linear dynamical system with a time-varying nonlinearity in the feedback path

Theorem 2 *Given a system as in figure 2 with a m-dimensional feedback. The linear part is a minimal realization of* $\mathbf{H}(s)$ *and each (scalar) component of the nonlinear feedback belongs*

[5] On one hand we are more restrictive, since we require the inequality $(\mathbf{g}(t, \mathbf{y}) - a\mathbf{y})' \cdot (b\mathbf{y} - \mathbf{g}(t, \mathbf{y}) \geq 0$ to hold componentwise. On the other hand (9) admits different sectors for the nonlinearity and is thus more general.

to a sector $[a_i, b_i]$ *with*

$$(g_i(t, \mathbf{y}) - a_i \cdot y_i) \cdot (b_i \cdot y_i - g_i(t, \mathbf{y})) \geq 0, \; i = 1, \dots m \tag{9}$$

and the transfer matrix

$$\mathbf{T}(s) = \mathbf{H}(s)[\mathbf{I} + Diag[a_i] \cdot \mathbf{H}(s)]^{-1} + Diag[1/(b_i - a_i)] \tag{10}$$

is positive real, then the origin of the system is asymptotically stable.

For m=1 this comes down to the consideration of positive realness of a scalar transfer function which has a nice geometrical interpretation in the Nyquist plot of the transfer function $\mathbf{H}(s)$. This is known as the circle criterium.

We emphasize: *The system (4) as well as the feedback setup of figure 2 generally can be put into the form (6). Therefore the above theorem or basically the KYL by assuring the existence of* a *suitable Lyapunov function essentially provides a common Lyapunov function for all possible matrices* $\mathbf{A}(t)$. This idea is the clue to the illustrations in section 3.

In order to apply theorem 2 we will put (4) or (6) into such a feedback frame in figure 2. If $\mathbf{A}(t)$ varies e.g. only with rank1 one can put this into a scalar time-varying feedback, i.e. m=1, and one applies the circle criterium.

The common application of the circle criterium decides which sector $[a, b]$ of the nonlinearity is admissible for a given transfer function $\mathbf{H}(s)$. In our setup, however, the sector might be determined by the system (1) and the transfer function depends on the coupling parameter k, the admissible range of which we want to find. Thus we have to apply the circle criterium in some sense in *reverse*. The following describes how this can be done in principle. First we recall some definitions and theorems on positive realness and positive definiteness.

Definition 2 (Positive Realness (pr.)) *A complex matrix* $\mathbf{H}(s)$ *is positive real if its hermitian part* $\mathbf{J}(s) = \mathbf{H}(s) + \mathbf{H}^*(s)$ [6] *is positive definite on the imaginary axis, i.e. for*

$$\mathbf{H}(s = j\omega): \; \mathrm{Re}(\mathbf{v}^*\mathbf{H}\mathbf{v}) = \mathbf{v}^*(\mathbf{H} + \mathbf{H}^*)\mathbf{v} > 0, \; \forall \, \mathbf{v} \neq \mathbf{0}, \; \forall \, \omega \in \mathrm{R} \tag{11}$$

* stands for the conjugated tranpose of a vector or a matrix and Re for the real part of the quadratic form.

Theorem 3 (pdf. $\Longleftrightarrow$ (Leading Principal Minors > 0)) *A hermitian matrix is positive definite if and only if all its leading principal minors are positive [12].*

The leading principal minors of an n-square matrix are its n 'north-westerly' subdeterminants. Thus, in order to establish positive realness of the hermitian part of a transfer matrix we have to prove the positiveness of n functions on the imaginary axis. These appear to be polynomials of ω^2:

$$P(\omega^2) = a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + \dots, \; a_i \in \mathrm{R} \tag{12}$$

The positiveness of a polynomial up to 4th order can be decided by more or less handy criteria on its coefficients. An obviously necessary condition for $P(\omega^2) > 0, \; \forall \, \omega$ is

$$a_0 > 0 \tag{13}$$

Given this one has to establish that $P(.)$ has no positive real roots. A powerful criterium to decide this is the so called:

[6] $\mathbf{J}$ is a hermitian matrix: $\mathbf{J} = \mathbf{J}^*$, i.e. it is a sum of a real symmetric matrix and a purely imaginary skew-symmetric matrix. All its eigenvalues are real. And (11) implies that they are all positive.

Lemma 1 (Descarte's Rule of Signs) *The number of positive real roots of a polynomial $P(.)$ with real coefficients a_i is either equal to the number of variations of signs between the a_i in successive order or less than this number by an even integer [13].* [7]

Thus a second necessary condition for a nth. order principal minor polynomial (12) to have no positive real roots is

$$a_n > 0 \tag{14}$$

and we would be finished if all $a_i > 0$. In case they aren't, the actual number of positive real roots, which has to be zero for our purpose, can be determined by sign changes of so called *Inners* of matrices built out of the polynomial coefficients a_i [13]. This leads to conditions on *characteristic expressions* of a_i such as the *Discriminants*, as known for 2nd and 3rd order polynomials. We will state these conditions for 3rd and 4th order polynomials later. The reason, why it becomes less handy for higher order polynomials is, that too many different cases of sign changes had to be considered.

Recall, our matrix $\mathbf{A}$ depends on the coupling parameter k and so does the transfer matrix to be investigated for positive realness and so do the coefficients of the principal minor polynomials of ω^2. Consequently we will detect the border of (analytically established) synchronization at the coupling parameter value(s), where such positiveness criteria cease to be fulfilled. All this will be demonstrated at examples in sections 3 and 4.

2.3 The Outweighing Approach

Recall Lyapunov's direct method requires decreasing energy with respect to a Lyapunov function always and everywhere. Our goal here is to admit outweighing of temporary divergence by appropriate convergence with respect to an energy-like function. That means to establish synchronization for coupling parameter values where Lyapunov's direct method fails. Although in this case the energy-like function is sometimes called Curduneanu function [9] we stick to the name Lyapunov function since the idea is related.

In [14] this idea was applied to a one dimensional difference system (4). Dimension=1 provides the advantage that divergence and convergence occur in the same (only one) direction of state space and their outweighting is therefore easy to estimate. Here we will regard the general case of n dimensional difference systems.

We point out: *In more-dimensional difference systems diverging and converging may occur in different 'directions' of state space and the estimation of their outweighting is non-trivial. One has to regard the* **worst-case of diverging** *over the whole difference state space. That means the maximum divergence with respect to a chosen Lyapunov function $V(\mathbf{z})$ for all states.*

The following quantity (we don't know a proper name yet) serves this purpose:

$$\mu_V(\mathbf{A}(t)) = \sup_{\mathbf{z} \in \mathrm{R}^n / \{\mathbf{0}\}} [\dot{V}(\mathbf{A}(t), \mathbf{z}) / V(\mathbf{z})] \tag{15}$$

In case V is a *norm on* R^n then the quantity defined above appears to be the *matrix measure induced by* V. [8]

[7] E.g. a 3rd order polynomial with coefficient signs $+, -, +, -$ has either 3 or 1 positive real roots.
[8] This relation was once nicely pointed out in [15].

Simple integration of (15) yields an upper bound of the systems energy:

$$V(\mathbf{z}(t)) \leq V(\mathbf{z}(0)) \exp[\int_0^t \mu_V(\mathbf{A}(\tau))d\tau] \tag{16}$$

(16) provides the possibility to estimate the outweighting of divergence and convergence:

Theorem 4 *If for some V exists a continous function $\mu_V(t)$ with (15) and*

$$\lim_{T\to\infty} \int_0^T \mu_V(\tau)d\tau = -\infty \tag{17}$$

then the trivial solution of (6) is asymptotically stable. (adapted from [9])

It follows that in order to tell whether synchronization takes place one needs merely the *mean value* of $\mu_V(\mathbf{A}(t))$. Surely, this requires knowledge about which proportion of time each possible matrix $\mathbf{A}(t)$ is *valid*. See section 5 for a discussion of related problems.

$\mu_V(\mathbf{A}(t))$ can be determined for the kind of Lyapunov function we chose in (7) in the following way:

Lemma 2 *μ_V defined in (15) with (7) and (8) is equal to the maximum generalized eigenvalue of $(-\mathbf{Q}, \mathbf{P})$.*

$$\lambda_{\max}(-\mathbf{Q}, \mathbf{P}) = \mu_V(\mathbf{A}) \tag{18}$$

Proof: According to (15) we want to determine the maximum value of the ratio of the two quadratic forms of $\dot{V}$ and V. This can be converted into the following conditional optimization problem to be solved by means of Lagrangian theory:

$$\text{maximize } \dot{V} = -\mathbf{z}'\mathbf{Q}\mathbf{z} \text{ over } \mathbf{z} \in \mathrm{R}^n/\{\mathbf{0}\} \text{ with } V = \mathbf{z}'\mathbf{P}\mathbf{z} = \text{constant} \tag{19}$$

We introduce the *Lagrangian* $L = \dot{V} - \lambda \cdot V$ and achieve as conditions for the extremal value:

$$-\mathbf{Q}\mathbf{z} = \lambda \cdot \mathbf{P}\mathbf{z} \tag{20}$$

$$\mathbf{z}'\mathbf{P}\mathbf{z} = \text{constant} \tag{21}$$

(20) defines λ to be a generalized eigenvalue of the pair of matrices $(-\mathbf{Q}, \mathbf{P})$. Clearly, multiplication of (20) with the eigenvector $\mathbf{z}'$ belonging to the largest generalized eigenvalue (which obeys (21)) reveals that $\lambda_{\max}(-\mathbf{Q}, \mathbf{P})$ is the maximum ratio of $\dot{V}$ and V which completes the proof.

Surely, in order to apply the outweighing approach (theorem 4) one still has to find a suitable Lyapunov function which leaves the mean value of $\mu_V(t)$ negative. Next we explain, how this problem can be brought back to the need of a common Lyapunov function in a special situation. Thus it can be solved by means of theorem 2. Concluding this section we will discuss, why a similar approach suggested in [10] must lead to weaker results than the direct application of theorem 4.

A Application of Theorem 4 in a Special Situation

Assume system (1) is piecewise linear and knowledge is availible about which proportion of time it stays in each linear region. We consider a linear coupling scheme and assume there are merely two different Jacobians of $\mathbf{F}$ in (1), lets say $D\mathbf{F}_1$ and $D\mathbf{F}_2$, which differ only by rank1. W.l.o.g.:

$$\begin{aligned} \dot{\mathbf{y}} = &\quad \mathbf{F}(\mathbf{y}) - k \cdot \mathbf{b}\mathbf{c}'(\mathbf{y} - \mathbf{x}), \quad \mathbf{b}, \mathbf{c} \in \mathrm{R}^n & (22) \\ D\mathbf{F}_2 = &\quad D\mathbf{F}_1 - g \cdot \mathbf{e}_1\mathbf{e}_1', \qquad g, k \in \mathrm{R} & (23) \end{aligned}$$

$\mathbf{e}_1$ denotes the first unity vector. Furthermore we only consider the linearized difference system of (4). Consequently $\mathbf{A}(t) \in \{\mathbf{A}_1, \mathbf{A}_2\}$ in (6) with

$$\mathbf{A}_1 = D\mathbf{F}_1 - k \cdot \mathbf{b}\mathbf{c} \text{ and } \mathbf{A}_2 = \mathbf{A}_1 - g \cdot \mathbf{e}_1\mathbf{e}_1' \tag{24}$$

Remember at this point we could already provide a common Lyapunov function for $\{\mathbf{A}_1, \mathbf{A}_2\}$ by application of the circle criterium since both matrices differ only by rank1. One had to decide the positive realness of a scalar transfer function depending on the coupling k. However, by application of the outweighing approach we can establish synchronization for a wider range of k.

Assume, $p_1, p_2 = 1 - p_1$ are the proportions of time each linear region is 'valid' in (1) resp. (4). By application of theorem 4 we have to require that:

$$p_1\mu_V(\mathbf{A}_1) + p_2\mu_V(\mathbf{A}_2) < 0 \text{ or } \mu_V(\mathbf{A}_2) < -\frac{p_1}{p_2}\mu_V(\mathbf{A}_1) = -p \cdot \mu_1 \tag{25}$$

For the sake of simplicity we droped a few indices and arguments at the end.

Thus, if we could find a Lyapunov function with $\mu_1 < \mu$ and $\mu_2 < -p \cdot \mu$ then we meet the condition in (25). This will be a common Lyapunov function of two modified matrices:

Lemma 3 *If* $V = \mathbf{z}'\mathbf{P}\mathbf{z}$ *is a common Lyapunov fuction for the matrices*

$$\begin{aligned} &\mathbf{B}_1 = \mathbf{A}_1 - m\mathbf{I} \quad \textit{and} \quad \mathbf{B}_2 = \mathbf{A}_2 + pm\mathbf{I}, \; m \in \mathrm{R} \;\; \textit{then} & (26) \\ &\mu_V(\mathbf{A}_1) < 2m \quad \textit{and} \quad \mu_V(\mathbf{A}_2) < -2pm & (27) \end{aligned}$$

Proof: If $V = \mathbf{z}'\mathbf{P}\mathbf{z}$ is a common Lyapunov fuction for $\mathbf{B}_i, i = 1, 2$ then

$$\begin{aligned} \mathbf{z}'(\mathbf{B}_i'\mathbf{P} + \mathbf{P}\mathbf{B}_i)\mathbf{z} &= -\mathbf{z}'\mathbf{Q}_i\mathbf{z} < 0, \; \forall\, \mathbf{z} \neq \mathbf{0} \;\text{ and} & (28) \\ \dot{V} = \mathbf{z}'(\mathbf{A}_1'\mathbf{P} + \mathbf{P}\mathbf{A}_1)\mathbf{z} &= -\mathbf{z}'\mathbf{Q}_1\mathbf{z} + 2m \cdot \mathbf{z}'\mathbf{P}\mathbf{z} < 2m \cdot V & (29) \end{aligned}$$

It follows $\dot{V}/V < 2m$ and thus $\mu_V(\mathbf{A}_1) < 2m$. A similar argument holds for $\mathbf{A}_2, \mathbf{B}_2$ which completes the proof. It is obvious that $2m$ serves the purpose required above for μ.

Figure 3 illustrates the different ranges of matrices required to have a common Lyapunov function if outweighing is admitted and if not. This should clarify that it is well possible that for a coupling value k with outweighing a common Lyapunov function exists while without outweighing it does not. Furthermore figure 3 indicates the additional difficulty we face now: the range of matrices, we have to provide a common Lyapunov function for, depends now on two parameters, k and m.

The idea to bring the outweighing approach back to the need of a common Lyapunov function for modified matrices could be extended to the case where the $\mathbf{A}(t)$ constitute a continuous range of matrices. Then we neeeded a common Lyapunov function for the matrices $\mathbf{A}(t) - m(t)\mathbf{I}$ with $\overline{m(t)} = 0$. $\overline{(.)}$ denotes the time average. However, at present we do not see a way to choose $m(t)$ appropriately.

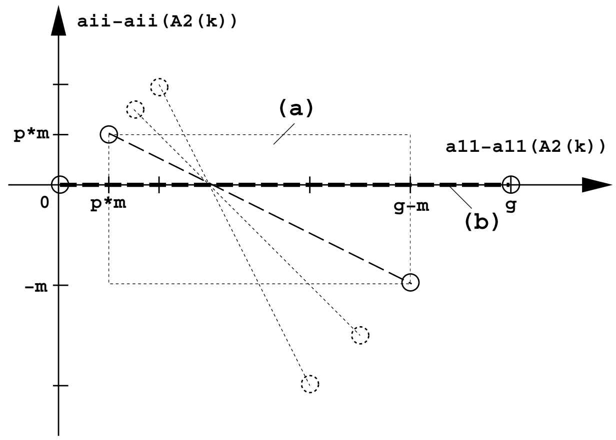


Figure 3: Comparison of ranges of $\mathbf{A}$ in terms of its components a_{11} and $a_{ii}, i > 1$ required to have a common Lyapunov function (a) with and (b) without outweighing

B Dicussion of the Approach in [10]

In [10] the authors complain about the necessary inspiration for derivation of a suitable Lyapunov function in conventional approaches but they merely replace this by the necessary inspiration of a suitable choice of a vector norm. This is even more restrictive, since not every Lyapunov function serves as a norm.

The authors also start with (6) but instead of dealing with $\mu_V(\mathbf{A})$ dircetly they separate $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$. However, in case V is a norm $\mu_V(\mathbf{A})$ is the *matrix measure* and obeys the triangular inequality $\mu_V(\mathbf{A}) \leq \mu_V(\mathbf{A}_1) + \mu_V(\mathbf{A}_2)$. This weakens the results which aim to make the average of μ_V negative.

In the approach of [10] $\mathbf{A}_1$ is constant with lets say $\mu_V(\mathbf{A}_1) = \mu_1$. Instead of $\mu_1 + \overline{\mu_V(\mathbf{A}_2)} < 0$ the authors require $\mu_1 + \overline{\|\mathbf{A}_2\|} < 0$, where $\|(.)\|$ denotes the *matrix norm.* However, for all matrix measures and matrix norms induced by the same vector norm V : $\mu_V(\mathbf{A}) \leq \|\mathbf{A}\|$ [7]. This weakens the results again.

We see that [10] though slightly weaker in the concept could still yield results in cases we don't provide a practical method for (except trial and error with the choice of the Lyapunov function). However, the authors also face difficulties and much need for intuition in the practical analytical realization of their approach. We merely want to point out the more general frame (theorem 4) all this embedds in and evoke deep understanding of the issues by means of the following illustrations.

3 Two-dimensional Illustration

3.1 Set of Possible Quadratic Lyapunov Functions for a Matrix

We restrict our discussion to quadratic Lyapunov functions of the form (7). Thus each Lyapunov function is represented by a symmetric 2-square matrix. [9] It is sensible to normalize p_{11} to 1. Consequently, a 2-dimensional quadratic form is described by two 2 parameters, we

[9]A skew-symmetric part contributes 0 in a quadratic form

call them p_1, p_2.

$$\mathbf{P} = \begin{pmatrix} 1 & p_1 \\ p_1 & p_2 \end{pmatrix} \tag{30}$$

As already emntioned $\mathbf{P}$ has to be pdf.

Lemma 4 *A matrix $\mathbf{M}$ ceases to be pdf. if one of its principal minors (not only the leading principal minors) ceases to be positive [9]. This means in the 2-dinensional case:*

$$m_{11} = 0 \text{ or } m_{22} = 0 \text{ or } m_{11}m_{22} - m_{12}^2 = 0 \tag{31}$$

Thus the region of pdf. matrices in the space of quadratic forms $(p_1, p_2) \in \mathrm{R} \times \mathrm{R}$ is described by a parabula $p_2 > p_1^2$.

From theorem 1 follows that a Hurwitz matrix $\mathbf{A}$ maps the whole space of pdf. matrices $\mathbf{Q}$ into a certain region of pdf. matrices $\mathbf{P}$. We call this the *Set of Possible Lyapunov functions of* $\mathbf{A}$. Given a Hurwitz matrix $\mathbf{A}$ this set can be determined by requiring the conditions (31) to hold for $\mathbf{Q} = -(\mathbf{A}'\mathbf{P} + \mathbf{PA})$, see (8). The first two equations in (31) define a straight line in the (p_1, p_2)-space and the last one an ellipse. The regions of positiveness belong to the corresponding halfspaces resp. to the region inside the ellipse.

We illustrate this for two matrices:

$$\mathbf{M} = \begin{pmatrix} 0 & 3 \\ -2 & -1 \end{pmatrix} \text{ and } \tilde{\mathbf{M}} = \mathbf{M} - 0.5\mathbf{I} = \begin{pmatrix} -0.5 & 3 \\ -2 & -1.5 \end{pmatrix} \tag{32}$$

Figure 4(a) shows the set of possible Lyapunov functions for $\mathbf{M}$. The first two constraints according to (31) lead to $4p_1 > 0$ and $q_{22} = -6p_1 + 2p_2 > 0$ i.e. the region right of the p_2-axis and left of the straight line. The third constraint (here omitted) describes the inner region of the ellipsis, which depicts the actual set for $\mathbf{M}$. All is nice above the parabola which limits the set of pdf. matrices in terms of p_1, p_2. We observed that these three regions have no common point when $\mathbf{A}$ is not Hurwitz.

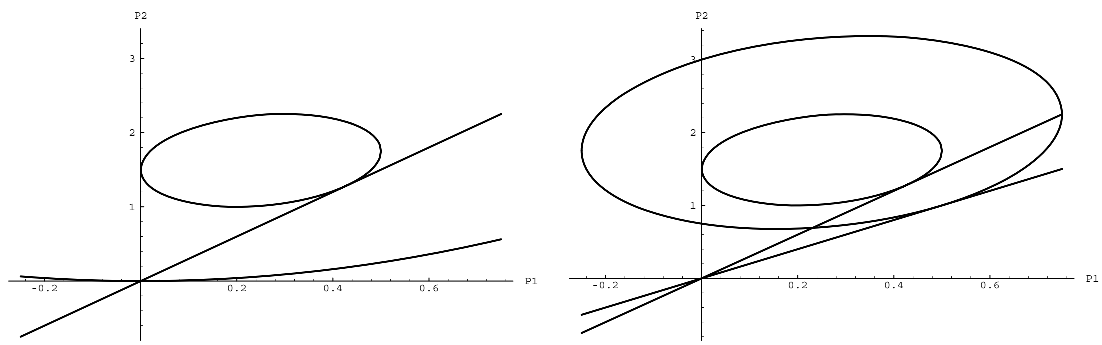
By an argument similar to the proof of lemma 3 it is clear, that every Lyapunov function of $\mathbf{M}$ is a Lyapunov function of $\tilde{\mathbf{M}}$ too, i.e. one set of possible Lyapunov functions contains the other one, see figure 4(b).

The advanced reader will now anticipate that the synchronization border we detect without outweighing corresponds to the coupling value k for which the matrices $\mathbf{A}(t)$ have just a Lyapunov function common in their sets. Whereas by outweighing (i.e. modifications as in lemma 3) we modify the sets which have no point in common for a certain value of k. One is enlarged the other is made smaller until they eventually overlap.

3.2 Without Outweighing

We chose a 2-dimensional example of the special situation described in section 2.3 A, i.e. we assume two matrices to differ only by rank1.

$$\mathbf{A}_1 = \begin{pmatrix} 2-k & 3 \\ -2 & -1 \end{pmatrix} \text{ and } \mathbf{A}_2 = \mathbf{A}_1 - 1 \cdot \mathbf{e}_1\mathbf{e}_1' = \begin{pmatrix} 1-k & 3 \\ -2 & -1 \end{pmatrix} \tag{33}$$



(a) (b)

Figure 4: Set of possible Lyapunov functions for (a) $\mathbf{M}$ and (b) $\mathbf{M}, \tilde{\mathbf{M}}$ of (32)

A The Hurwitz Border

It is necessary that both matrices are Hurwitz in order to have a common Lyapunov function, though not sufficient. The well known necessary and sufficient conditions for a characteristic polynomial of a matrix to be Hurwitz (e.g. in [9]) lead in the 2-dimensional case to trace $= a_{11} + a_{22} < 0$ and $\det(\mathbf{A}) > 0$. For the matrices in (33) this gives:

$$k > 1 \tag{34}$$

B Reverse Application of the Circle Criterium

Since both matrices in (33) differ only by rank1, theorem 2 comes down to $m = 1$ and thus to the application of the circle criterium. We put this into the feedback structure of figure 2 and have: $\mathbf{A} = \mathbf{A}_1,\ \mathbf{B} = \mathbf{e}_1,\ \mathbf{C} = \mathbf{e}_1',\ \mathbf{D} = \mathbf{0},\ a = 0,\ b = 1$ with

$$H(s) = \frac{1+s}{4+k-s+ks+s^2} \tag{35}$$

In this situation the sector of the nonlinearity is fixed (a, b) but the transfer function depends on the coupling parameter, we want to find the admissible values for, such that according to theorem 2:

$$T(s) = H(s) + 1 \text{ is pr., i.e. } \mathrm{Re}(T(s = j\omega)) > 0,\ \forall\, \omega \in \mathrm{R} \tag{36}$$

We multiply $T(s)$ with its denominator and its conjugate and require the real part of the resulting nominator, $J(s)$ to be positive on the imaginary axis:

$$J(s) = 20 + 9k + k^2 + 9s^2 + 3ks^2 - k^2s^2 + s^4 \overset{s^2 \to -v}{=} 20 + 9k + k^2 + (-9 - 3k + k^2)v + v^2 \quad (37)$$

Here we wrote the polynomial in ω^2, cf. (12) as a polynomial in v and require this to have no positive real roots. Conditions (13) and (14) are easily satisfied since

$$a_0 = 20 + 9k + k^2 > 0, \text{ for } k < -5 \text{ or } k > -4 \quad (38)$$

$$a_2 = 1 > 0 \quad (39)$$

(Due to the negative error feedback setup in (22) we are only interested in $k > 0$.) According to lemma 1 we would be finished if also $a_1 > 0$. In this case the number of variations of signs of the a_i would be zero and garantees no positive real roots. However:

$$a_1 = -9 - 3k + k^2 < 0, \text{ for } -1.85 < k < 4.85 \quad (40)$$

In this case the discriminant, dis $= a_1^2 - 4a_0a_2$, has to be negative in order to assure that there are no positive real roots.

$$\begin{aligned} &\text{dis} = 1 + 18k - 13k^2 - 6k^3 + k^4 = (-7.42 + k)(-1.04 + k)(0.05 + k)(2.41 + k) \\ \Rightarrow\quad & k > 1.04 \end{aligned} \quad (41)$$

Thus, only for (41) $J(v)$ has no positive real roots and $J(s = j\omega)$ is positive for all $\omega \in \mathrm{R}$ [10].

Figure 5(a) illustates that the sets of possible Lyapunov functions of $\mathbf{A}_1, \mathbf{A}_2$ just touch each other for the critical coupling value detected by means of the circle criterium, (41). Note, that this common Lyapunov function has by no means to be a diagonal one ($p_1 = 0$). Thus, if one pre-chooses the kind of Lyapunov function naively, e.g. to be a diagonal one as done in [1], one does not get as good results as by taking full advantage of the power of the circle criterium, which provides just any Lyapunov function.

Figure 5(b) demonstrates that is not sufficient to require all matrices to be Hurwitz, as done in [2], in order to provide the existence of a common Lyapunov function. Both sets have no common point for a coupling value between the Hurwitz border (34) and the critical value justified by the circle criterium (41) [11].

3.3 With Outweighing

We assume knowledge is available about which proportion of time each linear region is 'valid', be it $p_1 = 1/3, p_2 = 2/3$ thus $p = 0.5$ in (25). According to lemma 3 we now look for a common Lyapunov function of two modified matrices:

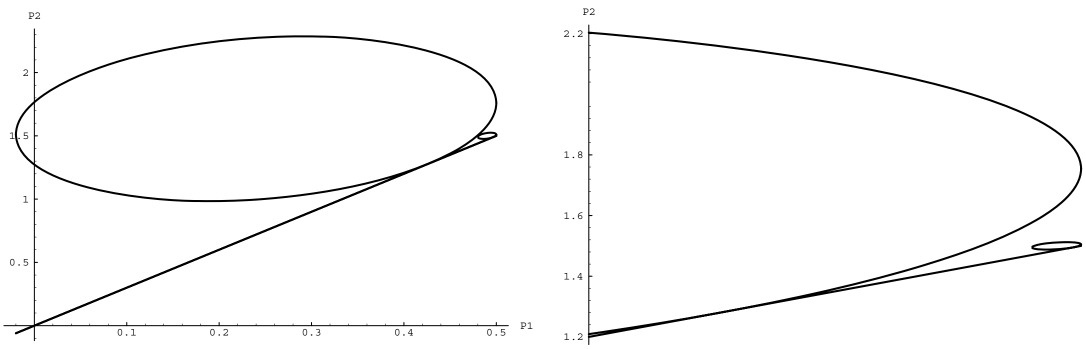
$$\mathbf{B}_1 = \begin{pmatrix} 2 - k - m & 3 \\ -2 & -1 - m \end{pmatrix} \text{ and } \mathbf{B}_2 = \begin{pmatrix} 1 - k + 0.5m & 3 \\ -2 & -1 + 0.5m \end{pmatrix} \quad (42)$$

Since both matrices in (42) now differ by rank2 theorem 2 leads to the check for positive realness of a 2-dimensional transfer matrix. We put this into the feedback structure of figure 2 and have: $\mathbf{B} = \mathbf{I},\ \mathbf{C} = \mathbf{I},\ \mathbf{D} = \mathbf{0},\ a_1 = 0,\ b_1 = 1 - 1.5m,\ a_2 = 0,\ b_2 = 1.5m$ and

$$\mathbf{A} = \begin{pmatrix} 2 - k - m & 3 \\ -2 & -1 + 0.5m \end{pmatrix} \quad (43)$$

[10] The region $k > 7.42$ with dis > 0 is outside the region we are concerned with according to (40)

[11] We cannot illustrate the situation at the Hurwitz border, because then one set melts to a point



(a) (b)

Figure 5: Set of possible Lyapunov functions of $\mathbf{A}_1, \mathbf{A}_2$ in (33) for (a) $k = 1.04$ and (b) $k = 1.02$

We chose $\mathbf{A}$ to be neither $\mathbf{B}_1, \mathbf{B}_2$ for that both b_1, b_2 are positive [12]. Now the transfer function depends on the coupling parameter k as well as the *outweighing parameter* m:

$$\mathbf{H}(s) = \frac{1}{d}\begin{pmatrix} 1 - 0.5m + s & 3 \\ -2 & -2 + k + m + s \end{pmatrix} \text{ with } d = \begin{array}{l} 4 + k + 2m - 0.5km - 0.5m^2 \\ -s + ks + 0.5ms + s^2 \end{array} \tag{44}$$

According to theorem 2 we now have to decide the positive realness of the matrix $\mathbf{T}(s) = \mathbf{H}(s) + \mathrm{Diag}[1/b_1, 1/b_2]$, respectively the positive definitness of its Hermitian part $\mathbf{J}(s) = \mathbf{H}(s) + \mathbf{H}^*(s)$ on the imaginary axis. Following theorem 3 this leads us to consider the positiveness of the its leading principal minors, namely j_{11} and $\det(\mathbf{J})$, which become after multiplication with the common denominator and it conjugate:

$$j_{11} \overset{s^2 \to -v}{=} \begin{array}{l} 20 + 9k + k^2 + 10m - 2.5km - 1k^2m - 1.5m^2 - 1.25km^2 + 0.25k^2m^2 + 0.5m^3 \\ +0.125km^3 - 0.125m^4 + (-9 - 3k + k^2 - 1m + 0.5km - 0.25m^2)v + v^2 \end{array} \tag{45}$$

We omit $\det(\mathbf{J})$ here and mention merely that it is a polynomial in even powers of ω up to power 8 i.e. a 4th-order polynomial in v. In both cases the coefficients of v^i depend on two

[12] Inspect figure 3 to see that only such m with $b_1 = g - (1+p)m > 0$ are sensible because every region, to provide a common Lyapunov function for, with $g - (1+p)m < 0$ would be bigger than another region with smaller m

parameters: k, m. Since the conditions for positiveness will lead up to 24th-order polynomials in k we do not see another way than scanning the sensible range of $m \in [0, g/(1+p)] = [0, 1/1.5]$[13] for the widest admissible range of k.

For fixed m the positiveness of (45) is established in the same way as in section 3.2 since it is also a 2nd-order polynomial in v. Here we sketch the way to decide the positiveness of $\det(\mathbf{J})$ and obtain the admissible range of k where both leading principal minors are positive.

We start as in section 3.2 with the conditions (13) and (14). $a_4 = 1 > 0$ is immediately fulfilled and a_0 is a 4th-order polynomial in k but positive in the interesting range of $k > 0$ and $m \in [0, 1/1.5]$. However, $a_1 < 0$ in the region $0 < k < 1.04$, which would correspond to an improvement of the synchronization border, for all sensible m. This destroys our hope, we could exclude any positive real roots by $a_i > 0,\ i = 1 \dots n$ and lemma 1.

Similar to the discriminant for 2nd-order polynomials there exist so called *Characteristic Expressions* for 4th-order polynomials [13]:

$$
\begin{aligned}
G &= a_3^3 - 4a_3a_2 + 8a_1 &(46)\\
H &= 8a_2 - 3a_3^2 &(47)\\
F &= 16a_2^2 + 3a_3^4 - 16a_2a_3^2 - 64a_0 + 16a_3a_1 &(48)\\
I &= a_2^2 + 12a_0 - 3a_3a_1 &(49)\\
J &= 72a_2a_0 + 9a_3a_2a_1 - 2a_2^2 - 27a_1^2 - 27a_0a_3^2 &(50)\\
\text{dis} &= -(4I^3 - J^2) &(51)
\end{aligned}
$$

The necessary and sufficient conditions for a certain root clustering of a 4th-order polynomial (real/imaginary/negative) are stated in terms of a_i and these characteristic expressions and can be combined to the case 'no positive real roots'. Here we merely state the conditions for 'no real roots', because this turned out to determine the critical value of k, for which $\det(\mathbf{J}(v))$ ceases to have no positive real roots and for which $\det(\mathbf{J}(s))$ ceases to be positive on the imaginary axis.

A 4th-order polynomial has no real roots if

$$
(H \geq 0,\ \text{dis} > 0) \text{ OR } (F < 0,\ \text{dis} > 0) \text{ OR } (H > 0,\ F = 0,\ , G = 0) \qquad (52)
$$

In our case $H, F$and dis are for fixed m polynomials in k of order $4, 6$ and 24. By inspection of their zeros we achieved the smallest admissible k for $m = 2/3$, i.e. for the upper bound of the sensible m-range.

For $m = 2/3$ the first principal minor $j_{11}(v)$ has no positive real roots for $2/3 < k < 6.9$ whereas $\det(\mathbf{J}(v))$ has no positive real roots for $0.69 < k < 7.1$. Thus we found $\mathbf{T}(s)$ to be positive real for

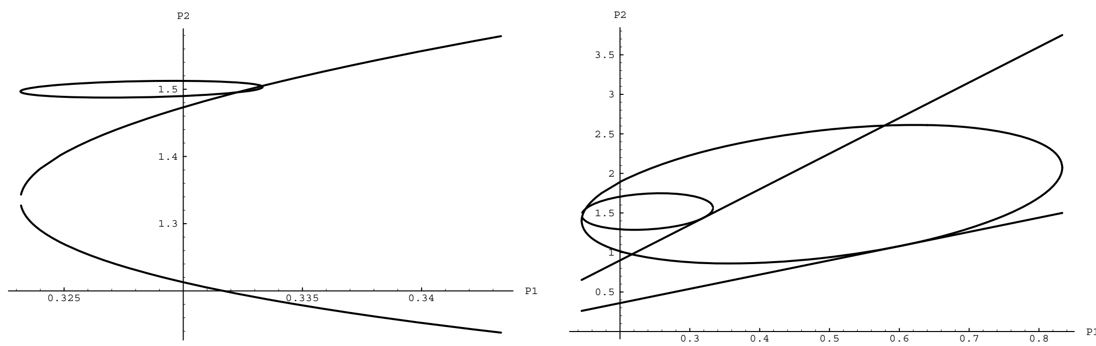
$$
0.69 < k < 6.9 \qquad (53)
$$

It is understood that the range of admissible coupling values k consists of the union of the admissible k- ranges for all m. Since, the range (53) overlaps the range, we have established in section 3.2 with $m = 0$, we had proven asymptotic stablity of (6) for $k > 0.69$. This is an improvement of the admissible k-border of more than 30%.

Figure 6 illustrates the situation for (a) the smallest admissible coupling value and (b) for the critical coupling value without outweighing (41). This should demonstrate how the sets

[13] The upper bound depicts the case when the square of required common Lyapunov function melts to a vertical line in figure 3

of possible Lyapunov functions, which just touch in figure 5(a), are modified by outweighing.



(a) (b)

Figure 6: Set of possible Lyapunov functions of $\mathbf{B}_1, \mathbf{B}_2$ in (3) for $m = 2/3$ and (a) $k = 0.69$ and (b) $k = 1.04$

We conclude, by application of the outweighing approach it is possible to establish stabiliy for a wider range of parameters.

4 Example: Coupled Chua's circuits

Two coupled Chua's circuits [2] also represent a case where the matrices $\mathbf{A}(t)$ in (6) vary only by rank1. The two extremal matrices are:

$$\mathbf{A}_1 = \begin{pmatrix} -\alpha(1+a)-k & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix} \text{ and } \mathbf{A}_1 = \begin{pmatrix} -\alpha(1+b)-k & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix} \tag{54}$$

with the usual set of parameters: $\alpha = 10, \beta = 14.87, a = -1.27, b = -0.68$.

Applying the Hurwitz conditions on both matrices leads to the Hurwitz border $k > 11.12$ [2]. However, this result is false in the sense that for this coupling value no common Lyapunov function of the form (7) exists, as we will see in the sequel.

Here we apply only the circle criterium since the outweighing approach would lead to higher than 4th order polynomials. By establishing a common Lyapunov function for $\mathbf{A}_1, \mathbf{A}_2$

we provide one for all $\mathbf{A}(t)$, since they lie in the convex hull of $\mathbf{A}_1, \mathbf{A}_2$. Again we put this into the structure of figure 2 and have: $\mathbf{A} = \mathbf{A}_1,\ \mathbf{B} = \mathbf{e}_1,\ \mathbf{C} = \mathbf{e}_1',\ \mathbf{D} = \mathbf{0},\ a = 0,\ b = 5.9$ with

$$H(s) = \frac{14.87 + s + s^2}{-40.149 + 14.87k + 2.17s + ks - 1.7s^2 + ks^2 + s^3} \tag{55}$$

We have to establish the positive realness of the transfer function $T(s) = H(s) + 1/5.9$. Multiplication with the common denominator and its conjugate leads to a real part,J, which is a polynomial in even powers of ω up to order 6 and thus a 3rd order polynomial in v:

$$J = -1910.45+110.56k+221.12k^2-(-267.03+34.37k+28.74k^2)v+(-17.38+0.5k+k^2)v^2+v^3 \tag{56}$$

$a_3 = 1 > 0$ and $a_0 > 0$ for $k < -3.2, k > 2.7$. Thus, the conditions (13) and (14) are well fulfilled above the Hurwitz border. Again, $a_1 < 0$ (for $k > 2.5$) destroys our hope to finish with $a_i > 0,\ i = 1 \ldots n$.

The Descarte's rule of sign lemma 1 apllied to the polynomial in $-v$ gives exactly one variation of sign and thus exactly one negative real root of J, no matter which sign a_2 has. If we could exclude that J has 3 real roots then we exclude that it has positive real ones. This can be decided by means of thediscriminant of 3rd order polynomials, defined in the following way:

$$Q = \frac{a_2^2 - 3a_1}{9},\ R = \frac{2a_2^3 - 9a_2a_1 + 27a_0}{54},\ \text{dis} = Q^3 - R^2 \tag{57}$$

If the discriminant is negative then the polynomial has not 3 real roots [13]. dis is a 8th order polynomial in k and is negative for $k < 2.68, k > 11.77$.

Thus, $\mathbf{T}(s)$ is positive real for

$$k > 11.77 \tag{58}$$

That means that there are some coupling values between the Hurwitz border and (58) for which no common Lyapunov function exists although all matrices are Hurwitz.

In [1] the analytically determined synchronization border was $k > -\alpha \cdot a = 12.7$ because the Lyapunov function was pre-chosen to be diagonal. As expected our result admits a bigger range of k since it provides a common Lyapunov function which can have any form.

5 Conclusions

We explained why the proof of synchronization is essentially a proof of unique asymptotic behaviour or a proof of asymptotic stability of the zero solution of a difference system.

The application of Lyapunov's direct method for this purpose leads basically to the need for a common Lyapunov function of a set of matrices $\mathbf{A}(t)$. We demonstrated the power of a theorem based on the Kalman-Yacubovitch Lemma in providing such common Lyapunov function by means of positive realness of a transfer function. Some tools of linear algebra suitable to establish this are recalled.

We demonstrated the application of the circle criterium in some sense in reverse. That is, the sector of the nonlinearity is fixed but the transfer function depends on a parameter that we derive the admissible ranges for. This way we achieved a sharper proof (in the sense of 'for a wider range of this parameter') than others [1] with a naively pre-chosen Lyapunov function.

We described the outweighing approach which admits outweighing of temporary divergence by appropriate convergence with respect to an energy-like function and we developed how this can be brought back to the need of a common Lyapunov function for modified matrices.

For a 2-dimensional (difference) state space we illustrated that Hurwitz matrices do not necessarily have a common Lyapunov function (i.e. the aproach in [2] can lead to wrong results) and that the common Lyapunov function provided by the circle criterium does not need to be a diagonal one (i.e to pre-chose this gives worse results).

Furthermore, we introduced the set of possible Lyapunov functions of a matrix and showed how it is modified by adding a multiple of the identity matrix. This illustrates why the outweighing approach, i.e. the search for a common Lyapunov function for modified matrices, can lead to better results.

The application of the outweighing approach requires the mean value of the quantity defined in (15) to be negative. However, such a mean value depends on the measure on the attractor. It could be negative for the natural measure of the chaotic attractor in the invariant synchronization manifold and positive if the measure is supported by an unstable periodic orbit of this manifold. Such a situation can lead to locally riddled basins of the synchronization manifold or on-off intermittency [16]. We see clearly that this limits the application of the approach to cases where one can assume a certain minimum proportion of time in a contracting region. This means considering a somehow worst-case measure, but we are far away from providing such a tool. However, we hope that the illustrated ideas help to evoke a deeper understanding of the issues in proofs of synchronization.

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