A PROOF OF SYNCHRONIZATION BY USE OF THE KALMAN- YACUBOVITCH LEMMA

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Abstract — **We propose an alternative proof of synchronization based on a necessary and sufficient condition for the existence of a quadratic Lyapunov function. This method theoretically detects the whole parameter range of a given synchronization structure, for which just any quadratic Lyapunov function exists. In principle we are able to exceed results of approaches which either pre- choose the shape of the Lyapunov function or try to find it by optimization.**

I. INTRODUCTION

Designing a synchronization scheeme of e.g. two identical chaotic systems for communication purposes one often has to find parameters for which synchronization can be proven.

One even might be interested in the whole range of admissible parameters in order to optimize the parameter choice with respect to other features like robustness against noise or mismatch.

Or one might need the information that for a certain synchronization structure (e.g. scalar error feedback) no admissible feedback parameter exists in order to decide that another structure (e.g. dynamic error feedback [1]) is necessary.

Without loss of generality we consider a Master- Slave configuration. **G** could depend on the feedback parameter to be designed.

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad \mathbf{x} \in \mathrm{R}^n \tag{1}$$

$$\dot{\mathbf{y}} = \mathbf{G}(\mathbf{x},\mathbf{y}), \quad \mathbf{y} \in \mathrm{R}^n \tag{2}$$

Conditions for synchronization are often based on the existence of a suitable quadratic Lyapunov function

$$V(\mathbf{z}) = \mathbf{z}'\mathbf{P}\mathbf{z}; \;\; \mathbf{P} \in \mathrm{R}^{(n\times n)}, \; \mathbf{P} > 0 \tag{3}$$

which proves the gobal asymptotic stability (e.g. in [5]) of an error system.

$$\dot{\mathbf{z}} = \mathbf{F}(\mathbf{x}) - \mathbf{G}(\mathbf{x},\mathbf{y}) = \mathbf{f}(\mathbf{x},\mathbf{z}) \tag{4}$$

Current proofs of synchronization either prechoose the shape of the Lyapunov function (e.g. a diagonal **P** as in [2]) or try to find it by solving a constrained nonlinear non- convex optimization problem [1].

Although the latter approach allows the usage of efficient LMI methods [3] we wish to provide an alternative tool overcoming the following drawbacks:

One possibly detects a parameter range smaller than the analytically provable one. And if one merely does not find a suitable quadratic Lyapunov function one cannot conclude that none exists.

Therefore we suggest to apply a theorem, well-known in control theory, which is based on the famous Kalman- Yacubovitch Lemma (KYL) [4].

The KYL provides a condition **necessary and sufficient** for the existence of a quadratic Lyapunov function without constructing it explicitely.

Since this approach leaves the form of **P** in (3) free one can prove synchronization (for a possibly larger parameter range) as long as any quadratic Lyapunov function exists. And one could detect if none exists.

Next we state the theorem based on the KYL and develop how its condition can be checked analytically. Finally we demonstrate its application with an example.

II. APPLICATION OF THE KYL

We do not cite the KYL itself but it is the basis of the theorem following below. This deals with a typical control theory setup:

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A linear dynamic system with an m-dimensional input and output and transfer matrix $\mathbf{H}(s)$ has a nonlinear static feedback. This feedback can be time-varying but has to obey a sector condition, see figure 1 for an illustration.

Especially when the synchronizing systems are Lur'e systems it is natural to bring (4) into this form. The matrix $\mathbf{A}$ could depend on the feedback parameter to be designed.

which has an elegant graphical interpretation in the Nyquist plot of the transfer function $\mathbf{H}(s)$. This is known as the circle criterium.

Definition 1 (Positive Realness (pr.)) *A complex matrix $\mathbf{H}(s)$ is positive real if its hermitian part $\mathbf{J}(s) = \mathbf{H}(s) + \mathbf{H}^*(s)$ [2] is positive definite (pdf.) on the imaginary axis, i.e. for*

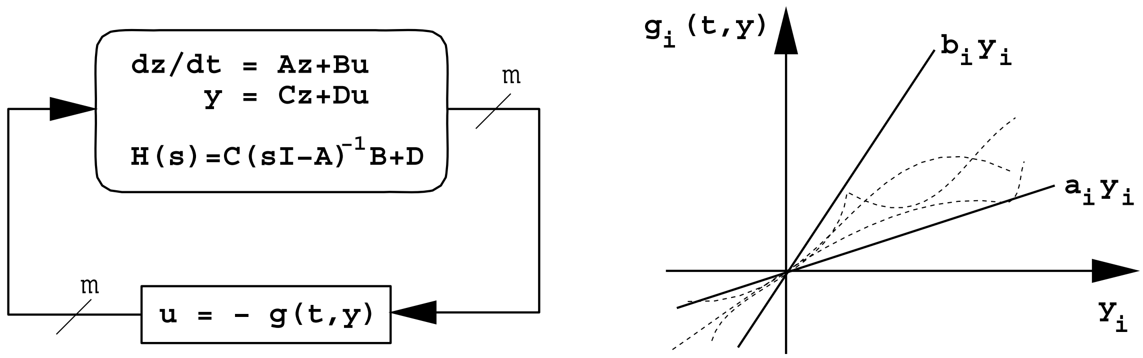

Fig. 1: Feedback scheme consisting of a linear dynamical system with a time-varying nonlinearity in the feedback path

$$\mathbf{H}(s = j\omega): \ \mathrm{Re}(\mathbf{v}^*\mathbf{H}\mathbf{v}) = \mathbf{v}^*(\mathbf{H} + \mathbf{H}^*)\mathbf{v} > 0 \quad \forall\, \mathbf{v} \neq \mathbf{0},\ \forall\, \omega \in \mathrm{R} \qquad \textbf{(8)}$$

* stands for the conjugated transpose of a vector or a matrix and Re for the real part of the quadratic form.

Theorem 1 *Consider the system in figure 1. Suppose (i) the pair $(\mathbf{A}, \mathbf{B})$ is controllable, (ii) the pair $(\mathbf{C}, \mathbf{A})$ is observable and (iii) each (scalar) component of the nonlinear feedback belongs to a sector $[a_i, b_i]$ with*

$$(g_i(t, \mathbf{y}) - a_i \cdot y_i) \cdot (b_i \cdot y_i - g_i(t, \mathbf{y})) \geq 0,\ i = 1, \ldots m \qquad \textbf{(5)}$$

Define

$$\mathbf{H}_a(s) = \mathbf{H}(s)[\mathbf{I} + Diag[a_i] \cdot \mathbf{H}(s)]^{-1} \qquad \textbf{(6)}$$

Suppose (iv) all poles of $\mathbf{H}_a(s)$ have negative real parts and (v) the **transfer matrix**

$$\mathbf{T}(s) \;=\; \mathbf{H}(s)[\mathbf{I} + Diag[a_i] \cdot \mathbf{H}(s)]^{-1} + Diag[1/(b_i - a_i)] \qquad \textbf{(7)}$$

is **positive real**. *Under these conditions, the origin of the system is globally asymptotically stable.*

We stated the theorem in a slightly different form than in [5] [1]. The proof follows straight the one given there.

For m=1 this comes down to the consideration of positive realness of a scalar transfer function

Theorem 2 (pdf. $\Longleftrightarrow$ (LPMs > 0)) *A hermitian matrix is positive definite if and only if all its leading principal minors (LPMs) are positive [6].*

The leading principal minors of an m-square matrix are its m 'north-westerly' subdeterminants.

Consequently, in order to establish positive realness of a transfer matrix we have to prove the positiveness of m functions, the LPMs of its hermitian part on the imaginary axis.

These appear to be polynomials of ω^2:

$$P(\omega^2) = a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + \ldots,\ a_i \in \mathrm{R} \qquad \textbf{(9)}$$

Next we cite some criteria suitable to establish the positiveness of such a polynomial.

An obviously necessary condition for $P(\omega^2) > 0,\ \forall\, \omega$ is

$$a_0 > 0 \qquad \textbf{(10)}$$

[1] On one hand we are more restrictive, since we require the inequality $(\mathbf{g}(t, \mathbf{y}) - a\mathbf{y})' \cdot (b\mathbf{y} - \mathbf{g}(t, \mathbf{y}) \geq 0$ to hold componentwise. On the other hand (5) admits different sectors for the nonlinearity and is thus more general.

[2] $\mathbf{J}$ is a hermitian matrix: $\mathbf{J} = \mathbf{J}^*$. All its eigenvalues are real.

Given this one has to establish that $P(.)$ has no positive real roots. A helpful criterium to decide this is the so called:

Lemma 1 (Descarte's Rule of Signs) *The number of positive real roots of a polynomial $P(.)$ with real coefficients a_i is either equal to the number of variations of signs between the a_i in successive order or less than this number by an even integer [7].* [3]

Thus a second necessary condition for a nth. order principal minor polynomial (9) to have no positive real roots is

$$a_n > 0 \qquad \mathbf{(11)}$$

and we could finish if all $a_i > 0$.

In case they aren't, the actual number of positive real roots, which has to be zero for our purpose, can be determined by sign changes of so called *Inners* of matrices built out of the polynomial coefficients a_i [7].

This leads to conditions on *characteristic expressions* of a_i such as the *Discriminants*, as known for 2nd and 3rd order polynomials.

For higher order polynomials more and more different cases of sign changes have to be considered. And this method becomes less handy.

Recall, the error system (4) depends on the feedback parameter (e.g. the matrix $\mathbf{A}$ in figure 1) and so does the transfer matrix to be checked for positive realness and so do the coefficients of the m principal minor polynomials of ω^2.

Consequently, we detect the border of (analytically established) synchronization at the feedback parameter value(s), where such positiveness criteria cease to be fulfilled. Next we demonstrate this at an example.

III. EXAMPLE

As sort of a benchmark example we consider coupled Chua's circuits [8]. k describes the scalar error feedback of the first component. And we want to detect an as large as possible range of admissible k.

We put the error system into the structure of figure 1 with $m = 1$ and have:

$$\mathbf{A} = \begin{pmatrix} -\alpha(1+g_0) - k & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix} \qquad \mathbf{(12)}$$

[3] E.g. a 3rd order polynomial with coefficient signs $+,-,+,-$ has either 3 or 1 positive real roots.

with the usual set of parameters: $\alpha = 10, \beta = 14.87, g_0 = -1.27$ and

$$\begin{aligned} H(s) &= (14.87 + s + s^2)/d \qquad \mathbf{(13)} \\ d &= -40.149 + 14.87k + 2.17s + ks \\ &\quad -1.7s^2 + ks^2 + s^3 \end{aligned}$$

$\mathbf{B} = \mathbf{e}_1,\ \mathbf{C} = \mathbf{e}_1',\ \mathbf{D} = \mathbf{0},\ a = 0,\ b = 5.9$ with

Applying a necessary (Hurwitz) condition to the scheeme leads to the Hurwitz border $k > 11.12$ [8]. However, this is not sufficient (as claimed in [8]) to establish synchronization.

We have to establish the positive realness of the transfer function $T(s) = H(s) + 1/5.9$. Multiplication with the common denominator and its conjugate leads to a real part, J, which is a polynomial in even powers of ω up to order 6 and thus a 3rd order polynomial in $v = \omega^2$:

$$\begin{aligned} J &= -1910.45 + 110.56k + 221.12k^2 \\ &\quad -(-267.03 + 34.37k + 28.74k^2)v \\ &\quad +(-17.38 + 0.5k + k^2)v^2 + v^3 \qquad \mathbf{(14)} \end{aligned}$$

Next we check the positiveness of $J(\omega^2)$ by determining for which parameter k the polynomial $J(v)$ has no positive real roots.

$a_3 = 1 > 0$ and $a_0 > 0$ for $k < -3.2, k > 2.7$. I.e. the conditions (10) and (11) are well fulfilled above the Hurwitz border. However, $a_1 < 0$ (for $k > 2.5$) destroys our hope to finish with $a_i > 0,\ i = 1 \ldots 3$.

The Descarte's rule of sign (lemma 1) applied to the polynomial in $-v$ gives exactly one variation of sign and thus exactly one negative real root of J, no matter which sign a_2 has.

If we could exclude that $J(v)$ has 3 real roots then we exclude that it has positive real ones.

This can be decided by means of the discriminant of 3rd order polynomials, defined in the following way:

$$Q = \frac{a_2^2 - 3a_1}{9},\ R = \frac{2a_2^3 - 9a_2a_1 + 27a_0}{54}$$
$$\text{dis} = Q^3 - R^2 \qquad \mathbf{(15)}$$

If the discriminant is negative then the polynomial has not 3 real roots [7].

dis is a 8th order polynomial in k and is negative for $k < 2.68, k > 11.77$.

Thus, $\mathbf{T}(s)$ is positive real for

$$k > 11.77 \qquad \mathbf{(16)}$$

In [2] the analytically determined synchronization border was $k > -\alpha \cdot a = 12.7$ where the Lyapunov function was pre-chosen to be diagonal.

As expected our result admits a bigger range of k since it provides the existence of a Lyapunov function which can have any form.

IV. CONCLUSION

We proposed an alternative proof of synchronization by application of a theorem, well- known in control theory, to the error system. This theorem is based on the Kalman-Yacubovitch Lemma and proves global asymptotic stability. It establishes the existence of a quadratic Lyapunov function by means of a frequency domain criterium, namely by the positive realness of a transfer marix.

We show a way to check positive realness analytically. This leads to the positiveness of m polynomials which are the leading principal minors of the hermitian part of the transfer matrix. We recall some criteria capable to establish the positiveness of such a polynomial.

The power of the approach arises from the fact that the KYL provides a condition necessary and sufficient for the existence of a quadratic Lyapunov function.

Since the shape of the Lyapunov function is free one can prove synchronization theoretically as long as just any such Lyapunov function exists and thus for the widest possible range of e.g. feedback parameters. And one can detect in principle if for a certain feedback structure no quadratic Lyapunov function exists.

We demonstrated the application of the proposed method at an example with $m = 1$. This corresponds to a special case, the well known circle criterium.

We apply it in some sense in reverse. That is, the sector of the nonlinearity is fixed but the transfer function depends on a parameter that we derive the admissible range for.

As expected we are able to establish synchronization for a wider range of this parameter than others [2] with a pre-chosen Lyapunov function.

REFERENCES

[1] J. A. K. Suykens, P. F. Curran, L. O. Chua, 'Master- Slave Synchronization using Dynamic Output Feedback', Int. J. Bifurcation & Chaos, 1997.

[2] M. P. Kennedy, private communication, and similar in M. Hasler, H. Dedieu, J. Schweizer, M. P. Kennedy, 'Synchronization of Chaotic Signals', Proc. Workshop on Nonlinear Dynamics of Electronic Systems, Dresden, July 1993, World Scientific, 1994, 244-261.

[3] S. Boyd et.al., Linear matrix inequalities in system and control theory, SIAM (Studies in Applied Mathematics), Vol. 15,1994

[4] K. S. Narendra, J. H. Taylor, 'Frequency Domain Criteria for Absolute Stability', Academic Press New York, 1973.

[5] M. Vidyasagar, 'Nonlinear Systems Analysis', 2nd Ed. Prentice-Hall, Englewood Cliffs N. J., 1993.

[6] M. Marcus, H. Minc, 'Introduction to Linear Algebra', Macmillan Company New York, 1965.

[7] E. I. Jury, 'Inners and Stability of Dynamic Systems', R. E. Krieger Publishing Co. Florida, 1982.

[8] L. O. Chua, M. Itoh, L. Kocarev, K. Eckert, 'Chaos Synchronization in Chua's Circuit', J. Circuits, Systems and Computers, vol. 3, no. 1, (1993) 83-108.