A Proof of Stability and the Outweighing Approach

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Abstract— **Stability proofs are important in many fields of science and engineering. A good example is the synchronization of chaotic systems. We propose the application a stability theorem based on the Kalman- Yacubovitch Lemma. Furthermore we introduce a method to improve such stability proofs - the outweighing approach. Illustrative examples are provided.**

Keywords— **Proof of global asymptotic stability, proof of synchronization of chaotic systems.**

I. Introduction

Prove of global asymptotic stability (g.a.s.) is an unbiquitous problem in science. Sometimes the applied criteria provide rather conservative conditions in the sense that the detected stability range (e.g. in terms of system parameters) is much smaller than the actual stability range. The idea throughout this paper is to enlarge the analytically detectable stability ranges i.e. to improve or sharpen the proofs.

A g.a.s. problem arises for instance with proofs of synchronization of chaotic systems, like for the following Master-Slave-configuration:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad \mathbf{x} \in \mathrm{R}^n \tag{1}$$

$$\dot{\mathbf{y}} = \mathbf{G}(\mathbf{x},\mathbf{y}), \quad \mathbf{y} \in \mathrm{R}^n \tag{2}$$

The system (2) synchronizes to (1) if:

$$\mathbf{z} = \mathbf{x} - \mathbf{y} \longrightarrow 0 \text{ as } t \to \infty \tag{3}$$

Thus in order to establish synchronization we have to show that the origin, $\mathbf{z} = 0$, of the *difference system*:

$$\dot{\mathbf{z}} = \mathbf{F}(\mathbf{x}) - \mathbf{G}(\mathbf{x},\mathbf{y}) = \mathbf{f}(\mathbf{x},\mathbf{z}) \tag{4}$$

is globally asymptotically stable. Sometimes it is useful to put the considered system into a time-variant, linear form:

$$\dot{\mathbf{z}} = \mathbf{A}(t) \cdot \mathbf{z}, \ \mathbf{A} \in \mathrm{R}^{n \times n} \tag{5}$$

where the Matrix $\mathbf{A}$ depends on $\mathbf{x}, \mathbf{z}$ and this way on the time t. A common way to prove g.a.s. is to apply Lyapunov's Direct Method [1]. That is one has to find a scalar, energy-like function, the Lyapunov function, on the state space with a minimum at the origin and establishes a strictly decreasing energy along the system flow. See figure 1 for an illustration.

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Quadratic forms have been proven to be handy as **Lyapunov functions**:

$$V = \mathbf{z}'\mathbf{P}\mathbf{z}, \ \mathbf{P} \in \mathrm{R}^{n \times n} \tag{6}$$

where $\mathbf{P}$ has to be positive definite (pdf.). $'$ stands for the transpose of a vector or a matrix.

Handwritten proofs of g.a.s., e.g. proofs of synchronization, often prechoose for the sake of simplicity the shape of the Lyapunov function as e.g. in [2]. While other approaches try to find a Lyapunov function by solving a nonlinear non-convex optimization problem [3]. Although the latter approach allows the usage of efficient LMI methods [4] we wish to provide an alternative tool overcoming the following drawbacks:

One possibly detects a smaller stability range than the analytically provable one. And if one does not find a suitable Lyapunov function one cannot conclude that none exists.

Therefore the first purpose of this paper is to recall a g.a.s. theorem which leaves the shape of the quadratic Lyapunov function free and to propose a method to apply this theorem analytically (section II).

Since this approach proves g.a.s. (or e.g. synchronization) as long as any quadratic Lyapunov function exists one can possibly detect a larger stability range than with a prechosen quadratic Lyapunov function. And one could in principle detect analytically if none exists.

Another drawback associated with Lyapunov's direct method is that it requires decreasing energy always and everywhere. This way one cannot establish g.a.s. of systems which outweigh temporary divergence by appropriate convergence with respect to an energy-like function.

Therefore the second purpose of this paper is to describe the outweighing approach. In stability theory this is in principle known as the *comparison principle* [5] and the energy-like function is then called Curduneanu function [6]. But we also propose a realization method of the outweighing approach (section III).

In section IV we illustrate the ideas of sections II and III which should convince that the described approaches can lead to better results. We hope this encourages the reader to apply our methods to other stability problems. In section V we demonstrate the application of the theorem of section II with a synchronization example.

II. A G.A.S. THEOREM AND ITS APPLICATION

A. The g.a.s. theorem based on the Kalman-Yacubovitch Lemma (KYL)

We do not cite the KYL ([6]) itself but it is the basis of the theorem following below. Its power arises from the fact that the KYL provides a condition **necessary and sufficient** for the existence of a quadratic Lyapunov function without constructing it explicitely.

The theorem deals with a typical control theory setup: A linear dynamic system with an m-dimensional input and output and transfer matrix $\mathbf{H}(s)$ has a nonlinear static feedback. This feedback can be time-varying but has to obey a sector condition, see figure 2. In general the system under consideration, e.g. (4) or (5) can be put into this form.

Theorem 1 (Circle criterium) Consider the system in figure 2. Suppose (i) the pair $(\mathbf{A},\mathbf{B})$ is controllable, (ii) the pair $(\mathbf{C},\mathbf{A})$ is observable and (iii) each (scalar) component of the nonlinear feedback belongs to a sector $[a_i, b_i]$ with

$$(g_i(t,\mathbf{y}) - a_i \cdot y_i) \cdot (b_i \cdot y_i - g_i(t,\mathbf{y})) \geq 0,\ i = 1,\dots m \quad (7)$$

Define

$$\mathbf{H}_a(s) = \mathbf{H}(s)[\mathbf{I} + \mathrm{Diag}[a_i] \cdot \mathbf{H}(s)]^{-1} \quad (8)$$

Suppose (iv) all poles of $\mathbf{H}_a(s)$ have negative real parts and (v) the **transfer matrix**

$$\begin{aligned}\mathbf{T}(s) &= \mathbf{H}(s)[\mathbf{I} + \mathrm{Diag}[a_i] \cdot \mathbf{H}(s)]^{-1} \\ &\quad + \mathrm{Diag}[1/(b_i - a_i)] \end{aligned} \quad (9)$$

is **positive real**. Under these conditions, the origin of the system is globally asymptotically stable.

We stated the theorem in a slightly different form than in [1] [1]. The proof follows straight the one given there.

Thus the KYL converts our search for a suitable Lyapunov function into a check of positive realness of a transfer matrix $\mathbf{T}(s)$.

For $m = 1$ condition (9) requires a scalar transfer function to be positive real which has a graphical interpretation in the Nyquist plot of the transfer function $\mathbf{H}(s)$. This is commonly known as the circle criterium.

Next we propose a method to check positive realness in general.

B. A method to check positive realness analytically

First we recall some definitions and theorems on positive realness and positive definiteness.

Definition 1 (Positive Realness (pr.)) A complex matrix $\mathbf{T}(s)$ is positive real if its hermitian part $\mathbf{J}(s) = \frac{1}{2}(\mathbf{T}(s) + \mathbf{T}^*(s))$ [2] is positive definite (pdf.) on the imaginary axis, i.e. for

$$\begin{aligned}\mathbf{T}(s = j\omega):\ \mathrm{Re}(\mathbf{v}^*\mathbf{T}\mathbf{v}) = \mathbf{v}^*\tfrac{1}{2}(\mathbf{T} + \mathbf{T}^*)\mathbf{v} > 0 \\ \forall\, \mathbf{v} \neq \mathbf{0},\ \forall\, \omega \in \mathrm{R}\end{aligned} \quad (10)$$

[1] On one hand we are more restrictive, since we require the inequality $(\mathbf{g}(t,\mathbf{y}) - a\mathbf{y})' \cdot (b\mathbf{y} - \mathbf{g}(t,\mathbf{y}) \geq 0$ to hold componentwise. On the other hand (7) admits different sectors for the nonlinearity and is thus more general.

[2] $\mathbf{J}$ is a hermitian matrix: $\mathbf{J} = \mathbf{J}^*$. All its eigenvalues are real.

* stands for the conjugated transpose of a vector or a matrix and Re for the real part of the quadratic form.

Theorem 2 (pdf. $\Longleftrightarrow$ (LPMs > 0)) A hermitian matrix is positive definite if and only if all its leading principal minors (LPMs) are positive [7].

The leading principal minors of an m-square matrix are its m 'north-westerly' subdeterminants.

Consequently, in order to establish positive realness of a transfer matrix we have to prove the positiveness of m functions, the LPMs of its hermitian part on the imaginary axis.

These appear to be polynomials of ω^2:

$$P(\omega^2) = a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + \dots,\ a_i \in \mathrm{R} \quad (11)$$

Next we cite some criteria suitable to establish the positiveness of such a polynomial.

An obviously necessary condition for $P(\omega^2) > 0,\ \forall\, \omega$ is

$$a_0 > 0 \quad (12)$$

If this is the case one has to establish that $P(.)$ has no positive real roots. A helpful criterium to decide this is the so called:

Lemma 1 (Descarte's Rule of Signs) The number of positive real roots of a polynomial $P(.)$ with real coefficients a_i is either equal to the number of variations of signs between the a_i in successive order or less than this number by an even integer [8]. [3]

Thus a second necessary condition for a nth. order principal minor polynomial (11) to have no positive real roots is

$$a_n > 0 \quad (13)$$

and we could finish if all $a_i > 0$.

In case they aren't, the actual number of positive real roots, which has to be zero for our purpose, can be determined by sign changes of so called *Inners* of matrices built out of the polynomial coefficients a_i [8].

This leads to conditions on *characteristic expressions* of a_i such as the *Discriminants*, as known for 2nd and 3rd order polynomials. For higher order polynomials more and more different cases of sign changes have to be considered. And this method becomes less handy.

Recall, we have to check the positive realness of a transfer matrix. This transfer matrix possibly depends on system parameters (the stability range of which we wish to detect) and so do the coefficients of the principal minor polynomials of ω^2.

Consequently we will detect the border of (analytically established) g.a.s. (or e.g. synchronization) at those parameter value(s), for which such positiveness criteria cease to be fulfilled. All this will be demonstrated at examples in sections IV and V.

[3] E.g. a 3rd order polynomial with coefficient signs $+,-,+,-$ has either 3 or 1 positive real roots.

III. The Outweighing Approach

A. Approach

We think the outweighing approach is best applicable if the system under consideration is in linear form (5). Therefore we will use it throughout this section although the idea could be formulated also more generally.

Recall: Lyapunov's direct method requires decreasing energy with respect to a Lyapunov function always and everywhere. This leads with (5) and (6) to:

$$\dot{V} = \mathbf{z}'(\mathbf{A}'\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{z} = -\mathbf{z}'\mathbf{Q}\mathbf{z} < 0, \; \forall\, \mathbf{z} \neq \mathbf{0} \tag{14}$$

Our idea now is to admit outweighing of temporary divergence by appropriate convergence with respect to an energy-like function, i.e. to relax the condition (14) and to prove g.a.s. also in cases where Lyapunov's direct method fails.

In [9] this idea was applied to a one dimensional (difference) system (4). Dimension=1 provides the advantage that divergence and convergence occur in the same (only one) direction of state space and their outweighing is therefore easy to estimate.

However, in more-dimensional systems diverging and converging may occur in different 'directions' of state space and the estimation of their outweighing is nontrivial. Therefore we consider the **worst-case of diverging**, i.e. the maximum divergence with respect to a Lyapunov function $V(\mathbf{z})$ *over all directions* in state space *at each moment of time.* And then we estimate its time average. [4]

Definition 2 ($\mu_V(\mathbf{A}(t))$) The following quantity represents the maximum divergence of a system (5) with respect to a Lyapunov function $V(\mathbf{z})$ at time t:

$$\mu_V(\mathbf{A}(t)) = \sup_{\mathbf{z} \in \mathrm{R}^n/\{\mathbf{0}\}} [\dot{V}(\mathbf{A}(t), \mathbf{z})/V(\mathbf{z})] \tag{15}$$

If V is a *norm on* R^n then the quantity defined above is the *matrix measure induced by* V([11]).

Simple integration of (15) yields an upper bound of the system energy:

$$V(\mathbf{z}(t)) \leq V(\mathbf{z}(0)) \exp[\int_0^t \mu_V(\mathbf{A}(\tau)) d\tau] \tag{16}$$

By means of (16) it is possible to estimate whether convergence compensates temporary divergence:

Theorem 3: Let V be a Lyapunov function and $\mu_V(t)$ be a function defined in (15). If

$$\lim_{T \to \infty} \int_0^T \mu_V(\tau) d\tau = -\infty \tag{17}$$

then the trivial solution of (5) is globally asymptotically stable. (adapted from [6])

Condition (17) holds if the *mean value* of $\mu_V(\mathbf{A}(t))$ is negative. In this case convergence predominates temporary divergence. Thus it is sufficient to inspect the mean value of $\mu_V(\mathbf{A}(t))$ in order to prove g.a.s. To do this requires of course knowledge about which proportion of time each possible matrix $\mathbf{A}(t)$ is *valid.* See section VI for a discussion of this problem.

[4] Another approach [10] is in some sense dual: it considers the maximum divergence *over all times* for *each state direction.* However this method works so far only in two-dimensional state space.

B. $\mu_V(\mathbf{A})$

$\mu_V(\mathbf{A}(t))$ can be determined for a quadratic Lyapunov function (6) in the following way:

Lemma 2: μ_V, defined in (15), with $V = \mathbf{z}'\mathbf{P}\mathbf{z}$ and $\dot{V} = -\mathbf{z}'\mathbf{Q}\mathbf{z}$ is equal to the maximum generalized eigenvalue of $(-\mathbf{Q}, \mathbf{P})$.

$$\lambda_{\max}(-\mathbf{Q}, \mathbf{P}) = \mu_V(\mathbf{A}) \tag{18}$$

Proof: $\mu_V(\mathbf{A})$ is the maximum ratio between the two quadratic forms $\dot{V}$ and V. We convert this into the following conditional optimization problem which can be solved by means of Lagrangian theory:

$$\begin{aligned} \text{maximize } \dot{V} &= -\mathbf{z}'\mathbf{Q}\mathbf{z} \text{ over } \mathbf{z} \in \mathrm{R}^n/\{\mathbf{0}\} \\ \text{with } V &= \mathbf{z}'\mathbf{P}\mathbf{z} = \text{constant} \end{aligned} \tag{19}$$

We introduce the *Lagrangian* $L = \dot{V} - \lambda \cdot (V - \text{constant})$ and obtain the following conditions for the extremal value:

$$-\mathbf{Q}\mathbf{z} = \lambda \cdot \mathbf{P}\mathbf{z} \tag{20}$$

$$\mathbf{z}'\mathbf{P}\mathbf{z} = \text{constant} \tag{21}$$

(20) defines λ to be a generalized eigenvalue of the pair of matrices $(-\mathbf{Q}, \mathbf{P})$. Multiplication of (20) with the eigenvector $\mathbf{z}'$ belonging to the largest generalized eigenvalue (which obeys (21)) reveals that $\lambda_{\max}(-\mathbf{Q}, \mathbf{P})$ is the maximum ratio of $\dot{V}$ and V which completes the proof.

C. Realization

In order to apply the outweighing approach (theorem 3) one still has to find a suitable Lyapunov function which leaves the mean value of $\mu_V(t)$ negative. Next we explain, how this problem can be converted into another problem, which can be solved by means of the circle criterium.

First we introduce the useful notion of a *common* Lyapunov function:

Definition 3: A Lyapunov function $V = \mathbf{z}'\mathbf{P}\mathbf{z}$ for which (14) holds for a set of matrices $\mathbf{A}$ is a **common Lyapunov function** of this set of matrices.

The circle criterium (theorem 1) essentially establishes the existence of such a common Lyapunov function for all possible matrices $\mathbf{A}(t)$, which occur when the system of figure 2 is transformed into the time-varying, linear form (5).

We modify the matrices $\mathbf{A}(t)$ such that a common Lyapunov function of the modified matrices leads to a $\mu_V(\mathbf{A}(t))$ that stays well below prescribed values:

Lemma 3: If $V = \mathbf{z}'\mathbf{P}\mathbf{z}$ is a common Lyapunov function for the matrices

$$\mathbf{B}(t) = \mathbf{A}(t) - m(t)\mathbf{I} \qquad m \in \mathrm{R} \;\text{ then} \tag{22}$$

$$\mu_V(\mathbf{A}(t)) < 2m \tag{23}$$

Proof: If $V = \mathbf{z}'\mathbf{P}\mathbf{z}$ is a common Lyapunov function for all $\mathbf{B}(t)$ then

$$\begin{aligned} \mathbf{z}'(\mathbf{B}(t)'\mathbf{P} + \mathbf{P}\mathbf{B}(t))\mathbf{z} &= -\mathbf{z}'\mathbf{Q}(t)\mathbf{z} < 0,\ \forall\ \mathbf{z} \neq \mathbf{0} \\ &\text{and} \\ \dot{V} = \mathbf{z}'(\mathbf{A}(t)'\mathbf{P} + \mathbf{P}\mathbf{A}(t))\mathbf{z} &= -\mathbf{z}'\mathbf{Q}(t)\mathbf{z} + 2m(t)\cdot \mathbf{z}'\mathbf{P}\mathbf{z} \\ &< 2m(t)\cdot V \end{aligned} \tag{24}$$

It follows $\dot{V}/V < 2m(t)$ and thus $\mu_V(\mathbf{A}(t)) < 2m$ which completes the proof.

If we choose $m(t)$ such that $\overline{m(t)} = 0$ ($\overline{(.)}$ denotes the time average) then condition (17) in theorem 3 holds. In section IV we demonstrate the realization if this method with a very simple example.

Concluding this section we remark that a similar approach [12] also applies the outweighing idea and we discuss briefly, why it must lead to weaker results than the direct application of theorem 3: 1st. In [12] the Lyapunov function is a vector norm. The authors also start with (5) but instead of dealing dircetly with $\mu_V(\mathbf{A})$ they separate $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$. However, since V is a norm, $\mu_V(\mathbf{A})$ is the *matrix measure* and it obeys the triangular inequality $\mu_V(\mathbf{A}) \leq \mu_V(\mathbf{A}_1) + \mu_V(\mathbf{A}_2)$. This weakens the result which aims to make the average of μ_V negative. 2nd. In [12] $\mathbf{A}_1$ is constant with lets say $\mu_V(\mathbf{A}_1) = \mu_1$. The authors demand $\mu_1 + \overline{\|\mathbf{A}_2\|} < 0$ instead of $\mu_1 + \overline{\mu_V(\mathbf{A}_2)} < 0$, where $\|(.)\|$ denotes the *matrix norm*. However, for all matrix measures and matrix norms induced by the same vector norm V : $\mu_V(\mathbf{A}) \leq \|\mathbf{A}\|$ [1]. This weakens the result of [12] again.

IV. Two-dimensional Illustration

In this section we first introduce the set of possible Lyapunov functions of a matrix. It allows an illustration of the methods of sections II and III. Then we demonstrate the application of theorems 1 and 3 and show that we can detect a larger g.a.s. range than with a prechosen quadratic e.g. diagonal Lyapunov function and that an even larger g.a.s. range can be detected if knowledge is available about which proportion of time each $\mathbf{A}(t)$ is valid in (5).

A. Set of Possible Quadratic Lyapunov Functions of a Matrix and its Modification

In two-dimensional state space a quadratic Lyapunov function (6) is represented by a symmetric, pdf. 2×2-matrix $\mathbf{P}$. If p_{11} is normalized to 1 then a two-dimensional quadratic form is described by two 2 parameters, p_1, p_2.

$$\mathbf{P} = \begin{pmatrix} 1 & p_1 \\ p_1 & p_2 \end{pmatrix} \tag{25}$$

We use the following lemma in order to detect the set of pdf. matrices, i.e. the set of Lyapunov functions $\mathbf{P}$:

Lemma 4: A matrix $\mathbf{M}$ ceases to be pdf. if one of its principal minors (not only the leading principal minors) ceases to be positive [6], i.e. for a two-square matrix:

$$(m_{11} = 0) \text{ or } (m_{22} = 0) \text{ or } (m_{11}m_{22} - m_{12}^2 = 0) \tag{26}$$

Thus the set of Lyapunov functions (25) is described by $p_2 > p_1^2$ with $(p_1, p_2) \in \mathrm{R} \times \mathrm{R}$.

The following theorem is the clue to the idea of possible Lyapunov functions of a matrix:

Theorem 4: For every Hurwitz matrix $\mathbf{A}$ and for each pdf. matrix $\mathbf{Q}$ exists a pdf. matrix $\mathbf{P}$ such that

$$\mathbf{A}'\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q} \tag{27}$$

This result is known as the Lyapunov Matrix Equation [1]. Notice the close relation of (27) to (14).

Definition 4: The **set of possible Lyapunov functions of the matrix A** is the set of pdf. forms $\mathbf{P}$ into which the whole set of pdf. matrices $\mathbf{Q}$ is mapped by (27).

We determine the set of possible Lyapunov functions of a given matrix $\mathbf{A}$ by application of the conditions (26) on $\mathbf{Q}$ with (27). We illustrate this for two matrices:

$$\begin{aligned} \mathbf{M} &= \begin{pmatrix} 0 & 3 \\ -2 & -1 \end{pmatrix} \text{ and} \\ \tilde{\mathbf{M}} = \mathbf{M} - 0.5\mathbf{I} &= \begin{pmatrix} -0.5 & 3 \\ -2 & -1.5 \end{pmatrix} \end{aligned} \tag{28}$$

The first two equations in (26) define a straight line in the (p_1, p_2)-space and the last one an ellipse. The regions of positiveness belong to the corresponding halfspaces, $4p_1 > 0$ and $q_{22} = -6p_1 + 2p_2 > 0$, resp. to the region inside the ellipse, which is also the intersection of the three regions. All is nice above the parabola $p_2 = p_1^2$ which limits the set of pdf. matrices $\mathbf{P}$, Figure 3(a).

By means of such illustration we will visualize the term common Lyapunov function of matrices $\mathbf{A}(t)$, cf. definition 3, namely a Lyapunov function which is common to the set of possible Lyapunov functions for all matrices $\mathbf{A}(t)$, i.e. a common point p_1, p_2 of the corresponding ellipses.

Figure 3(b) visualizes how modifications as in lemma 3 modify the set of possible Lyapunov functions of a matrix. By an argument similar to the proof of lemma 3 it is clear, that every Lyapunov function of $\mathbf{M}$ is a Lyapunov function of $\tilde{\mathbf{M}}$ too.

B. Application of the Circle criterium (Theorem 1)

We consider an example of system (5) where the set of matrices $\mathbf{A}(t)$ contains only two matrices $\mathbf{A}_1, \mathbf{A}_2$:

$$\begin{aligned} \mathbf{A}_1 &= \begin{pmatrix} 2-k & 3 \\ -2 & -1 \end{pmatrix} \text{ and} \\ \mathbf{A}_2 &= \mathbf{A}_1 - 1\cdot \mathbf{e}_1\mathbf{e}_1' = \begin{pmatrix} 1-k & 3 \\ -2 & -1 \end{pmatrix} \end{aligned} \tag{29}$$

We want to detect an as large as possible range of the parameter k for which the system is g.a.s.

We put the system into the feedback structure of figure 2: Since both matrices in (29) differ only by rank1 we have: $m = 1$ and $\mathbf{A} = \mathbf{A}_1,\ \mathbf{B} = \mathbf{e}_1,\ \mathbf{C} = \mathbf{e}_1',\ \mathbf{D} = \mathbf{0},\ a = 0,\ b = 1$ with

$$H(s) = \frac{1+s}{4+k-s+ks+s^2} \tag{30}$$

In this case conditions *(i)-(iii)* of theorem 1 hold. With $a = 0$ we have $H_a(s) = H(s)$ in (8) and condition *(iv)* holds if $\mathbf{A}_1$ is Hurwitz, i.e. if trace$(\mathbf{A}_1) < 0$ and $\det(\mathbf{A}_1) > 0$. This gives

$$k > 1 \tag{31}$$

According to condition *(v)* of the circle criterium the system is g.a.s. if

$$T(s) = H(s) + 1 \text{ is pr., i.e. } \mathrm{Re}(T(s = j\omega)) > 0,\ \forall\, \omega \in \mathrm{R} \tag{32}$$

We multiply $T(s)$ with its denominator and its conjugate and demand that the real part of the resulting nominator, $J(s)$, is positive on the imaginary axis:

$$\begin{aligned} J(s) &= 20 + 9k + k^2 + 9s^2 + 3ks^2 - k^2s^2 + s^4 \\ &\overset{s^2 \to -v}{=} 20 + 9k + k^2 + (-9 - 3k + k^2)v + v^2 \end{aligned} \tag{33}$$

Here we wrote the polynomial in $\omega^2 = -s^2$ as a polynomial in v which must not have positive real roots. Conditions (12) and (13) are satisfied for all $k > 1$, cf. (31), since:

$$a_0 = 20 + 9k + k^2 > 0, \text{ for } k < -5 \text{ or } k > -4 \tag{34}$$

$$a_2 = 1 > 0 \tag{35}$$

According to lemma 1 we could finish if also $a_1 > 0$. In this case the number of variations of signs of the a_i would be zero which would guarantee that there are no positive real roots. However:

$$a_1 = -9 - 3k + k^2 < 0, \text{ for } -1.85 < k < 4.85 \tag{36}$$

For the case $1 < k < 4.85$, cf. (31),(36), we have to inspect the discriminant, dis $= a_1^2 - 4a_0a_2$. It has to be negative in order to assure that there are no positive real roots.

$$\begin{aligned} \text{dis} &= 1 + 18k - 13k^2 - 6k^3 + k^4 \\ &= (-7.42 + k)(-1.04 + k)(0.05 + k)(2.41 + k) \\ \Rightarrow &\quad k > 1.04 \end{aligned} \tag{37}$$

Thus, for $k > 1.04$: $J(v)$ has no positive real roots, $J(s = j\omega)$ is positive for all $\omega \in \mathrm{R}$ and $T(s)$ is positive real.

Figure 4(a) illustrates that the sets of possible Lyapunov functions of $\mathbf{A}_1, \mathbf{A}_2$ just touch each other for the critical coupling value detected by means of the circle criterium, (37). Note, that this common Lyapunov function is not a diagonal one ($p_1 \neq 0$). Thus, if one pre-chooses e.g. a diagonal Lyapunov function one would not detect the whole g.a.s. range, (37).

Figure 4(b) demonstrates that is not sufficient to require all matrices $\mathbf{A}(t)$ to be Hurwitz, as done in [13], in order to establish the existence of a common Lyapunov function. Both sets have no common point for a coupling value between the Hurwitz border (31) and the value justified by the circle criterium (37)

C. Application of Theorem 3

We consider the same example as in section IV-B but assume additionally, that knowledge is available about which proportion of time the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ are *valid* in (5), be it $q_1, q_2 = 1 - q_1$. Under this condition we want to detect an as large as possible g.a.s. range of the parameter k. According to theorem 3 the system is g.a.s. if the mean value of $\mu_V(\mathbf{A}(t))$ is negative, i.e.:

$$\begin{aligned} q_1\mu_V(\mathbf{A}_1) + q_2\mu_V(\mathbf{A}_2) &< 0 \text{ or} \\ \mu_V(\mathbf{A}_2) &< -\frac{q_1}{q_2}\mu_V(\mathbf{A}_1) = -q \cdot \mu_V(\mathbf{A}_1) \end{aligned} \tag{38}$$

A Lyapunov function with $\mu_V(\mathbf{A}_1) < 2\mu$ and $\mu_V(\mathbf{A}_2) < -2q \cdot \mu$ meets the condition in (38). According to lemma 3 a common Lyapunov function $V = \mathbf{z}'\mathbf{P}\mathbf{z}$ of the two modified matrices

$$\mathbf{B}_1 = \mathbf{A}_1 - \mu\mathbf{I} \text{ and } \mathbf{B}_2 = \mathbf{A}_2 + q\mu\mathbf{I},\ \mu \in \mathrm{R} \tag{39}$$

serves this purpose.

(39) modifies the sets of possible Lyapunov functions of $\mathbf{A}_1, \mathbf{A}_2$, cf. figure 3. One is enlarged and the other is made proportionally smaller, i.e. (14) is relaxed for $\mathbf{A}_1$ (temporary divergence admitted) and $\mathbf{A}_2$ has to obey a more stringent condition (proportionally stronger convergence).

Again we detect the g.a.s. range by application of the circle criterium, i.e. we establish a common Lyapunov function for the modified matrices $\mathbf{B}_1, \mathbf{B}_2$. We assume $q_1 = 1/3, q_2 = 2/3$, i.e. $q = 0.5$ and have

$$\begin{aligned} \mathbf{B}_1 &= \begin{pmatrix} 2 - k - \mu & 3 \\ -2 & -1 - \mu \end{pmatrix} \text{ and} \\ \mathbf{B}_2 &= \begin{pmatrix} 1 - k + 0.5\mu & 3 \\ -2 & -1 + 0.5\mu \end{pmatrix} \end{aligned} \tag{40}$$

Figure 5 illustrates the influence of the modifications on the range of matrices, we need a common Lyapunov function for in terms of their diagonal elements. Since the matrices in (40) now differ by rank 2 we have $m = 2$ in theorem 1. In this case theorem 1 establishes the existence of a common Lyapunov function for all matrices depicted by the square in figure 5. It follows that the sensible range of μ is

$$\mu \in [0, 2/3] \tag{41}$$

Every other μ would lead to a bigger range of matrices in figure 5.

We put this example into the feedback structure of figure 2: $\mathbf{B} = \mathbf{I}$, $\mathbf{C} = \mathbf{I}$, $\mathbf{D} = \mathbf{0}$, $a_1 = 0$, $b_1 = 1 - 1.5\mu$, $a_2 = 0$, $b_2 = 1.5\mu$ and

$$\mathbf{A} = \begin{pmatrix} 2 - k - \mu & 3 \\ -2 & -1 + 0.5\mu \end{pmatrix} \tag{42}$$

Now the transfer function depends on two parameters k and μ:

$$\begin{aligned} &\mathbf{H}(s) = \tfrac{1}{d} \begin{pmatrix} 1 - 0.5\mu + s & 3 \\ -2 & -2 + k + \mu + s \end{pmatrix} \text{ with} \\ &d = 4 + k + 2\mu - 0.5k\mu - 0.5\mu^2 - s + ks + 0.5\mu s + s^2 \end{aligned} \tag{43}$$

In this case conditions *(i)-(iii)* of theorem 1 hold. And condition *(iv)* is fulfilled if $\mathbf{A}$ is Hurwitz. This yields

$$k > 1 - 0.5\mu \tag{44}$$

According to *(v)* of theorem 1 we now have to decide the positive realness of the matrix $\mathbf{T}(s) = \mathbf{H}(s) + \text{Diag}[1/b_1, 1/b_2]$, respectively the positive definitness of its Hermitian part $\mathbf{J}(s) = \frac{1}{2}(\mathbf{T}(s) + \mathbf{T}^*(s))$ on the imaginary axis. Following theorem 2 this leads us to consideration of the positiveness of the its leading principal minors, namely j_{11} and $\det(\mathbf{J})$.

Again we obtain after multiplication with the common denominator and its conjugate polynomials in even powers of ω, namely up to power 4, resp. 8, i.e. 2nd- resp. 4th-order polynomials in v. But now the coefficients of v^i depend on two parameters: k, μ, s. e.g. j_{11}:

$$\begin{aligned} j_{11} \overset{s^2 \to -v}{=} & \; 20 + 9k + k^2 + 10\mu - 2.5k\mu - 1k^2\mu - 1.5\mu^2 \\ & - 1.25k\mu^2 + 0.25k^2\mu^2 + 0.5\mu^3 + 0.125k\mu^3 - 0.125\mu^4 \\ & + (-9 - 3k + k^2 - 1\mu + 0.5k\mu - 0.25\mu^2)v + v^2 \end{aligned} \tag{45}$$

The positiveness of j_{11} and $\det(\mathbf{J})$ is established by anew application of the method proposed in section II-B and already once demonstrated in section IV-B. I.e. we apply the necessary and sufficient conditions for the root cluster 'no positive real roots' which are stated in terms of *characteristic expressions* of the polynomial coefficients a_i, [8].

It follows a scanning of the sensible range of μ, cf. (41). For fixed μ the characteristic expressions are again polynomials in k. By inspection of their zeros we obtained the largest g.a.s range of k for $\mu = 2/3$,i.e. for the upper bound of (41) [5]:

$$k > 0.69 \tag{46}$$

Figure 6 depicts the situation for the smallest admissible value of k (a) with and (b) without outweighing. This should demonstrate how the sets of possible Lyapunov functions, which just touch in figure 4(a), are modified by outweighing.

This illustrates: by application of the outweighing approach it is possible to establish stability for a wider range of parameters, compare (37),(46).

[5] The upper bound depicts the case when the square of required common Lyapunov function melts to a vertical line in figure 5

V. Example: Coupled Chua's circuits

As sort of a benchmark example we consider the synchronization of coupled chaotic Chua's circuits [13]. k describes the scalar error feedback of the first component. And we want to detect an as large as possible range of k, such that the systems synchronize.

$$\dot{\mathbf{x}} = \mathbf{M}\mathbf{x} + \mathbf{e}_1 g(\mathbf{e}_1'\mathbf{x}) \tag{47}$$

$$\dot{\mathbf{y}} = \mathbf{M}\mathbf{x} + \mathbf{e}_1 g(\mathbf{e}_1'\mathbf{x}) + k\mathbf{e}_1'(\mathbf{x} - \mathbf{y}) \tag{48}$$

with

$$\begin{aligned} g(x) &= -\alpha\{g_1 x + 0.5(g_1 - g_0)(|x-1| - |x+1|)\} \text{ and} \\ \mathbf{M} &= \begin{pmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix} \end{aligned} \tag{49}$$

and the usual set of parameters: $\alpha = 10, \beta = 14.87, g_0 = -1.27, g_1 = -0.68$.

The systems synchronize if the difference system, cf. (4), is g.a.s. We put the error system into the structure of figure 2 with $m = 1$ and have:

$$\mathbf{A} = \begin{pmatrix} -\alpha(1+g_0) - k & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix} \tag{50}$$

$\mathbf{B} = \mathbf{e}_1,\ \mathbf{C} = \mathbf{e}_1',\ \mathbf{D} = \mathbf{0},\ a = 0,\ b = 5.9$ and

$$H(s) = \frac{14.87 + s + s^2}{-40.149 + 14.87k + 2.17s + ks - 1.7s^2 + ks^2 + s^3} \tag{51}$$

In this case conditions *(i)-(iii)* of theorem 1 hold. And condition *(iv)* is fulfilled if $\mathbf{A}$ is Hurwitz. This yields $k > 11.12$.

According to condition *(v)* of the circle criterium (theorem 1) the system is g.a.s. if the transfer function $T(s) = H(s) + 1/5.9$ is positive real. Multiplication with the common denominator and its conjugate leads to a real part,J, which is a polynomial in even powers of ω up to order 6 and thus a 3rd order polynomial in $v = \omega^2 = -s^2$:

$$\begin{aligned} J \;=\; & -1910.45 + 110.56k + 221.12k^2 \\ & -(-267.03 + 34.37k + 28.74k^2)v \\ & +(-17.38 + 0.5k + k^2)v^2 + v^3 \end{aligned} \tag{52}$$

Next we determine for which k $J(s)$ is positive on the imaginary axis, i.e. for which k the polynomial $J(v)$ has no positive real roots.

$a_3 = 1 > 0$ and $a_0 > 0$ for $k < -3.2, k > 2.7$. I.e. the conditions (12) and (13) are well fulfilled above the Hurwitz border. However, $a_1 < 0$ (for $k > 2.5$) destroys our hope to finish with $a_i > 0,\ i = 1 \ldots 3$.

The Descarte's rule of sign (lemma 1) applied to the polynomial in $-v$ gives exactly one variation of sign and thus exactly one negative real root of J, no matter which sign a_2 has.

By inspection of a characteristic expression, the discriminant of a 3rd order polynomial, we can exclude that $J(v)$ has 3 real roots and thus that it has positive real ones:

$$Q = \frac{a_2^2 - 3a_1}{9},\ R = \frac{2a_2^3 - 9a_2a_1 + 27a_0}{54},\ dis = Q^3 - R^2 \tag{53}$$

If the discriminant is negative then the polynomial has not 3 real roots [8]. dis is a 8th order polynomial in k and is negative for $k < 2.68, k > 11.77$. Thus, $\mathbf{T}(s)$ is positive real for

$$k > 11.77 \tag{54}$$

In [2] the analytically determined synchronization border was $k > -\alpha \cdot a = 12.7$ where the quadratic Lyapunov function was pre-chosen to be diagonal.

As expected our result admits a bigger range of k since it provides the existence of a quadratic Lyapunov function which can have any form.

VI. Conclusions

We proposed the application of a g.a.s. theorem based on the Kalman-Yacubovitch Lemma. Its power (compared to other approaches) arises from the fact, that it provides the existences of a quadratic Lyapunov function of any form without constructing it explicitely. It converts the search for a suitable Lyapunov function into a check of positive realness of a transfer function. Some tools of linear algebra suitable to establish this are recalled.

We described the outweighing approach which admits outweighing of temporary divergence by appropriate convergence with respect to an energy-like function. Thus we relax criteria of conventional g.a.s. approaches. We also presented a method to bring the outweighing back to the need of a common Lyapunov function for modified matrices. In principle this can also be solved by means of the Kalman-Yacubovitch based g.a.s. theorem 1.

In order to demonstrate the power of the presented methods we showed some illustrations in two-dimensional state space. For this purpose we introduced the set of possible Lyapunov functions of a matrix. We hope this made the advantages of the presented ideas visible and it encourages the reader to apply the theorems also to other problems.

We demonstrated the application of the Kalman-Yacubovitch based g.a.s. theorem to an exapmle of synchronization of chaotic systems. We achieved better results than others.

The application of the outweighing approach requires the mean value of the quantity μ_V, s. (15), to be negative. In case the matrix $\mathbf{A}(t)$ in (5) is not primarily a function of time but basically a function of $\mathbf{x}(t)$ then the mean value amounts to the time average of μ_V along $\mathbf{x}(t)$. However, such a mean value depends on the measure on the attractor. It could be negative for the natural measure of the chaotic attractor in the invariant synchronization manifold and positive if the measure is supported by an unstable periodic orbit of this manifold. Such a situation can lead to locally riddled basins of the synchronization manifold or on-off intermittency [14].

We see that this limits the application of the outweighing approach to cases where one can assume a somehow worst-case measure. However, we defer this problem to future discussion.

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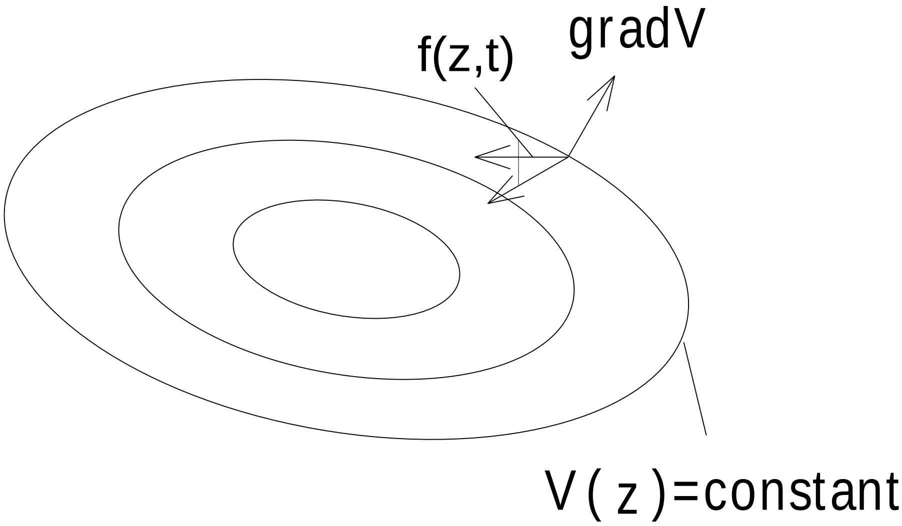


Fig. 1. Illustration of decreasing energy along a system flow

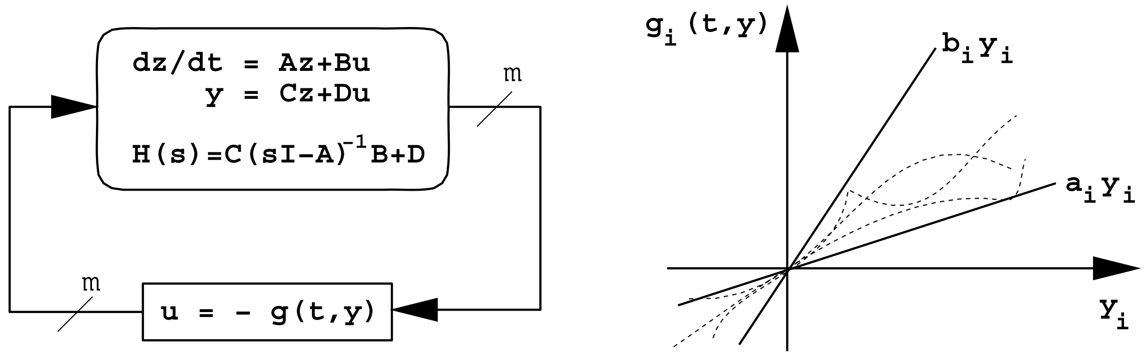
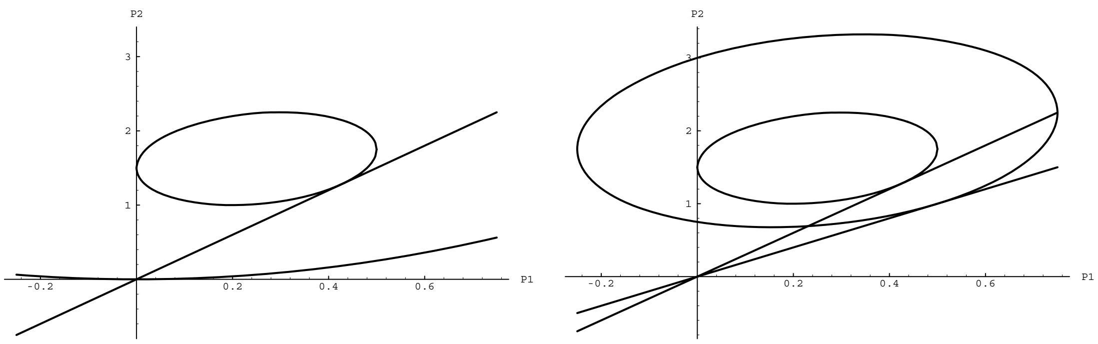
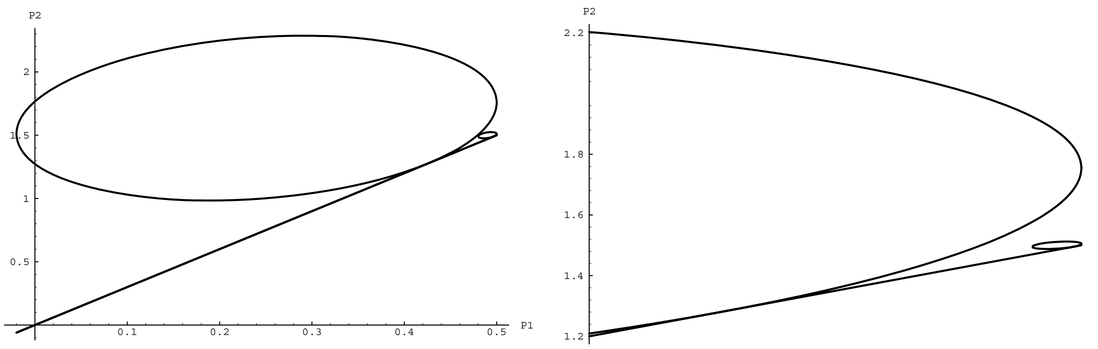


Fig. 2. Feedback scheme consisting of a linear dynamical system with a time-varying nonlinearity in the feedback path



(a) (b)

Fig. 3. Set of possible Lyapunov functions for (a) $\mathbf{M}$ and (b) $\mathbf{M}, \tilde{\mathbf{M}}$ of (28)



(a) (b)

Fig. 4. Set of possible Lyapunov functions of $\mathbf{A}_1, \mathbf{A}_2$ in (29) for (a) $k = 1.04$ and (b) $k = 1.02$

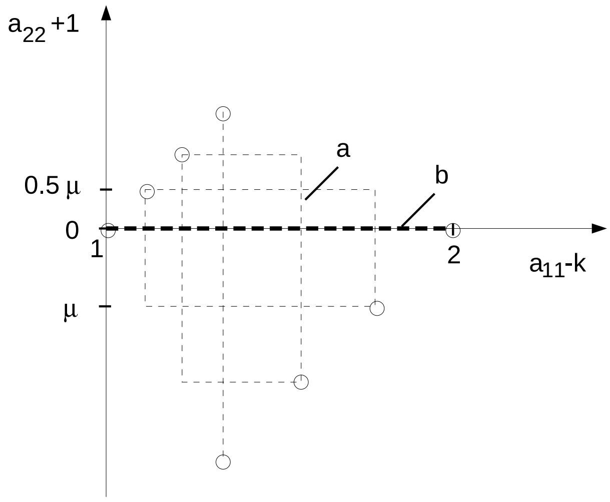


Fig. 5. Comparison of ranges of matrices in terms of their diagonal components required to have a common Lyapunov function (a) with and (b) without outweighing

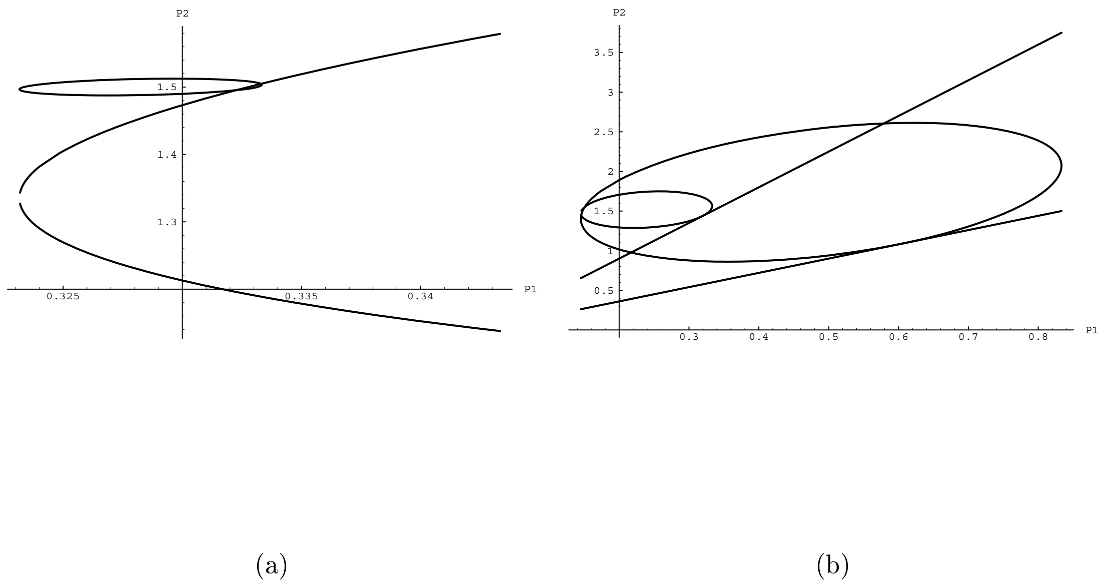


Fig. 6. Set of possible Lyapunov functions of $\mathbf{B}_1, \mathbf{B}_2$ in (40) for $\mu = 2/3$ and (a) $k = 0.69$ and (b) $k = 1.04$

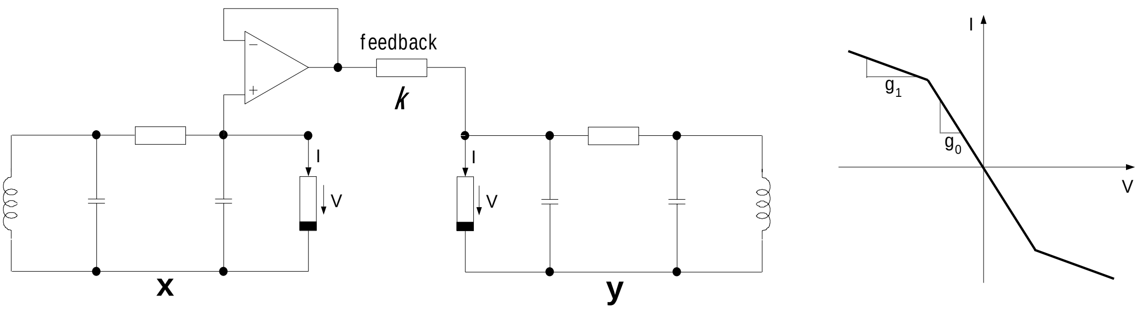


Fig. 7. Coupled chaotic Chua's circuits