Mixed Predictability as an Asymmetrical Measure of Interdependence in Multivariate Time Series

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Abstract. Mixed state analysis [1] has been proposed to detect interdependencies between time series by assessing the predictability of variables by means of embedded data using samples of more than one variable (mixed states). By this way bidirectional coupling could be distinguished from unidirectional one and the direction of the latter could be detected. The motivation of our paper goes beyond the detection of the nature of coupling (unidirectional or bidirectional) and aims at quantification of the degree of coupling in each direction. We propose to take the improvement of the predictability of one variable by additionally considering another variable as a quantitative measure of the coupling from the second variable to the first. This approach somehow meets the information theoretic concept of transfer entropy and it is also applicable to short time series. We demonstrate the use of this approach to coupled deterministic discrete maps and to EEG data.

INTRODUCTION

In many applications, especially in the field of physiological signal processing, one is interested in the detection of synchronization (or interdependencies in general) between systems (or subsystems) generating the signals. Recently different methods aiming to detect different kinds of synchrony (phase synchrony, generalized synchrony) have been applied to EEG signals [2], [3]. Synchrony, in a strict sense, is a symmetrical feature; once the systems are fully synchronized, drive and response systems are indistinguishable. Two systems also can both influence the dynamics of each other in an asymmetric fashion. Thus, the proper detection of such (uni- or bi-)directional dependence is important and can provide new insight into the characterization of underlying dynamics.

Mixed state analysis has been proposed [1] to detect interdependencies by assessing the predictability of variables by means of embedded data using samples of more than one variable (mixed states). The deterministic structure of multivariate data was detected in [4] by constructing the mixed states of embedded samples of all variables; interdependence was established by finding the embedding dimensions (no. of samples of each channel) out of all possible combinations which minimize the prediction error. This method is practically feasible only for small number of variables and does not quantify the interdependencies. In [5] transfer entropy was proposed as a measure of directional interdependence. Since estimation of entropies based on probability distributions in mixed state spaces requires a huge quantity of data, transfer entropy is not suitable for limited data sets.

The primary aim of this study is to devise a measure which allows both (i) the detection of the nature of hidden coupling (unidirectional or bidirectional), and (ii) the quantification of the degree of coupling in each direction. We propose an index, based on the concept of predictability improvement by the construction of mixed states[1]. By several examples including simulated and real-life signals, we demonstrate that this new index of predictability improvements successfully detects and quantifies the nature and degree of interdependencies, respectively.

PREDICTABILITY IMPROVEMENT BY MIXED STATES

Predictability is a sine qua non for determinism. For a deterministic time series, a continuous function maps the past values onto the future values. One way to check the existence of such a map is the following. The (scalar) time series, x_i, $i = 1, \dots, N$, is embedded into the state space X, i.e. subsequent samples form a vector, $\mathbf{x}_i = x_i, x_{i-l}, \dots, x_{i-(m-1)l}$ (m - embedding dimension, l - time lag). If neighboring vectors are followed by their corresponding post-images while maintaining the closeness in neighborhood sense, a continuous map (past to future) is assumed. The difference between the predicted value (by means of the neighborhood) and the (original) future value of a given point in state space is the prediction error. It is a measure of predictability, i.e. the smaller the prediction error, the more predictable the time series. The mean squared error, $\mathrm{MSEx}(X)^2$ (x predicted based on neighborhood in X) is defined as:

$$\mathrm{MSEx}(X) = \frac{1}{N_r} \sum_{i=1}^{N_r} (x_{i+h} - \overline{x_{nni_{1,\dots,k}+h}})^2 \tag{1}$$

Here $nni_{1,\dots,k}$ refers to the time indices of the k nearest neighbors in state space, and N_r, h are the number of considered vectors and the prediction horizon, respectively. $\overline{(.)}$ is the mean operator. A locally constant prediction model is employed by taking the mean of k nearest neighbors. This is computationally less expensive than the local autoregressive model as discussed in [7].

In a similar way, mixed predictability is assessed in terms of the prediction error where the prediction is based on the neighborhood in the mixed state spaces $X \oplus Y$. The state space vector in the mixed state space is formed out of samples from two time series: $x_i, x_{i-l}, \dots, x_{i-(m-1)l}, y_i, y_{i-l}, \dots, y_{i-(n-1)l}$ ($m+n$ - embedding dimension, theoretically two different lags for two time series are possible). The prediction error based on mixed states is $\mathrm{MSEx}(X \oplus Y)$.

We define predictability improvement ($\mathrm{PI}x(y)$) of x with the help of additional information from the state space Y as

$$\mathrm{PI}x(y) = \mathrm{MSEx}(X) - \mathrm{MSEx}(X \oplus Y) \tag{2}$$

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[1] This concept, in its basic form, was proposed in [6] to measure primarily linear interdependence.
[2] The time series x_i is either normalized to zero mean and unit variance or the MSEx has additionally to be normalized with respect to the covariance of x_i.

It is the difference between the (self-) predictability of x and the mixed predictability of x based on mixed states $X \oplus Y$.

We propose PI$x(y)$ as a measure of the interdependence, i.e. to which extent the dynamics of x is influenced by the dynamics of y. PI$y(x)$ can be measured in an analogous way, which indicates the dependence in the opposite direction. The more the system Y influences the future of system X (e.g. the higher the terms involving Y in the state equations of X are due to higher coupling strength), the more the future of x is clarified by additional inclusion of the past of y.

Remark: A similar idea holds for the transfer entropy [5], which measures the flow of information from signal y to x (and vice versa). It is the difference between the entropy rate of x alone and the entropy rate based on transition probabilities which additionally include the past values of y. Transfer entropy and predictability improvement, both being a difference of measures of uncertainty with and without a helping signal, measure how much the future is clarified by an additional knowledge of the past of another signal.

RESULTS

We demonstrate the use of predictability improvement with examples including simulated and real-life signals.

Consider an one dimensional ring lattice of 100 unidirectionally coupled tent maps

$$x_m(i+1) = f(\varepsilon x_{m-1}(i) + (1-\varepsilon)x_m(i)),\ m = 1,\ldots,100 \tag{3}$$

with $f(x) = 1-2|.5-x|$. The strength of the unidirectional coupling is varied from $\varepsilon = 0$ to $\varepsilon = 0.05$. Figure 1 shows the predictability improvements in both directions for the parameters setting $m = n = l = h = k = 1$ (results, computed with 10^4 points, are averaged over 10 runs with random initial conditions after discarding 10^4 points as transients). As expected, PI$x1(x2)$ is approximately zero since there is no direct coupling from $x2$ to $x1$, whereas PI$x1(x2)$ reflects the strength of the coupling in the other direction. The direct comparison with transfer entropy [5] shows that the index based on predictability improvement can produce equivalent results with one tenth of the data.

It is to be noted that the method based on predictability improvement cannot distinguish between fully synchronized and uncoupled systems. The same holds for the transfer entropy since both methods measure the degree of interdependence as long as the systems are not fully synchronized. Thus, the proposed index is sensitive to coupling strength up to perfect synchronization.

Next, consider two bidirectionally coupled identical Henon maps [1]

$$\begin{aligned} x(i) &= a - x(i-1)^2 + b * x(i-2) + c_{yx} * (x(i-1)^2 - y(i-1)^2) \\ y(i) &= a - y(i-1)^2 + b * y(i-2) + c_{xy} * (y(i-1)^2 - x(i-1)^2) \end{aligned} \tag{4}$$

with $a = 1.4, b = .3$. The coupling strengths in both directions are independently varied $c_{xy}, c_{yx} = 0,\ldots,0.36$. The predictability improvements with $m = 2, n = h = k = 1$ of 10^4 points after 10^4 transients are shown in Figure 2. The proposed index successfully

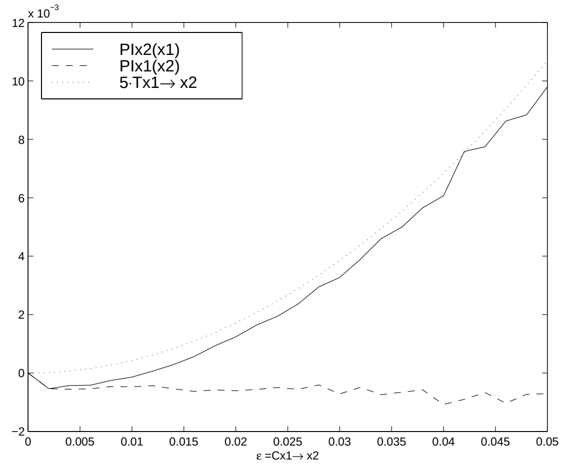

FIGURE 1. Predictability improvements PI$x2(x1)$-solid, PI$x1(x2)$-dashed and the theoretic transfer entropy $T_{x1\to x2}$-dotted as a function of unidirectional coupling strength.

reflects the coupling strengths in both directions up to synchronization. The difference of the predictability improvements in both directions reflects the asymmetry of the coupling.

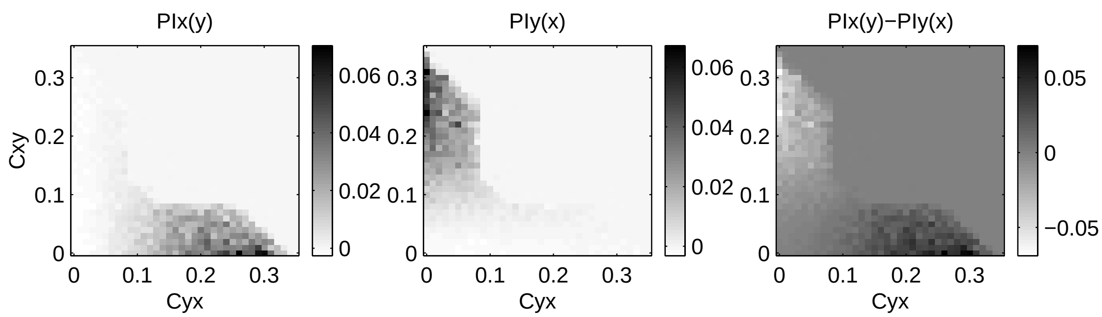

FIGURE 2. Predictability improvements, PI$x(y)$-left and PI$y(x)$-middle as a function of the coupling strengths in both directions. right: Difference of the predictability improvements reflecting the asymmetry of the coupling up to synchronization

Next, we applied the proposed index to multivariate EEG signals from 20 subjects, (10 musicians and 10 nonmusicians) recorded (at 128 Hz sample frequency) while listening to music (90s Bach, French Suite Nr. 5) and at resting condition (eyes closed) [2]. The predictability improvements were calculated in both directions for all pairs between 19 electrodes for subsequent and nonoverlapping windows of 10s length. We chose $l = 5, m = 10$ in accordance with usual embedding techniques for scalar time series (mutual information and false nearest neighbors) and $k = 10, n = 5$ free choice motivated by a case study of one subject. We have to report that (2) can acquire negative values (while transfer entropy is nonnegative). This refers to the worsening of prediction by forcefully adding information from other signal, which might be a signature of independence. Here only positive values of PI are considered. The results were statistically (Wilcoxon test, $p = 5\%$) compared between the groups as well as between task and resting condition.

The differences of the (self-)predictabilities between the two groups while listening to music (Figure 3) reflect the different degrees of determinism of each electrode region. The individual (self-)predictabilities provide the basis with respect to which the predictability improvements between different electrode channels are measured. The significant differences of PI between musicians and nonmusicians while listening to music are topographically presented in Figure 3. Higher information transfer took place between multiple cortical areas in the brain in musicians as compared with non-musicians while listening to music. The detailed analysis will be reported elsewhere.

It might be noted that the interpretation of the detected flow of information as measured by the proposed index should be different from the interpretations of other indices, which treat synchrony and interdependency in a common framework. In a recent result [8], the two features, (flow of information and synchrony) even get opposite signs.

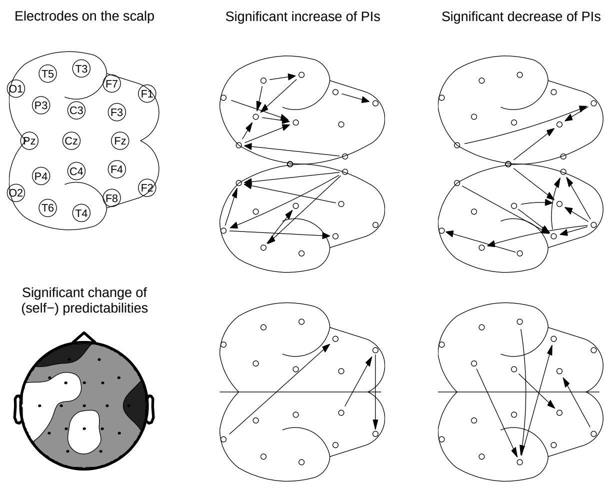

FIGURE 3. Upper left: Position of the 19 electrodes on the scalp according to 10-20 electrode-placement system (recordings with respect to the averaged ear-lobes signals). Lower left: Significant differences of (self-)predictabilities of musicians with respect to nonmusicians while listening to music (black-increase, white-decrease). Middle/right column: Topographical maps showing significant increase/decrease of predictability improvements of musicians compared to nonmusicians while listening to music. Arrowheads roughly represent the flow of information. Intra- and inter- hemispheric connections are presented separately (upper/lower part).

DISCUSSION

A novel index based on predictability improvement of one signal with the help of additional information from another signal by the reconstruction of mixed state spaces is proposed to detect both the nature and the degree of hidden interdependences between signals. In several examples, it has been shown that this index reveals asymmetries in coupling even when the data are few and noisy (like EEG). However, like in all methods based on state space reconstruction, proper choices of embedding and prediction parameters play an important role in the successful application of this new index. Also, by means of any directional measure of interdependence including the proposed one, we cannot prove the presence of actual coupling strengths, neither we can exclude the influences of many other systems. The method of surrogate data [9] might be helpful to provide a confidence limit, which can exclude the spurious couplings measured by the applied index - this is a scope of future study.

ACKNOWLEDGMENTS

The authors are thankful to Prof. H. Petsche for various supports.

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