Chapter 6

Network Structure versus the Number of States and Input Derivatives

In section 3.2.4 we already gave the idea that the number of input derivatives and the number of states describing the inverse system does not depend on the specific functions but on the system structure. Recall: we regarded which state of the original system is influenced by the input and by which other state. The minimal number of integrators working between input and output in the original system indicates its relative degree. The relative degree of the original system in turn determines the number of input derivatives and the number of states of the corresponding inverse system.

In this chapter an algorithm is given which allows to determine the number of input derivatives and states of a circuit being an inverse system or not, i.e. without regarding its original.

The definitions and results in the next section correspond to those of [18] while the sections following afterwards represent a new approach and tool.

6.1 State Equations of Circuits

In this section we will give a fast approach to the topic. This should provide a clear view on what a differential equation system describing a circuits behaviour is.

We are convinced that the ideas to be presented in the sequel provide a useful tool which could make things easier or more obvious. Therefore this section is understood also as a short and easy introduction for those who do not want to read a book or were perhaps deterred by tedious formalisms sometimes found in network theory.

6.1.1 Circuits and Their Equations

In the sequel we always deal with circuits of the following type which provides a considerably generality.

Definition 6.1 (Circuits) *A circuit consists of one-ports and ideal operational amplifiers. The admitted one-ports, to be represented by one branch in the network graph, are resistors, time dependent voltage or current sources, capacitors and inductors. The ideal operational amplifier is the only admitted two-port, to be represented by two branches in the network graph, namely the nullator-norator pair. The capacitors (inductors) are charge- (flux-) controlled and the resistors are at least voltage or current controlled (cf. definition 6.2).*

Remark 6.1 Since all controlled sources (which are two-ports as well) can be modelled by two-ports consisting of only resistors and ideal operational amplifiers the definition 6.1 is not really restrictive, see Fig. 6.2 for an example.

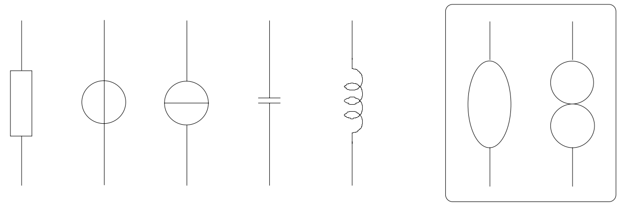

Figure 6.1: Basis circuit elements: nullators and norators appear always in pairs; the resistors, capacitors and inductors can be nonlinear although we do not mark it especially

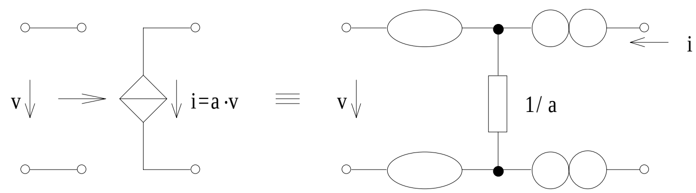

Figure 6.2: Modelling of a voltage controlled current source by means of ideal op-amps. and a resistor

Remark 6.2 The part of the definition concerning the control is necessary (since we need it later) and not too restrictive since in general real capacitors, inductors and resistors fit in this.

Remark 6.3 Of course, real circuits consist of real elements. It is understood that we deal with circuit models and regard solutions of circuit modells only, which are supposed to reflect the reality.

Definition 6.2 (Circuit Equations) *The circuit equations are the set of all equations describing the network and its elements. Let b, N_C, N_L, N_S resp. N_R be the number of network branches, capacitors, inductors, time dependent sources resp. Resistors.*
There are

- *$2b$ non differential circuit equations:*

  b Kirchhoff equations, namely the set of linearly independent loop and cutset equations,
  N_C constitutive relations of capacitors, the voltage controlled by the charge

$$v_C = v_C(q_C) \tag{6.1}$$

  N_L constitutive relations of inductors, the current controlled by the flux

$$i_L = i_L(\phi_L) \tag{6.2}$$

  N_S constitutive relations of the time dependent sources,

$$v_S = e(t) \text{ resp. } i_S = e(t) \tag{6.3}$$

  N_R constitutive relations of resistors, voltage or current controlled

$$i = g(v) \text{ or } v = r(i) \tag{6.4}$$

  $b - N_C - N_L - N_S - N_R$ constitutive relations of ideal operational amplifiers,

$$i_{nullator} = 0, v_{nullator} = 0 \tag{6.5}$$

  and

- *$N_C + N_L$ differential circuit equations:*

$$\dot{q}_C = i_C \text{ and } \dot{\phi}_L = v_L \tag{6.6}$$

The circuit equations represent DAEs (circuit DAEs). However, we are used to regard a system of ODEs in order to analyse the circuit behaviour. Usually at this point the engineer starts writing down the circuit equations, tries to eliminate certain terms and sometimes stops being amazed for it is seemingly impossible to achieve ODEs. A powerful result of [18] is that one can tell, whether appropriate ODEs exist, by a pure inspection of the circuit structure in advance without any calculation. This section is supposed to evoke deep understanding of the background of the crucial theorem to be presented at the end.

6.1.2 States

First we define what appropriate ODEs are:

Definition 6.3 (State Equations of a Circuit) *A system of ordinary differential equations*

$$\dot{\mathbf{x}} = f(\mathbf{x}, t) \tag{6.7}$$

is called state equations of a circuit (circuit SEs) if the following two conditions hold:

1. *If the vector of time functions* $(\mathbf{v}(t), \mathbf{i}(t), \mathbf{q}_C(t), \Phi_L(t))$ *is a solution of the circuit equations (definition 6.2) then* $(\mathbf{q}_C(t), \Phi_L(t))$ *represent a solution* $\mathbf{x}(t)$ *of the circuit SEs (6.7). That is that the charges and fluxes serve as* states.

2. *If* $(\mathbf{q}_C(t), \Phi_L(t))$ *is a solution of the circuit SEs (6.7) then there exists* **exactly one** *solution* $(\mathbf{v}(t), \mathbf{i}(t), \mathbf{q}_C(t), \Phi_L(t))$ *of the circuit equations with these charges and fluxes.*

If SEs exist only for a certain range of charges (fluxes) or capacitor voltages (inductor currents) we call them **local** *state equations, otherwise* **global** *state equations.*

Remark 6.4 Often one is interested in state equations in terms of other circuit variables than the charges and fluxes. This corresponds to consideration of the same system motion in other coordinates, which is possible if there exists a diffeomorphism between these circuit variables and the charges and fluxes. However, such diffeomorphism does not necessarily exist. It follows that, assumed SEs with charges and fluxes exist, one cannot rely on finding state equations in terms of other network variables.

Remark 6.5 Often people consider the system motion on a certain differential manifold (namaly, the configuration space [18]) which provides a certain freedom of choice of coordinates. One could tempt to change to other coordinates if charges and fluxes do not serve as coordinates on this differential manifold. However, since the charges and fluxes are the only network variables which occur as differential in the circuit equations (definition 6.2) **state equations exist only if they exist in terms of charges and fluxes as well**. Therefore, in order to determine whether SEs exist at all one has to establish whether they exist in terms of charges and fluxes.

Remark 6.6 Since the charges and fluxes serve as states every capacitor and inductor can be regarded to represent a state. It is well known that capacitors which form loops (inductors which form cutsets) do not provide the full number of states N_C (N_L) because their charges (fluxes) are dependent on each other. For example, the initial charge value of a loop capacitor as well as its time function is fixed by those of the rest capacitors of the loop.

We are interested in SEs in terms of only those charges (fluxes) wich really represent independent states. Therefore, it is suggested to eliminate in a first step all capacitor loops (inductor cutsets). Following the ideas of [18] this is done by replacing them with multiple access capacitors (inductors). An N-port of dependent capacitors is replaced by an *equivalent N-port* consisting of N branches forming a tree (i.e. no loop). The constitutive relations of these N branches involve charge and voltage of each branch but as opposed to Equ. (6.1) the branches are coupled with each other. Nevertheless, they represent N indepentent states. Analog to definition 6.1 one has to demand that the N port voltages are functions of the N port charges, i.e. that the multiple access capacitor is charge controlled. An analogous approach serves for inductor cutsets.

This way some spurious states, i.e. capacitor (inductor) branches can be eliminated. However, due to the circuit structure still some capacitor (inductor) branches can turn out to not represent a state. Therefore, we will call all capacitors (inductors) left at this point *supposed states* in the sequel and we will give a criterion to determine how many of them really represent states.

The crucial point of the definition 6.3 is that all circuit variables must be a unique function of the states (and the source values at each moment). If this is the case then in particular the unknown variables of the differential circuit equations, Equ. (6.6), i_C and u_L are uniquely determinded as a function of the states, which in turn ensures the existence of circuit SEs.

This is the clue to how to take the decision whether there exist SEs in terms of the supposed states or not:

Proposition 6.1 (Existence of Global State Equations) *Every capacitor (inductor) supposed to represent a state is replaced by a voltage (current) source.*
If the resulting resistive network has **exactly one** *solution for all (admitted) values of the sources, then there exist global state equations in terms of the charges (fluxes) of the capacitors (inductors).*
(For a proof see [18])

This way the decision whether there exist global state equations is reduced to the proof of existence and uniqueness of the solution of a resistive network.This can be done, as already mentioned, by a pure inspection of the circuit structure for a huge class of circuits.

6.1.3 Existence and Uniqueness of the Solution of a Resistive Network

First we introduce the terms 'pair of conjugate trees' and 'uniform partial orientation of resistors'. In the following definitions as well as explanations of their meaning are given. The latter are not supposed to prove something but should provide deep understanding. Those who are only intersted in tools could skip over them. The final conclusions represent extracts from theorems proved in [18].

Pair of Conjugate Trees

Definition 6.4 (Pair of Conjugate Trees) *A* tree *is a set of network branches including all nodes without forming a loop. (A* cotree *is the complementary set to a tree which does not form a cutset.)*
A pair of conjugate trees *consists of* **one** *tree which contains*

1. **all** *voltage sources,*
2. **no** *current source,*
3. **some** *resistors and*
4. **all** *nullators*

while the **other** *tree contains the* **same** *branches* **but instead** *of the nullators all norators.*

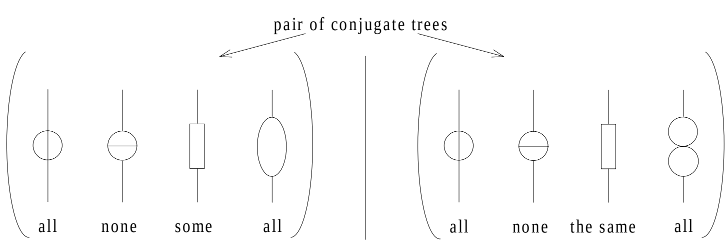

Figure 6.3: Elements which have to be contained in conjugate trees

Remark 6.7 A pair of conjugate trees cannot exist if there are loops which consist exclusively of voltage sources and nullators (or exclusively of voltage sources and norators) or branch compositions which **can** turn out to represent nullators (norators), cf. Fig. 6.4. We call these loops *bad loops.* We emphisize 'can turn out' because the existence of conjugate trees does not concern the kind (vc. or cc.) of resistors.

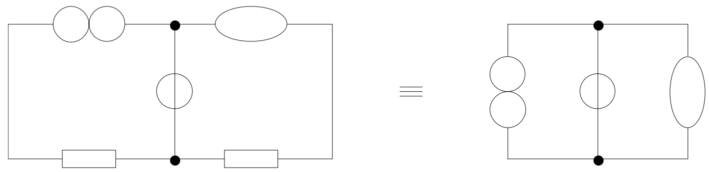

Figure 6.4: As there is a bad loop of a voltage source with a branch which turns out to be a norator (nullator) the network has no pair of conjugate trees.

Branch compositions which actually turn out to be nullators consist of a nullator
in pure series with a current controlled (cc.) resistor or a voltage source or
in parallel with a voltage controlled (vc.) resistor or a current source.
Branche compositions which turn out to be norators consist of a norators
in pure series with a resistor or a voltage source or
in parallel with a resistor or a current source.
Branch compositions consisting of a voltage source in parallel with a vc. resistor turn out to represent voltage sources.
Branch compositions consisting of a current source in series with a cc. resistor turn out to represent current sources, cf. Fig. 6.5.
We always supposed that the appropriate voltage or current is admissible to the resistor, which forms a composition. Some of the mentioned branches actually turn out to be fixators. However, nothing is changed with respect to the uniqueness of a solution when regarding them as a nullator.

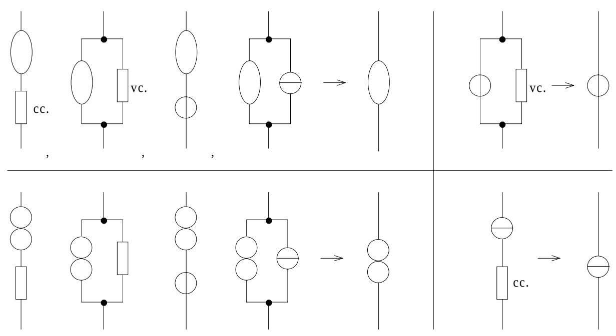

Figure 6.5: Branch compositions which turn out to represent nullators, norators, voltage resp. current sources

A pair of conjugate trees also cannot exist if there are cutsets which consist exclusively of current sources and nullators (norators) or branch compositions which **can** turn out to represent nullators (norators). We call these cutsets *bad cutsets.*

In case of bad loops (cutsets) without norators the constitutive relations of this loop (cutset) are linearly dependent with the corresponding Kirchhoff equation. We call them *dependent loops (cutsets).* In case of bad loops (cutsets) without nullators all circuit equations (as far as there exists a solution) are fulfilled with an arbitrary current in the *bad loop* or an arbitrary voltage of the *bad cutset.* We call them *free loops (cutsets)*, cf. Tab. 6.1.

Consequently, those bad loops and cutsets without nullators and norators are free and dependent, i.e. the free additional loop current (cutset voltage) occurs in the same loop (cutset) which provides the linearly dependent constitutive relations.

If we suppose all resistors to be vc. and cc. resistors and to admit every voltage resp. current, then the following statement holds:
The non existence of conjugate trees indicates that already a subset of circuit equations is linearly dependent irrespective of the network resistors. It follows that the non existence of conjugate trees is equivalent to non uniqueness of the solution of the resistive network. (However, the converse is not valid.)

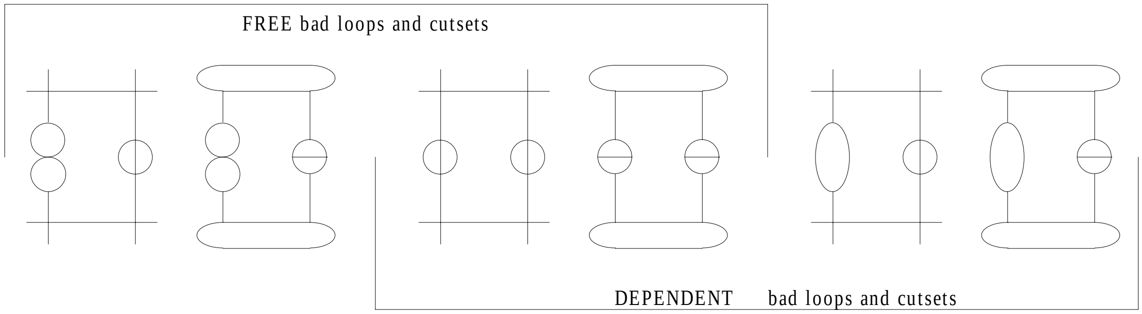

Table 6.1: Branch compositions which make the existence of conjugate trees impossible: bad loops and cutsets, which indicate that the circuit equations are linearly dependent irrespective of the network resistors; bad sets without norators are called **dependent** and bad sets without nullators are called **free** bad loops and cutsets

This fact will be of crucial importance in the next section.

Remark 6.8 Since it might be tedious to check whether there exist bad loops or cutsets the detection of a pair of conjugate trees should be a useful tool in order to decide whether the solution can be unique or not. The result of this check is not changed when all voltage (current) sources are replaced by short (open) circuits. This is due to the fact, that a tree (cotree) is also a tree (cotree) of the remaining network when one branch of the tree (cotree) is shortened (opened).

If a pair of conjugate trees exists then at least a part of circuit equations is linearly independent, namely the Kirchhoff equations and the constitutive relations of sources and nullators (may be in connection with some resistors). In this case the set of $2b$ circuit equations can be reduced to N_R equations involving only the voltages of the tree resistors $\mathbf{v}_{tr}$ and currents of the cotree resistors $\mathbf{i}_{co}$ as performed in the following steps:
Departing from the tree containing the nullators, every cotree branch closes a loop. This leads to the set of $b-n+1$ linearly independent loop equations. (n) is the number of nodes) I.e. every cotree voltage is a linear combination L_{loop} of the voltages of tree resistors and voltage sources.

$$(\mathbf{v}_{co}, \mathbf{v}_{i_S}, \mathbf{v}_{nor}) = L_{loop}(\mathbf{v}_{tr}, \mathbf{e}_{tr}(t)) \tag{6.8}$$

Departing from the cotree containing the nullators, every tree branch forms a cutset with the cotree branches. This leads to the set of $n-1$ linearly independent loop equations. I.e. every tree current is a linear combination L_{cut} of the currents of cotree resistors and current sources.

$$(\mathbf{i}_{tr}, \mathbf{i}_{v_S}, \mathbf{i}_{nor}) = L_{cut}(\mathbf{i}_{co}, \mathbf{e}_{co}(t)) \tag{6.9}$$

Now the resistor equations (6.4) lead to the mentioned *reduced equation system.* If additionally the tree (cotree) resistors are current (voltage) controlled, then it has the following explicit form:

$$\begin{aligned} \mathbf{i}_{co} &= g(\mathbf{v}_{co}) = g(\mathbf{v}_{tr}, \mathbf{e}_{tr}(t)) \\ \mathbf{v}_{co} &= r(\mathbf{i}_{tr}) = r(\mathbf{i}_{co}, \mathbf{e}_{co}(t)) \end{aligned} \tag{6.10}$$

If this reduced equation system has exactly one solution then all voltages (currents) of resistors, current (voltage) sources and norators are uniquely determined by Equs. (6.8) resp. (6.9).

We conclude: The existence of a pair of conjugate trees indicates that at least a part of circuit equations is linearly independent irrespective of the resistive part. But the network still may have a unique solution, multiple solutions or none solution due to different resistors. One can tell whether the uniqueness of a solution depends on the specific resistors or not again by consideration of the network structure. This does the following approach capable to establish the uniqueness of a solution irrespective of specific resistors.

Partial Orientation of Resistors

The partial orientation approach assumes that there exists at least one solution of a network and considers whether another solution could exist. For this one regards only increments of voltages (currents). Actually, only the signs of these increments are considered i.e. whether a branch voltage (current) of a possible second solution is bigger, equal or smaller compared with those of the first solution. These signs are expressed in branch orientations, where 'equal' corresponds to non orientation. The term 'partial' stands for the fact that not all resistors are oriented. A trivial orientation contains no oriented branch.

It is clear that a resistive network has at most one solution if there is no nontrivial partial orientation possible with respect to its equations. Since only signs of voltage and current increments are considered und not specific values one can check this way only the existence of a possible orientation with respect to Kirchhoffs laws. *But if all resistors have increasing characteristics then already the check with respect to Kirchhoffs laws allows to establish the uniqueness of the solution irrespective of resistor values.* Namely, one has to exclude the existence of a nontrivial partial orientation.

The possible orientations depend only on the resistive network structure, i.e. on the position and type (cf. definition 6.2) of resistors, i.e. on the loops and cutsets they form. in it but not on the specific values of resistors. Therefore one obtains this way results not just for a specific circuit but for a huge circuit class having the same structure.

In the sequel a definition of a partial orientation is given (which is called uniform for some reason). The equivalent conditions in the following remark will clarify that a uniform partial orientation does not contradict the Kirchhoff equations.

Definition 6.5 (Uniform Partial Orientation of Resistors with Strictly Increasing Characteristics) *A partial orientation of resistor branches in a resistive network is uniform if:*

1. *Every oriented resistor is part of a uniform loop composed of oriented resistors, voltage sources and norators.*

2. *Every oriented resistor is part of a uniform cutset composed of oriented resistors, current sources and norators.*

Uniform indicates that all branches of a loop (cutset) have the same orientaion with respect to this loop (cutset).

Remark 6.9 (Equivalent Conditions)
As can be easily checked by application of the colored branch theorem (CBT) the conditions of definition 6.5 are **equivalent** to:

1. No oriented resistor is part of a uniform cutset composed of oriented resistors, current sources, nullators and non oriented resistors.

2. No oriented resistor is part of a uniform loop composed of oriented resistors, voltage sources, nullators and non oriented resistors.

These equivalent conditions guarantee that the orientations do not lead to a contradiction with the Kirchhoff equations Fig. 6.6. Indeed, if any oriented resistor formed a uniform loop with other oriented resistors, voltage sources, nullators and non oriented resistors, this would indicate that all voltage increments with respect to the assumed first solution in this loop are non negative and not all are zero. This in turn is a contradiction to Kirchhoffs law, which demands that the sum of loop voltages of every solution is zero.

Remark 6.10 (How to check the existence of a uniform partial orientation)
In order to exclude the existence of a nontrivial partial orientation one has to regard three possible cases for each resistor: the orientation in both directions and the non orientation.

In a first step one marks all resistors which cannot be oriented, because they form loops (cutsets) with only nullators and voltage (current) sources (cf. remark 6.9). Those can be treated as nullators from now

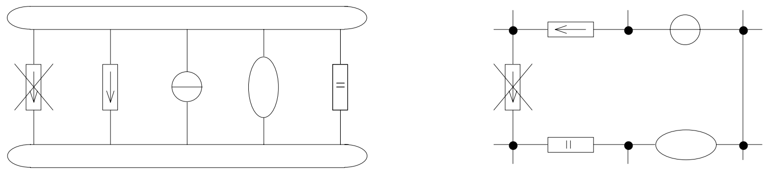

Figure 6.6: Parial orientations leading to contradictions with Kirchhoff equations: not admitted uniform loops and cutsets

on. This step can be repeated. The resistors left after this procedure must be tested by systematical tries to be oriented or not. One could first orientate one resistor and consider whether the orientation of the next is fixed due to the conditions of remark 6.9 or not. This amounts to avoiding inappropriate uniform loops and cutsets. If a chosen orientation leads to a contradiction then the next has to be examined. Finally, by systematic trials and errors one can exclude any orientation of all resistors or not.

Since the result of this check does not depend on specific source values, all voltage (current) sources could be replaced by short (open) circuits. This could make contradictions to Kirchhoffs laws easier to detect.

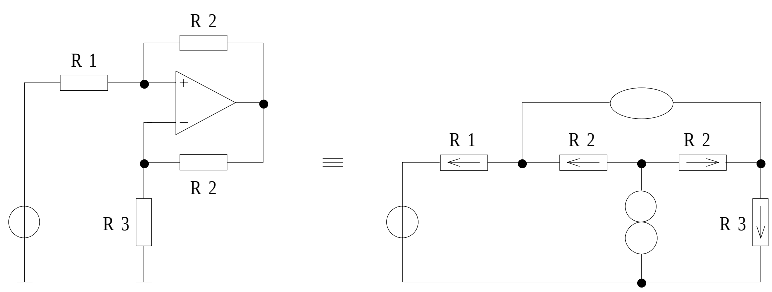

Figure 6.7: Example for an existing nontrivial partial orientation which indicates that at least for some resistor values the solution if it exists is not unique, namely for $R_1 = R_3$

Remark 6.11
(Uniform Partial Orientation of Resistors with only monoton increasing characteristics)
If the resistor characteristics are not strictly monoton, then a voltage orientation (i.e. increment) is not necessarily identical with the current orientation. For instance, a voltage orientation of a voltage controlled monoton increasing resistor does not imply a current orientation. Therefore one had to regard two orientations of each branch. Namely, one had to check whether there exist a nontrivial voltage or current orientation of a graph, which does not contradict Kirchhoffs laws. In proposition 6.2 the necessity of consideration of two orientations is overcome by a more constrained condition on the pair of conjugate trees.

We conclude:
If a network structure (only such with a pair of conjugate trees is considered) has no nontrivial partial orientation then the solution of all networks with this structure (as far as it exists) is unique.

We know two powerful tools so far which allow to establish uniqueness resp. non uniqueness of network solutions irrespective of network resistors or their dependence on network resistors. While in case of *linear* resistive networks uniqueness of the solution is equivalent to its existence for the case of *nonlinear* resistors we always added *'as far as it exists'*. It is the power of the following proposition which establishes both the existence and the uniqueness of resistive network solutions.

Concluding Proposition and its Application

Proposition 6.2 (Existence and Uniqueness of the Solution of a Resistive Network)
All resistors of a network are assumed to have monoton increasing characteristics with all voltages (currents) admissible when being voltage (current) controlled. All nonlinear resistors are replaced by linear resistors with positiv values.
If the resulting linear resistive network

1. *has* **no nontrivial uniform partial orientation** *and*
2. *has a* **special pair of conjugate trees** *where all resistors of the tree are current controlled and all resistors of the cotree are voltage controlled*

then the resistive network has **exactly** *one solution.*
(For a proof see [18])

Remark 6.12 If only the condition concerning the voltage (current) admissibility of proposition 6.2 does not hold then the uniqueness of the solution is still guaranteed but not its existence. I.e. the solution is unique 'as far as it exists'.

Remark 6.13 This result reduces the decision of a unique solution of a nonlinear resistive network to a simple check of the existence of appropriate conjugate trees and of the existence of no nontrivial orientation on the linear network graph. This should be quickly finished for not too large networks by systematical tries. Since the only possible trivial partial orientation has to be established for the associated linear resistive network, it is sufficient to regard only one orientation as suggested in remark 6.10 although the network under consideration might have resistors with only monoton increasing characteristics.

Remark 6.14 Proposition 6.2 is a useful tool not only to establish the existence of global state equations but also the uniqueness of the dc operating point. For this one has to consider the resistive network resulting when capacitors (inductors), which represent states, are replaced by open (short) circuits.

In Tab. 6.2 we sum up the tools presented in this section.

<table>
<tr><td colspan="2" rowspan="2"></td><td colspan="2">a pair of special conjugate trees</td></tr>
<tr><td>exists</td><td>does not exist</td></tr>
<tr><td rowspan="2">a nontrivial partial orientation</td><td>exists</td><td>0, 1, finitely many or ∞ in dependence of network resistors</td><td>0 or ∞ solutions irrespective of network resistors</td></tr>
<tr><td>does not exist</td><td>1 solution irrespective of network resistors</td><td>0 or ∞ solutions irrespective of network resistors</td></tr>
</table>

Table 6.2: Uniqueness resp. non uniqueness of solutions of resistive networks with monoton increasing characteristics

6.1.4 Example: RC-Ladder Oscillator

In order to demonstrate the use of the provided tools we apply them to the RC-Ladder oscillator Fig. 6.8. The transistor is supposed to work only in the forward region and is therefore modelled by a basis-emitter diode the current of which controls the collector-emitter current source. In order to establish the existence of state equations in terms of the capacitor charges we have, according to proposition 6.1, to prove the existence of a unique solution of the associated resistive network. The associated resistive network, obtained by replacing the supposed state capacitors by voltage sources and by modelling the current controlled current source of the transistor by means of diodes and a nullator-norator pair, is shown in Fig. 6.9. Next, we check whether the conditions of proposition 6.2 hold.

Assume both diodes have an exponential characteristic $i = i_s \exp \frac{v}{v_T}$ where v_T resp. i_s is the temperatur voltage resp. the saturation current. Due to the forward current amplification b_F the saturation current of the collector-emitter diode is b_F-times larger then this of the basis-emitter diode. This implies that the diodes represent voltage and current controlled resistors. However, while all voltages are admissible this does not hold for all currents.

The network with all nonlinear resistors replaced by linear ones is depicted in Fig. 6.10. According to remark 6.10 we can check the existence of a nontrivial partial orientation when all voltage sources are shortened:

1. Assume the resistor branch 1 is oriented as depicted in Fig. 6.10. Avoiding uniform loops results in the same orientation of the resistor branches 2, 3, 4 and eventually of branch 5. This, however, results in a uniform cutset, which is a contradiction to condition 1. in remark 6.9 and proves the assumption to be false.

2. Assuming the resistor branch is oriented in the opposite direction leads in the same way to a contradiction.

3. Assume the branch 1 is not oriented. This results in the only possible orientation, where no resistor branch is oriented.

Since there does not exist a nontrivial partial orientation condition 1. of proposition 6.2 holds.

As depicted in Fig. 6.9 there exist a pair of conjugate trees with all tree resistors current controlled and all cotree resistors voltage controlled.

Since therefore all conditions of proposition 6.2 hold the existence and uniqueness of the solution of the associated resistive network is shown and this way the existence of state equations.

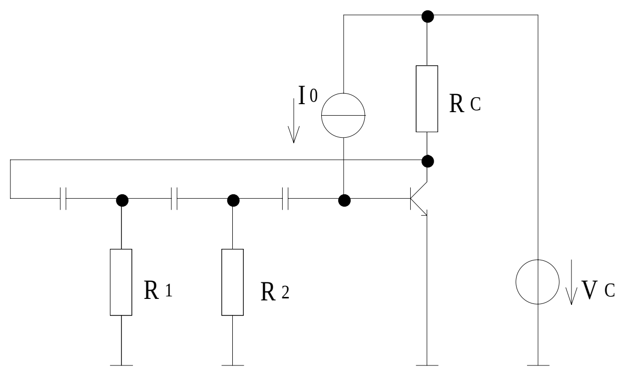

Figure 6.8: RC-Ladder oscillator

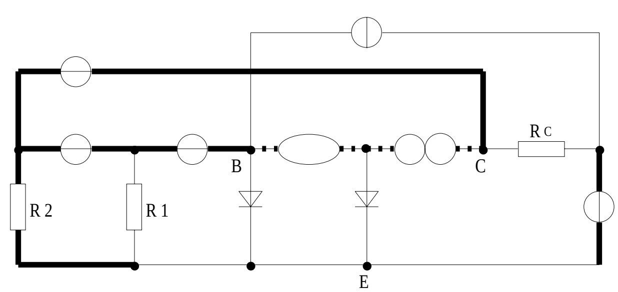

Figure 6.9: Associated resistive network: the supposed state capacitors are replaced by voltage sources, the current controlled current source of the transistor is modelled by means of diodes and a nullator-norator pair: there exists a pair of conjugate trees with all tree (cotree) resistors curent (coltage) controlled with all currents (voltages) admissible

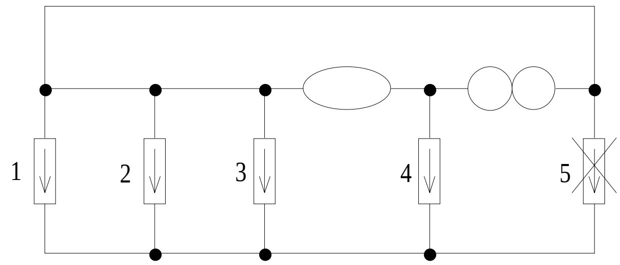

Figure 6.10: Associated linear resistive network: all voltage (current) sources shortened (opened): every try to orientate a resistor branch leads to a contradiction: there does not exist a partial orientation

The following sections are devoted to the case when the application of the mentioned propositions fails because condition 2. of proposition 6.2 does not hold.

6.2 Obliteration of Bad Loops and Cutsets

For the sake of simplicity we present the following approach under the assumption that all resistors are vc. and cc., i.e. as soon as there exists a pair of conjugate trees it is also a pair of special conjugate trees. We consider networks whithout loops (cutsets) of capacitors (inductors), i.e. there are no spurious states. If there were some, they should be replaced according to remark 6.6. Furthermore, we require the network to have no bad loop (cutset) in the sense of remark 6.7 which are composed of voltage (current) sources and nullators or norators but which do not include branches of supposed states. We call such bad loops (cutsets) *bad loops (cutsets) without states*. One can check the existence of such bad loops and cutsets without states with the network where capacitors and inductors are replaced by resistors. This is due to the fact that all bad loops (cutsets) remain loops (cutsets) when one branch is replaced by a resistor except those including this branch and no bad cutset (loop) is added.

A circuit which fits in this is depicted in Fig. 6.11a. (The circuit could be understood as an inversion of the RLDiode circuit, when the capacitor voltage becomes the input) As can be verified in Fig. 6.11b there is no bad loop (cutset) in the sense of remark 6.7 when the capacitor and inductor are replaced by resistors since there exists a pair of conjugate trees.

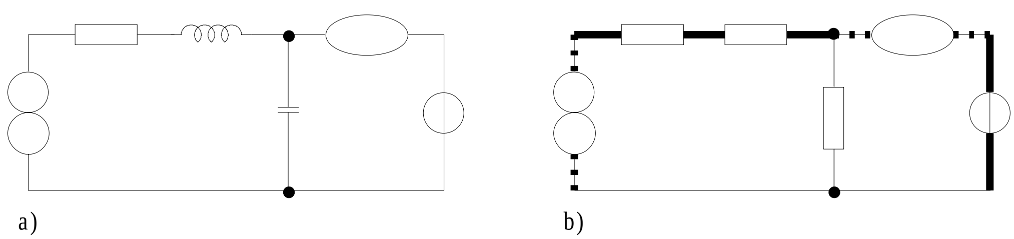

Figure 6.11: (a) Circuit which has no spurious states and no bad loops or cutsets without states (b) Network with all supposed states replaced by resistors: a pair of conjugate trees is depicted

However, the resistive network with all supposed states replaced by appropriate sources (cf. proposition 6.1) fails to satisfy the condition 2. of proposition 6.2 irrespective of the type of resistors. Therefore, one cannot expect the network to have a state representation in terms of all supposed states. Indeed, as we will derive in the sequel some supposed state elements can turn out to represent differentiators.

Remark 6.15 (Open Question)

We conjecture: In case no pair of conjugate trees exists there exist bad loops or cutsets without norators, i.e. dependent bad loops (cutsets) cf. Tab. 6.1.

The opposite case would indicate that there are infinitely many solutions irrespective of network resistors but no loop or cutset equation turns out to be linearly dependent with others. Indeed, the linear dependence can be rather implicit and thus difficult to locate in the network. As a possible approach we suggest to replace branch compositions which turn out be something else according to Fig. 6.5 repeatedly until dependent bad loops or cutsets appear obviously as depicted in Tab. 6.1. For many examples this method worked.

In the sequel we rely on that the conjecture holds, i.e. all results are understood in the sense: 'as far as this conjecture holds'.

Proposition 6.3 (Obliteration of Dependent Bad Loops and Cutsets)
Consider a network which has no spurious states and no bad loops and cutsets without states.
All capacitors (inductors) supposed to represent states are replaced by voltage (current) sources.
If the resulting resistive network has no pair of conjugate trees, then

1. *In every dependent bad loop (cutset) one capacitor (inductor), provided it is voltage (current) controlled, does not represent an independent state. It represents a time dependent* imposed *current (voltage) source whose time function depends on the derivative of the signals of the sources which are involved in this dependent bad loop (cutset).*
   If additionally in every dependent set one supposed state can be chosen such that it is not involved in another dependent set then
2. *by the introduction of the imposed sources all former dependent bad loops and cutsets are 'obliterated'.*

Proof:
If there exists a bad loop (cutset) including capacitors (inductors) but not without them then one capacitor $\tilde{C}$ (inductor $\tilde{L}$) has closed this loop (cutset). Therefore its voltage (current) is a sum S of the other (in the following called: rest) loop voltages (cutset currents) including at least one source value, otherwise it would represent a spurious state.

In general the rest loop voltages (cutset currents) can be functions of voltages and currents of sources (including nullators), of the *remaining capacitors and inductors* and even resistors due to branch compositions which 'turn out to be ...' cf. remark 6.7 (Fig. 6.12). However, for sake of simplicity the following notion corresponds to the case that the considered bad loops (cutsets) are composed exclusively of voltage (current) sources, capacitors (inductors) and nullators. Although we deal with signal values we omit the argument (t) for easy readability.

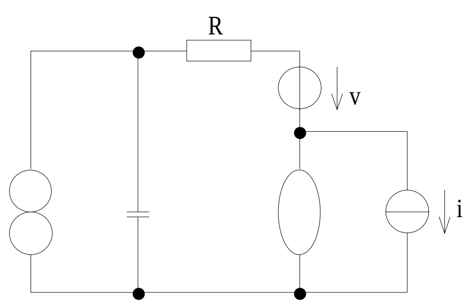

Figure 6.12: The voltage of the capacitor is fixed by a dependent bad loop with a branch composition which turns out (in two steps) to represent a fixator, namely $i = i_1$ and $v = v_1 + R \cdot i_1$

$$
\begin{aligned}
v_{\tilde{C}} &= S(\mathbf{v}_C, \mathbf{v}_S) \\
&\text{or} \\
i_{\tilde{L}} &= S(\mathbf{i}_L, \mathbf{i}_S) \qquad (6.11)
\end{aligned}
$$

If this capacitor (inductor) is voltage (current) controlled, then the initial value of its charge (flux) as well as its time function is determined via the voltage (current) by Equ. (6.11) as function of the rest loop voltages (cutset currents). It is clear, as far as there exists a solution of the circuit equations it must fulfill the differential equation of the capacitor $\tilde{C}$ (inductor $\tilde{L}$) too. This means that the current (voltage) of the capacitor (inductor) branch is determined as a function of the rest loop voltages (cutset currents) namely as a function of their derivatives.

$$\begin{aligned} i_{\tilde{C}} = \dot{q}_{\tilde{C}} &= c_d(v_{\tilde{C}}) \cdot S(\dot{\mathbf{v}}_C, \dot{\mathbf{v}}_S) = i_{imp}(\mathbf{i}_C, \dot{\mathbf{v}}_S) \\ &\text{or} \\ v_{\tilde{L}} = \dot{\phi}_{\tilde{L}} &= l_d(i_{\tilde{C}}) \cdot S(\dot{\mathbf{i}}_L, \dot{\mathbf{i}}_S) = v_{imp}(\mathbf{v}_L, \dot{\mathbf{i}}_S) \end{aligned} \tag{6.12}$$

where $c_d(\cdot)$ resp. $l_d(\cdot)$ is the differential capacitance resp. inductance.

Consider the *modified* resistive network which is obtained by replacing the remaining capacitors (inductors) by voltage (current) sources and the one capacitor (inductor) in **each** dependent bad loop (cutset) replaced by the imposed current (voltage) source. In this network every former dependent bad loop (cutset) is *obliterated*.

The obliteration of all present dependent bad loops and cutsets by introduction of imposed sources is called *replacement procedure* in the sequel. We see the problem that it is possibly difficult to know how many dependent bad loops and cutsets there are, because they can be difficult to find.

In the circuit example of Fig. 6.13 the capacitor forms a dependent bad loop with the voltage source and the nullator. This way its voltage is fixed and it can be replaced by a current source the value of which depends on the derivative of the voltage source signal. The current source corresponding to the inductor forms a bad cutset with the norator i.e. no dependent bad cutset. Therefore it is left unchanged in this step.

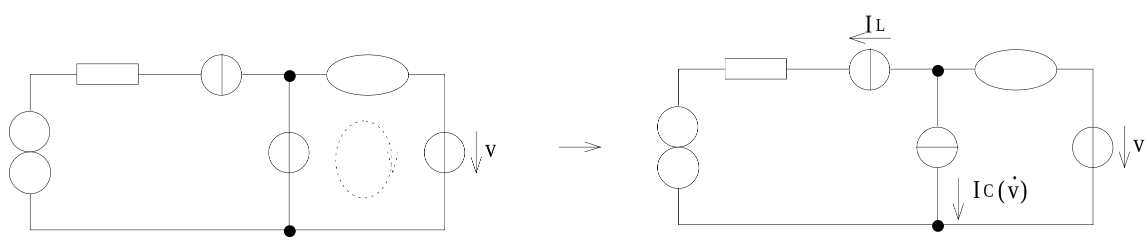

Figure 6.13: Obliteration of a dependent bad loop: the voltage source corresponding to the capacitor is replaced by an imposed current source

If additionally the resistive network has a unique solution $U(\cdot)$ (in the example of Fig. 6.13 this is not the case), which necessarily depends also on the imposed source, then the current (voltage) of each remaining capacitor (inductor) is a unique function of the remaining state variables and the imposed source values:

$$\begin{aligned} \mathbf{i}_C &= U_{\mathbf{i}C}(\mathbf{v}_C, \mathbf{i}_L, \mathbf{e}(t), \mathbf{i}_{imp}(\mathbf{i}_C, \dot{\mathbf{v}}_S), \mathbf{v}_{imp}(\mathbf{v}_L, \dot{\mathbf{i}}_S)) \\ \mathbf{v}_L &= U_{\mathbf{v}L}(\mathbf{v}_C, \mathbf{i}_L, \mathbf{e}(t), \mathbf{i}_{imp}(\mathbf{i}_C, \dot{\mathbf{v}}_S), \mathbf{v}_{imp}(\mathbf{v}_L, \dot{\mathbf{i}}_S)) \end{aligned} \tag{6.13}$$

If the implicit equations (6.13) have a unique solution then the remaining capacitors and inductors really serve as states. The state equations involve the derivative of the signals of the hitherto existing sources.

We conclude: The existence of a dependent bad loop (cutset) involving a supposed state indicates that this state is fixed by other states and sources, i.e. there is no free choice of its initial value. The system motion is constrained to a lower dimensional space by the non differential circuit equations. Only there one is free to choose initial conditions. One could say: *A state has lost his freedom.* Nevertheless, its constitutive relations have to be fulfilled by any solution. This is given when regarding it as a source whose value correspomds to the differential of the former state value which is fixed by other states and sources. Therefore the lost state works as differentiator. It is clear that the appearence of an input derivative might be symbolic because the corresponding sources are autonomous. The existence and uniqueness of the solution of the modified resistive network can be checked again by application of proposition 6.2.

In case, every bad loop (cutset) contains only one capacitor (inductor) the equations (6.13) are already explicit. In case a bad loop (cutset) contains more than one capacitor (inductor) a certain freedom of choice of the state to be obliterated is provided.

Proposition 6.4 *If N_{badC} resp. N_{badL} is the number of dependent bad loops (resp. cutsets) to be obliterated according to proposition 6.3 then the maximum number of independent states left after this first step is:*

$$N_C^1 + N_L^1 = N_C + N_L - N_{badC} - N_{badL} \tag{6.14}$$

The proof follows directly from the fact that none imposed source can represent an independent state because it is fixed by the sources and the remaining supposed states,

As we will see in the following section, although all former bad loops (cutsets) are obliterated now, there can be new ones formed by the imposed sources.

6.3 Algorithm to Determine the Number of Input Derivatives and States

The replacement procedure described in the last section serves for the obliteration of bad loops and cutsets present at the same time. Fig. 6.13 represents an example where the obliteration of all dependent bad loops and cutsets present at the beginning, namely a dependent bad loop, leads to a *new* dependent bad cutset, namely this of the current sources i_L and i_C corresponding to the inductor and the replaced capacitor and the nullator. Therefore the current of the inductor is fixed by the current of the imposed source, which already depends on the first derivative of the voltage signal $\dot{v}(t)$. If we replace the inductor by a voltage source, its value depends on the derivative of $i_C(t)$ and therefore necessarily on the second derivative of the voltage signal $\ddot{v}(t)$ (Fig. 6.14).

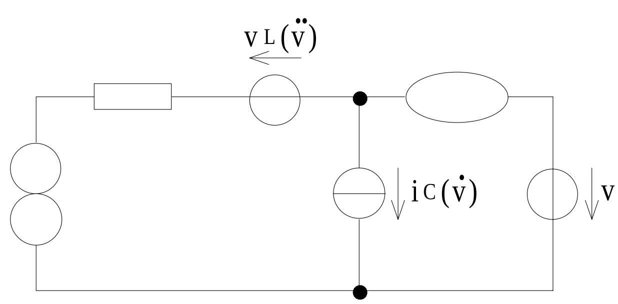

Figure 6.14: By the second application of the replacement procedure an imposed source is introduced which depends on the second derivative of source signals

Note, obviously none of the dynamic elements represents a state in this case and the signals of the circuit as far as there exists a solution depend on up to the second derivative of the voltage source signal.

Proposition 6.5 *With every application of the replacement procedure the number of occuring source derivatives is increased by one.*

Proof:
Assume there is a dependent bad loop (cutset) without imposed sources introduced by the last procedure, then it was a dependent bad loop (cutset) before the application of the procedure too. This is a

contradiction to the fact that all present bad loops (cutsets) are obliterated by the procedure. Therefore, the assumption was false which proves the proposition.

Proposition 6.6 *If the repeated application of the replacement procedure leads to a network whose motion is described by ODEs and if N^i_{badC} resp N^i_{badL} is the number of dependent bad loops (resp. cutsets) obliterated by the i-th. replacement procedure, then*

1. *the number of repeated applications r is the number of input derivatives describing the system motion.*
2. *the number of independent states is:*

$$\hat{N}_C + \hat{N}_L = N_C + N_L - \sum_{i=1}^{r}(N^i_{badC} + N^i_{badL}) \tag{6.15}$$

The proof of this proposition follows directly from proposition 6.5 resp. results directly from repeated application of proposition 6.4.

The repeated application of this replacement procedure stops

1. when the modified network with imposed sources has a pair of conjugate trees or
2. when there is still a dependent bad loop (cutset) but which does not contain an original supposed state which could be replaced in a next step.

In the second case one could go back and check whether the situation changes when another supposed state is obliterated as demonstrated in a rather peculiar example in Fig. 6.15.

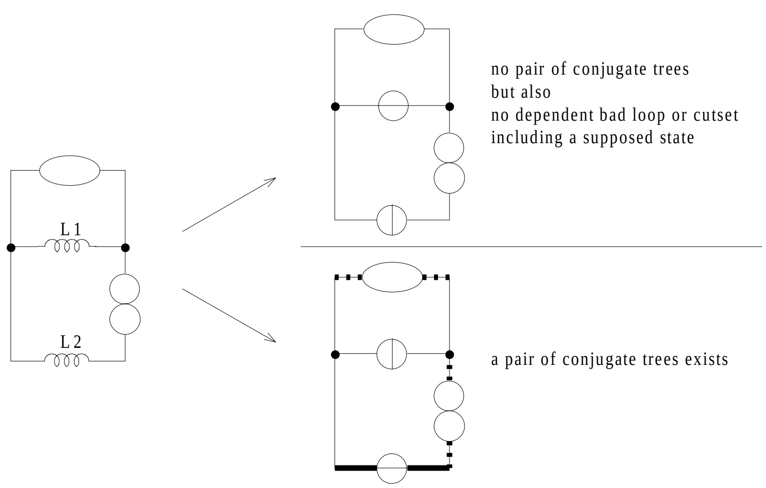

Figure 6.15: Example where the replacement algorithm stops for one choice of the imposed state without a pair of conjugate trees whereas this is achieved for another choice

In the first case it is left to check whether the system motion can be described in terms of the remaining supposed states. This results in proving the existence of a unique solution of the modified network with imposed states and the check whether equations 6.13 have an explicit solution.

6.3.1 Example: Demonstration of the Algorithm

Fig. 6.16a shows a circuit with three supposed states. As the associated resistive network (Fig. 6.16b) reveals, there exists no pair of conjugate trees. Namely, the dependent bad cutset involving the inductor current source is obvious. The successive application of the replacement procedure is demonstrated in Fig. 6.16c,d,e.

The dependent bad loop of Fig. 6.16d is not that obvious. This is due to the fact that it is formed by a branch composition which turns out to be a fixator similar to Fig. 6.12. Such dependent bad loop can be made obvious by repeated application of the replacing steps depicted in Fig. 6.5. Another method is to mark step by step every branch whose voltage or current is fixed by other sources including nullators. This way in the first step the resistor of the marked bad loop was marked, because it forms a cutset with only current sources and the nullator. Thus from now on it can be treated as a fixator because by the current the voltage is fixed as well. In the next step the voltage source corresponding to the capacitor C_1 was marked, because it forms a loop with only the nullator, the imposed voltage source and the fixator corresponding to the resistor. Thus the dependent bad loop including the capacitor is obvious.

Finally, all supposed states are replaced by imposed sources which depend on up to the third derivative of the current source signal. As proved by the existence of conjugate trees and the only possible trivial orientation of the associated linear resistive network in Fig. 6.16e and f the network has a unique solution. Note that this solution is a static function of the current source signal and its derivative. Since all supposed states turned out to be imposed the network has no own dynamic.

The bad loop including the norator indicates that the algorithm cannot lead to a pair of conjugate trees until the voltage source corresponding to the capacitor C_1 becomes a part of a dependent bad loop and is replaced by an imposed source. This happened in the third step.

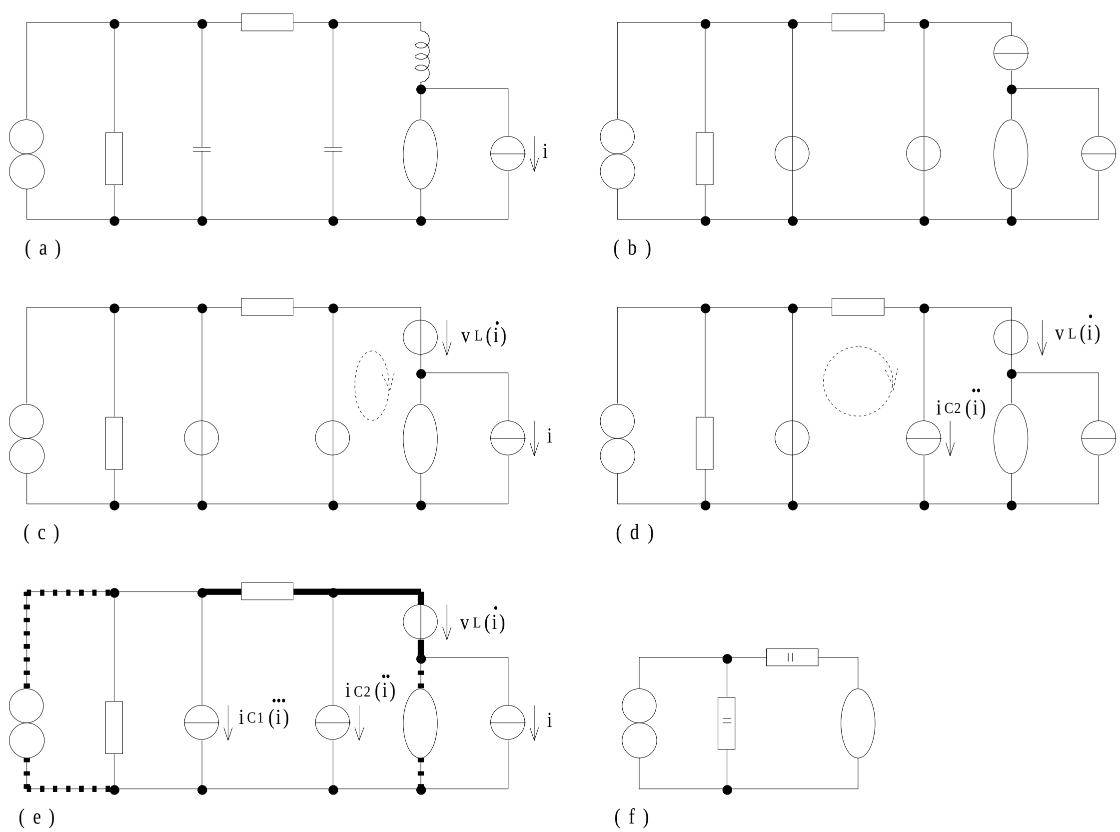

Figure 6.16: Successive application of the replacement procedure: (a) network with three supposed states, (b) supposed states replaced by sources, (c) dependent bad cutset of (b) is obliterad leading to a new dependent bad loop, (d) dependent bad loop of (c) is obliterad forming a new one (e) there is no more bad loop or cutset: there exist conjugate trees, (f) the associated linear resistive netvork has no nontrivial partial orientation

6.3.2 Relation to the Reative Degree

The comparison with chapter 3 reveals that the relative degree of a circuit, assumed to have a global state representation, can be determined for any choice of input and output by the application of the replacement algorithm to the inverted network.

Namely, the number of repeated applications of the replacement procedure to the inverted network, provided it stops successfully, (i.e. a pair of conjugate trees is achieved and the modified resistive network has a unique solution) is equal to the relative degree of the original circuit. This follows directly from the system equivalence (definition 3.5) and proposition 6.6.

As explained in-depth in section 6.1.3 the existence of a pair of conjugate trees is by no means sufficient for the existence and uniqueness of the network solution. Namely, by a singular resistor situation another bad loop or cutset could be formed.

We emphasize the proposed method is not comprehensive but bases on the conjecture of remark 6.15. However in the most considered (not especially tricky chosen) examples the algorithm worked.