Chapter 5

Inverse System Design

Recall: In order to key information in a chaotic signal we need a chaotic original system. Its inverse has to have unique asymptotic behaviour in order to synchronize. At first glance this seems to be contradictory, because a unique steady state is sort of the opposite of chaotic behaviour. The general structure to be presented here elucidates how these features can be changed into each other by system inversion. Next we consider the realization of circuit inversion for any input-output situation, i.e. if input and output are not just the two variables of one port - two-port inversion. Finally, we design novel inverse system examples which represent two-port realizations and which apply either the idea of dimension reduction by a high relative degree or the general structure.

5.1 General Structure

In this section we present a general structure, capable of inversion and synchronization. This structure serves for continuous and discrete-time systems. Furthermore we will extend this structure to a still more general one.

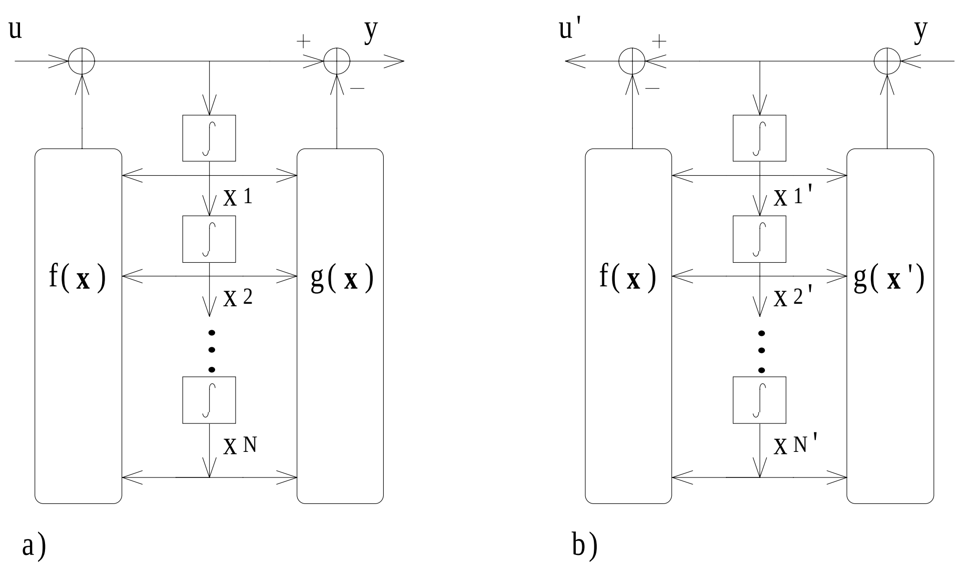


Figure 5.1: General structure capable of inversion and synchronization a) original and b) its inverse (continuous time system) with $f(\mathbf{x}), g(\mathbf{x}) \, : \, I\!\!R^n \rightarrow I\!\!R$ continuous

The structure in Fig. 5.1 has zero relative degree and therefore it is also applicable to discrete-time systems (Fig. 5.2).

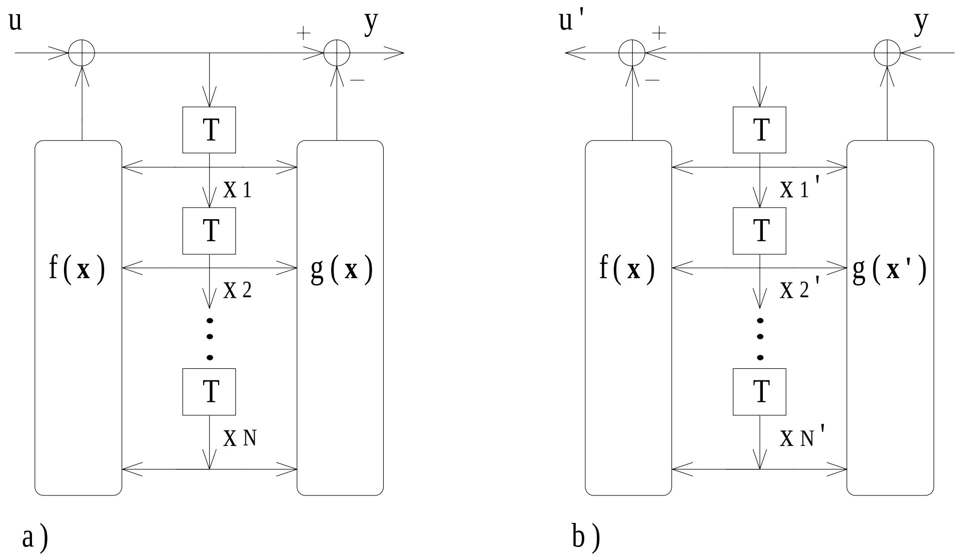


Figure 5.2: General structure of a discrete-time system a) original and b) its inverse with $f(\mathbf{x}), g(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ continuous

The state equations of the original system (Fig. 5.1a) are:

$$\begin{aligned} \dot{x}_1 &= u + f(\mathbf{x}) \\ \dot{x}_{i+1} &= x_i \text{ for } i = 1, \ldots, N-1 \\ y &= u + f(\mathbf{x}) - g(\mathbf{x}) \end{aligned} \tag{5.1}$$

and for the inverse system (Fig. 5.1b) we get:

$$\begin{aligned} \dot{x'}_1 &= y + g(\mathbf{x}') \\ \dot{x'}_{i+1} &= x'_i \text{ for } i = 1, \ldots, N-1 \\ u' &= y - f(\mathbf{x}') + g(\mathbf{x}') \end{aligned} \tag{5.2}$$

Similar equations hold for discrete-time systems, e.g. the state equation for the system of Fig. 5.2a) are:

$$\begin{aligned} x_1(k+1) &= u(k) + f(\mathbf{x}(k)) \\ x_{i+1}(k+1) &= x_i(k) \text{ for } i = 1, \ldots, N-1 \\ y(k) &= u(k) + f(\mathbf{x}(k)) - g(\mathbf{x}(k)) \text{ with } k \in \mathbb{N}_+ \end{aligned} \tag{5.3}$$

The only constraints we impose on the functions f and g are that they are continuous and that the solutions of (5.1) and (5.2) exist for all positive times and are uniquely determined by the initial condition of the states, $x(0)$ and $x'(0)$ respectively.

Now consider the solution $\mathbf{x}(t)$ and $y(t)$ of (5.1) with a given input $u(t)$ and with an initial condition $x(0)$. Then clearly $\mathbf{x}(t) \equiv \mathbf{x}'(t)$ and $u(t) \equiv u'(t)$ is a solution of (5.2), which proves that the system of Fig. 5.1b) is indeed an inverse system of the system of Fig. 5.1a) (cf. definition 3.2).

5.1.1 Chaotic Bahaviour $\rightleftharpoons$ Unique Asymptotic Behaviour

The general structure is a key to the understanding of how the feature of chaotic motion (in the autonomous case u=0) can be converted into the feature of asymptotic uniqueness by system inversion. Consider the different roles of the functions $f(\mathbf{x})$ and $g(\mathbf{x})$.

In the original system $f(\mathbf{x})$ represents the recursive part whereas $g(\mathbf{x})$ is the 'feed-forward' part. The function $g(\mathbf{x})$ does not influence the motion of the original system. It simply provides a certain output function. In the inverse system the roles of $f(\mathbf{x})$ and $g(\mathbf{x})$ are exchanged. The function $g(\mathbf{x})$ represents now the recursive part while $f(\mathbf{x})$ does not influence the motion of the inverse system but provides the output function.

Obviously, certain feedbacks are 'enabled' resp. 'disenabled' by system inversion. It is understood that the qualitative system behaviour can depend crucially on the kind of the static feedback.

It follows that by choosing functions $f(\mathbf{x})$ and $g(\mathbf{x})$ of different nature, we can obtain a different dynamic behaviour for the original and the inverse system. In our case, the goal is to have an original system that is chaotic and an inverse system with unique asymptotic behaviour, in order to achieve synchronization.

Assume that $f(\mathbf{x})$ is a nonlinear function of states leading to chaotic motion of the original system. In case $g(\mathbf{x})$ is a linear function, i.e. $g(\mathbf{x}) = \sum_{i=1}^{N} c_i x_i$, asymptotic uniqueness of the inverse system and therefore synchronization is easy to design or to establish.

Although the inverse system has a nonlinear output function, because of $f(\mathbf{x})$, the motion of its states is described by a linear system. If the natural frequencies of this system are in the left complex halfplane, its asymptotic behaviour is unique, and therefore the inverse system synchronizes with the original system.

5.1.2 Realization of Synchronization by Pole Setting

The difference between two solutions $\mathbf{x}(t)$ and $\mathbf{x}'(t)$, be they of the original and the inverse system or both of the inverse system, satisfies for a general $g(\mathbf{x})$:

$$\begin{aligned} \Delta\dot{x}'_1 &= g(\mathbf{x}) - g(\mathbf{x}') \\ \Delta\dot{x}'_{i+1} &= x_i - x'_i = \Delta x_i \text{ for } i = 1, \ldots, N-1 \\ \Delta u &= u - u' = f(\mathbf{x}') - f(\mathbf{x}) + g(\mathbf{x}) - g(\mathbf{x}') \end{aligned} \tag{5.4}$$

and in case of a linear $g(\mathbf{x})$:

$$\begin{aligned} \Delta\dot{x}'_1 &= \sum_{i=1}^{N} c_i x_i - \sum_{i=1}^{N} c_i x'_i = \sum_{i=1}^{N} c_i \Delta x_i \\ \Delta\dot{x}'_{i+1} &= x_i - x'_i = \Delta x_i \text{ for } i = 1, \ldots, N-1 \\ \Delta u &= u - u' = f(\mathbf{x}') - f(\mathbf{x}) + \sum_{i=1}^{N} c_i \Delta x_i \end{aligned} \tag{5.5}$$

The difference equations for discrete-time systems are similar. Note that the differential equations in (5.5) are linear. They have the form

$$\Delta\dot{\mathbf{x}} = \mathbf{A} \cdot \Delta\mathbf{x} \tag{5.6}$$

and for discrete-time systems

$$\Delta\mathbf{x}(k+1) = \mathbf{A} \cdot \Delta\mathbf{x}(k) \tag{5.7}$$

with

$$\mathbf{A} = \begin{bmatrix} c_1 & c_2 & \cdots & c_N \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} \tag{5.8}$$

The coefficients c_i have to be such that the eigenvalues of the matrix $\mathbf{A}$ are in the left complex halfplane in case of continuous-time systems and inside the unit circle for discrete-time systems. Since the coefficients c_i in this special structure, which is called control canonical form (CCF) in [12], represent the coefficients of the characteristic polynomial $P(s)$ resp. $P(z)$ certain eigenvalues can be easily realized by the appropriate choice of c_i, i.e. by pole setting. s and z are considered to be the complex variable for analogue resp. discrete-time systems.

Setting the poles sufficiently far from the imaginary axis or inside the unit circle, respectively, accelerates the synchronization of the inverse system. The characteristic polynomial has the form

$$P(s) = s^N - \sum_{i=1}^{N} c_i s^{N-i} \text{ resp. } z^N - \sum_{i=1}^{N} c_i z^{N-i} \tag{5.9}$$

For instance, the deadbeat property, mentioned for discrete-time systems in section 3.2.2, could be achieved by setting all poles to zero, i.e. by choosing all $ci = 0$ for $i = 1, \ldots, N$. In this case the inverse is non recursive. Obviously the influence of initial states vanishes and the inverse system synchronizes after N time steps.

Remark 5.1 (Confusion about obvious synchronization)
Some confusion could arise if the obvious synchronization for discrete-time systems (i.e. the deadbeat or non recursivity property) is assumed to be the analogy to the immediate synchronization of analogue systems (cf. Figs. 3.4, 3.5). This is absolutely **false**.

The latter is due to $r = N$ i.e. the fact that the inverse system has no states. For this a discrete-time analogy does not exist (cf. 3.2.2).

The structural time-continuous analogy to the non recursivity (which is a $r = 0$ case) is an inverse system with no feedback. However, while for discrete-time systems a zero eigenvalue indicates immediately vanishing influence of initial states, for analogue systems a zero pole does not even provide asymptotically vanishing influence. Hence, the continuous-time analogy to the immediate synchronization of discrete-time systems does not even synchronize.

5.1.3 More General Structure

Examples of chaotic discrete-time systems with the recursive structure of Fig. 5.1 for the states are known (e.g. Henon map) and one only has to choose an appropriate output function. However, one cannot expect all chaotic circuits to be of this nature.

The advantage of a linear difference system (5.6) can be preserved also for a more general structure. The properties responsible for a linear difference system are:

1. The state equations of the original system contain a single nonlinear real-valued function $f(\mathbf{x}, u) : I\!R^{N+1} \to I\!R$ and a strictly linear function of states, i.e. in the autonomous case $u = 0$ it is a Lur'e system [41], which includes several known chaotic systems.
2. The output of the original system is a linear combination of states and the function $f(\mathbf{x}, u)$.

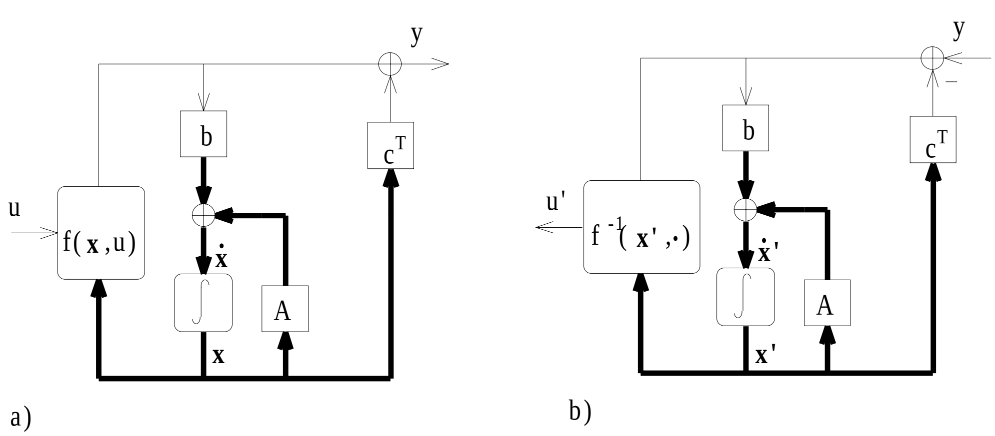


Figure 5.3: a) More general structure b) its inverse; The bold lines represent vectors in stead of scalars

Fig. 5.3 shows the more general structure and its inverse. In order to obtain a suitable inverse system, we have to require that the function $f(\mathbf{x}, u)$ is invertible with respect to u and that this inverse function

$f^{-1}(\mathbf{x}, \cdot)$, is also continuous. We still could use a general nonlinear output function $g(\mathbf{x})$ instead of the linear combination of states (Fig. 5.4). But in this case the difference system would not be linear. Note the functions f and g play similar roles as those in Fig. 5.1. The system has a linear kernel while the different nature of feedback functions (f in the original system and g in the inverse system) can lead to qualitatively different asymptotic bahaviour, namely a chaotic or a unique one.

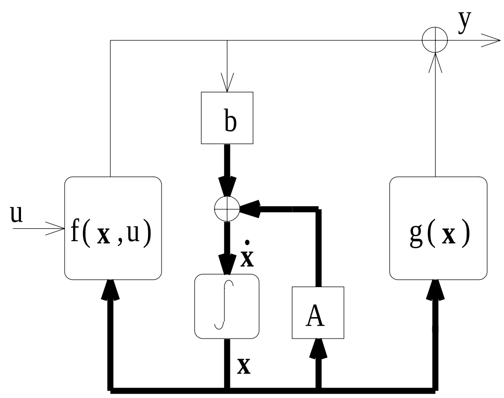


Figure 5.4: More general structure with nonlinear feedback and nonlinear output function; The bold lines represent vectors in stead of scalars

In order to achieve synchronization easily the discussion will be limited to a linear output function. The state equations of the systems of Fig. 5.3a) and b) are

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{b} \cdot f(\mathbf{x}, u) \\ y &= \mathbf{c}^T \cdot \mathbf{x} + f(\mathbf{x}, u) \end{aligned} \tag{5.10}$$

$$\begin{aligned} \dot{\mathbf{x}}' &= \mathbf{A}\mathbf{x}' + \mathbf{b} \cdot (y - \mathbf{c}^T \cdot \mathbf{x}') \\ u' &= f^{-1}(\mathbf{x}', y - \mathbf{c}^T \cdot \mathbf{x}') \end{aligned} \tag{5.11}$$

Consequently, the difference equations are:

$$\begin{aligned} \Delta\dot{\mathbf{x}} &= \dot{\mathbf{x}} - \dot{\mathbf{x}}' = (\mathbf{A} - \mathbf{b} \cdot \mathbf{c}^T)\Delta\mathbf{x} \\ \Delta u &= u - u' = f^{-1}(\mathbf{x}, y - \mathbf{c}^T \cdot \mathbf{x}) - f^{-1}(\mathbf{x}', y - \mathbf{c}^T \cdot \mathbf{x}') \end{aligned} \tag{5.12}$$

Since the systems of Fig. 5.3 have again a zero relative degree they can also be realized as discrete-time systems.

5.1.4 Realization of Synchronization by Application of the Ackermann Formula

Again the difference system is linear, and $\Delta u \to 0$ if $\Delta\mathbf{x} \to 0$ because of the continuity of the function $f^{-1}(\mathbf{x}, \cdot)$. In order to guarantee that $\Delta\mathbf{x} \to 0$, we have to choose the parameters of the linear part in such a way that the eigenvalues of $(\mathbf{A} - \mathbf{b} \cdot \mathbf{c}^T)$ are in the left complex halfplane. Pole setting is not so obvious as for the system of Fig. 5.1. But an appropriate choice of $\mathbf{c}^T$ can be found by applying the Ackermann formula, provided the pair $(\mathbf{b}, \mathbf{A})$ is controllable [12].

$$\mathbf{c}^T = e_N^T \mathbf{C}^{-1} P(\mathbf{A}) \tag{5.13}$$

Here, e_N^T is the transpose of the N-th unity vector, $\mathbf{C}^{-1}$ is the inverse of the controllability matrix $\mathbf{C} = (\mathbf{b}, \mathbf{A}\mathbf{b}, \ldots \mathbf{A}^{N-1}\mathbf{b})$ which is regular for $(\mathbf{b}, \mathbf{A})$ controllable. $P(\mathbf{A})$ is the (by pole setting) desired characteristic polynomial applied to the matrix $\mathbf{A}$, cf. (5.9). See [12] for a proof.

For instance, in order to make discrete-time difference system deadbeat, i.e. in order to set all poles of $(\mathbf{A} - \mathbf{b} \cdot \mathbf{c}^T)$ to zero, one has to choose:

$$\mathbf{c}^T = e_N^T \mathbf{C}^{-1} \mathbf{A}^N \tag{5.14}$$

Thus by application of the Ackermann formula we do not only achieve synchronization but can also choose the synchronization speed.

For discrete-time systems a free choice of output coefficients c_i should always be possible. An important result is that one can by application of (5.14) achieve perfect synchronization after a finite time.

On the other hand for analogue systems the circuit realization does hardly admit any linear combination of states, because the restriction to accessible branch voltages and currents is sensible.

5.1.5 Another Way to Achieve a Linear Inverse System

We are not only restricted to a set of possible circuit variables to be combined with the value of the nonlinear static function. Worse, $f(\mathbf{x}, u)$ can even be inacessible for any combination with other circuit variables. This is the case e.g. for the basis-emitter current, i_{BE}, of a bipolar transistor, clearly a nonlinear function of the basis-emitter voltage. i_{BE} is inaccessible for any combination with other currents *before* it controls the collector-emitter current (cf. also section 5.3.2).

For such cases there is still another way to achieve a linear inverse system departing from the general structure in Fig. 5.3. It is to transmit not the value of the nonlinear function but its argument. This idea was already realized in [43]. The situation is depicted in Fig. 5.5. By choosing this argument as input the feedback due to the nonlinear function is cut out. Obviously, this method does not leave any freedom of output choice and can only be applied, if the linear kernel is stable i.e. the matrix **A** is Hurwitz.

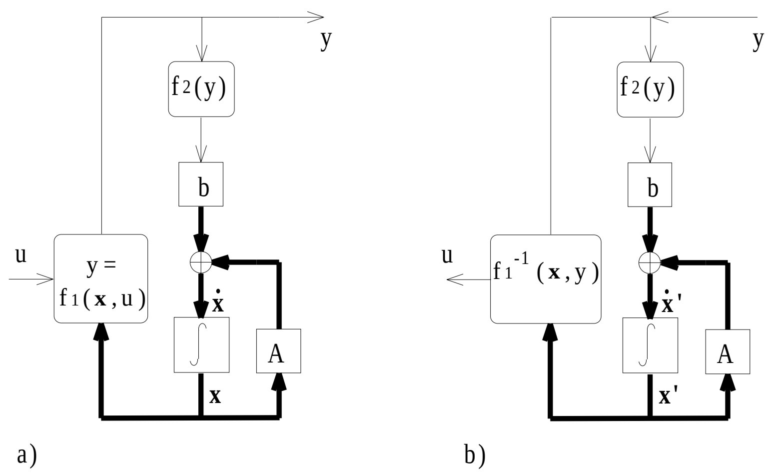


Figure 5.5: General structure using a Lur'e type system where the argument of the nonlinear static function is transmitted b) its inverse; The bold lines represent vectors in stead of scalars

Although we are in general restricted to only a set of possible outputs, we can put the general structure idea into practice as will be demonstrated by examples in section 5.3.

5.2 Two Port Realizations

The circuit realization of the inverse system examples in section 3.3.1 uses both for the transmitter and the receiver basically a nonlinear dynamic one-port. In this section we will show that the restriction to one-ports is not necessary. The inversion of a two-port requires the use of *ideal* operational amplifiers. Finally, a criterion is given which allows to establish the correct functioning of *nonideal* op-amps.

5.2.1 Two Port Realizations by Means of Ideal Operational Amplifiers

Indeed, the information signal can be injected into the circuit by, say, a voltage source and any other voltage or current, not just the source current, can serve as chaotic signal to be transmitted. In general the input branch, k, is described by:

$$v_k(t) = u(t) \text{ resp. } i_k(t) = u(t) \tag{5.15}$$

The output branch, l, can assumed to be an open circuit in case its voltage serves as output and is a short circuit if its current serves as output signal. Thus the ouput branch is described by:

$$i_l = 0 \text{ resp. } v_l = 0 \tag{5.16}$$

Provided the network with these constitutive relations of branches k and l has a solution, we obtain an output signal:

$$y(t) = v_l(t) \text{ resp. } y(t) = i_l(t) \tag{5.17}$$

In order to invert this situation we have to give up Equ. (5.15), to maintain Equ. (5.16) and to add:

$$v_l(t) = y(t) \text{ resp. } i_l(t) = y(t) \tag{5.18}$$

This is we leave the branch k free of any constitutive relation - a *norator* - whereas for the branch l we maintain the open resp. short circuit property **and** additionally fix its voltage resp. current. Therefore it is a branch with **two** constitutive relations -a *fixator*. Roughly speaking, we take a degree of freedom away from branch l and give it to the branch k.

The combination of Equs. (5.16) and (5.18) can be realized by a *nullator in series* with a voltage source resp. by a *nullator in parallel* with a current source.

Provided the network with these constitutive relations of branches k and l has also a solution, we obtain an output signal, $u'(t)$, of the inverse circuit, namely the voltage resp. current of the norator branch:

$$u'(t) = v_k(t) \text{ resp. } u'(t) = i_k(t) \tag{5.19}$$

The four possible situations of two-port inversion are depicted in Fig. 5.6. The one-port inversion can be embedded here if an extra output branch is introduced.

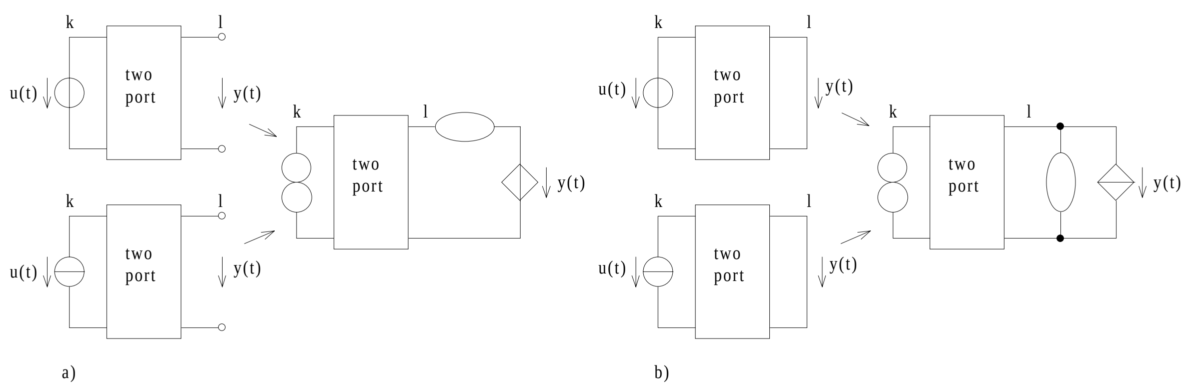


Figure 5.6: Two-port inversion: a) when the voltage of branch l served as output signal: the $v \to v \to v$ resp. $i \to v \to i$-method, b) when the current of branch l served as output signal: the $v \to i \to v$ resp. $i \to i \to i$-method; the voltage resp. current of the norator $u'(t)$ are supposed to be a copy of the original input signal $u(t)$

The two-port represented by a nullator-norator pair is the *ideal operational amplifier*, Fig 5.7.

In the sequel we take it for granted that the inverse system synchronizes with the original system in case of an ideal op-amp., i.e. the inverse circuit with an ideal op-amp. has the unique asymptotic behaviour property (cf. section 3.1.3).

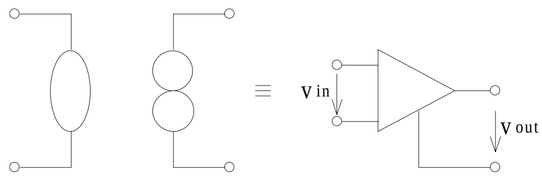


Figure 5.7: A nullator-norator pair: model of the ideal operational amplifier

But we still have to check that the non ideal characteristics of the operational amplifier do not destroy the correct functioning of the inverse system. (The question of the correct functioning of circuits containing op-amps. is a general one and not restricted to inverse system design. Although our results are applicable to all circuits with one nonideal op-amp. we will stick to inverse systems here) The purpose of the next section is to give conditions which ensure that the output of the inverse system deviates only slightly from the original information signal when the operational amplifier saturates and has a single pole frequency characteristic.

5.2.2 Influence of the Finite Gain and the Frequency Characteristic of the Op-Amp. - Criteria of Network Structure

If too large an input signal is applied to the inverse system, it will be driven into saturation and the retrieved signal will be heavily distorted. Therefore, we suppose that the dynamic range of the input signal is limited in such a way that the output of the operational amplifier, when its ideal model is taken, stays well below the limits imposed by the saturation effect of its non ideal model. This is necessary for correct functioning, but not sufficient.

It could still happen that the dynamic effects of the operational amplifier drive it into saturation. In order to discuss the dynamic effects, we take a simplifying point of view which could be justified on the basis of the complete system of circuit equations.

State Equations

We use the following model of the nonideal op-amp.:

$$\tau \dot{v}_{out} + v_{out} = f(v_{in}) \tag{5.20}$$

where $f(\cdot)$ indicates a saturation characteristic depicted in Fig. 5.9a, i.e. the op-amp. is now regarded as an open circuit at the input port and an *inertial* voltage controlled voltage source at the output port, Fig. 5.8.

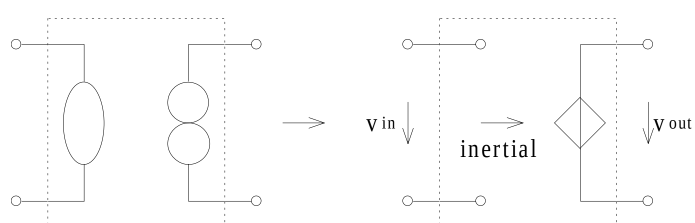


Figure 5.8: Ideal op-amp.: a nullator-norator pair; model of the nonideal op-amp.: an open circuit-inertial vc. voltage source pair

The inverse system with ideal op-amp. is described by:

$$\begin{aligned} \dot{\mathbf{x}} &= h(v_{out}, \mathbf{x}, s) \\ v_{in} &= g(v_{out}, \mathbf{x}, s) = 0 \end{aligned} \tag{5.21}$$

whereas the system with model (5.20) of the op-amp. is described by:

$$\begin{aligned} \dot{\mathbf{x}} &= h(v_{out}, \mathbf{x}, s) \\ \tau \dot{v}_{out} + v_{out} &= f(g(v_{out}, \mathbf{x}, s)) \end{aligned} \tag{5.22}$$

Here $\mathbf{x} \in I\!R^N$ is the state vector, $s(t)$ is the input signal and $v_{in} = g(v_{out}, \mathbf{x}, s)$ represents a *two port characteristic of the rest-network* (see below). The second equation in (5.21) is one constitutive relation of the nullator of the ideal op-amp. This *fixes* v_{out} as a function of the state and source values, provided the two-port characteristic $g(\cdot, \mathbf{x}, s)$ is invertible. But the second equation in (5.22) *fixes only the time derivative* of v_{out}.

Although we used the notion y for the input signal of the inverse system so far, we choose here s instead in order to indicate the validity of the approach for op-amp. circuits in general.

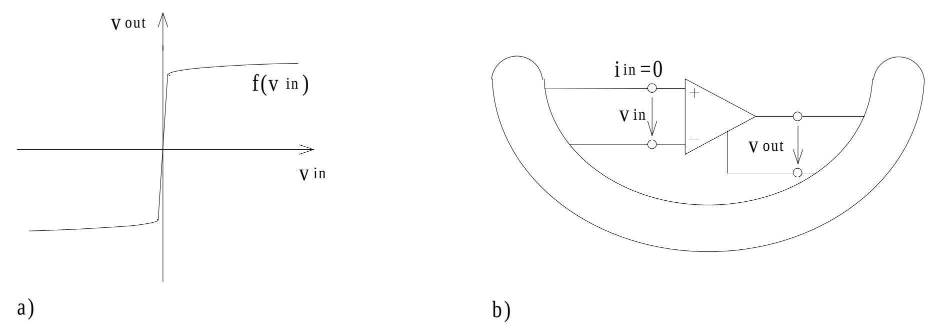


Figure 5.9: (a) Saturation characteristic of the op-amp. $f(v_{in})$ (b) Inverse system as a connection of the op-amp. two-port with the rest-network

Dimension of the System with Nonideal Op-Amp.

The Equs. (5.21, 5.22) require both rest-networks to obey the same state equations. This implies that the number of states does not change if the nullator-norator pair is replaced by an nonideal op-amp. except for the additional state caused by the dynamic model of the op-amp.

In chapter 6 we explain in detail that the dimension of the state space (i.e. the number of independent capacitor charges and inductor fluxes) depend strongly on the circuit structure, namely on the position of nullators and norators. It is clear that the circuit structure can change crucially when a nullator-norator pair is replaced by a pair of an open circuit and an inertial vc. voltage source. Thus also the dimension of the state space can change as is the case in the special example (not an inverse but an op-amp. circuit) proposed by [39], Fig. 5.10. By the methods described in chapter 6 one can show that the network with ideal op-amp. is zero-dimensional while the network with nonideal op-amp. is described by a two-dimensional state equation.

($*$) We require the rest-network when connected with the ideal and the nonideal op-amp. to have the same dimension.

Furthermore, the appearence of input derivatives in the system description makes our slow-fast approach difficult to apply. Input derivatives occur in the circuit description for inverse systems with

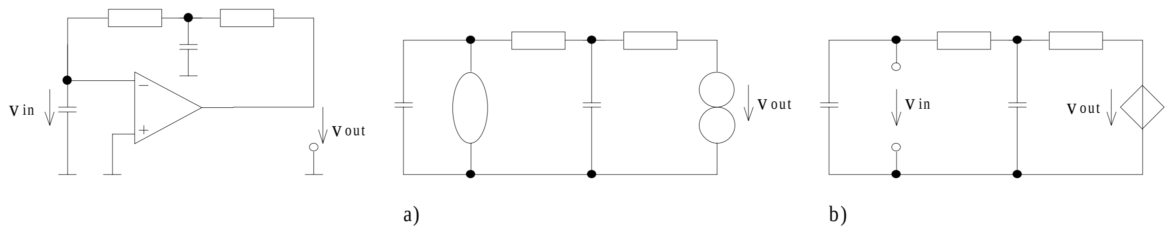


Figure 5.10: Circuit example a) model with ideal op-amp. -a zero-dimensional circuit; b) model with nonideal op-amp. -a two-dimensional circuit

reduced state space due to $r > 0$, cf. section 3.2, i.e. when not all capacitors and inductors represent states.

($**$) We apply our aproach only to inverse systems with zero relative degree resp. require the circuit with ideal op-amp. to have the *full dimension* of state space i.e. it is equal to the number of capacitors and inductors.

Remark 5.2 In general one can expect $*$ and $**$ to hold for circuits with only one op-amp., inverse to a zero relative degree system. This is due to the fact that the existence of a full-dimensional system description inplies the existence of conjugate trees with all capacitors on the tree and all inductors on the cotree (cf. chapter 6). One of these contains the norator while the nullator belongs to the cotree. Consequently, the norator replaced by a voltage source and the nullator replaced by an open circuit forms also a tree resp. cotree. This is a crucial condition for the circuit with nonideal op-amp. to have a state space of full dimension.

Slow-Fast Approach

We have to show that the system (5.22) produces nearly the same signals as the ideal system (5.21) which moves on a hypersurface $\mathcal{H}_{id}$ in $I\!R^{N+2}: v_{in} = g(v_{out}, \mathbf{x}, s) = 0$.

We take the following simplifying point of view: The dynamics of the ideal circuit are supposed to be much slower than the additional dynamics caused by the frequency roll-off of the operational amplifier. This is due to the very large slope A of $f(\cdot)$ at $v_{in} = 0$ abd holds at least if v_{in} is not very very close to zero or large.

Therefore, we analyse the fast dynamics in the limit of constant $\mathbf{x}$ and constant s. This means that we consider the inverse system as the connection of the operational amplifier, with its dynamic model, to the *static rest-network*, Fig. 5.9b. The remaining dynamics of this pseudo-static circuit are described by:

$$\tau \dot{v}_{out} = f(g(v_{out}, \mathbf{x}, s)) - v_{out} \tag{5.23}$$

The set of dc-operating points of Equ. (5.23) $\overline{v}_{out}(\mathbf{x}, s)$ for all $\mathbf{x}$ and s represents a hypersurface $\mathcal{H}_{dc}$ in $I\!R^{N+2}$ too, which is described by:

$$\begin{aligned} f^{-1}(v_{out}) &= g(v_{out}, \mathbf{x}, s) && (5.24)\\ &\approx \frac{v_{out}}{A} \text{ for } v_{in} \approx 0 && (5.25) \end{aligned}$$

We call this hypersurface the *slow manifold*.

Next we show that this hypersurface is close to the hypersurface $\mathcal{H}_{id}$ the system with ideal op-amp. is moving on. Then we derive conditions to ensure that the hypersurface $\mathcal{H}_{dc}$ is globally asymptotically stable under the dynamics (5.23).

$\mathcal{H}_{dc}$ is close to $\mathcal{H}_{id}$

The dc-operating point can for specific circuits be shown to be unique and to depend continuously on the the inverse of the slope A by applying criteria of [19]. Note, the dc-operating point of (5.23) is a solution of a resistive circuit, Fig. 5.11a, which consists of a nonlinear vc. voltage source: $v_{out} = f(v_{in})$ and the static rest-network where capacitors and inductors are replaced by voltage resp. current sources. Due to the continuity of the solution in $\frac{1}{A}$ the hypersurface $\mathcal{H}_{dc}$ with $\frac{1}{A} \approx 0$, i.e. the solution of (5.25), is close to $\mathcal{H}_{id}$ with $\frac{1}{A} = 0$. This result can also be obtained by inspection of the intersection of the two-port characteristics under the constraint $i_{in} = 0$, Fig. 5.11b.

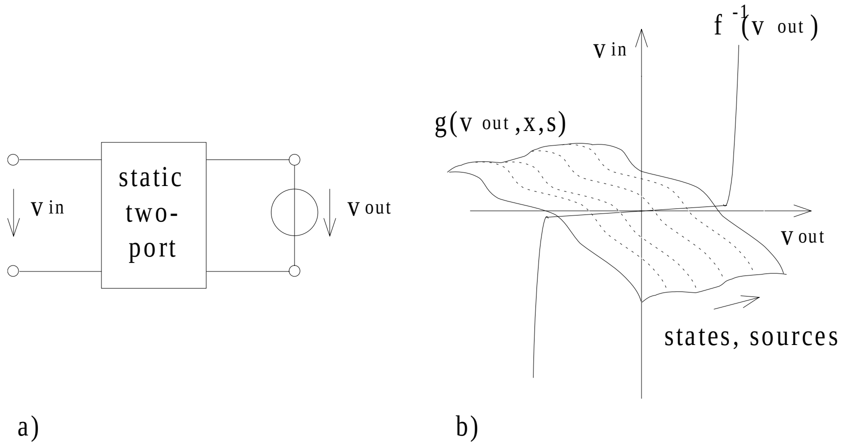


Figure 5.11: (a) Circuit to detect the static two-port characteristics under the constraint $i_{in} = 0 : \ v_{in} = g(v_{out}, \mathbf{x}, s)$ (b) Inverse saturation characteristic of the op-amp. $f^{-1}(v_{out})$ and two-port characteristic of the rest-network $v_{in} = g(v_{out}, \mathbf{x}, s)$, the intersection of both determines the dc-operation point

Two-port characteristics

In general n-port characteristics represent an n-dimensional manifold in $\mathbb{R}^{2n}$. Thus we had to consider a two-dimensional manifold in the four-dimensional space of $(v_{in} \times i_{in} \times v_{out} \times i_{out})$ for the static two-port. However, the constraint $i_{in} = 0$ imposed by the connection with the op-amp. allows to consider only a one-dimensional manifold in $(v_{in} \times v_{out} \times i_{out})$. Actually we are only interested in its projection on $(v_{in} \times v_{out})$ described by $v_{in} = g(v_{out}, \mathbf{x}, s)$ as depicted in Fig.5.11b.

Note there should be no confusion because the resistive network for the detection of the dc-operating point is accidentally the same as for the investigation of the two-port characteristic: The static two port is connected with an open cicuit-voltage source pair, which represents in the first case the dc-op-amp. (actually it should be a vc. voltage source) and in the second case a *test circuit* for the two-port characteristics under the constraint $i_{in} = 0$. Another possible test circuit was a fixator-norator pair, suitable for the calculation of $v_{out} = g(v_{in}, \mathbf{x}, s)$ under the constraint $i_{in} = 0$, which provides the same result concerning the monotonic dependence.

Conditions for the stability of $\mathcal{H}_{dc}$

From our slow-fast point of view, the dc-operating point, cf. Equ. (5.25), varies slowly and the solution of the inverse system will track it, provided it is asymptotically stable. In order to show this one can use a Ljapunov function:

$$V(v_{out}, \mathbf{x}, s) = \frac{(v_{out} - \overline{v}_{out}(\mathbf{x}, s))^2}{2} \tag{5.26}$$

$$\dot{V} = (v_{out} - \overline{v}_{out}(\mathbf{x}, s)) \cdot (f(g(v_{out}, \mathbf{x}, s)) - v_{out}) \leq 0 \tag{5.27}$$

If $g(v_{out}, \mathbf{x}, s)$ is a strictly decreasing function then, because $f(\cdot)$ is strictly increasing, $f(g(v_{out}, \mathbf{x}, s)) - v_{out}$ is strictly decreasing as well. Both multipliers of (5.27) are zero at $v_{out} = \overline{v}_{out}(\mathbf{x}, s)$. This and the fact that they are strictly increasing, resp. decreasing functions of v_{out} proves the validity of (5.27).

It remains to show that $g(v_{out}, \mathbf{x}, s)$ is a strictly decreasing function. In case $v_{in} = g(v_{out}, \mathbf{x}, s)$ is strictly increasing one can change the op-amp. polarity to obtain correct functioning.

Proof of strict monotonicity of $g(v_{out}, \mathbf{x}, s)$

The following proposition allows to establish strictly monotonic dependence of a circuit variable on a source value by mere inspection of the network structure.

Proposition 5.1 (Monotonic Dependence) *Consider resistive networks:*

1. *which consist of resistors with strictly monotonic characteristics, independent sources and ideal op-amps.*
   and

2. *which have for all source values exactly one solution.*

If all admissible partial orientations of a circuit structure, where exactly one source is oriented, lead to the same orientation of a branch voltage (current) then all resistive networks with this structure have a strictly monotonic dependence of the branch voltage (current) with respect to the source value.

An admissible orientation is an orientation of network branches which does not contradict Kirchhoffs Laws. The branch orientations represent voltage and current differences, namely their signs, of the solutions with different source values, see also chapter 6. Consequently, every branch can be oriented in both directions resp. nonoriented, which corresponds to incresing, decreasing resp. constancy of the considered circuit variable with respect to the value of the oriented source. This proposition is derived from [19] where a proof is given. Proposition 6.2 provides a tool to check whether condition 2 of the above proposition holds.

In our slow-fast approach we are interested in the dependence $v_{in}(v_{out}) = g(v_{out}, \mathbf{x}, s)$. Therefore, we have to treat the two-port as a resistive network according to Fig. 5.11a and to check the mentioned conditions. For this all slowly varying states are replaced by sources, namely, capacitors (inductors) by voltage (current) sources. Condition 2. of proposition 5.1 holds if there exist circuit state equations in terms of capacitor charges and inductor fluxes as required in $(**)$.

If now all possible partial orientations, where the only oriented source is v_{out}, lead to the same voltage orientation of the branch v_{in} then the circuit has the monotonicity property necessary to stabilize the slow manifold.

The result of this orientation check does not depend on specific source values as it corresponds to a circuit structure. Therefore all voltage (current) sources can be replaced by short (open) circuits as will be demonstrated for two examples in the next section. Similarly, the admissible orientations of resistors depend only on the kind of the strictly monotonicity (increasing/decreasing) of the resistor characteristic. This means that strictly increasing characteristics allow only voltage and current orienatations in the same direction, while strictly decreasing resistor characteristics allow only voltage and current orientations directed against each other. Consequently, we distinguish only two types of resistors in the network structure.

We conclude:
Under reasonable assumptions on the system and its inverse, the circuit theoretic property of monotonic dependence as discussed in [19] enables to assure the correct functioning of the inverse system and of circuits with one nonideal op-amp. in general when saturation and dynamic behaviour of the operational amplifier is taken into account.

5.3 Novel Inverse System Design Examples

If a system with relative degree $r = N$ (e.g. Fig. 3.4) is used for information hiding, then the inverse immediately extracts the correct information, since there is no influence of initial states. Because of this property it seems desirable to choose input and output of a system in such a way that its relative degree is N. However, in this case any added channel noise leads to disastrous errors in signal recovery because it is several times differentiated. Obviously, one has to make a compromise between the low dimensionality of the inverse system (and the difference system) and the number of integrations the transmitted signal undergoes.

First we will present a continuos time, $r = N$ system example and then we will apply our general structure, cf. section 5.1 in order to design $r = 0$ examples with a linear inverse system. In all cases we extend the circuit realizations published so far (cf. section 3.3.1) to two-port realizations.

5.3.1 Relative Degree $r = N$ Example

A quick look at the block diagram of Chua's circuit driven by a current source in parallel with the capacitor C_1 Fig. 3.14a) reveals that in order to obtain a system with relative degree $r = N = 3$, the output should be x_3, i.e. the inductor current. Indeed, in this case, the input signal has to go through 3 integrators to reach the output. This somewhat intuitive reasoning can also be established by the algebraic calculations proposed in appendix A. Fig. 5.12 shows the block diagram of the inverse system. A common method to find an output resulting in r=N for a given system with a certain input is described in [21] pp. 165.

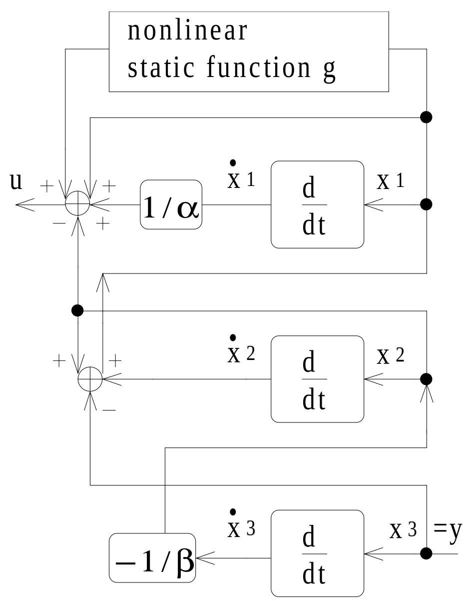


Figure 5.12: Block diagram of the inverse system of Chua's circuit in case the original is the $r = N = 3$ system with input: current source in parallel with C_1 and output: inductor current $i_L \sim x_3 = y$

According to section 3.2.1 the inverse is not a dynamical system. It does not contain any memory element and therefore synchronizes immediately. For those readers who prefer explanations in terms of electric circuits instead of block diagrams the following reasoning is given.

In the inverse system the inductor is driven with the transmitted current i_L. It therefore produces by differentiation the appropriate voltage v_{C2} of the capacitor C_2. In turn, the capacitor produces by differentiation the appropriate current i_{C2}. The voltage v_{C1} is a linear combination of i_L, i_{C2} and v_{C2}. The capacitor C_1 produces again by differentiation i_{C1}. Finally, the retrieved current is a function of i_{C1}, v_{C1} and v_{C2}. Obviously all reactances work as differentiators in the inverse system.

The just described procedure of how states of reactances are fixed as functions of successive fixed signals is exactly the algorithm of obliteration of bad loops and cutsets to be introduced in section 6.2.

In our example the input-output pair is not a current-voltage pair of a one-port. Fig. 5.13 shows the two-port realization of the inverse system a) by means of a nullator - norator circuit and b) by the corresponding operational amplifier circuit. Note that we have to realize a current input in series with the inductor. The current source must have zero voltage in order to produce the right voltage v_{C2}. Therefore a nullator in parallel with the current source is necessary. At the output of the inverse system we have to detect the current independently of voltage. Therefore we have to use a norator. This way we realize exactly the ideas described in section 5.2.1.

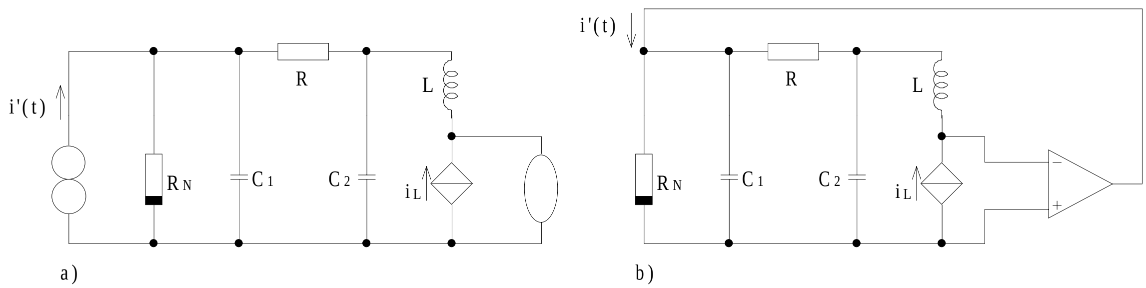


Figure 5.13: Two-port realization of the inverse system of the $r = 3$ Chua's circuit example a) by means of ideal network elements and b) by the appropriate use of an operational amplifier. The input signal is i_L and $i'(t)$ is the current to be retrieved

In practical realizations the nonideal characteristics of the operational amplifier have to be taken into account. This will lead to inaccurate signal recovery because this realization is highly sensitive to channel noise because of the differentiators.

5.3.2 Relative Degree $r = 0$ Examples - Application of the General Structure

As already described in section 5.1 there are two possibilities to invert systems with Lur'e type structure (i.e. with only one static nonlinear function) such that the inverse system is linear. These are the transmission of the value of the nonlinear static function in linear combination with states or the transmission of its argument, Fig. 5.3 and 5.5. For both cases we give an example here.

Since the application of the general structure implies a zero relative degree, we can confirm the correct functioning of the op-amp. realization by inspection of the circuit structure as described in the last section.

The zero relative degree examples overcome the disadvantages of non zero relative degree systems as discussed above

Example using Chua's circuit

From the block diagram of Chua's circuit in Fig. 3.14a) it is clear that this system has Lur'e type feature. Choosing again an additional current source in parallel with C_1 as input, the comparison with the general structure transmitter equation (5.10) leads to:

$$\begin{aligned} f(\mathbf{x}, u) &= u - g(x_1) \\ \mathbf{b}^T &= (\alpha, 0, 0) \\ \mathbf{A} &= \begin{pmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix} \end{aligned} \tag{5.28}$$

We use again the customary normalization of Chua's circuit, cf. e.g. [8]. The output function $y = \mathbf{c}^T \cdot \mathbf{x} + f(\mathbf{x}, u)$ has to be chosen so that the matrix $(\mathbf{A} - \mathbf{b} \cdot \mathbf{c}^T)$ is Hurwitz, i.e. it has all eigenvalues in the left complex halfplane.

Design:
Actually, we have no big choice in order to combine $f(\mathbf{x}, u)$ with linear functions of states. Since the value of the nonlinear static function represents a current, only the combination with currents of other branches makes sense. Such a combination branch should have at least one node in common with the nonlinear resistor branch. Then by introduction of an additional branch, a short circuit, one obtains the appropriate combination of currents in an output branch.

Since the matrix $\mathbf{A}$ is already Hurwitz we choose $\mathbf{c}^T = (0, 0, 0)$. The circuit realization which is again two-port realization is represented in Fig. 5.14.

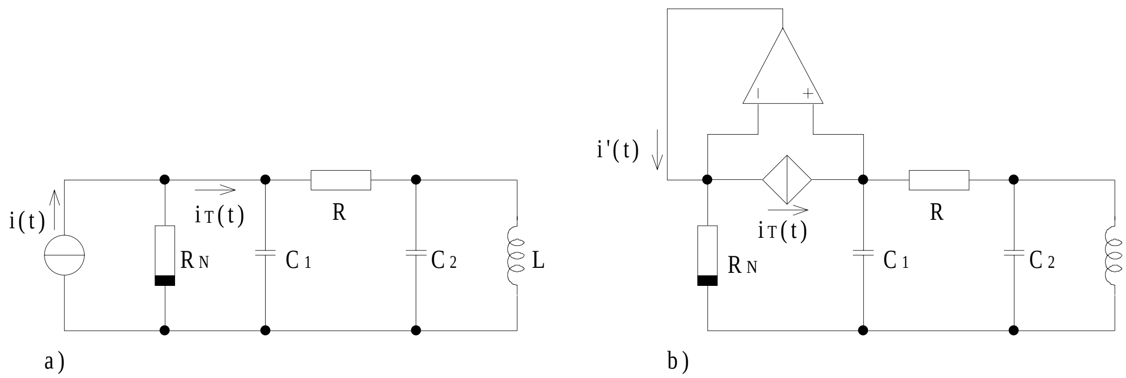


Figure 5.14: Realization of the $r = 0$ Chua's circuit example a) the original system with the input signal $i(t)$ and output signal $i_T(t)$ corresponding to $f(\mathbf{x}, u)$ in Equ.(5.28) b) the inverse system with exchanged roles of $i(t)$ and $i_T(t)$ -a two-port inversion

Synchronization:
For this inverse system synchronization, i.e. the unique asymptotic behaviour property, is obvious because the current source drives a linear passive one-port. This provides an asymptotic unique oneport voltage, which in turn fixes asymptotically the current of the nonlinear resistor and finally also the current $i'(t)$.

Monotonic dependence:
As in the previous example, an nonideal operational amplifier leads to a synchronization error. But one can show the network to fulfill the criterion of section 5.2.2: Fig. 5.15a shows the corresponding

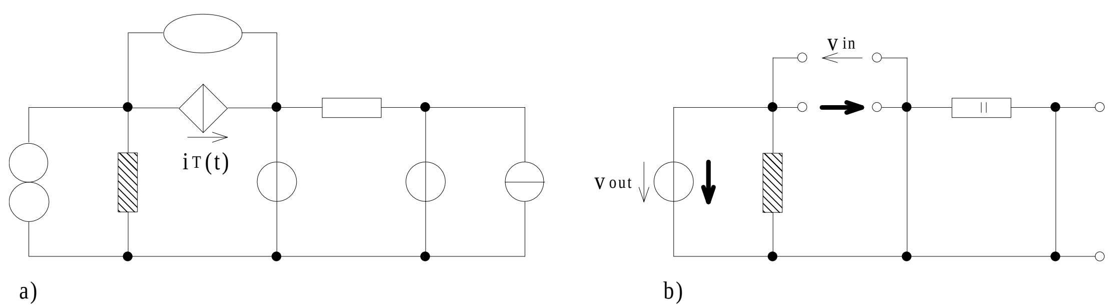


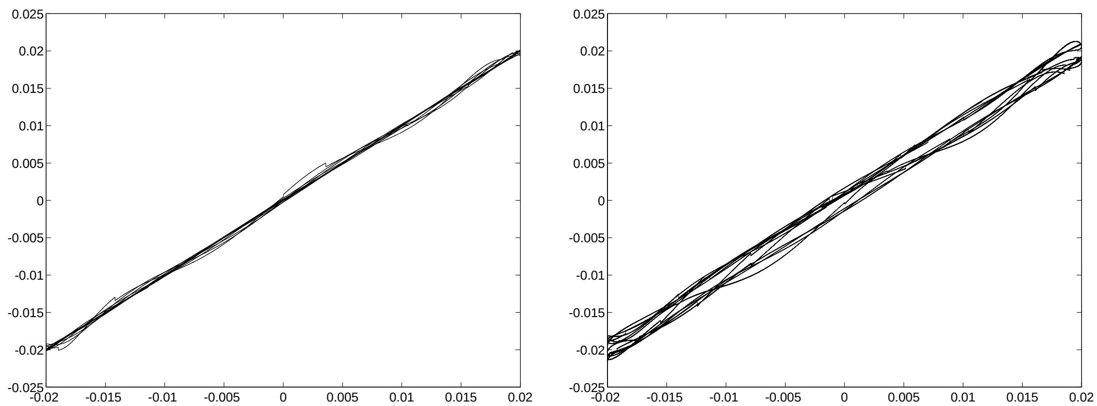
Figure 5.15: a) Structure of the resistive network corresponding to Fig. 5.14b with ideal op-amp.; b) resistive network structure for the detection of monotonic dependence of $v_{in}(v_{out})$ with the only possible voltage orientation: -bold arrows- with respect to the reference orientation -thin arrows-; '=' indicates that the branch is not oriented

resistive network with all meory elements replaced by sources. The resistive network for the test of monotonic dependence of $v_{in}(v_{out})$ with all current sources opened and all voltage sources shortened except v_{out} is depicted in Fig. 5.15b. Since there is only one possible voltage orientation of v_{in} the necessary monotonicity property is shown. In order to guarantee correct functioning it remains to choose the op-amp. polarity such that v_{in} is directed against the orientation caused by v_{out}. This provides the strictly decreasing dependence of $v_{in}(v_{out})$ which is necessary in order to stabilize the slowly varying dc-point near $v_{in} = 0$ (cf. section 5.2.2). Hence the retrieved signal $i'(t)$ can axpected to deviate only slightly from the original information signal $i(t)$.

Simulation Results with Ideal and Nonideal Op-Amp.:
We have performed simulations with an ideal operational amplifier and with an amplifier with a sigle pole frequency characteristic. The usual normalisations for the circuit parameters have been used with $\alpha = 10$

and $\beta = 14$. The input current was a sinusoid with amplitude 0.02 and frequency 0.5. The frequency dependent amplification of the operational amplifier was chosen to be $A_0/(1+\tau s)$, with $A_0 = 1000$ and $\tau = 0.1$. Fig. 5.16a) and b) show $i'(t)$ versus $i(t)$ after the transient behaviour has settled down, with an ideal operational amplifier and with a non-ideal op-amp., respectively.

The synchronization error introduced by the mediocre amplification of the op-amp and its frequency dependence is clearly visible, but synchronization is still sufficiently good for the transmission of information.



(a) (b)

Figure 5.16: $i'(t)$ versus $i(t)$ after the transient behaviour has settled down, under assumption of an (a) ideal operational amplifier, (b) with nonideal operational amplifier

Example using Colpitts Oscillator

We designed a transmission system using the Colpitts oscillator. The circuit behaviour can be modelled according to [23] by state equations which contain again only one nonlinear static function. Thus it is also a Lur'e type system. But as opposed to the last example the value of the nonlinear static function is inaccessible for any combination with states. It is an internal variable of the transistor, its basis-emitter current. Therefore this example corresponds the situation considered in section 5.1.5.

Design:
We choose a decomposition of the the function $f(\mathbf{x}, u)$ according to Fig. 5.5, i.e. we influence the argument of the nonlinear static function by the information signal and transmit the resulting argument. This is we manipulate its basis-emitter voltage with an information voltage signal. The original and inverse circuit are depicted in Fig. 5.17.

Synchronization:
The driven linear network contains a transistor, the model of which involves also a nullator-norator pair for the current controlled current source in the collector-emiter branch. Hence the criteria of unique asymptotic behaviour of section 4.1 are not applicable. But the inspection of the eigenvalues of the linear system reveals that the matrix describing the linear kernel of the system is Hurwitz, i.e. the inverse system synchronizes and recovers the information signal at least if an ideal op-amp. is used for the two-port inversion. This has been confirmed by simulation experiments, Fig. 5.18.

Monotonic dependence:
In Fig. 5.19a is the circuit structure shown, which corresponds to the inverse Colpitts system with an ideal op-amp. i.e. a nullator-norator pair. In Fig. 5.19b the only possible partial voltage orientation

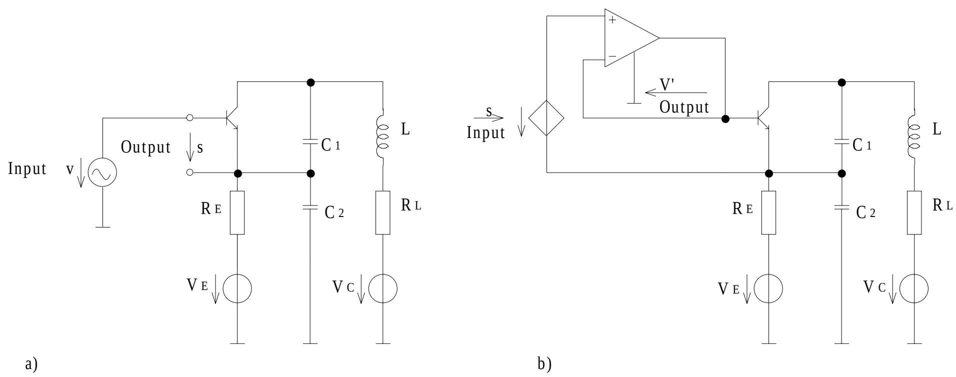


Figure 5.17: Tranmission system using the Colpitts oscillator (a) Transmitter (b) Receiver

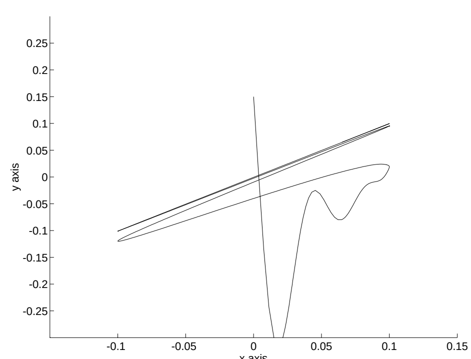


Figure 5.18: Overall transfer characteristic: $v'(t)$ versus $v(t)$

where the branch v_{out} is oriented is shown. This indicates that the inverse system has the necessary monotonicity property. And it is possible to choose the op-amp. polarity so that the slowly varying dc-point near $v_{in} = 0$ is stabilized. In this case the retrieved signal will be close to the desired information signal.

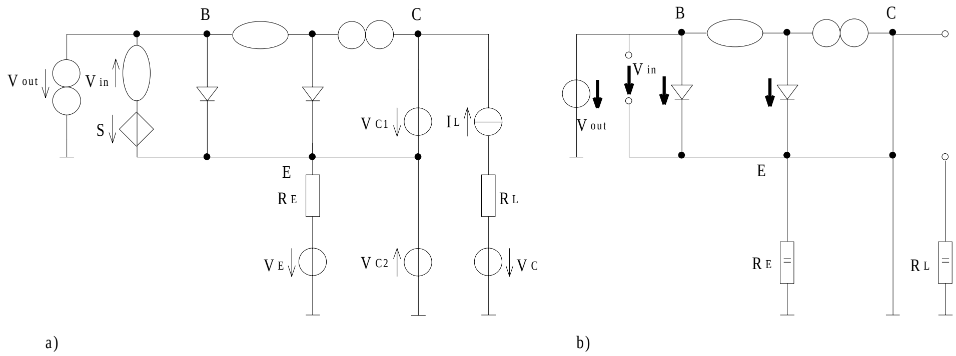


Figure 5.19: (a) Circuit structure corresponding to the inverse Colpitts system of Fig. 5.17b) the controlled current source of the transistor is modelled by an additional nonlinear resistor and a nullator-norator pair; (b) Network structure for consideration of monotonicity with the only possible voltage orientation when v_{out} is oriented, '=' indicates that the branch is not oriented

We conclude that our general structure provides a way to choose the output of a Lur'e type system so that its inverse has unique asymptotic behaviour. For the case of continuous time systems we presented new inverse systems which are, as opposed to the circuit examples published so far, two-port realizations. Typically, the circuit realizations of the inverse systems require an operational amplifier. We presented a criterion capable of establishing correct functioning of op-amp. circuits. In one example, we simulated the influence of the non ideal frequency characteristic of the operational amplifier. The resulting synchronization error was acceptable. For relative degree > 1 examples we observed by simulations that a much higher synchronization error was caused by non-ideal op-amps.