Chapter 4

Proofs of Synchronization

We are only interested in inverse systems that synchronize with the original system. According to section 3.1.3 this means that the inverse system has unique asymptotic behaviour. In other words, the asymptotic solution of the inverse system does not depend on the initial values of the $N - r$ rest states (cf. section 3.2). Asymptotic uniqueness of the solutions of the inverse system is equivalent to global asymptotic stability of the origin of the difference system, which describes the dynamics of the difference between any two solutions of the inverse system (cf. definition 3.4).

In order to prove this property one can apply either criteria of circuit theory (in case of analogue systems), system theory or simulation and practical experiments. While for linear systems unique asymptotic behaviour can be uniquely established the idea of this chapter is to sum up systematically some criteria suitable for nonlinear systems as well as to present a new approach.

In fact, for **all** synchronizaton principles vanishing influence of initial conditions has to be guaranteed. Therefore, although we lay emphasis on application for inverse systems the following criteria might be useful for other applications too which concern nonlinear systems.

4.1 Network Theory Criteria

The use of network criteria provides the possibility to establish unique asmptotic behaviour by pure inspection of the network structure. All these criteria are based on the construction of a Ljapunov function for the difference of any two solutions. Its derivative with respect to time can be shown to be negative definite by means of only the network structure, i.e. without calculation.

Next we cite two powerful propositions which guarantee unique asymptotic behaviour. Then we extend the results to connections of circuits with this feature.

4.1.1 Propositions

First we need to know the definition of some circuit features:

Definition 4.1 (Strictly Locally Passive Resistors) *A resistor is strictly locally passive if for any two points of its characteristic $(v_1, i_1) \neq (v_2, i_2)$ the following condition holds:*

$$(v_1 - v_2) \cdot (i_1 - i_2) > 0 \tag{4.1}$$

Definition 4.2 (Uniformly Locally Passivity) *A resistor (capacitor resp. inductor) is uniformly locally passive if there are constants $R_m, R_M (C_m, C_M$ resp. $L_m, L_M)$ such that for any two points of its characteristic $(v_1, i_1) \neq (v_2, i_2)$ $((v_1, q_1) \neq (v_2, q_2)$ resp $(\phi_1, i_1) \neq (\phi_2, i_2))$ the following condition holds:*

$$R_m \leq \frac{\Delta v}{\Delta i} \leq R_M \tag{4.2}$$

resp.

$$C_m \leq \frac{\Delta q}{\Delta v} \leq C_M \tag{4.3}$$

resp.

$$L_m \leq \frac{\Delta \phi}{\Delta i} \leq L_M \tag{4.4}$$

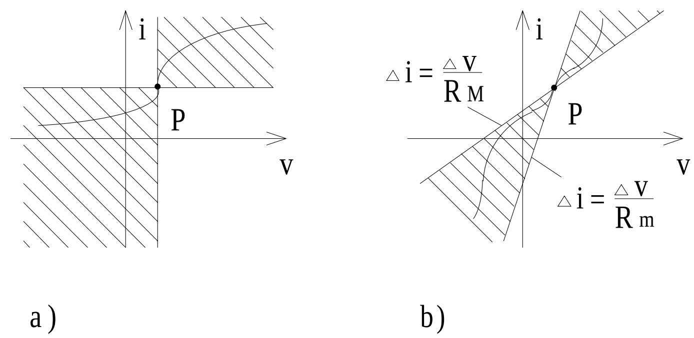

Figure 4.1: Resistor characteristics: (a) strictly locally passive, (b) uniformly locally passive: changing the axes to (q, v) resp. (ϕ, i) gives the appropriate characteristics of a capacitor and an inductor

Since all inverse systems represent driven circuits, i.e. circuits with time-dependent sources, we cite only those criteria applicable to non autonomous circuits. Note that these criteria establish a unique steady state for circuits excited by any voltage or current signal therefore also for a chaotic signal.

We assume that the circuit under consideration has a solution which exists for $t \longrightarrow \infty$. Section 2.2 of [18] provides theorems which establish this.

Proposition 4.1 (Unique Asymptotic Behaviour for Circuits with Linear Reactances) : *Consider a circuit composed of*

positive linear capacitors and inductors

resistors which are strictly locally passive

time-dependent and constant voltage and current sources

Suppose that

1. *all solutions are bounded*
2. *there are no loops of voltage sources, capacitors and inductors*
3. *there are no cutsets of current sources, capacitors and inductors*

then the circuit has a unique steady state. (For a proof see [18])

Remark 4.1 (Linear Passive Circuits)
If additionally the resistors are linear, then the circuit could be also analyzed by means of the Laplace transformation. There the feature of unique asymptotic behaviour is identical to poles only in the left hand plane. Although not all linear circuits with this feature fit in the conditions of proposition 4.1 for many of them the test whether all poles have negative real parts can be reduced to simple consideration of network elements. Actually, this we had in mind when we used the notion of linear passive circuits in section 3.3.1.

Proposition 4.2 (Unique Asymptotic Behaviour for Circuits with Nonlinear Reactances) : *Consider a circuit composed of*

uniformly locally passive capacitors and inductors with constants C_{mk}, C_{Mk} *and* L_{mk}, L_{Mk}

uniformly locally passive resistors with constants R_{mk}, R_{Mk}

time-dependent and constant voltage and current sources

Suppose that

1. *all solutions are bounded*
2. *there is no loop of voltage sources, capacitors and inductors and*
3. *there is no cutset of current sources, capacitors and inductors*

If

$$\left[\sum_{k=1}^{N_C} \frac{C_{Mk} - C_{mk}}{C_{Mk} + C_{mk}} + \sum_{k=1}^{N_L} \frac{L_{Mk} - L_{mk}}{L_{Mk} + L_{mk}}\right] \cdot \sqrt{RG} < \gamma \qquad (4.5)$$

where

$$R = \sum_{k=1}^{N_R} \sqrt{R_{Mk} - R_{mk}} \qquad (4.6)$$

$$G = \sum_{k=1}^{N_R} \frac{1}{\sqrt{R_{Mk} - R_{mk}}} \qquad (4.7)$$

$$\gamma = \min_{\forall R} \frac{1}{\sqrt{\frac{R_{Mk}}{R_{mk}}} + \sqrt{\frac{R_{mk}}{R_{Mk}}}} \qquad (4.8)$$

then the circuit has a unique steady state. (For a proof see [18])

Remark 4.2 Both proposition are only applicable to circuits without ideal op-amps., without controlled sources and therefore also without transistors. For such circuits network criteria for a unique steady state are missing so far and one can only try to apply criteria of system theory (cf. section 4.2).

4.1.2 Extension

If a voltage driven circuit possesses unique asymptotic behaviour this property remains true when another one-port with this feature, e.g. a nonlinear voltage controlled resistor, is added in parallel with the signal injecting voltage source. This is clear because the current flowing into each port and therefore also their sum depends asymptotically only on the voltage signal, Fig. 4.2. A similarly argument holds for the dual case, e.g. a current controlled nonlinear resistor is added in series with a signal injecting current source. This way the synchronization of the inverse system examples in section 3.3.1 was proven.

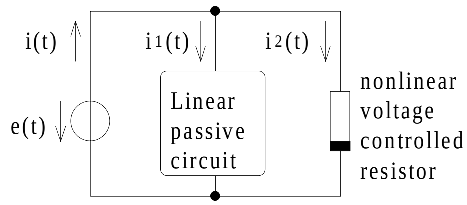

Figure 4.2: Circuit with unique asymptotic behaviour: a nonlinear voltage controlled resistor in parallel with a linear passive circuit excited by a voltage source

Remark 4.3 (Necessary Conditions for Chaotic Behaviour)
Since unique asymptotic behaviour excludes chaotic behaviour one could be tempted to derive design conditions for chaotic circuits from the above propositions. However, since all criteria based on a Ljapunov function (as propositions 4.1 and4.2) are only sufficient but not necessary and the same is valid for the relation between unique asymptotic behaviour and nonchaotic behaviour we have the following situation:

conditions of props. 4.1 **or** 4.2 **hold**	$\longrightarrow$	unique asymptotic behaviour	$\longrightarrow$	**no** chaotic behaviour

The inversion leads to:

conditions of props. 4.1 **and** 4.2 **do not hold**	$\longleftarrow$	**no** unique as. behaviour	$\longleftarrow$	chaotic behaviour

This way we have some necessary conditions for circuits to behave chaotically. May be, although they are rather week, they help to save some time for those who seek for chaotic parameters.

4.2 System Theory Criteria

While network theory criteria only consider the network structure the use of system theory criteria requires in general explicite differential equations. The cited network theory criteria establish directly unique asymptotic behaviour of driven circuits irrespective of the driving signal. In terms of system theory this amounts to prove that the difference between any two solutions of a non autonomous system vanishes irrespective of the input signal. This is to show that the zero solution of the difference system, i.e. its origin, is globally asymptotically stable.

Globally asymptotic stability implies that every solution converges to zero while asymptotic stability implies only a non empty basin of attraction. In many cases we must be satisfied by the proof of asymptotic stability. However one can hope that systems, which are not synchronized yet, enter this basin of attraction (i.e. their difference does) any time. Therefore the criteria to be cited henceforth basically establish asymptotic stability of a zero solution.

By Ljapunov's second method global asymptotic stability can be derived. However this method often fails, because it demands too restrictive ('too sufficient') conditions. Ljapunov's first method as well as our new approach guarantees only asymptotic stability. In many cases this should serve the purpose to establish synchronization, unless the basin of attraction becomes too small. This happens at the border (e.g. of a system parameter) between synchronization and desynchronization. Actually, intermittent outbursts have been observed at such border [20].

The idea of our new approach, although providing still sufficient conditions, is to determine the border between synchronization and desynchronization sharper than analytically done so far.

First we approach the term stability, then we present Ljapunovs first and second method and finally we present our new method.

4.2.1 Strong Term: Stability

There are several concepts of stability [36]:

Stability of solutions in the sense of Ljapunov: The well known δ, ε-criteria concerning the initial values and solutions of ODEs. (Additional attractivity implies asymptotic stability of solutions)

Orbital stability states the stability of an *orbit* (also called trajectory); refers to the stability of invariant sets

Stability of invariant sets concerns neighborhoods of an invariant set similar to the δ, ε-criteria. (An attractor is an invariant set. Since the definition of an attractor already implies its stability (even more!) the notion e.g. stable attractor or stable chaotic attractor is superfluous)

Note, all these definitions concern systems whose (possibly non autonomous) vector field is fixed. But the system notion we used admits a set of input signals, i.e. the right side of the describing ODEs is not fixed. In engineering applications we are also interested in features similar to the solution stability but irrespective of the input signal.

Such a notion which leaves the input signal free is the one of the **unique asymptotic behaviour**, cf. definition 3.4. Occasionally a system with this feature is called stable. Usually one is only interested in bounded unique solutions. Therefore often the solutions are additionally required to be bounded. For linear systems this can be achieved by the restriction to bounded input signals whereas for nonlinear systems one has to prove that bounded input signals imply bounded solutions.

For **linear systems** the term **stability** corresponds to poles only in the left complex plane and coincides with many features. Some of them are sometimes used as definition for stability of systems.

1. The trivial (zero) solution of the autonomous system is asymptotically stable, i.e. the 'free' system response converges to zero.
2. The solution of the forced system with zero initial states is asymptotically stable.
3. The forced system has unique asymptotic behaviour.
4. The system response is bounded for every bounded input signal.

Only some of these features are equivalent in termes of nonlinear systems too. Therefore, the term 'stable system' probably causes confusion about what it actually implies. We recommend, especially for those who grew up with linear system theory, to use the notion of an asymptotically stable solution when the ODEs, i.e. the input is fixed and to stick otherwise to the notion 'system with unique asymptotic behaviour'.

4.2.2 Ljapunov's Indirect Method - Linearization Method - (first method)

Ljapunovs first method considers a linearized system. Hence it can provide only local results.

Proposition 4.3 *Consider the system*

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) \text{ with } \mathbf{f} \text{ is } C^1 \text{ and } \mathbf{f}(\cdot, 0) = 0 \tag{4.9}$$

i.e. we have the trivial solution $\mathbf{x} = 0$ *- a fixed point. (This could correspond to the zero solution of the difference of synchronizing systems) Define*

$$\mathbf{A}(t) = \left[\frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial \mathbf{x}}\right]_{\mathbf{x}=0} \tag{4.10}$$

Assume

1. $$\lim_{\|\mathbf{x}\| \to 0} \sup_{t \geq 0} \frac{\|\mathbf{f}(t, \mathbf{x}) - \mathbf{A}(t)\mathbf{x}\|}{\|\mathbf{x}\|} = 0 \tag{4.11}$$

2. $\mathbf{A}(\cdot)$ *is bounded.*

3. 0 *is an uniformly asymptotically stable solution of the* **linearized system**

$$\dot{\mathbf{z}}(t) = \mathbf{A}(t)\mathbf{z}(t) \tag{4.12}$$

then 0 *is also a uniformly asymptotically stable fixed point of (4.9).*
(A proof is given in [41])

Remark 4.4 *Uniform* asymptotic stability means that the conditions for attractivity and stability (which imply by definition asymptotic stability) do not refer to the initial time. The conclusion of the above proposition holds for this case but not for asymptotic stability. However, this is not too restrictive, since the following criteria guarantee uniform asymptotic stability as well.

In the autonomous case the Jacobian matrix $\mathbf{A}$ is constant and the analysis stops here, because the uniform asymptotic stability can be established by negative real parts of the eigenvalues of $\mathbf{A}$.

In general we have to expect a time-variant linearized system. Its stability can be established for periodic $\mathbf{A}$ by means of characteristic (Floquet-) multipliers [36] and in general by negative conditional Ljapunov exponents or by the construction of a Ljapunov function.

4.2.3 Conditional Ljapunov Exponents

The conditional Ljapunov exponents are the Ljapunov exponents of a system *under the condition* that it is excited by a certain signal, namely in our case a chaotic signal. That means that they are the Ljapunov exponents of (4.12) evaluated along a chaotic solution.

The conditional Ljapunov exponents provide the sharpest border between synchronization and dysynchronization. Negative conditional Ljapunov exponents are necessary and sufficient for part 3 of proposition 4.3, [33]. Indeed if one Ljapunov exponent becomes zero the basin of attraction of a neighbouring solution is empty.

However, since this concept refers to the linearization along the trajectory it requires knowledge of the chaotic solution. In the sequel we will consider approaches which overcome this drawback but do not provide sharp criteria.

4.2.4 Ljapunov's Direct Method (second method)

Ljapunov's second method considers directly the differential equation (4.9) (and not its linearization). The manifest advantage is that it does not require the knowledge of solutions but inspects the vector field only.

Scalar Ljapunov Function

The basic idea of the use of a Ljapunov function is the following:
On the state space is an energy-like function $V(\mathbf{x})$ defined which assigns to each point a real value, its 'energy'. This way the statespace is characterized by equi-energy-hypersurfaces. (Remember equipotential lines or equi-altitude lines on geographic maps.) Additionally, the energy function can depend on the time $V(t, \mathbf{x})$(this corresponds to moving energy surfaces). The energy function is required to have a global minimum. (This is the countryside has one deepest point.) If at each point of state space the vector field of (4.9) $\mathbf{f}(t, \mathbf{x})$ is always directed such that the energy decreases along the system flow then the point of lowest energy is asymptotically stable. The latter fact is expressed by: The derivative of V with respect to time along the vector field is negative -or the geometrical interpretation for the autonomous case:- The inner product of the gradient of V with the vector field $\mathbf{f}(\mathbf{x})$ is negative.

First we need to know the notion of positive definite, decrescent resp. quasimonoton nondecreasing functions.

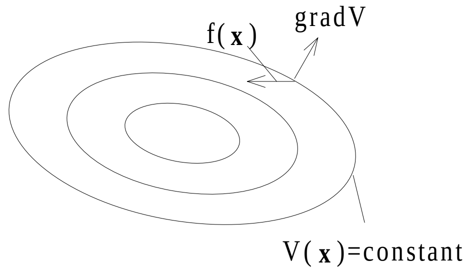

Figure 4.3: Visualization of decreasing energy along a system flow

Definition 4.3

class $\mathcal{K}$**:** *The* **class** $\mathcal{K}$ *is the class of continuous, strictly increasing functions* $f : I\!R_+ \rightarrow I\!R_+$ *with* $f(0) = 0$.

positive definite function: *A continous function* $V : I\!R_+ \times I\!R^N \rightarrow I\!R$ *with* $V(t, 0) \equiv 0$ *is said to be a* **positive definite function (pdf)** *if:*

$$\alpha(\|\mathbf{x}\|) \leq V(t, \mathbf{x}) \ \ \forall t \geq 0, \ \ \forall \mathbf{x} \in I\!R^N \tag{4.13}$$

with $\alpha(\cdot) \in \mathcal{K}$. *If the condition (4.13) holds only in a* **non empty ball** $\mathcal{B}_r$ *of* $I\!R^N$ *with* $\|\mathbf{x}\| \leq r, r > 0$ *then* V *is only a* **locally positive definite function (lpdf)**. *The feature* V *is lpdf is equivalent to* $V(t, 0) \equiv 0$ *and* $V(t, \mathbf{x}) > 0 \ \forall \mathbf{x} \neq 0 \in \mathcal{B}_r$ *[41]*

decrescent function: V *is* **decrescent** *if:*

$$V(t, \mathbf{x}) \leq \beta(\|\mathbf{x}\|) \ \ \forall t \geq 0, \ \ \forall \mathbf{x} \in I\!R^N \tag{4.14}$$

with $\beta(\cdot) \in \mathcal{K}$. *[41]* This feature is occasionally called 'V converges uniformly to zero'

quasimonotone nondecreasing function: *A continous function* $F : I\!R^m \rightarrow I\!R^m$ *is said to be* **quasimonotone nondecreasing in x** *if:*

$$\text{for } i = 1, \ldots, m : (\mathbf{x} \leq \mathbf{y} \text{ and } x_i = y_i) \longrightarrow (F_i(\mathbf{x}) \leq F_i(\mathbf{y})) \tag{4.15}$$

The inequality between vectors are understood to be componentwise inequalities [27].

Proposition 4.4 (Basic Result of Ljapunov's Direct Method)
Consider a system described by 4.9 with $\mathbf{f}(\cdot, 0) = 0$ *i.e.* $\mathbf{x} = 0$ *is a fixed point. If there exists a decrescent, called a scalar Ljapunov function,* $V(t, \mathbf{x}) : I\!R_+ \times I\!R^N \rightarrow I\!R_+ \subseteq C^1(I\!R^{N+1})$ *such that:*

$$V(t, 0) \equiv 0 \text{ and } V(t, \mathbf{x}) > 0 \ \ \forall \mathbf{x} \neq 0 \in \mathcal{B}_r \tag{4.16}$$

with:

$$\begin{aligned} \dot{V} &= \frac{\partial V(\mathbf{x}, t)}{\partial \mathbf{x}} \bullet \dot{\mathbf{x}} + \frac{\partial V(\mathbf{x}, t)}{\partial t} \\ &= \frac{\partial V(\mathbf{x}, t)}{\partial \mathbf{x}} \bullet \mathbf{f}(\mathbf{x}, t) + \frac{\partial V(\mathbf{x}, t)}{\partial t} \\ &< 0 \ \ \forall t \in I\!R_+, \ \ \forall \mathbf{x} \neq 0 \in \mathcal{B}_r \end{aligned} \tag{4.17}$$

then the fixed point $\mathbf{x} = 0$ *is uniformly asymptotically stable.* (A proof is given in [27])

There are several versions of the above proposition which require weaker or stronger conditions thus achieve weaker or stronger results e.g. global uniform asymptotic stability, cf. [41], [27].

In section 3.3.1 we already demonstrated the use of a Ljapunov function in order to establish unique asymptotic behaviour by proving the origin of a difference system to be asymptotically stable.

Remark 4.5

(a) The use of a Ljapunov function is a powerful tool. The already cited network criteria as well as well-known criteria of control theory as the famous circle criterion or the Popov criterion [41] are based on the construction of a scalar Ljapunov function the derivative of which is shown to be negative always and everywhere except in the origin (4.17).

(b) The above criterion is only sufficient and rather restrictive. The crucial point of this proposition is that the strict inequality (4.17) has to hold for all $\mathbf{x}$ and all time. This does not admit the temporary increasing of the systems energy. Therefore one cannot establish this way synchronization of difference systems which outweigh temporary expansion by appropriate contraction. Some progress (in the sense of sharper criteria compared with the mentioned conventional ones) is to be expected if such compensation is concerned as was e.g. already achieved in [5]. For temporarily expanding and converging cases the *Comparison Principle* [27] provides a suitable approach.

4.2.5 Comparison Principle for a Scalar AND a Vector Ljapunov Function

The comparison principle combines the concept of Ljapunov functions with the theory of differential inequalities. Thus a scalar or a vector equation, possibly of lower dimension than $\mathbf{x}$, is assigned to (4.9). The stability properties (e.g. uniform asymptotic stability) of the zero solution of this *comparison system* imply those of (4.9). This way much less restrictive conditions for asymptotic stability can be derived than those provided by proposition 4.4.

Comparison Principle for a Scalar Ljapunov Function

Again an energy-like function $V(t, \mathbf{x})$ is assigned to the state space. But as opposed to proposition 4.4 its derivative along the vector field is not supposed to be negative always and everywhere. Namely, a sort of worst case with respect to $\mathbf{x}$ is estimated. Hence, one still needs no knowledge of solutions but allows temporary increasing of system energy.

Proposition 4.5 (Scalar Comparison System)
Consider a system secribed by 4.9 with $\mathbf{f}(\cdot, 0) = 0$. Assume that:

1. *There exists a decrescent, positive definite function $V(t, \mathbf{x}) \,:\, I\!R_+ \times I\!R^N \to I\!R_+ \subseteq C^1(I\!R^{N+1})$ and*
2. *a function $g(t, V) \,:\, I\!R_+ \times I\!R_+ \to I\!R \subseteq C^0(I\!R^2)$ with $g(t, 0) \equiv 0$*

If

$$\dot{V} \le g(t, V(t, \mathbf{x})) \;\; \forall t \in I\!R_+, \;\; \forall \mathbf{x} \in I\!R^N \tag{4.18}$$

then the stability properties of the trivial solution of the comparison system:

$$\dot{u} = g(t, u), \; u(t_0) = u_0 \tag{4.19}$$

imply the appropriate stability properties of $\mathbf{x} = 0$ of (4.9) [27].

Note, $g(t, V(t, \mathbf{x}))$ represents an upper bound of $\dot{V}$ i.e. the worst case of energy decreasing resp. increasing for all $\mathbf{x}$. If the comparison system is a linear one:

$$\dot{u} = g(t, u) = a(t) \cdot u \tag{4.20}$$

then the uniform asymptotic stability of its trivial solution can be established by proving that positives parts outweigh negative parts of a(t).

Proposition 4.6 *Consider a scalar linear differential equation (4.20). If*

$$\int_{t_0}^{t_0+t} a(\tau)d\tau \text{ is bounded and } \longrightarrow -\infty \text{ as } t \to \infty \text{ uniformly in } t_0 \tag{4.21}$$

then the trivial solution of (4.20) is uniformly asmptotically stable.
This proposition is derived from [41].

The application of these ideas in order to establish synchronization is performed in [10].

Comparison Principle for a Vector Ljapunov Function

The use of a vector of Ljapunov functions provides the possibility to establish stability properties (e.g. uniform asymptotic stability) although each scalar Ljapunov function obeys less restrictive conditions than this in proposition 4.5. Actually, there are systems the stability of which cannot be established by a single Ljapunov function but by a set of them [27].

Proposition 4.7 (Vector Comparison System)
Consider a system secribed by 4.9 with $\mathbf{f}(\cdot, 0) = 0$*. Assume that:*

1. *There exists a vector function* $\mathbf{V}(t, \mathbf{x}) = (V_1, \ldots, V_m)^T \; : \; I\!R_+ \times I\!R^N \to I\!R^m_+ \; \subseteq C^1(I\!R^{N+1})$.
2. *There exists a nondecreasing norm* $\|\cdot\|$ *on* $\mathbf{V}(t, \mathbf{x})$*, e.g.*

$$\|\mathbf{V}(t, \mathbf{x})\| = \sum_{i=1}^{m} V_i(t, \mathbf{x}) \tag{4.22}$$

   which is positive definite and decrescent in $\mathbf{x}$.

3. *The function* $\mathbf{g}(t, \mathbf{V}) \; : \; I\!R_+ \times I\!R^m_+ \to I\!R^m \; \subseteq C^0(I\!R^2)$ *with* $\mathbf{g}(t, 0) \equiv 0$ *is quasimonotone nondecreasing in* $\mathbf{V}$ *(cf. definition 4.3)*

If

$$\dot{\mathbf{V}} \leq \mathbf{g}(t, \mathbf{V}(t, \mathbf{x})) \;\; \forall t \in I\!R_+, \;\; \forall \mathbf{x} \in I\!R^N \tag{4.23}$$

then the stability properties of the trivial solution of the vector comparison system:

$$\dot{u} = \mathbf{g}(t, u), \; u(t_0) = u_0 \geq 0 \tag{4.24}$$

imply the appropriate stability properties of $\mathbf{x} = 0$ *of (4.9) [27].*

In [27] an example is given where it is impossible to prove stability by a scalar Ljapunov function. But by the choice of a vector Ljapunov function $\mathbf{V} = (V_1, V_2)^T$ of $\mathbf{x} = (x_1, x_2)^T$ with $V_1 = \frac{1}{2}(x_1 + x_2)^2$ and $V_2 = \frac{1}{2}(x_1 - x_2)^2$ it is possible to prove stability. Note, although each function V_i is not a Ljapunov function, cf. (4.16), their sum is and thus condition 2 of proposition 4.7 is satisfied by this choice.

4.2.6 New Approach

We do not provide a new criterion but we combine several results to a hopeful tool:

1. Ljapunovs indirect method in order to deal with a linearized system
2. the comparison principles which allow expansion to be outweighed by contraction
3. the matrix measure to provide a scalar worst case estimation of contraction resp. expansion with respect to a Ljapunov function

Application of Ljapunovs indirect method $\longrightarrow$ linearized difference system

In synchronization problems we have mostly smooth right sides. Therefore one can assume a linear difference system. It describes the evolution of a small difference along the trajectory.

$$\Delta\dot{\mathbf{x}} = \mathbf{A}(t) \cdot \Delta\mathbf{x} \tag{4.25}$$

According to proposition 4.3 it is sufficient to show the uniform asymptotic stability of the trivial solution of this linearized system in order to establish a non empty basin of synchronization.

Further we assume the knowledge is available about which fraction of time the original system stays in every linearization region i.e. which part of time the difference system therefore obeys a certain matrix $\mathbf{A}(t)$. For piecewise linear systems this should be a promising technique. Recent efforts target on analytic expressions of these time fractions [30].

Application of the comparison principle

In order to apply the comparison principle we need a worst case estimation of divergence, i.e. energy increasing. In [10] this idea was applied for a one dimensional difference system. A system with dimension=1 provides the advantage that divergence and convergence occur in the same (only one) direction of state space. Thus it is easy to estimate whether contraction outweighs expansion. Here we will regard the general case of N-dimensional difference systems.

It is clear as soon as we have a worst case estimation, i.e. an upper bound of $\frac{\dot{V}}{V}$ for all $\mathbf{x}$ then we can apply proposition 4.6 with $a(t)$ replaced by this upper bound and can establish synchronization by its outweighed positive parts. In fact, the matrix measure amounts to serve this worst case estimation.

Matrix Measure

In moredimensional difference systems diverging and converging may occur in different 'directions' of state space and the estimation whether they outweigh each other is nontrivial. Therefore one regards the worst case of expansion for the whole difference state space. That means one determines the maximal divergence with respect to a chosen Ljapunov function $V(\mathbf{x})$ for all states. The *Matrix Measure induced by V* [42] serves this purpose.

Definition 4.4 (Matrix Measure) *The matrix measure μ induced by a matrix norm $\|\cdot\|$ is defined as [41]:*

$$\mu(\mathbf{A}) = \lim_{\varepsilon\to 0^+} \frac{\|\mathcal{I} + \varepsilon\mathbf{A}\| - 1}{\varepsilon} \tag{4.26}$$

Definition 4.5 (Matrix Norm) *The matrix norm $\|\cdot\|$ induce by a vector norm $\|\cdot\|$ is defined as [41]:*

$$\|\mathbf{A}\| = \sup_{\mathbf{x}\neq 0 \in \mathbb{R}^N} \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|} \tag{4.27}$$

We are familar with several vector norms such as the Euclidean or maximum norm. If a Ljapunov function $V(\mathbf{x})$ is used as norm the matrix measure turns out to be [42]:

$$\mu_V(\mathbf{A}) = sup_{\mathbf{x}\neq 0 \in \mathbb{R}^N} \left[\frac{\dot{V}(\mathbf{A},\mathbf{x})}{V(\mathbf{x})}\right] \tag{4.28}$$

where $\dot{V}(\mathbf{A},\mathbf{x})$ is the Lie derivative of V along the vector field $\mathbf{A}\mathbf{x}\,:\,L_{\mathbf{A}\mathbf{x}}V(\mathbf{x})$ (cf. appendix A). This shows that the matrix measure is exactly what we need in order to estimate the maximal expansion of a linear system described by the matrix $\mathbf{A}$ like (4.25).

Simple integration of Equ. (4.28) yields an upper bound of the difference systems energy [42]:

$$V(\Delta\mathbf{x}(t)) \leq V(\Delta\mathbf{x}(t_0)) \exp\left[\int_{t_0}^{t} \mu_V(\mathbf{A}(\tau))d\tau\right] \tag{4.29}$$

Equ. (4.29) provides the possibility to estimate whether convergence outweighs divergence: If the value of the integral in Equ. (4.29) is bounded and tends to $-\infty$ as $t \to \infty$ uniformly in t_0 then the trivial solution of the difference system is uniformly asymptotically stable. Namely, this is the essence of proposition 4.6.

I follows that in order to tell whether synchronization takes place one only needs the *mean value* of the matrix measure induced by a Ljapunov function. For one dimensional systems the matrix measure is not a worst case estimation with respect to the state space. Thus in this case the mean value provides the sharpest synchronization criterion, namely it is equal to the conditional Ljapunov exponent if the Euclidian norm were chosen as Ljapunov function. Only for a one dimensional system the mean value of the eigenvalue converges to the Ljapunov exponent (due to the only one direction of state space).

Estimation of the Matrix Measure

Departing from a quadratic Ljapunov function $V(\Delta\mathbf{x}) = \frac{1}{2}\Delta\mathbf{x}^T Q \Delta\mathbf{x}$ with the symmetric positive definite (pdf.) matrix $\mathbf{Q}$ leads to the derivative $\dot{V}(\mathbf{A}, \Delta\mathbf{x}) = \Delta\mathbf{x}^T(\mathbf{A}^T\mathbf{Q}+\mathbf{Q}\mathbf{A})\Delta\mathbf{x}$. According to (4.28) the matrix measure is the maximum value of the ratio of two quadratic functions $I\!R^N \to I\!R$. Since this is not easy to determine we suggest another upper bound estimation by means of the maximum (resp. minimum) eigenvalue $\overline{EV}$ (resp. $\underline{EV}$) of the symmetric matrices $\mathbf{S} = \mathbf{A}^T\mathbf{Q} + \mathbf{Q}\mathbf{A}$ resp. $\mathbf{Q}$.

$$\mu_V(\mathbf{A}) \leq \overline{\mu}_V(\mathbf{A}) = \overline{EV}(\mathbf{S})/\underline{EV}(\mathbf{Q}) \tag{4.30}$$

What is left to do in order to establish synchronization?

1. Best suitable choice of a Ljapunov function. This could be in case of piecewise linear systems a pdf. matrix $\mathbf{Q}$ with the smallest eigenvalue belonging to an eigenvector near to the direction of maximal expansion during 'bad' times, i.e. the eigenvector of $\mathbf{A}(t)$ with the largest (possibly positive) eigenvalue. This way the matrix measure is kept not too big during 'bad' times and has a good chance to be outweighed during 'good' times.

2. Estimation of the matrix measure by Equ. (4.30) for 'all' times i.e. for all possible matrices $\mathbf{A}(t)$.

3. Determine the mean value of the matrix measure by knowledge about the proportions of time the system Equ. (4.25) obeys the appropriate matrices $\mathbf{A}(t)$.

4. Decision whether synchronization takes place by the sign of the mean value of the matrix measure (if negative then yes).

We see that this method has several shortcomings for it is probably difficult to apply in case of not piecewise linear systems, the choice of $V(\mathbf{x})$ is still rather intuitive and it provides still only a sufficient conditition for synchronization. However, we think that it helps to determine the border between synchronization and desynchronization sharper as done so far analytically.

At least for the 'autonomous' synchronization principles (cf. section 2.2.1) this technique should be of use. To apply this method to inverse systems is to rely on that the signal features (especially this concerning the apropriate proportions of time) of the original chaotic system are not fundamentally changed by the additional influence of the input signal. This is a rather strong constraint on the input signals of the original system and according to remark 3.1 also on its output signals which are in turn the input siganls of the inverse system. However this assumption seems to be reasonable, since otherwise the information is probably not very good hidden in the chaotic signal.

Anyway it is still an open question how to choose an input signal so that the original system stays relyable in a chaotic regime. And for the change of signal features under influence of an input signal any analytically theory is missing as well (as far as I know).

4.3 Simulation and Measurements

Synchronization of the inverse system may still occur even if it cannot be proved by the criteria mentioned so far. This is due to he fact that they provide only sufficient but not necessary conditions. In this case one has to rely on simulation, even though simulation can never prove synchronization with certainty. However, some credibility is given to simulations by the fact that the statistical properties of a particular solution of a chaotic system are usually typical for almost all solutions. Therefore, a few simulation runs should give a good idea whether the inverse system synchronizes or not.

In any case, the synchronization of the inverse system should be tested by laboratory experiments in order to establish a minimal robustness against element variations, noise etc.