Chapter 3

Inverse System

3.1 Approach

Some of the transmission system examples can be treated from the general viewpoint of the inverse system concept. The idea is to control a chaotic system, the transmitter, with an information signal. The output of the transmitter, a chaotic broad band signal where the information is hidden, becomes after transmission the input of the receiver which has to retrieve the information signal. In order to do this, the receiver has to have an input-output relation inverse to that of the transmitter. Therefore we call it the inverse system. Note that both the transmitter and the receiver are nonlinear dynamic systems, the former hiding the information in chaos and the latter extracting the information from chaos.

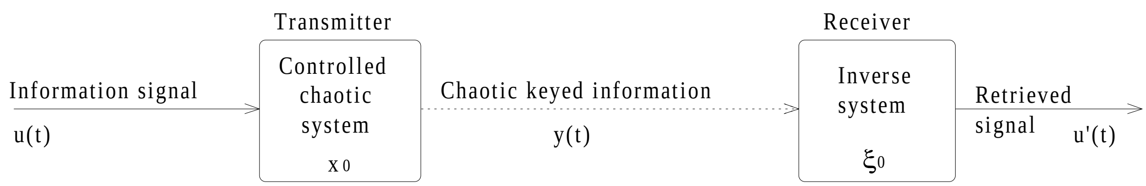


Figure 3.1: Inverse system principle

In practice, the information can only be retrieved, if the inverse system reproduces the input of the original system, at least asymptotically in time, irrespective of the initial conditions of the receiver. In this case, we say that the inverse sysnchronizes with the original system. This principle provides the exact retrieval of the original input signal under ideal transmission conditions as opposed to the other proposed methods which only approximately recover the information signal (chaotic masking) or can transmit multi-level discrete signals only (chaotic switching).

The inverse system concept applies to analog, discrete-time and digital systems as well. In this chapter a unified view on and a classification of known nonlinear inverse systems is presented.

3.1.1 Relation to the Pecora-Carroll Scheme

In section 2.2.1 we already considered the Pecora-Carroll synchronization scheme. Since according to Fig. 3.1 the inverse system is **also** driven by a transmitted signal the reader might be tempted to compare the synchronization of inverse systems with the Pecora-Carroll *driving scheme*.

We emphisize: There are fundamental differences between the excitation of an inverse system by a controlled source and the driving of a slave as proposed by [33], which are summed up in Tab. 3.1. (By Pecora-Carroll driving we refer to driving of a subsystem **and** to the error feedback approach.

<table>
<tr><th></th><th>Pecora-Carroll Scheme</th><th>Inverse System Principle</th></tr>
<tr><td>Transmitter</td><td>has no input
is autonomous</td><td>has an input
is non autonomous</td></tr>
<tr><td>The receiver
reconstructs</td><td>only the states
common to
slave and master</td><td>the non transmitted states
and retrieves
the transmitter input signal</td></tr>
<tr><td colspan="3">In terms of circuit realizations:
If an emitter state (e.g. a capacitor voltage) is transmitted</td></tr>
<tr><td>at the
receiver</td><td>the corresponding memory element
is replaced by a controlled source
or its motion is influenced
without imposing its state</td><td>the state of the memory element
is imposed by a controlled source
without replacing it</td></tr>
</table>

Table 3.1: Relation between the Pecora-Carroll scheme and the inverse system principle

3.1.2 Definitions

For the purpose of this chapter, we need a general definition of the inverse system and of its synchronization. In order to keep the concepts at a suitable level of generality, we choose an input - initial state - output description of systems rather than directly the state equations. The signals are defined on the real time interval $I\!R_+ = [0, \infty)$ for analogue systems and on $I\!N = \{0, 1, 2, \ldots\}$ for discrete-time systems. We limit the discussion to single-input single-output systems. For a given system not all signals are admissible and not any arbitrary signal can be produced at the output.

In words, the inverse system can be described as follows. The original system transforms an input signal u into an output signal y. This transformation depends on the initial state vector $\mathbf{x}(0)$ of the system at time 0. The inverse system retrieves u from y if a suitable initial state $\xi(0)$ is chosen.

Definition 3.1 (System) *A system is a transformation*

$$\Sigma \,:\, \mathcal{D} \times I\!R^N \to \mathcal{R} \tag{3.1}$$

where $\mathcal{D}$ is the set of admissible input signals, the **signal domain** *of Σ, and $\mathcal{R}$ is the set of output signals, the* **range** *of Σ. In case of an* analogue system*, the elements of $\mathcal{D}$ and $\mathcal{R}$ are continuous functions $u \,:\, I\!R_+ \to I\!R$, whereas in the case of* discrete systems*, they are arbitrary functions $u \,:\, I\!N \to I\!R$.*

In both cases, for $(u, \mathbf{x}(0))$ in the domain of Σ, $u \in \mathcal{D}$ is called the input signal*, $\mathbf{x}(0)$ the* initial state*, and y, its image under Σ, is called the* output signal *or* response *of the system. Finally, N is the* order *of the system.*

This definition is illustrated in Fig. 3.2. Instead of continuous signals for analogue systems, we could extend the signal space to discontinuous signals and even to distributions [28].

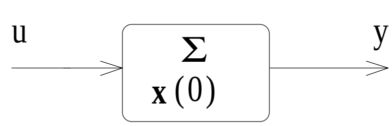


Figure 3.2: Definition of a system

Remark 3.1 According to the definition above each system has its set of admissible signals $\mathcal{D}$. This could obey any admissibility criteria one is free to specify for the system. (Obviously, this specifies also the range $\mathcal{R}$) Such admissibibility creteria could require e.g. the existence and uniqueness of the soltution for the whole future in case the system is described by ordinary differential equations (ODEs), cf. section 3.2.1. The results to be derived in this section are valid for all such specifications, because they refer only to this system definition.

Definition 3.2 (Inverse System) *A system Σ^{-1} with signal domain $\mathcal{D}'$, range $\mathcal{R}'$ and order N' is an* inverse *of the system Σ of order N, if the following conditions are satisfied*
a) $\mathcal{D}' = \mathcal{R}$ and $\mathcal{R}' = \mathcal{D}$
b) For every input signal $u \in \mathcal{D}$ and every initial state $\mathbf{x}(0) \in I\!R^N$ of Σ there exists an initial state $\xi(0)$ of Σ^{-1} such that

$$\Sigma^{-1}(y, \xi(0)) = u \tag{3.2}$$

where y is the output signal of Σ, i.e.

$$y = \Sigma(u, \mathbf{x}(0)) \tag{3.3}$$

c) For every input signal $v \in \mathcal{D}'$ and every initial state $\xi(0) \in I\!R^{N'}$ of Σ^{-1} there exists an initial state $\mathbf{x}(0)$ of Σ such that

$$\Sigma(z, \mathbf{x}(0)) = v \tag{3.4}$$

where z is the output signal of Σ^{-1}, i.e.

$$z = \Sigma^{-1}(v, \xi(0)) \tag{3.5}$$

This definition is illustrated in Fig. 3.3

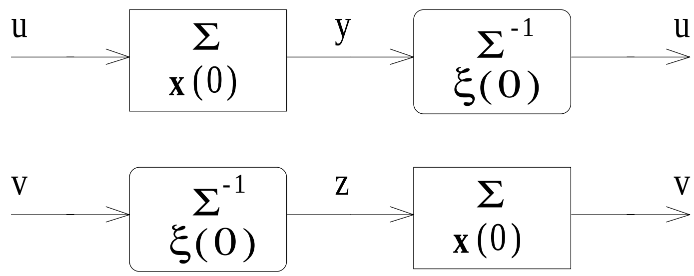


Figure 3.3: Definition of the inverse system

Remark 3.2

a) Definition 3.2 is symmetric in Σ and Σ^{-1}. Therefore, Σ is always an inverse system of Σ^{-1}.

b) One could assume that in order to possess an inverse, the system should be given by an injective mapping Σ. However, this is not necessary. Consider the differentiator. Its order is 0 and its signal domain is composed of the continuously differentiable functions on $I\!R_+$. Signals that differ only by a constant lead to the same response, and thus the differentiator is not injective. Its inverse is the integrator, a system of order 1 (and olso of relative degree 1, cf. section 3.2):

$$\int \; : \; (y, \xi(0)) \to z \tag{3.6}$$

where

$$z(t) = \xi(0) + \int_0^t y(\tau) d\tau \tag{3.7}$$

Given an output y of the differentiator, the integrator can reproduce the right input signal of the differentiator by choosing the appropriate initial condition $\xi(0)$.

c) Even though by definition the signal domain and the range of the inverse system are uniquely defined, the inverse system itself is not unique. The relation between different inverses will be clarified later in this section.

d) It is not unusual that the order of an inverse system is different from the order of the original system. The next section discusses this point in detail.

e) Note that the inverse system reproduces the input signal of the original system only if the correct initial state is chosen. For any other initial state, the output of the inverse system $u'(t)$ will be different, at least for some time. If, after some transient oscillations have died out, the inverse system always reproduces the input signal, we say that it synchronizes with the original system.

3.1.3 Synchronization versus Unique Asymptotic Behaviour of the Inverse System

Definition 3.3 (Synchronization of Inverse Systems) *The inverse system* Σ^{-1} synchronizes *with the original system* Σ *if for every input signal* $u \in \mathcal{D}$, *every initial state* $\mathbf{x}(0) \in I\!R^N$ *of* Σ *and every initial state* $\xi(0) \in I\!R^{N'}$ *of* Σ^{-1}

$$|u(t) - u'(t)| \longrightarrow 0 \text{ as } t \to \infty \tag{3.8}$$

where

$$u' = \Sigma^{-1}(y, \xi(0)) \tag{3.9}$$

and y *is given by Equ. (3.3).*

Remark 3.3

a) Definition 3.2 is *not symmetric* in Σ and Σ^{-1}. If Σ^{-1} synchronizes with Σ, it may well be that Σ does not synchronize with Σ^{-1}. This actually is the case for the systems we are interested in.

b) Synchronization can be expressed in terms of the behaviour of the inverse system alone, without extra reference to the original system. This will be the essence of proposition 3.1. This proposition relies strongly on part a) and c) of definition 3.2, namely that only such signals, which can be 'produced' by the original system, are by definition admissible to the inverse system. Thus no extra reference to the original system is needed, in order to specify the input. This and the fact, that for every admissible input at least one initial state leads to perfect signal reconstruction, allows to give the amazing statement above (cf. also to remarks 3.1 and 2.5).

Definition 3.4 (Unique Asymptotic Behaviour) *A system* $\Sigma : \mathcal{D} \times I\!R^N \to \mathcal{R}$ *has* unique asymptotic behaviour *if for all input signals* $u \in \mathcal{D}$ *and any two initial states* $\mathbf{x}(0), \mathbf{x}'(0) \in I\!R^N$ *the corresponding output signals* $y, y' \in \mathcal{R}$ *satisfy*

$$|y(t) - y'(t)| \longrightarrow 0 \text{ as } t \to \infty \tag{3.10}$$

Proposition 3.1 (Synchronization $\Longleftrightarrow$ Unique Asymptotic Behaviour) *An inverse system* Σ^{-1} *synchronizes with its original* Σ if and only if Σ^{-1} *has unique asymptotic behaviour.*

Proof:

Let $\Sigma : \mathcal{D} \times I\!R^N \to \mathcal{R}$ and $\Sigma^{-1} : \mathcal{R} \times I\!R^{N'} \to \mathcal{D}$

'Only if' part of the proposition:

Suppose that Σ^{-1} synchronizes with Σ. Consider an input signal $v \in \mathcal{R}$ and two initial states $\xi(0), \xi'(0) \in I\!R^{N'}$ of Σ^{-1}. Let the corresponding output signals of Σ^{-1} be z and z'. According to part c) of definition 3.2 for the inverse system there exists an initial state $\mathbf{x}(0) \in I\!R^N$ of Σ such that

$$\Sigma(z, \mathbf{x}(0)) = v \tag{3.11}$$

Furthermore, according to definition 3.3 for synchronization, applied to the initial state $\xi'(0)$ of Σ^{-1}, it follows that

$$|z(t) - z'(t)| \longrightarrow 0 \text{ as } t \to \infty \tag{3.12}$$

This proves that Σ^{-1} has unique asymptotic behaviour.

'if' part of the proposition:

Suppose that Σ^{-1} has unique asymptotic behaviour. Consider an input signal $u \in \mathcal{D}$ and an initial state $\mathbf{x}(0) \in I\!R^N$ of Σ and an initial state $\xi(0) \in I\!R^{N'}$ of Σ^{-1}. Let y be the output signal of Σ corresponding to u and $\mathbf{x}(0)$ and z the output signal of Σ^{-1} corresponding to y and $\xi(0)$. According to part b) of definition 3.2 for the inverse system, there exists an initial state $\xi'(0) \in I\!R^{N'}$ of Σ^{-1} such that the output signal of Σ^{-1} corresponding to y and $\xi'(0)$ is precisely u. On the other hand, according to definition 3.4 for unique asymptotic behaviour,

$$|z(t) - u(t)| \longrightarrow 0 \text{ as } t \to \infty \tag{3.13}$$

This proves that Σ^{-1} synchronizes with Σ.

In the next section, we will analyse the inverse by using a certain system transformation. For this purpose, we need the notion of equivalent systems.

3.1.4 Equivalence of Systems

Definition 3.5 (Equivalence of Systems) *Two systems* $\Sigma : \mathcal{D} \times I\!R^N \rightarrow \mathcal{R}$ *and* $\Sigma' : \mathcal{D}' \times I\!R^{N'} \rightarrow \mathcal{R}'$ *are* equivalent*, if the following conditions are satisfied a) D = D'*

b) For each input $u \in \mathcal{D}$ *and initial state* $\mathbf{x}(0) \in I\!R^N$ *of* Σ *there exists an initial state* $\mathbf{x}'(0) \in I\!R^{N'}$ *of* Σ' *such that* Σ' *excited by* u *produces the same output as* Σ*, i.e.*

$$\Sigma(u, \mathbf{x}(0)) = \Sigma'(u, \mathbf{x}'(0)) \tag{3.14}$$

c) For each input $u' \in \mathcal{D}'$ *and initial state* $\mathbf{x}'(0) \in I\!R^{N'}$ *of* Σ' *there exists an initial state* $\mathbf{x}(0) \in I\!R^N$ *of* Σ *such that* Σ *excited by* u' *produces the same output as* Σ' *, i.e.*

$$\Sigma'(u', \mathbf{x}'(0)) = \Sigma(u', \mathbf{x}(0)) \tag{3.15}$$

In this case, we shall write $\Sigma \equiv \Sigma'$ *.*

Remark 3.4

a) Conditions b) and c) imply the existence of a bijective transformation between the states of Σ and Σ'. Conversely, any transformation of the states defines an equivalent system. Note that this transformation may depend on the input signal.

For any reasonable example, this implies that the orders N and N' of the systems are the same. Otherwise the map between the states were not differentiable and the existence of a system motion describing vector field is questioned. However, often the purpose of such transformation is to elucidate certain properties of the system by features of a special equivalent system, which is required to be described by a ODEs. Actually, this is what we intend too, cf. section 3.2.1.

b) Equivalent systems have the same behaviour modulo the state transformation equivalence implies. Proposition 3.2 is one aspect of this fact. Its proof is obvious.

Proposition 3.2 *If two systems* Σ *and* Σ' *are equivalent, then* Σ *has unique asymptotic behaviour if and only if* Σ' *has unique asymptotic behaviour.*

Proposition 3.3 *All inverses of a system are equivalent. Furthermore, they either all synchronize with the original system, or they all fail to synchronize.*

Proof:

Let $\Sigma : \mathcal{D} \times I\!R^N \rightarrow \mathcal{R}$ be a system. Let $\Sigma^{-1} : \mathcal{R} \times I\!R^M \rightarrow \mathcal{D}$ and $\Sigma^{'-1} : \mathcal{R} \times I\!R^{M'} \rightarrow \mathcal{D}$ be two inverses of Σ. Consider an input signal $v \in \mathcal{R}$ and an initial state $\xi(0) \in I\!R^M$ of Σ^{-1} and let the corresponding output signal be z, i.e.

$$z = \Sigma^{-1}(v, \xi(0)) \tag{3.16}$$

We have to show that there exists an initial state $\xi'(0) \in I\!R^{M'}$ of $\Sigma^{'-1}$ such that

$$z = \Sigma^{'-1}(v, \xi'(0)) \tag{3.17}$$

According to condition c) of definition 3.2, there exists an initial state $\mathbf{x}(0)$ of Σ such that

$$v = \Sigma(z, \mathbf{x}(0)) \tag{3.18}$$

Applying now part b) of definition 3.2, there exists an initial state $\xi'(0)$ of $\Sigma^{'-1}$ such that Equ. (3.17) holds.

A similar argument can be given when the roles of Σ^{-1} and $\Sigma^{'-1}$ are interchanged. This proves that Σ^{-1} and $\Sigma^{'-1}$ are equivalent.

To prove the second part of proposition 3.3, we note that proposition 3.2 implies that either Σ^{-1} and $\Sigma^{'-1}$ both have unique asymptotic behaviour or they both do not. Finally, proposition 3.1 implies that either Σ^{-1} and $\Sigma^{'-1}$ both synchronize with Σ or they both do not.

3.2 Relative Degree

In order to find out whether an inverse system synchronizes, i.e. whether it recovers the original input u, the usual method is to investigate whether the states of the inverse system converge to those of the original system. It is natural to consider the difference system, i.e. the difference between the transmitter and the receiver states, and the global asymptotic stability of its origin. The reader might suppose that the dimension of the difference system is N, the order of the original system. However in the following we will show that the inverse system can be of lower dimensionality than the original system. In such a case, the correspondance between the states of the original and the inverse system is not obvious. Fortunately, by proposition 3.1, synchronization can be shown by establishing the unique asymptotic behaviour of the inverse system. This corresponds to considering the difference system for two solutions of the inverse system. Thus, establishing synchronization may lead to a lower dimensional difference system.

3.2.1 Analogue Systems

The notion of the relative degree and the later used state transformation originate from [21] whereas the other results are derived by us.

First Idea

As a first example, consider the system of Fig. 3.4. It consists of a pure integrator chain, a feedback and an input. Assume that the last state of the system is the output then the inverse of this system has the form depicted in Fig. 3.5. Note that it obviously has no proper dynamics anymore since it does not contain any integrator. The inverse simply realizes a static function of the input and its derivatives.

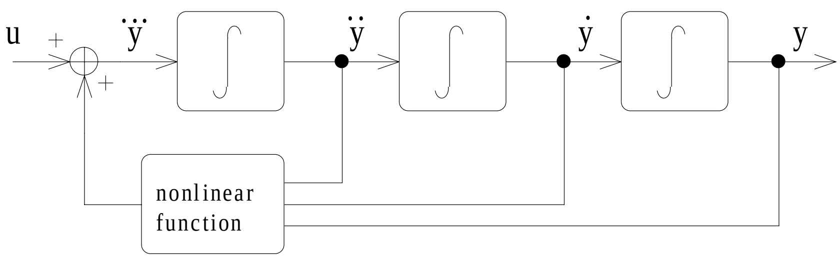


Figure 3.4: Analogue system consisting of a pure integrator chain

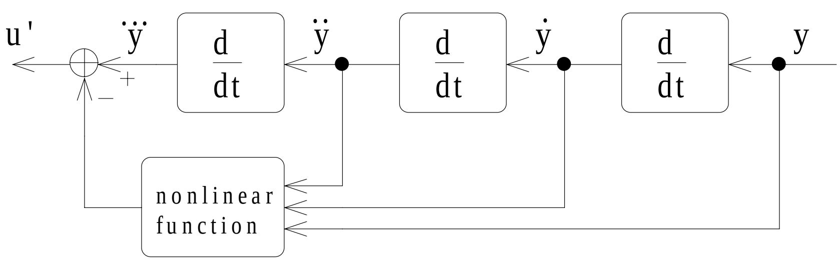


Figure 3.5: Inverse of the system in Fig. 3.4 realizing a static function of the input and its derivatives

Usually chaotic analogue systems have not such an ideal integrator chain structure. In the sequel we will determine how many integrators from a given chaotic system with chosen input and output are converted into differentiators in the inverse system. This is also the number by which the dimension N of the original system is decreased in the inverse system. It turns out that this number is independent of the particular realization of the original and the inverse system. We will show that it is the relative degree r of the original system.

Definition of the Relative Degree

In order to have a precise mathematical framework, we consider systems that are given by global state equations and we restrict our analysis to single-input single-output systems. The relative degree is defined for systems, the ODEs of which are linear with respect to the input [21]. Therefore, we will call them *inputlinear systems* in the sequel. Most examples published so far in the context of communication with chaotic signals belong to this category. The state equations of an analogue inputlinear system of order N are of the form

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \cdot u \\ y &= h(\mathbf{x}) \\ & u(t), y(t) \in I\!R^1,\ \mathbf{x}(t) \in I\!R^N \end{aligned} \tag{3.19}$$

where u is the input, y the output and $\mathbf{x}(t)$ is the state vector.

Definition 3.6 (Relative Degree) *An inputlinear system has the* relative degree r *at the point* $\mathbf{x}_0$ *if*

$$L_{\mathbf{g}} L_{\mathbf{f}}^k h(\mathbf{x}) = 0 \tag{3.20}$$

for all $\mathbf{x}$ *in a neighborhood of* $\mathbf{x}_0$ *and all* $k < r - 1$ *and*

$$L_{\mathbf{g}} L_{\mathbf{f}}^{r-1} h(\mathbf{x}_0) \neq 0 \tag{3.21}$$

where $L_{\mathbf{a}} b$ *denotes the Lie-derivative.*
See appendix A for a more detailed consideration. It turns out that:
The relative degree, r, indicates exactly which is the lowest output derivative that is directly influenced by the input, i.e. the number of times one has to differentiate the output $y(t)$ to have the input $u(t)$ explicitely appearing. Equivalently, r is the minimal number of integrations the input signal undergoes until it reaches the output.

Remark 3.5 (About the Definition of the Relative Degree)
(a) One could extend the definition of the relative degree, r, to other systems in the sense that it still indicates the lowest output derivative that is directly influenced by the input. But if the system motion does not depend linearly on the input, then r can depend on the input value. The restriction to inputlinear systems provides the possibility to determine the lowest, input-influenced output derivative irrespective of the specific input, i.e. as a feature of the system at all.
(b) Because $\mathbf{f}, \mathbf{g}$ and h are function of $\mathbf{x}$ the lowest, input-influenced output derivative, can depend on the system state $\mathbf{x}$. Namely, a 'dependence factor' can vanish at only a singular point of state space. Therefore the definition of the relative degree considers not just single points but neighborhoods. Consequently, there can be points of state space, where the relative degree is not defined. However, for sake of simplicity we restrict our further considerations to systems, the relative degree of which is defined for all $\mathbf{x}$.
(c) In case the output function h depends directly on the input signal we say the relative degree is 0.

Well-Defined Systems

In order to assure that Equs. (3.19) determine a *well-defined system*, i.e. it serves our purposes, we suppose that:

- $u \in \mathcal{D} = C^0(I\!R_+)$, i.e. that the input signals are continuous functions defined on $I\!R_+$; additionally we require them to be bounded,
- that the functions $\mathbf{f}, \mathbf{g} : I\!R^N \to I\!R^N$ and $h : I\!R^N \to I\!R$ belong to $C^\infty(I\!R^N)$, the functions that can be differentiated any number of times,
- and that the solutions of Equs. (3.19) exist for all times $t \in [0, \infty)$.

Under these conditions, the system Σ realizes a map $\mathcal{D} \times I\!R^n \to \mathcal{R} \subseteq C^1(I\!R_+)$ (continuously differentiable functions) defined by $\Sigma: (u, \mathbf{x}(0)) \to y$, where the output signal y is obtained from the unique solution $\mathbf{x}$ of the state equations with initial condition $\mathbf{x}(0)$. This follows directly from the theory of the existence and uniqueness of solutions of ordinary differential equations (ODEs) [3]. We could relax the requirements on $\mathbf{f}$and $\mathbf{g}$ to simple differentiability, or Lipschitz continuity, but the determination of the relative degree, which we require to be defined, uses repeated differentiation (cf. appendix A).

Remark 3.6 Note, the conditions we require to hold for a well-defined system ensure that the system solution exists for $t \in [0, \infty)$ and that the relative degree is defined for any input-output situation. Since the condition concerning the input signal is sufficient but not necessary it possibly constrains the domain $\mathcal{D}$ which would serve our purposes too. This constrains necessarily the range $\mathcal{R}$ as well. But since the propositions of the last section hold for any specification of the domain nothing is changed with respect to the application of the propositions.

Remark 3.7 Practical realizations of analogue systems are often circuits. But circuits are described by differential algebraic equations (DAEs). The above requirements for a well-defined system imply that the circuit motion is described by ODEs which neither have forward impass points nor have a changing order. In chapter 6 we will describe how the dimension of the state space depends on the network structure. It follows that the order of the describing ODEs can change, when circuit elements provide characteristics, the parts of which lead to different structures. Fig. 3.6 depicts a circuit example where the order of the system changes resp. where the system has a forward impass point [34]. Namely, it is an C^1-impass point, i.e. the ciruit possesses a solution but which is not bounded differentiable. In both cases the circuit system does not meet our requirements.

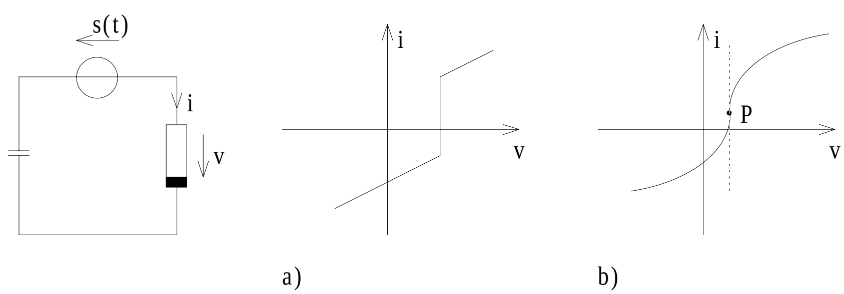


Figure 3.6: Circuit example: in case the nonlinear resistor has the characteristic (a): the outer parts of the characteristic provide a one dimensional system, while the inner part implies a zero dimensional system (the signals of which depend also on $\dot{s}(t)$); case (b): the vertical tangent in P (which corresponds to a voltage source) provides, that the dimension of the linearized circuit changes at this point. This indicates an impass point

Chain Structure and Its Inverse

Proposition 3.4 *The system of Equs. (3.19) with relative degree r is* equivalent *to the system in Fig. 3.7 where the output and its first r-1 derivatives are states.*
This proposition is a crucial result of [21]. It is based on the state transformation given in appendix B.

The transformation of states given in appendix B is a clue to the understanding why under system inversion the dimension is decreased by r. Assuming that this transformation is a diffeomorphism, an equivalent system (cf. Definition 3.5) can be considered. It has the structure of Fig. 3.7.

Since the system of Fig. 3.7 contains a pure integrator chain of length r we call it the *chain structure.* The system of Fig. 3.8 is obviously an inverse of the system of Fig. 3.7. It shows that r integrators of the original system are converted into differentiators. The non vanishing term of Equ. (3.21) is the real valued function $a(\mathbf{z})$ in Figs. 3.7, 3.8, as derived in appendix B. Thus the invertibility is guaranteed

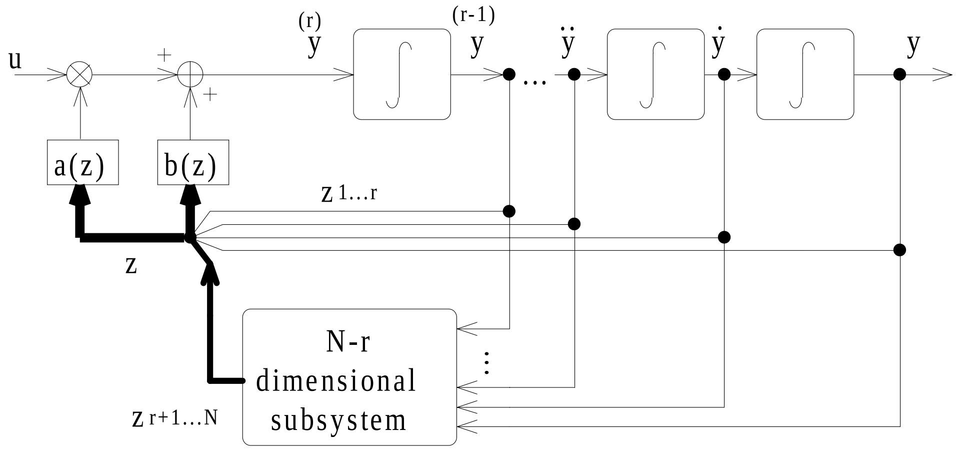


Figure 3.7: The structure into which every inputlinear system can be transformed

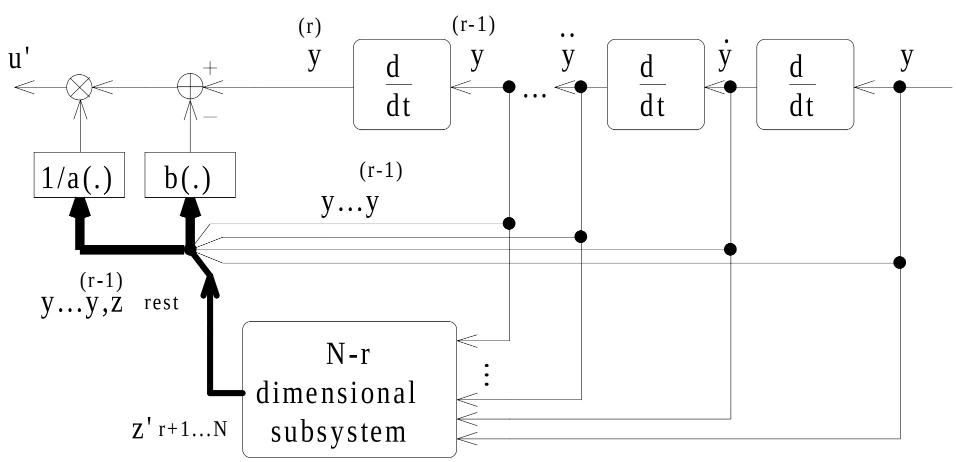


Figure 3.8: Inverse of the chain structure in Fig. 3.7 realizing a $N - r$ dimensional dynamic system

if Equ. (3.21) holds in the whole state space. Therefore we required the relarive degree to be globally defined (remark 3.5b).

The time evolution of the inverse chain structure depends on y and its first $r-1$ derivatives. This corresponds to the generalized state representation in [11]. The input is to be recovered as a function of the first r derivatives of y and the $n-r$ *rest states.* We conclude:

Proposition 3.5

1. *The inverse of a relative degree equal to r system is $N-r$ dimensional and it is therefore sufficient to consider an $N-r$ dimensional difference system in order to decide whether the systems synchronize or not.*

2. *If the relative degree of an analogue system (assumed to be represented by state equations) is not zero then its inverse system has a generalized state representation, in which the state derivatives depend also on derivatives up to the $r-1$-th order of the input y. The output is a function of the rest states and the first r derivatives of y.*

The proof follows directly from proposition 3.4, the fact, that the system structure of Fig. 3.8 is according to definition 3.2 obviously an inverse of the structure in Fig. 3.7 and proposition 3.3.

3.2.2 Discrete-time Systems

The state equations (3.19) become, in the discrete-time case

$$\begin{aligned} \mathbf{x}(n+1) &= \mathbf{f}(\mathbf{x}(n)) + g(\mathbf{x}(n)) \cdot u(n) = \mathbf{x}_{n+1}(\mathbf{x}_n, u_n) \\ y(n) &= h(\mathbf{x}(n)) = y_n(\mathbf{x}_n) \\ & u(n), y(n) \in I\!R^1,\ \mathbf{x}(n) \in I\!R^N \text{ and } f, g, h \in C^1(I\!R^N) \end{aligned} \tag{3.22}$$

Remark 3.8 (Translation of the term 'relative degree' to discrete-time systems)
Translated to discrete-time systems the relative degree gives the number of time steps the current input is delayed until it directly influences the output.

For analogue systems the relative degree is determined by repeated derivatives of h with respect to **f** and **g** at each point of state space. But in order to determine the relative degree for discrete-time systems one derives the output after repeated mapping steps $y_{n+i}(\mathbf{x}_{n+i}(\mathbf{x}_{n+i-1}(\mathbf{x}_{n+i-2}(\ldots), u_{n+i-2}), u_{n+i-1}))$ with respect to u_n. Therefore, the requirements on $\mathbf{f}, \mathbf{g}$ and h are relaxed to simple continuous differentiability in (3.22).

Note, the derivatives on $\mathbf{f}, \mathbf{g}$ were to be calculated at different points of state space, namely those following each other under the system flow. It clearly depends on the input signal and can therefore not be assigned to one point. In any case these conditions are sufficient to provide the possibility to determine the relative degree. They are by no means necessary for systems to serve our purposes (information coding - decoding).

However, we omit further consideration of these things since we will stick to zero relative degree discrete-time systems for the reasons explained below.

In the following we will show that discrete-time systems with non zero relative degree cannot be directly inverted, since, as opposed to analogue systems, there is no practical realization of an inverse of a memory element, i.e. there does not exist a causal inverse of a time delay.

Consider the simple example given in Fig. 3.9a. It has the relative degree one and clearly the inverse (Fig. 3.9b) requires an inverse of a time delay. Of course, from the application point of view, we can be satisfied by recovering a time-delayed version of the input. For this purpose, we delay in the original system all signals (Fig. 3.10) except the output. This corresponds to a shift of the time delay element over the operation node in Fig. 3.9a. The inverse of this modified system is depicted in Fig. 3.11b and does not require an inverse of a time delay. Therefore, this system is also an inverse of a system with relative degree zero (Fig. 3.11a). It follows:
Having inversion in mind it is reasonable to consider only zero relative degree discrete-time systems.

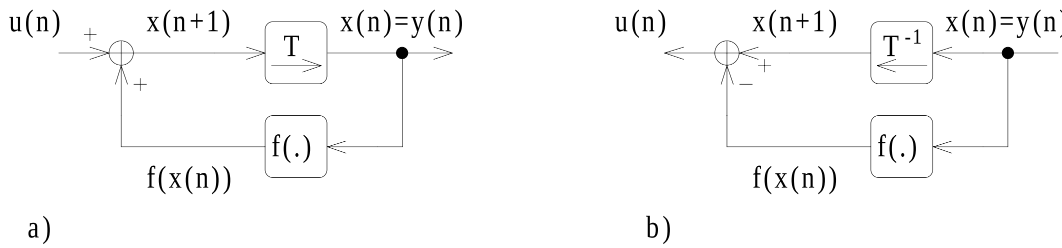


Figure 3.9: (a) A discrete-time system with relative degree 1 (b) Inverse of the system in (a), requiring the inverse of the time delay

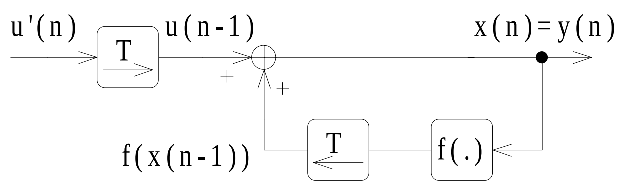


Figure 3.10: System of Fig. 3.9a with the time delay shifted over the summation node and thereby decreasing the time index of the signals at the summation node

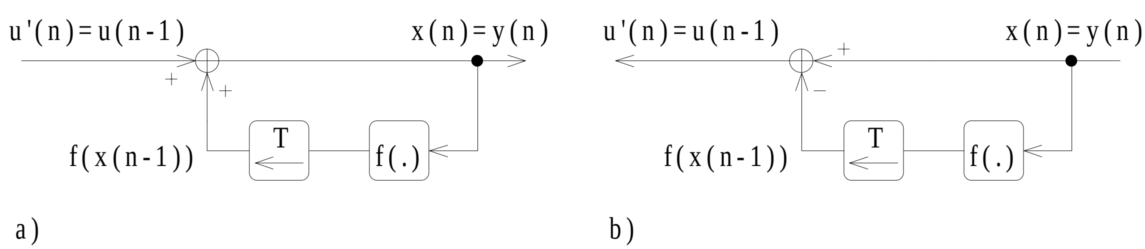


Figure 3.11: Systems with relative degree zero (a) *inverse of the inverse* of the system in Fig. 3.10 with input $u'(n) = u(n-1)$; (b) inverse of the system in Fig. 3.10 with output $u'(n) = u(n-1)$

3.2.3 Relation to the Inversion of Linear Systems

Linear systems can be described (provided they are controllable and observable) by transfer functions in the complex domain. It is well known that the inverse of a linear system is described by the inverse of the transfer function. We demonstrate this with one realization of a linear system with the transfer function:

$$G(s) = \frac{-\sum_{i=0}^{m} b_i s^i}{\sum_{i=0}^{N} a_i s^i} \tag{3.23}$$

Without loss of generality we can assume $a_N = 1$. The corresponding obeserver canonical form (all feedback comes from the output) is depicted in Fig. 3.12a.
In terms of linear systems the relative degree is exactly the difference between the degree of the denominator and the numerator polynomial of the transfer function, $r = N - m$. Clearly the system in Fig. 3.12b is an inverse of the system in a). An equivalent structure for discrete-time systems is obtained when all intergrators are replaced by time-delays.

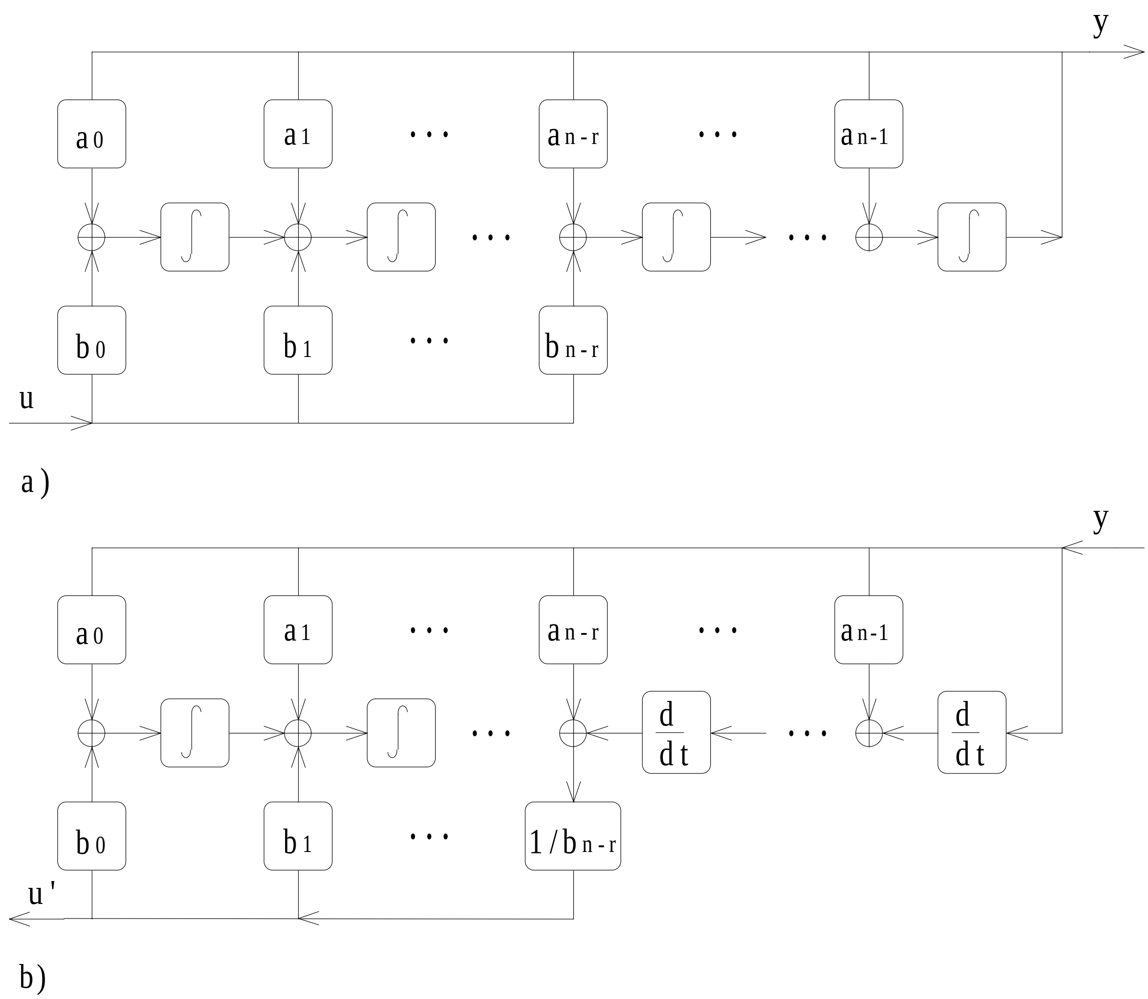


Figure 3.12: a) Observer canonical form of a linear system, b) its inverse; all integrators replaced by time delays gives the structure for discrete-time systems

The realization of the linear inverse system has the same features as just derived for the general case:
If the relative degree, r, of the original system is > 0, i.e. the degree of the denominator polynomial is larger than the degree of the nominator, then:

analogue systems: the inverse system contains r differentiators.

discrete-time systems: the inverse system is not directly realizable because of the dead time property implied by $r > 0$.

As for the relative degree in general, the difference of the degrees of the transfer function polynomials indicate the number of differentiations an input siganl undergoes until it influences the output, whereas for discrete time systems it is the number of time delays until the actual input value influences the output -its *dead time*. Consequently, the inversion of the transfer function demands non causal elements in case of discrete-time systems. The mentioned dead time property cannot occur for analog systems which are described by ODEs, because it demands systems with distributed parameters.

Unique asymptotic behaviour of the inverse system corresponds (provided it is observable) in terms of linear systems to the fact, that the **zeros** of the transfer function of the original system are situated in case of

analogue systems in the left complex half-plane, and in case of

discrete-time systems inside the unit circle.

(Such systems are minimum phase systems.) This guarantees that the inverse system is *asymptotically stable* (cf. also section 4.2.1).

3.2.4 Equivalent Approach for the Determination of the Relative Degree

Since the relative degree is the *minimal number* of integrations the input signal undergoes until it influences the output it can be usually recognised from the block diagram of the original system. As an example we consider the block diagram of [14] which represents the Chua's circuit with an input u realized by a current source in parallel to the capacitotor C_1 (Fig. 3.13).

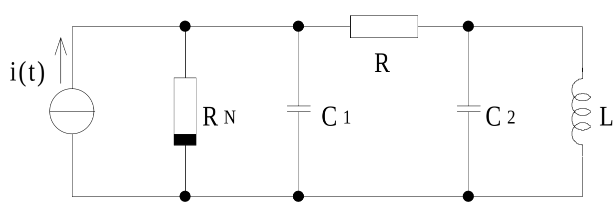


Figure 3.13: Example for a transmitter system: Chua's circuit driven by a current source

Clearly, choosing the capacitor voltage $v_{C1} \sim x_1$ as output, the minimal number of integrators between input and output is one. But choosing the other capacitor voltage $v_{C2} \sim x_2$ resp. the inductor current $i_L \sim x_3$ as output leads to $r = 2$ resp. 3 (Fig. 3.14). (We use here the customary normalisation of Chua's circuit, cf. e.g. [8]); The nonlinear static function corresponds to the nonlinear resistor)

This gives an idea that the relative degree is a feature of the *structure* of a system. That means not the specific functions on the right side of the ODE of the original system determine the relative degree but the structure, i.e. which state is influenced by the input and by which other states. In chapter 6 we will give an approach suitable for determination of the relative degree of analogue systems in terms of the structure of the circuit realization.

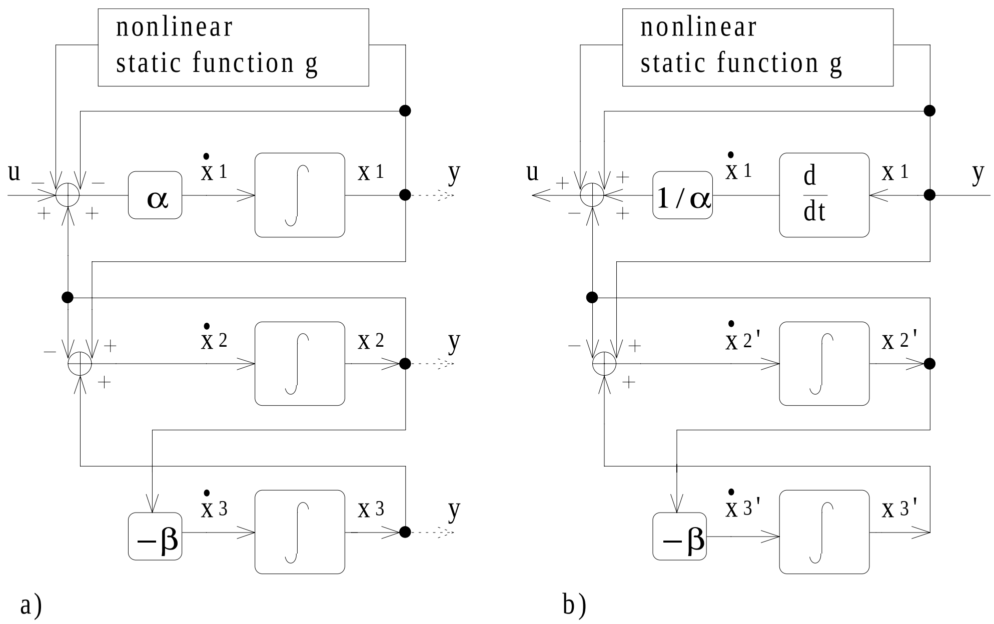


Figure 3.14: Example showing that the minimal number of integrators between input and output is converted into differentiators by system inversion, a) original, b) inverse system in case x_1 was the output of (a); It is easy to check that in case x_2 resp. x_3 were the output the inverse has 2 resp. 3 differentiators

3.2.5 Extension to Multi-Input Multi-Output Systems

Actually, it is possible to extend the approach to multi-input multi-output systems. The results are absolutely equivalent to those presented so far. The systems under consideration are assumed to be inputlinear as well, i.e. Equs. (3.19) hold with

$$\mathbf{g}(\mathbf{x}) = (\mathbf{g}_1(\mathbf{x}), \ldots, \mathbf{g}_m(\mathbf{x})) \tag{3.24}$$

$$\mathbf{h}(\mathbf{x}) = (h_1(\mathbf{x}), \ldots, h_m(\mathbf{x}))^T \tag{3.25}$$

$$\mathbf{u}(t), \mathbf{y}(t) \in I\!R^m,\ \mathbf{x}(t), \mathbf{f}(\mathbf{x}), \mathbf{g}_i(\mathbf{x}) \in I\!R^n,\ h_i(\mathbf{x}) \in I\!R^1\ i = 1, \ldots, m$$

The relative degree $\mathbf{r}$ is due to the multi-output a vector which assigns to **each output** y_i $i = 1 \ldots m$ a real number r_i which indicates that up to the $r_i - 1$-th derivative of y_i none is influenced by **any** of the input signals u_i $i = 1 \ldots m$. But the r_i-th derivative of y_i is influenced by **at least one** of the input signals u_i (the definition of the relative degree requires even more) [21].

Definition 3.7 (Relative Degree of a MIMO System) *An inputlinear system has the* vector relative degree $\mathbf{r} = (r_1, \ldots, r_m)$ *at the point* $\mathbf{x}_0$ *if for* $i = 1, \ldots, m$

1.

$$L_{\mathbf{g}_j} L_{\mathbf{f}}^k h_i(\mathbf{x}) = 0 \ \text{ for } j = 1, \ldots, m \text{ and } k < r_i - 1 \tag{3.26}$$

for all $\mathbf{x}$ *in a neighborhood of* $\mathbf{x}_0$

2. the $m \times m$*-Matrix*

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} L_{\mathbf{g}_1} L_{\mathbf{f}}^{r_1 - 1} h_1(\mathbf{x}) & \cdots & L_{\mathbf{g}_m} L_{\mathbf{f}}^{r_1 - 1} h_1(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ L_{\mathbf{g}_1} L_{\mathbf{f}}^{r_m - 1} h_1(\mathbf{x}) & \cdots & L_{\mathbf{g}_m} L_{\mathbf{f}}^{r_m - 1} h_1(\mathbf{x}) \end{bmatrix} \tag{3.27}$$

is nonsingular at $\mathbf{x} = \mathbf{x}_0$

Equivalent to the SISO-system case there exists a transformation of states leading to a chain structure depicted in Fig. 3.15, if additionally the *span*$\{\mathbf{g}_1, \ldots, \mathbf{g}_m\}$ is *involutive* [21]. Every output y_i corresponds to a chain of length r_i and there are a few rest states.

The Matrix $\mathbf{A}(\mathbf{x})$ represents under the condition (3.26) the Jacobian matrix $\frac{\partial(\overset{(r_1)}{y_1},\ldots,\overset{(r_m)}{y_m})^T}{\partial \mathbf{u}}$. Its position in the chain structure clarifies why the definition of the relative degree requires $\mathbf{u}, \mathbf{y}$ to be of the same dimension m. In fact the nonsingularity of $\mathbf{A}(\cdot)$ provides sort of a controllability of the original system. On the other hand it allows the inversion of the matrix and therefore of the whole system, Fig. 3.16.

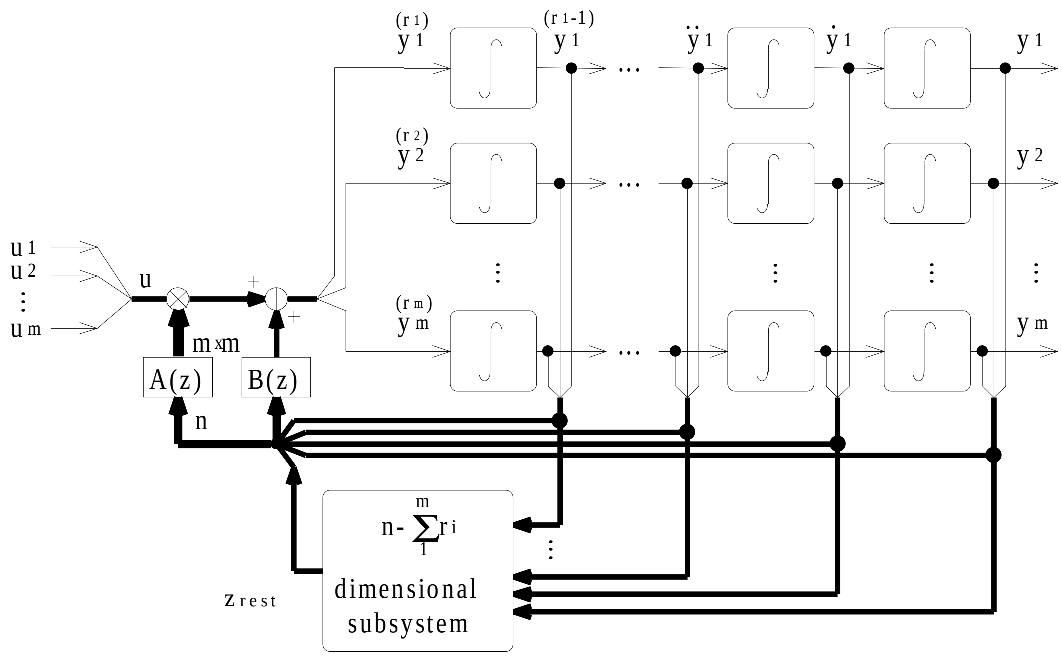


Figure 3.15: System structure into which every inputlinear MIMO system with relative degree $\mathbf{r} = (r_1, \ldots, r_m)$ can be transformed

If the number of inputs N_u were smaller than the number of outputs m or which is equivalent the $rank(\mathbf{A}) < m$ then not every *control signal* $\in \mathbb{R}^m$ could be 'produced' by $\mathbf{A} \cdot \mathbf{u}$.

If N_u was bigger than m several input signals produce the same *control signal*. In terms of information coding this corresponds to a loss of information.

In Fig. 3.17 we present a MIMO-circuit and its inverse. It is an $\mathbf{r} = (0, 1)$ system. In the inverse system every original input branch is converted into a norator and every original output branch involves a nullator. This clarifies in terms of circuits why system inversion requires the number of inputs to be equal to the number of and outputs. Otherwise the inverse network had a different number of nullators and norators which implies its singularity.

3.3 Classification and Analysis of Inverse System Examples

In this section we classify inverse system examples published so far with respect to their relative degree, the kind of their inverse system and their circuit realization method (in case of analogue systems). The corresponding state equations can be read out of block diagrams or one could consult the references.

3.3.1 Circuit Realizations

We dealt with dynamical systems by means of block diagrams so far. We now show how the inverse system can be realized by electrical circuits. All examples of this section realize the inversion by treating

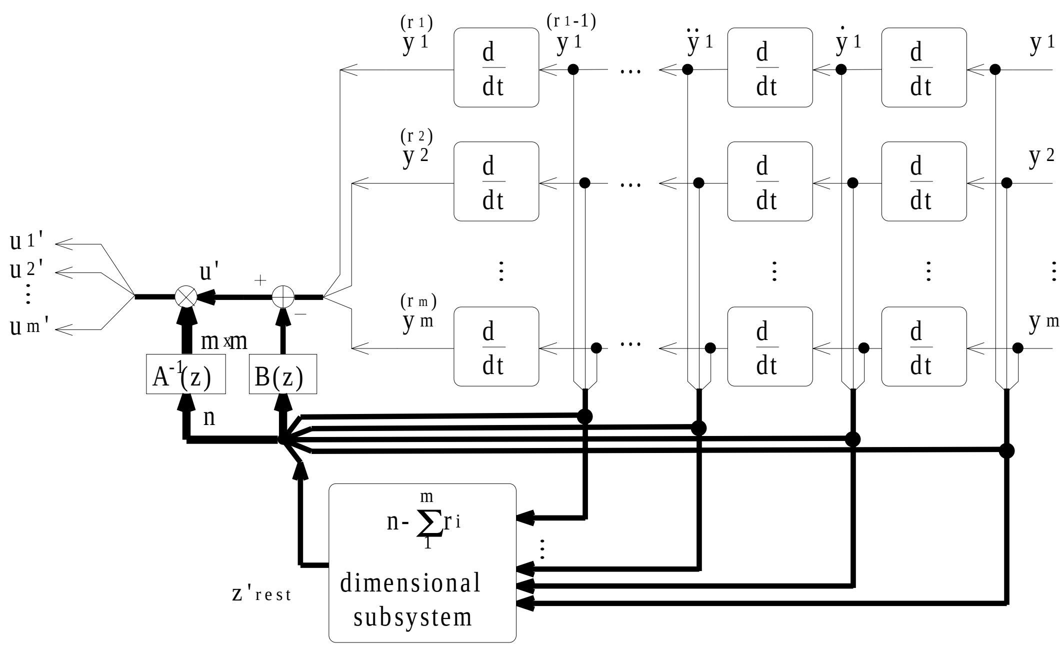


Figure 3.16: Inverse of the system in Fig. 3.15 realizing a $N - \sum r_i$-dimensional system

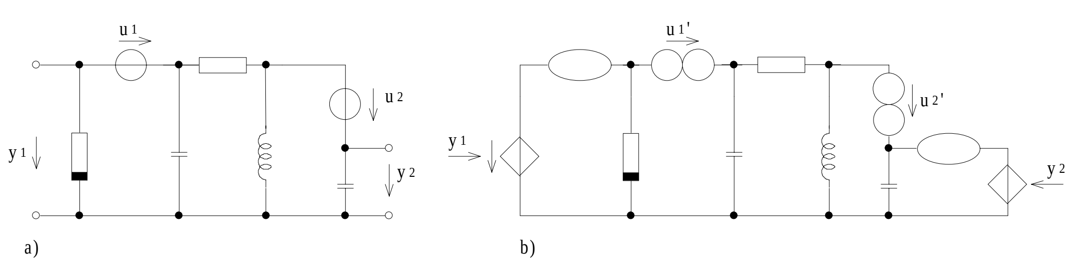


Figure 3.17: Example of a MIMO system and its inverse: Chua's circuit with $\mathbf{r} = (0, 1)$

current and voltage of a one-port alternatively as input and output. A one-port which is driven with a (information bearing) voltage as input and whose current is taken as output can be inverted by driving it with a current source and by using the port voltage as output (Fig. 3.18a). We call this the $v \to i \to v$ method and the opposite case the $i \to v \to i$ method. If synchronization takes place, the voltage on the current source is a copy of the information signal.

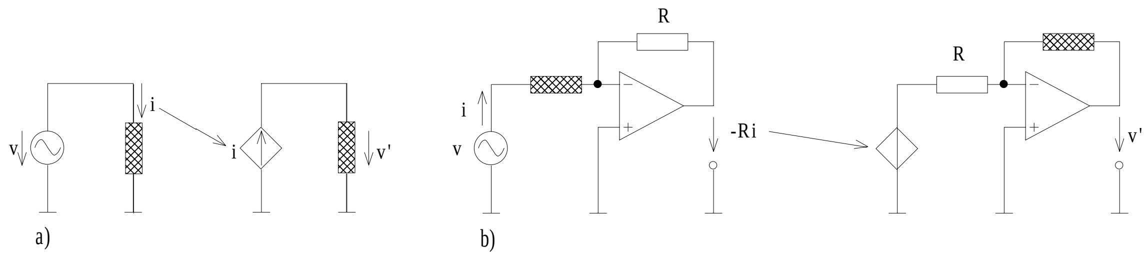


Figure 3.18: one-port Inversion (a) the $v \to i \to v$ method (b) circuit realization: The hatched box represents the nonlinear dynamical one-port, whereas the white box is a reference resistor R

Remark 3.9 Note, circuit inversion is achieved by use of op-amps, e.g. the op-amps in Fig. 3.18b serve as current $\to$ voltage resp. as voltage $\to$ current converters. Each op-amp. is supposed to work as an ideal operational amplifier, i.e. as a nullator-norator pair. In this section we consider synchronization only for the *ideal inversion* i.e. the ideal op-amp. case, whereas section 5.2.2 is devoted to the influence of nonideal op-amps.

RLDiode Circuit Example - Experimental Results

The RLDiode one-port (Fig. 3.19) is used, which produces a chaotic current if excited by a periodical voltage [29]. In order to transmit information the periodic driving voltage can be modulated. A signal corresponding to the chaotic one-port current is to be transmitted. This example illustrates the $v \to i \to v$ method a circuit realization of which is depicted in Fig. 3.18b.

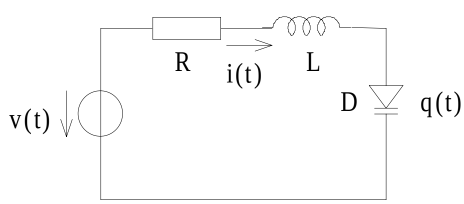


Figure 3.19: RLDiode-circuit

Next we determine the relative degree of this example, establish its synchronization and present experimental results.

Relative Degree:
The diode is modelled by a nonlinear resistor and a nonlinear capacitor connected in parallel. We suppose that both the nonlinear resistor characteristic $i_d(v_d)$ and the nonlinear capacitor characteristic $v_d(q)$ are strictly increasing. The circuit motion is described by the state equations:

$$\begin{aligned} \dot{q} &= i - i_d(v_d(q)) \\ \dot{i} &= \frac{v - v_d(q) - R \cdot i}{L} \end{aligned} \tag{3.28}$$

(3.29)

where the one-port voltage v is the input and its current i is the output.

Here we will determine the relative degree by pure inspection of the block diagram. The determination according to the definition by means of Lie-derivatives is performed in appendix A.1 and leads to the same result.

The block diagrams of the original and the inverse system are depicted in Fig. 3.20 and Fig. 3.21. They show clearly that this example has relative degree $r = 1$, since between input and output there is one integrator, namely the inductor, which is converted into a differentiator in the inverse system. Therefore the inverse system is a nonlinear dynamic one-port of order $n - r = 1$.

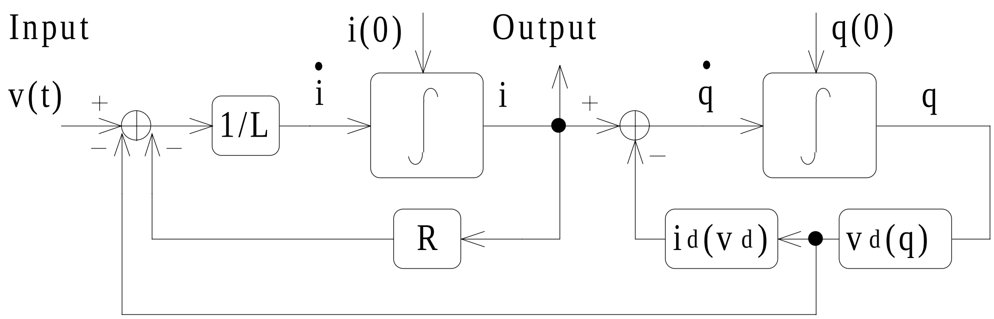


Figure 3.20: Block diagram of the RLDiode circuit where i is the inductor current, q is the charge of the diode capacity and $v_d(q), i_d(v_d))$ are the characteristics of the nonlinear capacitor and the nonlinear resistor of the diode

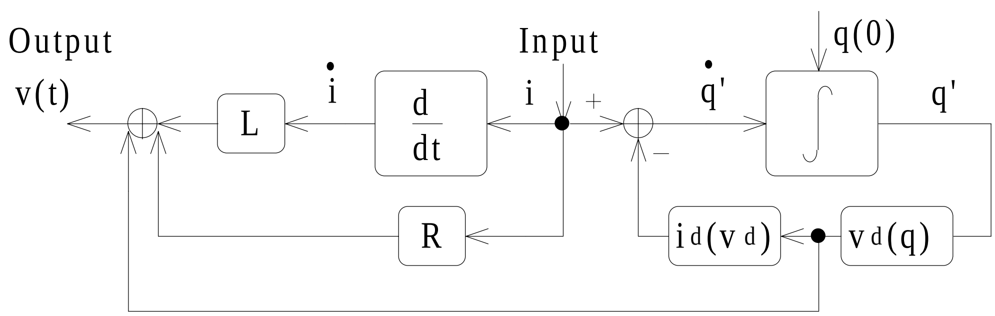


Figure 3.21: Inverse system of Fig. 3.20

Synchronization:
According to proposition 3.1 we have to establish unique asymptotic behaviour of the inverse system in order to assure synchronization, i.e. the perfect recovering of the information signal.

In order to prove unique asymptotic behaviour we simply apply a Ljapunov function for the one dimensional difference system:

$$V(q_1, q_2) = \frac{(q_1 - q_2)^2}{2} \tag{3.30}$$

$$\dot{V}(q_1, q_2) = -(q_1 - q_2)[i_d(v_d(q_1)) - i_d(v_d(q_2))] < 0 \text{ for } q_1 \neq q_2 \tag{3.31}$$

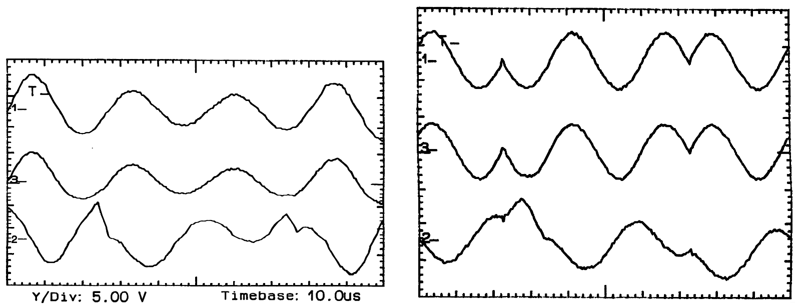
Since the nonlinear resistor characteristic $i_d(v_d)$ and the nonlinear capacitor characteristic $v_d(q)$ are strictly increasing, the function $i_d(v_d(q))$ is also strictly increasing, which implies (3.31)

Experimental Results:
We present some experimental resuts which will show that:

1. The principle works with a certain robustness against parameter mismatch between transmitter and receiver.

2. The information can be fairly well hidden in the transmitted broad band signal.

Fig. 3.22 shows experimental results under laboratory conditions. The signal retrieved at the receiver coincides quite good with the original input signal. But the transmitted chaotic signal seems to be uncorrelated to the information signal. These facts are also confirned by Fig. 3.23. The frequency spectra of the information and the transmitted signal are depicted in Fig. 3.24. Although the main frequency is still distinguishable in the transmitted signal the due to the amplitude modulation information bearing part is not detectible there.



(a) (b)

Figure 3.22: Experimental results of the RLDiode inverse system realization: wave forms: ch 1: input (a) AM-signal, (b) PSK-signal; ch2 (below): transmitted chaotic signal; ch3 (middle): retrieved signal

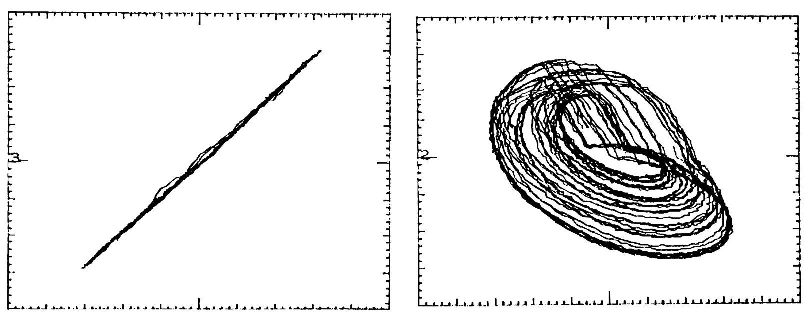


Figure 3.23: left: a fairly good transfer characteristic: retrieved signal versus input signal; right: transmitted signal versus input signal, when the information was an AM signal

Saito and Chua's Circuit Example

Saito Circuit Example:
Another inverse system realization is proposed in [17]. It treats the Saito-circuit as a one-port (Fig. 3.25).
The output of the transmitter is the one-port voltage v_{out}:

$$v_{out} = -r(i - i_L) \tag{3.32}$$

Equ. (3.32) indicates clearly that it is a zero relative degree system since the input i directly influences the output (cf. remark 3.5c).

The circuit as given in Fig. 3.25 does not have a conventional state representation because the hysteretic element does not provide a single valued function. For such systems the synchronization

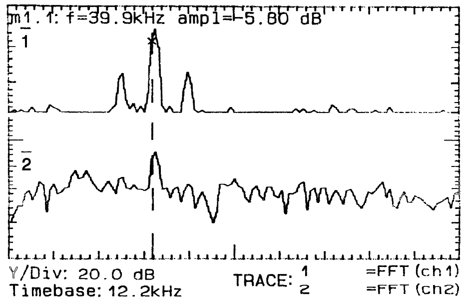


Figure 3.24: Frequency spectra showing a fairly well hidden information in the tansmitted signal: ch1-input AM-signal, ch2-transmitted chaotic signal (representing the current)

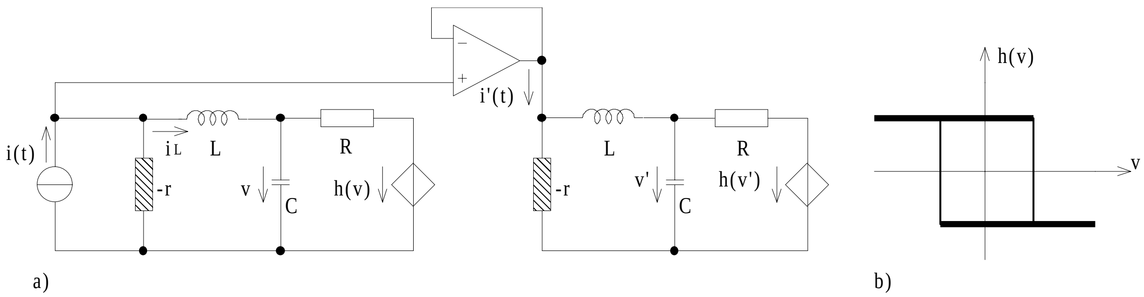


Figure 3.25: (a) The Saito circuit treated as one-port and its inverse from [31] (b) the nonlinear element of the Saito circuit: the hysteretic voltage controlled voltage source

cannot be established directly by means of a unique asymptotic solution of an ODE because it is not described by such. However, one can consider the system with an additional state introduced by a one-pole model of the operational amplifier which realizes the hysteretic 'function' h(v) (cf. [37]). But even for this extended system synchronization is difficult to prove analytically. Nevertheless, synchronization has been observed by simulation in this example.

Examples using the Chua's Circuit:
The following three realizations of inverse systems use Chua's circuit. Their characteristics are given in Tab. 3.2.

Example	Relative degree	Realization method	Kind of the inverse system
[14]	1	$i \to v \to i$ one-port	passive linear circuit ∥ nonlinear vc.resistor - voltage driven -
[22]	1	$i \to v \to i$ one-port	passive linear circuit ∥ nonlinear vc. resistor - voltage driven -
[43]	0	$i \to v \to i$ $v \to i \dots i \to v$ 2 times one-port:	1. nonlinear vc. resistor; 2. passive linear circuit in series with a nonlinear cc. resistor 1. - voltage driven - ; 2. - current driven -

Table 3.2: Features of the examples using the Chua's circuit; vc., cc. resp.∥ stand for voltage-, current controlled resp. for 'in parallel with'

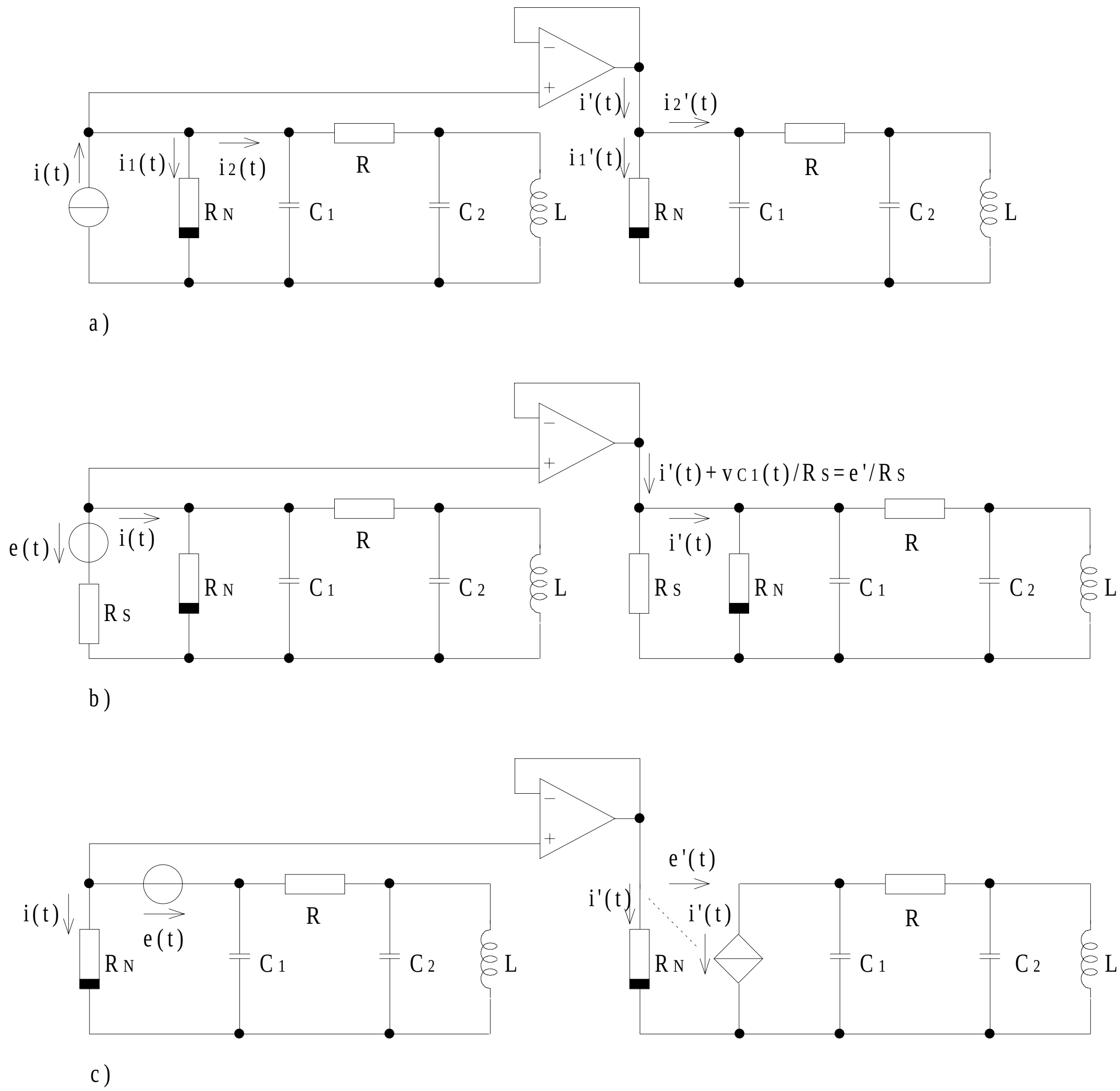


Figure 3.26: Three realizations of inverse systems with Chua's circuit: a) from [14], b) from [22] and c) from [43]

Relative Degree:
All examples represent one-port realizations of inverse systems (see below for detailed argumentation). But the relative degree is either 1 or 0. This is due to the different relations between one-port voltage and current in the system descriptions.

As already discussed in section 3.2.4 the block diagram of example [14], Fig. 3.14, reveals that there is one intergrator between the one-port current and its voltage. The [22] example is somewhat equivalent. The signal is injected by a source voltage $e(t)$. The current through the voltage source and the resistor R_s is

$$i(t) = \frac{e(t) - v_{C1}}{R_s} \tag{3.33}$$

Imagining that this current is injected by a source, we have the same situation as before. The $i \to v \to i$-method is applied at the same one-port. Thus the relative degree is also 1. The determination of the relative degree according to its definition 3.6 stops at the fist step, i.e. already $L_{\mathbf{g}}h(\mathbf{x}) \neq 0$.

The [43] example is not an inputlinear system. However, the extension of the definition of the relative degree according to remark 3.5a) allows to establish a zero relative degree. It is obvious again with the output equation: $y(t) = v_{C1}(t) + e(t)$, i.e. the input directly influences the output. This holds also when the one-port current was the output because it is related via the static nonlinear resistor characteristic to the voltage.

Synchronization:
According to the zero resp. one relative degree of these original three dimensional systems the inverse systems are 3 resp. 2 dimensional. However here we do not consider state equations in order to establish synchronization but will argue with circuit theoretic ideas.

For the first two examples, which represent the $i \to v \to i$- method, the inverse system is a voltage driven one-port, which consists of a voltage controlled resistor *in parallel* with a linear passive circuit. Thus, the currents of both parallel branches have a unique steady state and their sum, the one-port current as well. This proves that the system synchronizes and the current i is asymptotically retrieved. In case of the [22] example by adding the resistor R_s in parallel to the one-port, the one-port current becomes $e'(t)/R_s$ (cf. Equ. (3.33)) which is proportional to the desired information signal e(t).

The [43] example can be considered as the dual case to the [14] example in the sense that a one-port consisting of a nonlinear current controlled resistor (if the well known 3-segment characteristic is assumed) *in series* with a linear passive circuit is excited by a voltage signal $e(t)$. The direct inverse would be the same series circuit with a signal injecting current source $i(t)$, replacing the voltage source $e(t)$. However, the practical realization of the nonlinear resistor has a 5-segment characteristic which is not current controlled. Therefore the current that is to be transmitted in the $v \to i \to v$ method is first converted into a voltage by another one-port, namely the nonlinear resistor. This realizes a unique map when the system evolves within the range of the 3-segment charcteristic as is provided in this realization. This voltage is transmitted and reconverted into a current at the receiver by the nonlinear resistor. Again, the output of the inverse system is a sum of signals that are asymptotically identical to the corresponding signals in the original system, as discussed above for the dual case (currents instead of voltages).

Note, the nonlinear resistor in Chua's circuit is supposed to be voltage and current controlled under the above mentioned restriction. We noted always only that property, which was just necessary to establish unique signals.

3.3.2 Discrete-time System Realizations

All discrete time system examples to be considered here contain the chain structure of Fig.3.27. Since the input immediately influences the output the zero relative degree is evident.

According to section 3.2.2 it must be a zero relative degree structure in order to be invertible. There is only one exception: [26] with relative degree 1, cf. Fig. 3.28.

As explained in section 3.2.2 the system from [26] is invertible by converting it into a zero relative degree structure, i.e. by delaying the input u and the state x_{N-1} another time in the original system. This corresponds again to a shift of the delay element between input and output over the summation

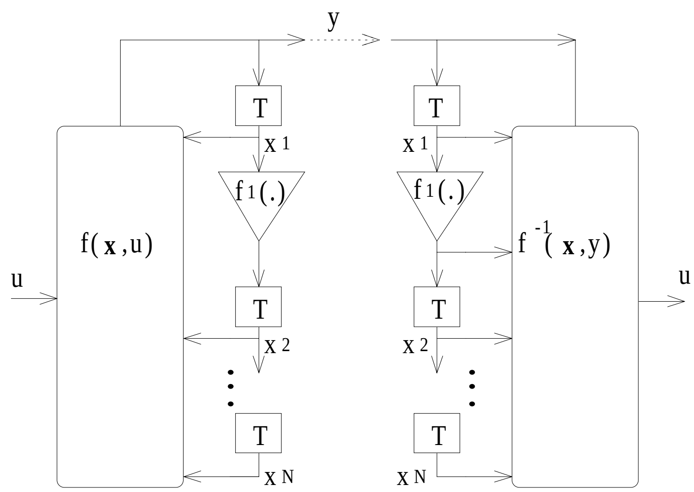


Figure 3.27: Chain structure and its inverse of the discrete time system examples; $f(\mathbf{x}, u)$ has to be invertible with respect to u

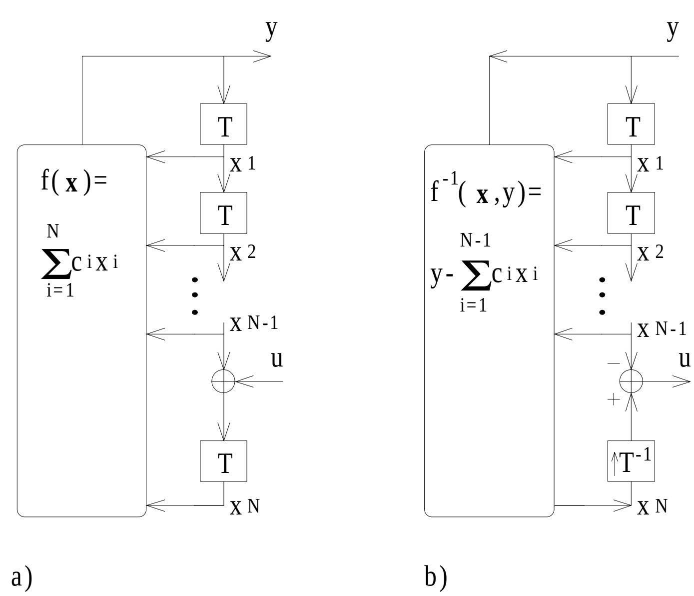


Figure 3.28: (a)Structure from [26] with relative degree 1 (therefore not directly invertible) (b) its 'inverse'

node thereby decreasing the time index of the signals there. This way only the time delayed input is retrieved in the inverse system and it would have been reasonable to choose a zero relative degree system right from the beginning.

In Tab. 3.3 the features of the discrete-time system examples are listed.

Example	Relative degree	Inverse system	State space	Used map: $f(\mathbf{x},u)=y$
[7]	0	non recursive	$I\!R^1$	logistic map $(x)+u$
[4]	0	non recursive	$I\!R^2$	mod. Henon map: $bx_2+1-ux_1^2$
[13]	0	non recursive	$\{I\!N(\mathrm{mod}\,2^n)\}^2$	$\{u+\sum_{i=1}^{2} c_i x_i\}(\mathrm{mod}\,2^n)$
[38]	0	non recursive	$\{I\!N \bmod p\}^N$	$\{u+\sum_{i=1}^{N} c_i x_i\} \bmod p$
[26]	1	non recursive	$\{I\!N \bmod p\}^N$	$\{\sum_{i=1}^{N} c_i x_i\} \bmod p$

Table 3.3: Features of the discrete time systems; p is a natural number

In the last three examples the transmitted signal y(k) cannot be chaotic due to the finiteness of the state space. However, when the state space is sufficiently large, these pseudo-random signals are very similar to chaotic signals [24]. A special case which is interesting for digital realization is the use of binary signals leading to a shift register structure. The coefficients $c_i (i = 1...N)$ have to be chosen in such a way that a maximum sequence is obtained in the autonomous case $u \equiv 0$. This aspect and the spectral properties of the transmitted signal y are discussed in detail in [38].

The main feature of all these systems is that their inverse is non recursive, i.e. what is called FIR systems when they are linear. The extension of FIR to nonlinear systems might be called deadbeat systems according to [1]. This means that after a *finite time* the influence of the initial conditions vanishes and the signal u is exactly recovered.