Chapter 2

Synchronization

2.1 Conventional and Chaotic Synchronization

Let us understand the word SYNCHRONIZATION as its direct translation Syn = with or common and Chronos = time. This could mean that two systems, each with its own time regime when uncoupled, come by any interaction between them to a common time regime.

Remark 2.1 It is understood that as soon as there is an interaction between the two systems it is impossible to speak about *two* separated systems any more. However, for sake of simple readability we will call the two interacting *partial systems* of the *composite system* in the sequel 'the two synchronized systems'.

Here the word synchronization has two meanings: the process and its result. The synchronization process i.e. the interaction between the two systems may be realized via unidirectional or bidirectional influence of the systems. This corresponds to external and mutual synchronization. In case of unidirectional interaction the terms *Master-Slave* are suitable. The slave turns out to be driven by a certain signal. In general the slave does not only perform a static function of the driving signal. We assume the slave to have its own dynamics i.e. the slave is a dynamic system.

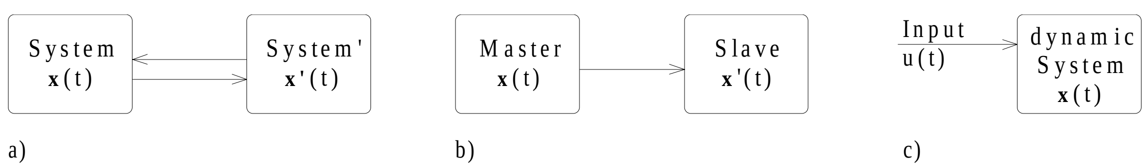

Figure 2.1: Illustration of (a) bidirectional and (b) unidirectional interaction between systems which corresponds to mutual resp. external synchronization; (c) driving of a dynamic system

We are familiar with synchronization of periodic signals so far. There synchronization leads to a certain relation between the characteristics of the periodic motions of the synchronized systems. This relation could be e.g. a constant phase difference (PLL) or a certain scaling between the system variables, Fig 2.2.

It follows that *conventional synchronization* stands for the fact that two systems get the same basic frequency. In other words: The time regime of the slave is effected by the influence of the master so that a certain periodic solution is stabilized. Or in case of mutual synchronization: The composite system has a stable periodic solution.

Remark 2.2 To have the same basic frequency does not mean that both systems have necessarily the same or similar time wave forms. For instance a periodic impulse signal could force a slave oscillator to a saw tooth signal.

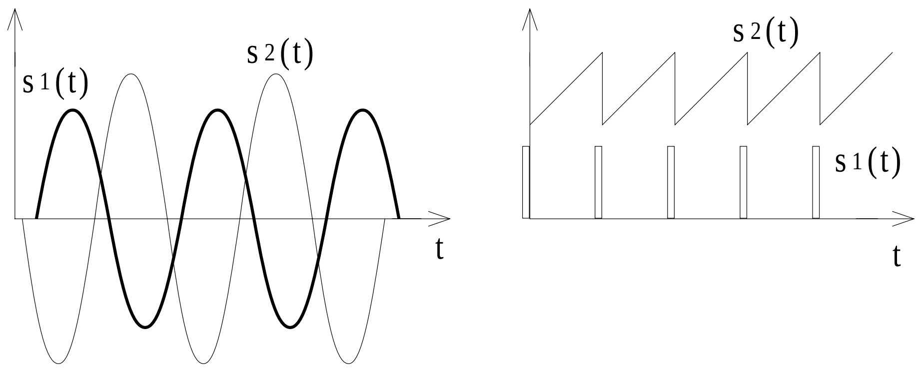

Figure 2.2: Examples for relations between signals $s_1(t)$ and $s_2(t)$: a) constant phase difference, b) constant scaling

However, in case of chaotic, i.e. nonperiodic, signals the terms basic frequency and phase loose their meaning. In the following we will explain, how the term synchronization applies to chaotic systems.

In [2] it is extended to chaotic motions which still have a sort of 'basic frequency' (a singular frequency which is distinguishable in the frequency spectrum). There synchronization indicates that the 'basic frequency' of a chaotic system is influenced.

In general it is delicate to speak about 'basic frequencies' in chaotic signals. However, it is not unusual that the power spectrum of a chaotic signal contains peaks at a certain frequency and its harmonics either due to a harmonic driving signal of a nonautonomous chaotic system (as for the RLDiode circuit cf. section 3.3.1) or due to certain eigenfrequencies (as for the modified oscillator with inertial nonlinearity [2]).

Remark 2.3 (Two basic types of chaotic attractors)
There exist from the viewpoint of oscillatory behaviour and attractor geometry at least two basically different types of chaotic attractors in 3-dimensional autonomous systems both of which have an eigenfrequency detectable in their signal spectra. (Beside these two exist certainly many other attractor types.)

The first, called double scroll type, has **two** oscillation planes (not necessarily parallel) where in both increasing oscillations take place. Here the third dimension provides the possibility to change between the oscillation planes which also serves as the (for bounded motion) necessary 'reset' of oscillation. The pure *jump and reset principle* is performed by the Saito circuit [37].

The second, called Rössler type, has **one** oscillation plane to which the oscillation is always more or less parallel. Here the third dimension (may be a state) serves as control of this oscillation and provides interchanging of dampening and undampening i.e. increasing and decreasing oscillations (as a continous function of this state). Therefore the notion *amplidude controlled oscillator* describes perfectly the principle which is realized by the 'oscillator with inertial nonlinearity' [2].

May be, many chaotic attractors of the mentioned system class can be classified this way. However, further consideration of this attractor structure point of view is beyond this work. We only wanted to explain the 'basic frequency' property.

Being forced by a periodic signal the basic frequency of a chaotic motion can be changed into the driving frequency. This process can be called synchronization although the motion becomes not necessarily periodical but can also stay chaotical (nonsynchonous $\rightarrow$ synchronous chaos) [2]. In general, the driving amplitude necessary to achieve synchronization is the bigger the higher the detuning between drive and eigenfrequency as is usual for conventional synchronization too.

There is a large variety of possible behaviour if two chaotic systems interact. They can compose a hyperchaotic system. Subharmonic solutions can be stabilized. The coexistence of several attractors in the uncoupled systems provides the possible multistability in the composite system as well ..., cf. [2]. This is a huge open field and next we will focus on a special case.

From now on we concentrate on interacting systems which are identical or contain identical subsystems. That means if there is a distinguishable basic frequency it can be the same for both systems yet without any interaction between them. In the conventional sense there is therefore no stronger synchronization possible.

However, one can additionally achieve something what is called in [2] *symphasation* i.e. the complete coincidence between the time wave forms. In terms of chaotic systems we will call the exact reconstruction of time wave forms synchronization which is much more restrictive than the conventional meaning of synchronization.

Definition 2.1 (Synchronization in terms of chaotic signals) *A system Σ_2 (receiver) synchronizes with a system Σ_1 (transmitter) on a* **set** *of signals $\mathbf{s}_1(t)$ of Σ_1 if the signals $\mathbf{s}_2(t)$ of Σ_2 asymptotically copy them.*

$$\mathbf{s}_1(t) - \mathbf{s}_2(t) \longrightarrow 0 \ \textit{as}\ t \to \infty \tag{2.1}$$

Remark 2.4

a) Definition 2.1 fixes neither the kind of interaction between the systems nor which signals of Σ_1 have to be identically reconstructed by Σ_2. According to different synchronization prinziples $\mathbf{s}_1(t)$ can be the set of all state variables of Σ_1, a subset of this or just an input signal. This will be clarified in section 2.2.1 resp. chapter 3.

b) Definition 2.1 simply presents a relation between two signals which is nearly the most restrictive one can demand.

c) Opposed to conventional synchronization there is not necessarily a periodic solution stabilized by the interaction. But it is quite unusual if not false to say 'a certain chaotic motion is stabilized by system interaction'. If synchronization takes place the systems move on a *diagonal subspace* of the whole statespace with identical signals $\mathbf{s}_1 = \mathbf{s}_2$. We call this subspace *synchronization manifold.* Thus by synchronization the appropriate part of the diagonal manifold becomes stable (cf. stability of invariant sets, section 4.2.1). In other words: The trivial (zero) solution of the difference between the partial systems is stabilized.

Notice, even if one system does not exactly produce the signals of the other the external or mutual influence can lead to somewhat similar motion of them. This suggests to introduce the notion of approximate synchronization as a continuous matter in the sense that two systems not only may or may not synchronize but also may approximately synchronize. In order to estimate up to which extent both systems synchronize in [2] the ordinary coherence is proposed, i.e. achieving exact synchronization is increasing the coherence.

In [35] it was proposed to regard synchronization as decreasing of the dimension of the composite system attractor. Consider two chaotic systems with attractors of dimension D (for instance Hausdorff or geometrical dimension). Clearly, the composite of both systems without interaction has the attractor dimension 2D. The composite of totally synchronized systems which move only on the diagonal subspace of the composite state space has again the attractor dimension D. That means synchronization is dimension decreasing.

We conjecture: There is a rather simple relation between the degree of mutual coupling of two systems and the composite attractor dimension and: The minimal coupling necessary for the onset of synchronization is in direct relation to some quantitative feature of the single chaotic system. (The term coupling will be clarified in section 2.2.1.)

Such a relation to a quantitative characteristic has been already confirmed by Yu. Kusnetzow et al. [31]. There the minimal amplitude a of an external harmonic signal necessary to achieve a periodic solution of a chaotic system at any driving frequency was found to be in a surprisingly simple relation with the Kolmogorow entropy K (normalized to the driving frequency) of the system:

$$a = c \cdot K^{\chi} \tag{2.2}$$

where c represents a scaling factor and $\chi = 0.33$ was astonishingly constant for several chaotic systems.

This shows that even in chaotic systems some useful precise quantitative relations can be proved.

Now we will consider chaotic synchronization under the aspect of practical use.

2.2 Communication by Chaotic Signals (state of the art)

Different methods for the modulation of information with a chaotic signal at a transmitter have been published so far. In order to retrieve this information the receiver has to synchronize. Two basically different cases of the transmitter must be distinguished:

1. The information does not influence the chaotic motion of the transmitter. It can be an autonomous chaotic systems. The corresponding synchronization principles are described in the sequel.

2. The transmitter is a chaotic system whose motion is controlled by the information signal. It is nonautonomous. This leads to the inverse system approach considered comprehensively in the remaining chapters.

2.2.1 Synchronization Prinziples

Due to the autonomous transmitter system we call the synchronization principles to be described here *autonomous synchronization principles*. The relation to the situation depicted in Fig. 2.1 b is obvious. The task to be realized here is shown in Fig. 2.3. This means the receiver has to be an identity system

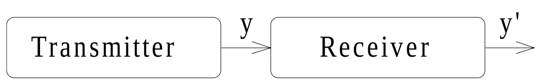

Figure 2.3: Task to be realized by autonomous synchronization principles in order to serve for communication with chaotic signals

for the transmitted chaotic signal. It might seem senseless to reproduce a received signal. However, in section 2.2.2 we explain how this ability is exploited for information transmission.

There are two ways to realize the reconstruction of a transmitted signal:

1. repeated driving of a subsystem

2. error feedback

Both ways can be treated by the Pecora-Carroll approach [33] which is in fact more generally applicable than for the driving of a subsystem.

Pecora-Carroll approach or Master-Slave Principle

The Pecora-Carroll approach allows to establish asymptotically stable solutions of systems which are excited by a chaotic signals. We consider an autonomous chaotic system -the *master*- and a driven system -the *slave*-. The master system is described by:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}) \end{aligned} \tag{2.3}$$

whereas the slave is decribed by:

$$\dot{\mathbf{x}'} = \mathbf{f}'(\mathbf{x}', \mathbf{y}) \tag{2.4}$$
$$\text{with } \mathbf{x} \in I\!R^N, \mathbf{y} \in I\!R^m, \mathbf{x}' \in I\!R^{N'}$$

Actually, in [33] the drive signal is supposed to be a subset of the master states: $\{y_i, i = 1, \ldots, m\} \subseteq \{x_i, i = 1, \ldots, N\}$ but this is not a necessary restriction.

The basic idea of the Pecora-Carroll approach is to establish the asymptotic stability of the solution of 2.4 by means of *conditional Ljapunov exponents*. Consider the difference system between any two solutions of the slave. Obviously a solution of 2.4 is asymptotically stable if the trivial (zero) solution of the difference system is asymptotically stable. This can be examined by linearization of 2.4 along the system flow (cf. section 4.2.2). Due to the chaotic driving the linearized system of 2.4 is neither constant nor periodical. Thus an asymptotically stable solution cannot be proved by negative real parts of eigenvalues nor by Floquet multipliers inside the unit circle but by means of negative conditional Ljapunov exponents (cf. section 4.2.3). This way at least a non empty basin of attraction of each solution can be shown.

In practical applications one does not want an *arbitrary* slave system to produce *any* asymptotically stable solution. Namely, the slave states are in general a subset of the master states: $\{x'_i, i = 1, \ldots, N'\} \subseteq \{x_i, i = 1, \ldots, N\}$.

Here synchronization means that the states, common to both systems, asymptotically copy each other. Since the interaction is unidirectional it is more suitable to say the motion of the slave converges to that of the master.

Driving of a Subsystem or Transmission of One State

Here we consider the case when the slave contains a subsystem of the master, i.e. $\{x'_i, i = 1, \ldots, N'\} \subset \{x_i, i = 1, \ldots, N\}; \{f'_i, i = 1, \ldots, N'\} \subset \{f_i, i = 1, \ldots, N\}$ and $\{y_i, i = 1, \ldots, m\} \subseteq \{x_i : x_i \notin \{x'_i\}\}$.

In a practical realization one is only interested in the transmission of one signal. Therefore we assume: $y(t) \in I\!R^1$. If we use the notion $\mathbf{x}$ only for the states common to master and slave, and $\mathbf{z}$ for master states which are neither part of the slave nor transmitted we obtain:

$$\dot{\mathbf{x}} = \mathbf{f}_{\mathbf{x}}(\mathbf{x}, y) \tag{2.5}$$

$$\dot{y} = \mathbf{f}_y(\mathbf{x}, y, \mathbf{z}) \tag{2.6}$$

$$\dot{\mathbf{z}} = \mathbf{f}_{\mathbf{z}}(\mathbf{x}, y, \mathbf{z}) \tag{2.7}$$

for the master system and:

$$\dot{\mathbf{x}}' = \mathbf{f}'_{\mathbf{x}}(\mathbf{x}', y) \tag{2.8}$$

for the slave system.

Note in case $\mathbf{x}'(0) = \mathbf{x}(0)$ the slave copies the signals common to the master right from the beginning. Of course in practical cases it is impossible to adjust the initial states. Therefore we require this solution to be globally asymptotically stable. Then every solution of (2.4) converges to that of (2.5).

A circuit realization of this method is shown in Fig. 2.4. We remark that the slave circuit is identical to a 'subcircuit' of the master where the memory element which represents the transmitted states (the capacitor C_1) is replaced by a source. This drives the slave with the voltage signal of the v_{C1} of the master.

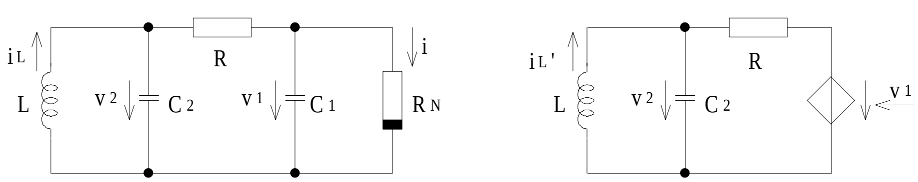

Figure 2.4: Circuit realization of driving of a subsystem with Chua's circuit

The situation in terms of vector fields is depicted in Fig. 2.5. The slave represents the general case of a controlled system, i.e. its vector field depends on the input signal. This should clarify the difference between the autonomous chaotic system and the driven subsystem, which is not chaotical at all. Obviously, the vector field of the slave, when excited with a certain value of y, is the projection of the vector field of the master on the hypersurface : $y = constant$.

Once again it is clear, that identical initial conditions lead to identical signals. It remains to show, that the solution with any initial condition converges to this. This can be easy in case the subsystem is linear as for the example in Fig. 2.4. Otherwise synchronization of the subsystem may be difficult to prove (see chapter 4 for several methods to prove an asymptotically stable solution). In some cases one needs knowledge about the driving signal for this purpose.

Remark 2.5 ('Conditionally')
Some people [40] call this feature 'conditionally synchronizing'. But notice the term synchronization already refers to the master system, namely the copy of its states. Thus a subsystem synchronizes (with its master) or not, but it does not synchronize 'conditionally'. Conversely the term unique asymptotic behaviour admits in general any input signal, not necessarily a signal of the master. Thus in case one can prove unique asymptotic behaviour only under the condition that the input signal obeys certain constraints the notion of conditionally unique asymptotic behaviour makes sense.

For the inverse system approach the notion synchronization and unique asymptotic behaviour of the receiver coincide. But this is due to the fact, that as opposed to subsystems every signal, admissible to the receiver, can be produced by the transmitter and the term synchronization refers to this (cf. chapter 3).

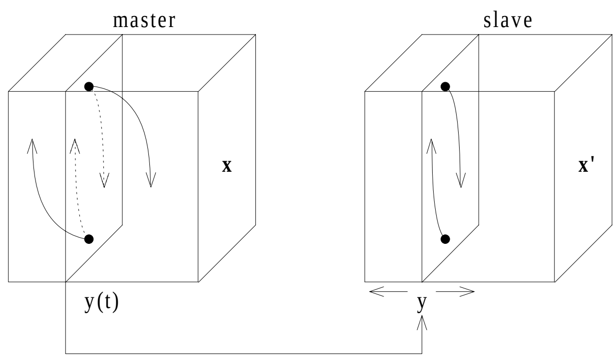

Figure 2.5: Illustration of the driving of a subsystem by a master signal in terms of vector fields: 'Driving is to place a master signal to the slaves disposal and to look what he does'

Since by driving of a subsystem only non transmitted states are reconstructed, one needs another step in order to reconstruct the transmitted signal. Actually, any other subsystem with negative conditional Ljapunov exponents serves this purpose provided it is driven by one of the states, already reconstructed in the first subsystem, say x_1 and it contains the state to be finally reconstructed y, Fig. 2.6.

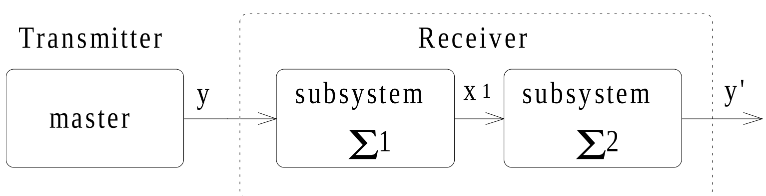

Figure 2.6: Scheme to realize the task of Fig. 2.3 by means of repaeted driving of different subsystems

Decomposition into subsystems:
Although the decomposition into subsystems is a rather restrictive realization of Fig. 2.6 it is the only one considered so far in the literature, e.g. [40], [15]. Namely, it requires that the two subsystem $\Sigma 1$ and $\Sigma 2$ have no common state and their composition gives the whole master system, Fig. 2.7. Clearly, this scheme can only be realized if there is a decomposition of the chaotic master system into two subsystems $\Sigma 1$ and $\Sigma 2$ which:

1. interact only by scalar signals, namely x_1 and y_1, i.e.

$$\dot{\mathbf{x}} = \mathbf{f_x}(\mathbf{x}, y_1) \quad (2.9)$$
$$\dot{\mathbf{y}} = \mathbf{f_y}(x_1, \mathbf{y}) \quad (2.10)$$

and

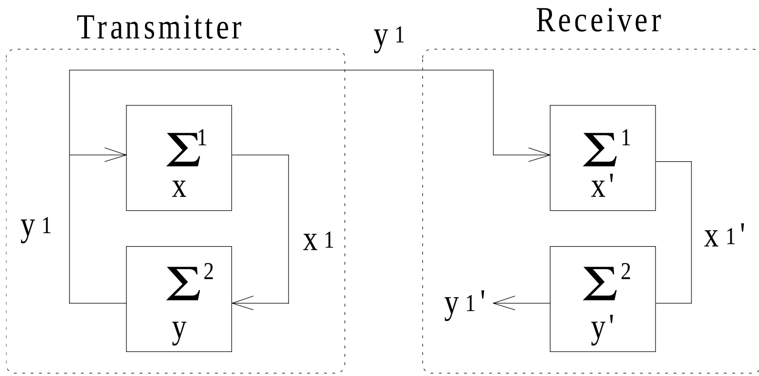

Figure 2.7: Scheme to realize the task of Fig. 2.3 by decomposition into two subsystems

2. both have negative conditional Ljapunov exponents.

Synchronization by Error Feedback

By error feedback all master states can be reconstructed in one step, thus also the transmitted signal. This implies that the slave contains the whole original chaotic system (all states: $\{x'_i, i = 1, \ldots, N'\} = \{x_i, i = 1, \ldots, N\}$) and is additionally driven with an error feedback, Fig. 2.8. The transmitter resp.

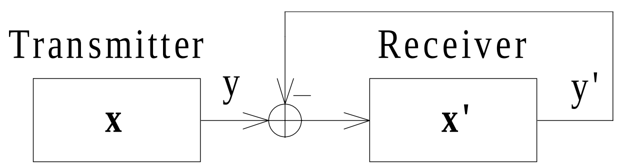

Figure 2.8: Scheme to realize the task of Fig. 2.3 by error feedback

receiver is described by:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) \\ y &= h(\mathbf{x}) \end{aligned} \tag{2.11}$$

resp.

$$\begin{aligned} \dot{\mathbf{x}'} &= \mathbf{f}(\mathbf{x}') + \mathbf{g}(y - y') \\ y' &= h(\mathbf{x}') \end{aligned} \tag{2.12}$$

The expression in (2.12) emphasizes the error feedback property. A more general expression like $\mathbf{g}(\mathbf{x}, y)$ also serves the purpose.

Again it is obvious that identical initial states lead to identical signals, i.e. the error is zero right from the beginning. The error feedback has to be chosen such that the trivial solution of the difference system is stabilized by it. Since the transmitter behaves chaotically the difference system cannot be linear. Thus synchronization is not that easy to show as possibly for a driven subsystem.

One way to design a feedback resp. to prove synchronization is the use of a Ljapunov function as performed in [16]. The basic idea is to eliminate any expansion of the difference system with respect to a Ljapunov function by adding 'dissipative terms', i.e. terms which make neighbouring solutions moving towards each other.

Error feedback is similar to *coupling* described in [8].

Relation to Coupling of Chaotic Systems:
Coupling leads (as opposed to the master-slave approach discussed so far) to bidirectional interaction of systems described by:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) - \mathbf{g}(y - y') \\ y &= h(\mathbf{x}) \end{aligned} \tag{2.13}$$

resp.

$$\begin{aligned} \dot{\mathbf{x}}' &= \mathbf{f}(\mathbf{x}') + \mathbf{g}(y - y') \\ y' &= h(\mathbf{x}') \end{aligned} \tag{2.14}$$

The only difference to the description of the error feedback is that the feedback term $\mathbf{g}(y - y')$ occurs in both systems, i.e. there is a bidirectional interaction between the systems and consequently mutual synchronization. We call $\mathbf{g}(y - y')$ the coupling term and say the faster it makes neighbouring solutions converging towards each other the higher is the *degree of coupling*.

The difference of the variables of coupled systems obeys the same differential equation as the difference between slave and master in the feedback case up to the fact, that $\mathbf{g}(\Delta y)$ appears twice. Thus in case a feedback serves for synchronization of a master-slave system half of its value should be sufficient to synchronize coupled systems.

Fig. 2.9 indicates the difference between error feedback and coupling. From Fig. 2.9 it is obvious that

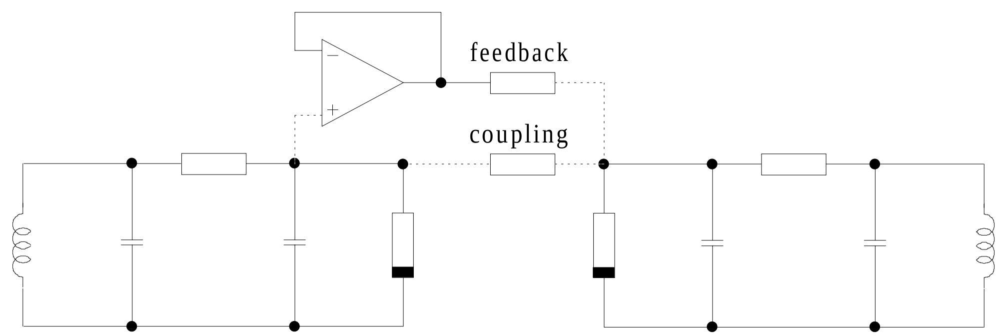

Figure 2.9: Close relation and difference between error feedback and coupling

the limit case when the feedback resistor goes to zero corresponds to the driving of a subsystem (also [6]).

Remark 2.6 (Convergence Rate - Dissipativity)
The convergence rate indicates how fast two neighbouring solutions tend to each other and thus how fast the slave converges to the master. One could also say it indicates how dissipative the driven system is.

Note for the driving of a subsystem the convergence rate is fixed by the choice of the subsystem. But one is free to manipulate the convergence rate by the choice of the feeedback in case of the error feedback approach.

2.2.2 Modulation/Demodulation Methods

Here we will explain how synchronizing chaotic systems which realize the task of Fig. 2.3 can be exploited for the transmission of information. At the transmitter the information signal is either added to a chaotic signal -chaotic masking- or a multi-level discrete signal controls the switching between different chaotic systems, namely between different parameters -chaotic switching-.

Chaotic Masking

In this method [25], [32] the information signal is simply added to a chaotic signal. The information is probably well hidden if it is small enough compared with the chaotic signal. For this it is called chaotically masked. The principle relies on the fact that the receiver reconstructs only the original chaotic part of the transmitted signal, although excited by a signal which is a bit 'disturbed' by the information signal. This means: The receiver works as a matched filter for the chaotic signal or: The synchronization is robust against 'disturbances'. Finally, the approximate information is retrieved by subtraction of the reconstructed chaotic signal from the transmitted signal, Fig. 2.10.

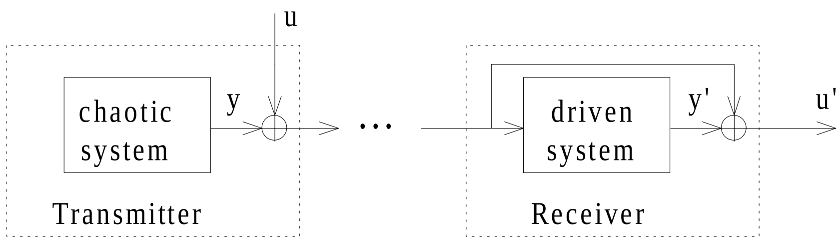

Figure 2.10: Chaotic masking -principle

All examples of this modulation method use the synchronization via decomposition into subsystems. Notice the amazing fact that although the master signal is not exactly supplied to the first subsystem and also to the second subsystem only an approximately reconstructed state is supplied, the original master state is sufficiently well recovered.

In [25] the separation of chaotic and information part is actually performed by a linear filter because the first subsystem is a pure linear system and it is required that the information frequency band is far away from the chaotic band of the system. Of course, from the point of view of secure communication such a demand is disastrous.

But in [9] chaotic masking is demonstrated by a circuit based on the Lorenz system. Both its subsystems are nonlinear and although the frequency band of the added information is inside the chaotic band of the system the information is fairly well retrieved.

An obvious disadvantage of this modulation method is that the receiver cannot distinguish the information from channel noise.

Chaotic Switching

While chaotic masking relies on robust synchronization, the chaotic switching method is based on desynchronization when a signal different from the original master signal is supplied. Thus this method is somewhat converse of chaotic masking. The chaotic switching principle is suitable for transmission of multi-level discrete signals. But most of the published examples concentrate on the binary case as we do.

A binary signal controls at the transmitter the switching between two different chaotic systems, namely between different parameters. At the receiver two systems are mounted with two choices of parameters. The parameters have to be chosen such that only one of the receiver systems synchronizes, namely the one which has the parameter equal to the value which is just switched on at the transmitter. The other system, with the different parameter value has to desynchronize. This means that exactly one of the error signals $e(t)$ and $e'(t)$ converges to zero until the next switching instant. The information is retrieved by detecting which system synchronizes, Fig. 2.11.

In this case synchronization via decomposition into subsystems and error feedback are possible and performed so far.

While converging to zero for the 'right parameter system' is ensured by synchronization one has to rely on a considerable error signal for the 'wrong parameter system'. The latter is the bigger 1st) the less dissipativ the driven system is and 2nd) the more the driving signal deviates from the signal produced by the equal parameter system, i.e. the bigger the difference between the parameters is.

To choose a large parameter difference implies that the corresponding wave forms are obviously distinguishable. This is not favourable for secure communications.

If the signals corresponding to the two parameters are very similar the feedback has to be chosen so close to the synchronization border that it is just sufficient to ensure synchronization in case of driving with the right signal and to allow desynchronization for the driving with the wrong signal.

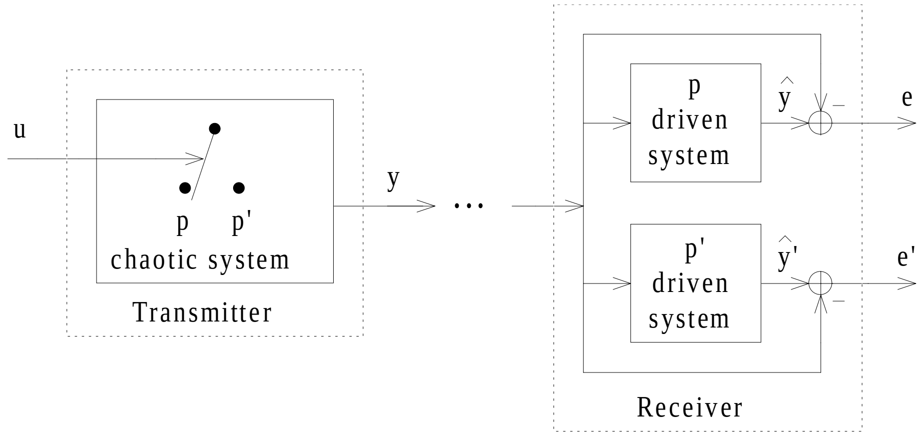

Figure 2.11: Chaotic switching -principle

It is clear that the faster synchronization takes place, the more time it takes until the error signal is sufficiently large. Consequently one has to find a compromise between synchronization and desynchronization speed.