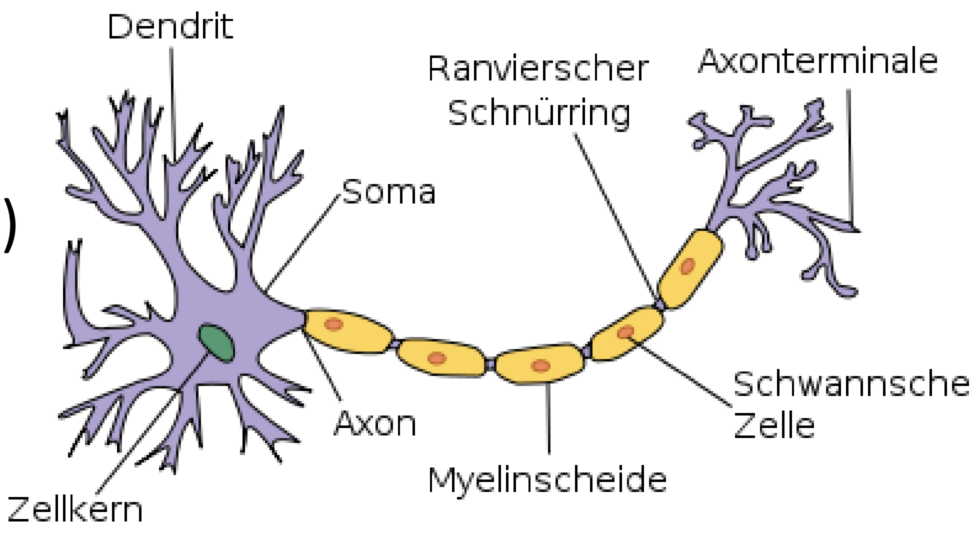
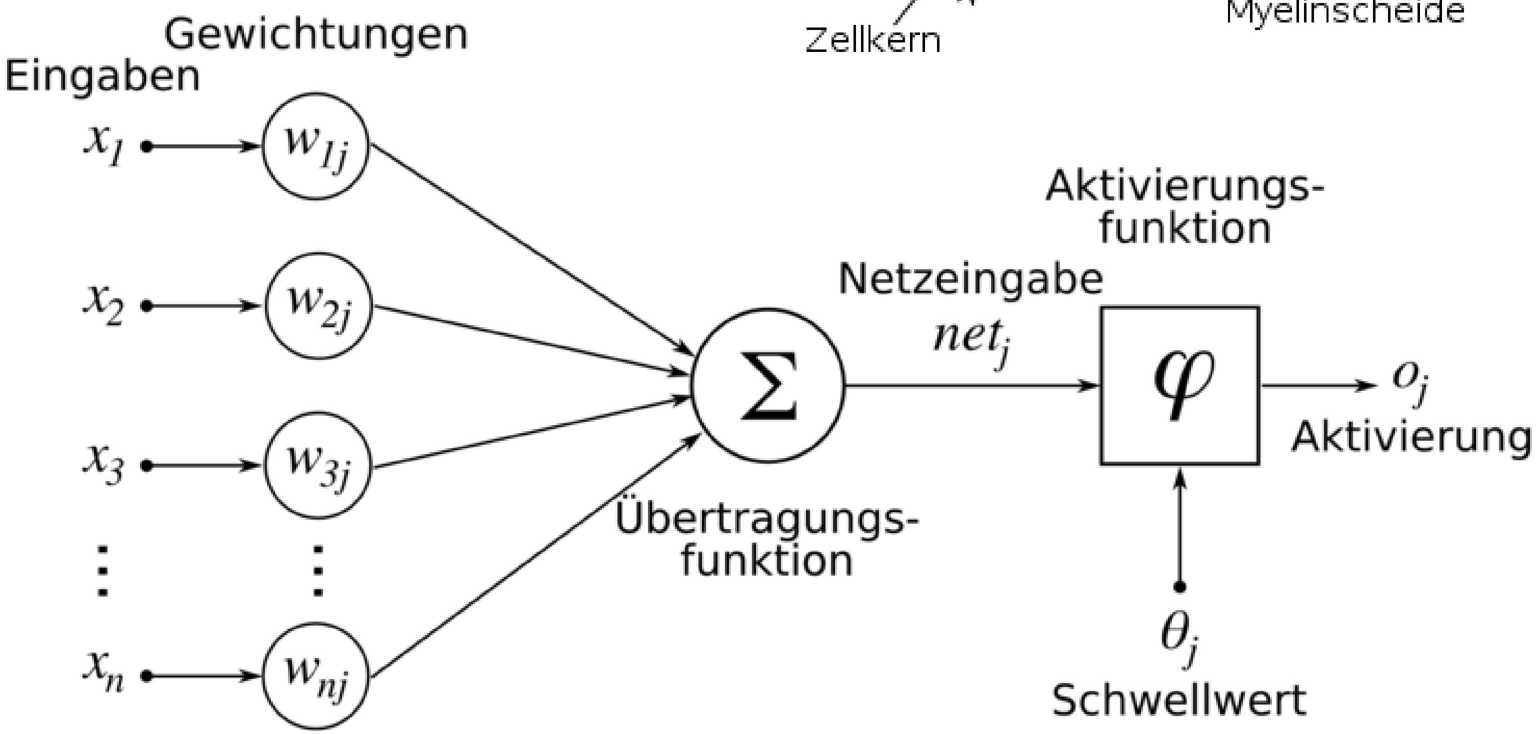
Pattern Recognition

Neuron

Neuron

Hunan vs. Computer
(two nice pictures from Wikipedia)





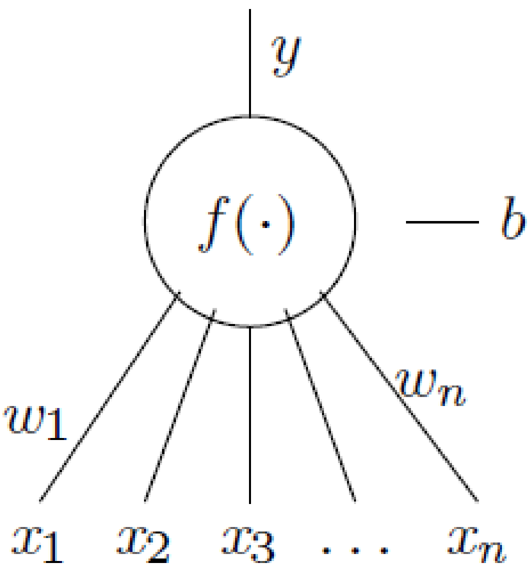
Neuron (McCulloch and Pitt, 1943)

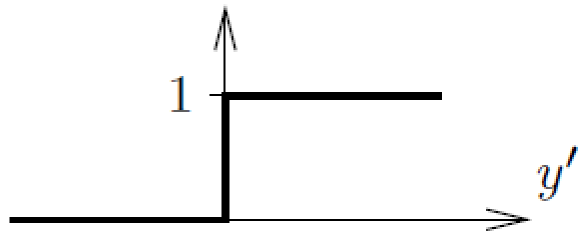
Input: $x \in \mathbb{R}^n$

Weights: $w \in \mathbb{R}^n$

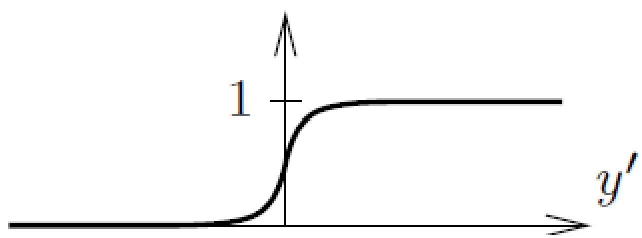
Activation: $b \in \mathbb{R}$

Output: $y = f(y' - b) = f(\langle w, x \rangle - b)$





Step-function $f(y') = \begin{cases} 1 & \text{If } y' > 0 \\ 0 & \text{otherwise} \end{cases}$

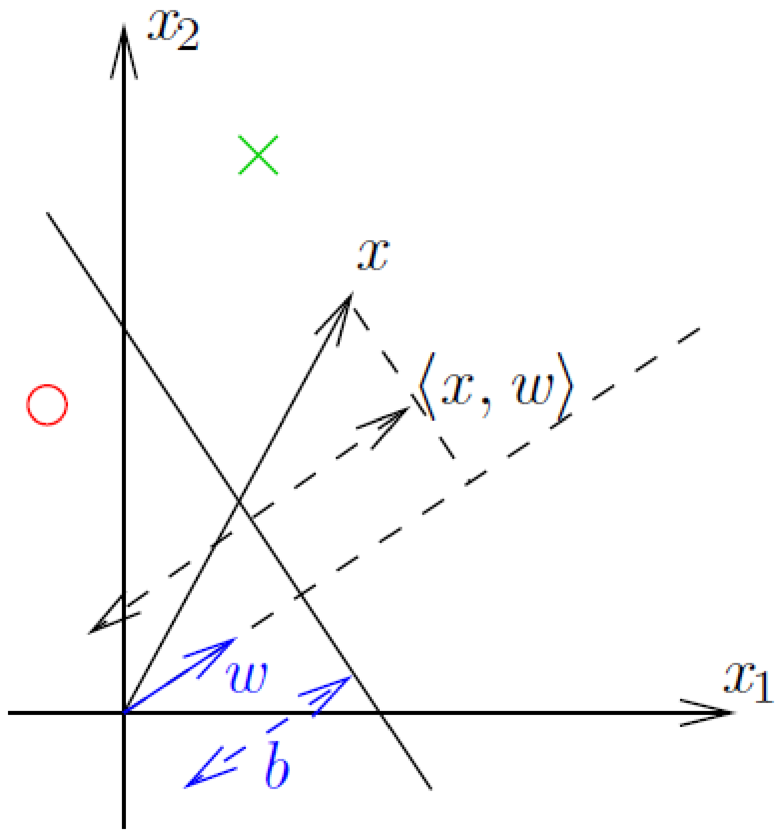


$\langle x, w \rangle \lessgtr b$

Sigmoid-function
(differentiable!!!)

$f(y') = \frac{1}{1+\exp(-y')}$

Geometric interpretation



$$\langle x, w \rangle = \|x\| \cdot \|w\| \cdot \cos\phi$$

Let w be normalized, i.e. $\|w\| = 1$

$\Rightarrow \|x\| \cdot \cos\phi$ the length of the projection of x onto w.

Separation plane: $\langle x, w \rangle = const$

Neuron implements a linear classifier

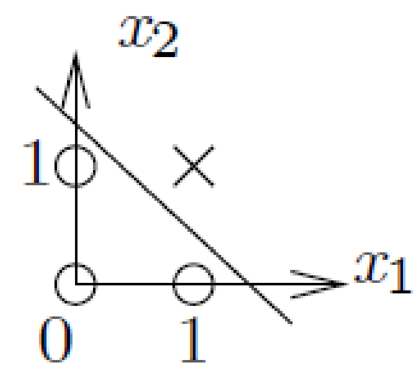
Special case – Boolean functions

Input: $x = (x_1, x_2),\ x_i \in \{0, 1\}$

Output: $y = x_1 \& x_2$

Find w and b so, that $\mathrm{step}(w_1 x_1 + w_2 x_2 - b) = x_1 \& x_2$

x_1	x_2	y
0	0	0
0	1	0
1	0	0
1	1	1



$$w_1 = w_2 = 1,\ b = 1.5$$

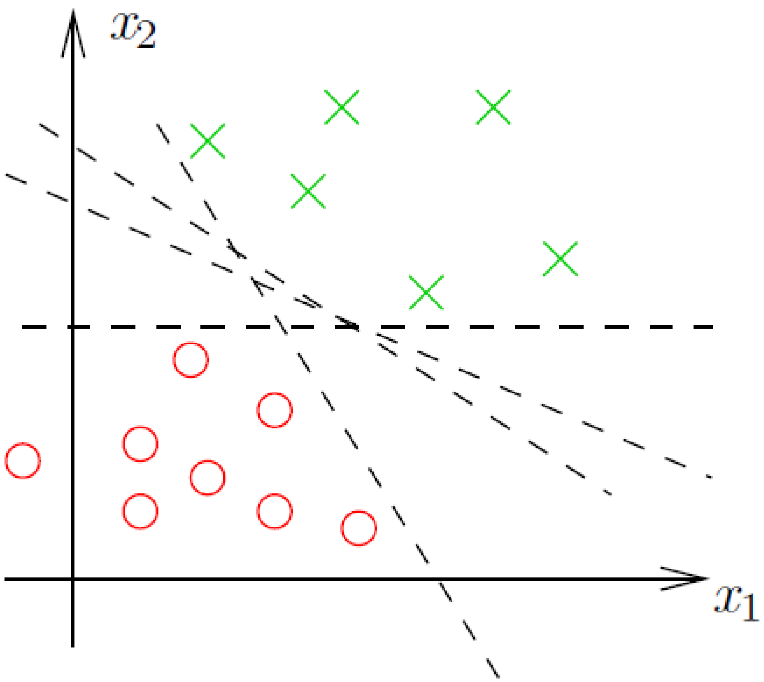
Disjunction, other Boolean functions, but XOR

Learning

Given: training data $\Big((x^1, y^1), (x^2, y^2), \ldots, (x^L, y^L)\Big),\ x^l \in \mathbb{R}^n,\ y^l \in \{0, 1\}$

Find: $w \in \mathbb{R}^n,\ b \in \mathbb{R}$ so that $f(\langle x^l, w\rangle - b) = y^l$ for all $l = 1, \ldots, L$

For a step-neuron: system of linear inequalities



$$\begin{cases} \langle x^l, w\rangle > b & \text{if} \quad y^l = 1, \\ \langle x^l, w\rangle < b & \text{if} \quad y^l = 0. \end{cases}$$

Solution is not unique in general !

"Preparation 1"

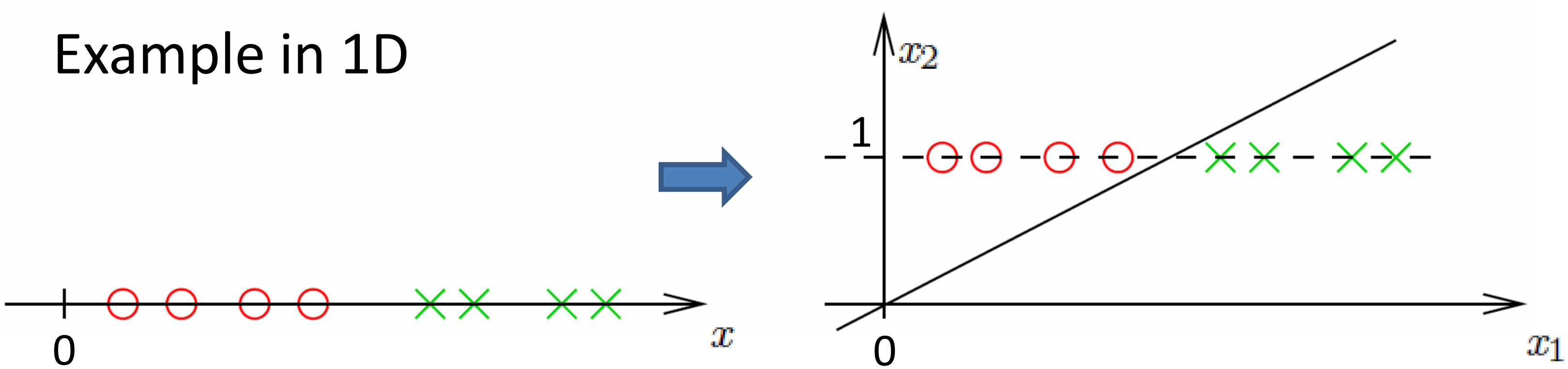
Eliminate the bias:

The trick – modify the training data

$$x = (x_1, x_2, \ldots, x_n) \Rightarrow \tilde{x} = (x_1, x_2, \ldots, x_n, 1)$$

$$w = (w_1, w_2, \ldots, w_n) \Rightarrow \tilde{w} = (w_1, w_2, \ldots, w_n, -b)$$

$$\langle x^l, w \rangle \gtrless b \Rightarrow \langle \tilde{x}^l, \tilde{w} \rangle \gtrless 0$$



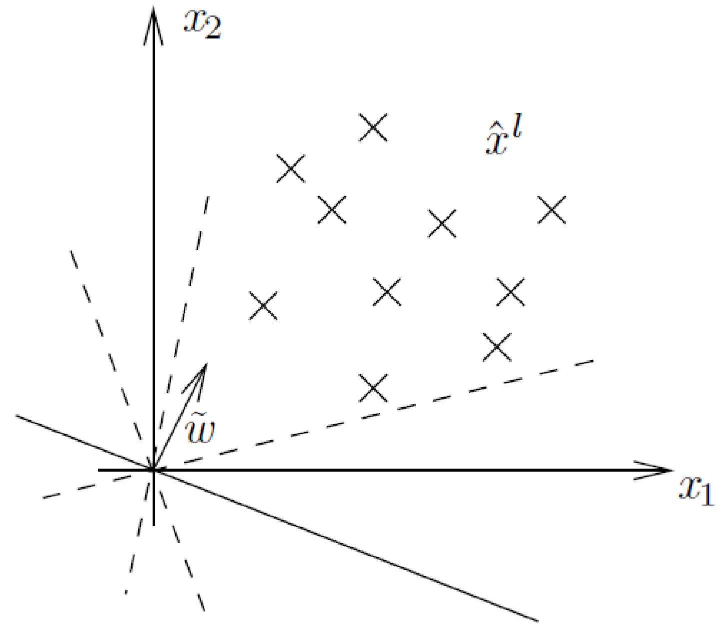
non-separable without the bias

separable without the bias

"Preparation 2"

Remove the sign:

The trick – the same



$\hat{x}^l = \tilde{x}^l$ for all with $y^l = 1$

$\hat{x}^l = -\tilde{x}^l$ for all with $y^l = 0$

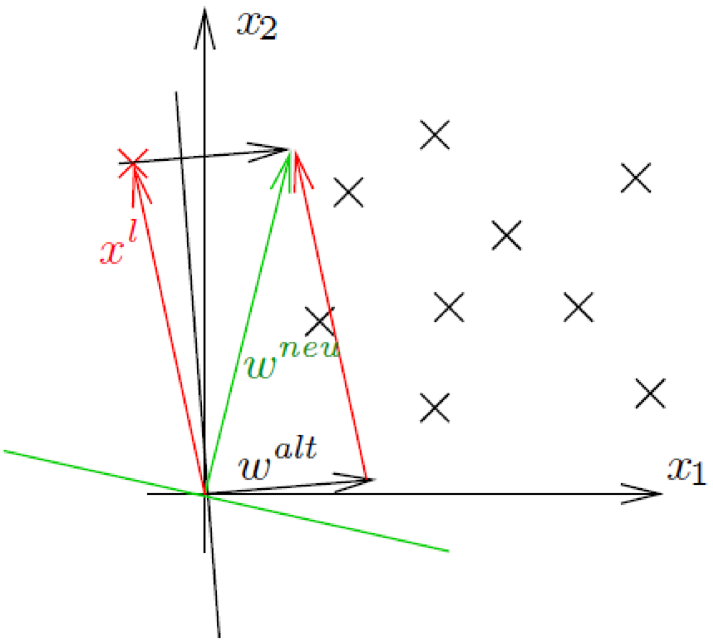
All in all:

$$\begin{cases} \langle x^l, w \rangle > b & \text{if} \quad y^l = 1 \\ \langle x^l, w \rangle < b & \text{if} \quad y^l = 0 \end{cases} \Rightarrow \boxed{\langle \hat{x}^l, \tilde{w} \rangle > 0 \quad \forall l}$$

Perceptron Algorithm (Rosenblatt, 1958)

Solution of a system of linear inequalities:

1. Search for an equation that is not satisfied, i.e.
$$\langle x^l, w \rangle \leq 0$$
2. If not found – Stop
else update $w^{neu} = w^{alt} + x^l$
go to 1.



- The algorithm terminates if a solution exists (the training data are separable)
- The solution is a convex combination of the data points

Proof of convergence

The idea: look for quantities that

a) grow/decrease quite fast,

b) are bounded.

Consider the length of $w^{(n)}$ at n-th iteration:

$$\|w^{(n+1)}\|^2 = \|w^{(n)} + x^i\|^2 = \|w^{(n)}\|^2 + 2\langle w^{(n)}, x^i\rangle + \|x^i\|^2 \leq \|w^{(n)}\|^2 + D^2$$

with $D = \max_l \|x^l\|$

<0, because added by the algorithm

$$\boxed{\|w^{(n)}\| \leq \sqrt{n}D}$$

Proof of convergence

Another quantity – the projection of $w^{(n)}$ onto the **solution** w^*.

$$\frac{\langle w^{(n+1)}, w^* \rangle}{\|w^*\|} = \frac{\langle w^{(n)}, w^* \rangle}{\|w^*\|} + \frac{\langle x^i, w^* \rangle}{\|w^*\|} \geq \frac{\langle w^{(n)}, w^* \rangle}{\|w^*\|} + \epsilon$$

>0, because of the solution

With $\epsilon = \min_l \langle x^l, w^* \rangle / \|w^*\|$ – the **Margin**

$$\boxed{\frac{\langle w^{(n)}, w^* \rangle}{\|w^*\|} \geq n\epsilon}$$

Proof of convergence

All together:

$$\boxed{\|w^{(n)}\| \leq \sqrt{n}D} \quad \text{and} \quad \boxed{\frac{\langle w^{(n)}, w^* \rangle}{\|w^*\|} \geq n\epsilon} \quad \Rightarrow \quad \frac{\langle w^{(n)}, w^* \rangle}{\|w^*\| \cdot \|w^{(n)}\|} \geq \sqrt{n}\frac{\epsilon}{D}$$

But $1 \geq \dfrac{\langle w^{(n)}, w^* \rangle}{\|w^*\| \cdot \|w^{(n)}\|}$ (Cauchy-Schwarz inequality)

So $1 \geq \sqrt{n}\dfrac{\epsilon}{D}$ and finally $n \leq \dfrac{D^2}{\epsilon^2}$

If the solution exists,

the algorithm converges after D^2/ϵ^2 steps at most.

An example problem.

Consider another decision rule for a real valued feature $x \in \mathbb{R}$:

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = \sum_i a_i x^i \gtrless 0$$

It is not a linear classifier anymore but a polynomial one.

The task is again to learn the unknown coefficients a_i
given the training data $\big((x^l, y^l)\ldots\big),\ x^l \in \mathbb{R},\ y^l \in \{0,1\}$

Is it also possible to do that in a “Perceptron-like” fashion ?

An example problem.

The idea: reduce the given problem to the Perceptron-task.

Observation: although the decision rule is not linear with respect to x, it is still linear with respect to the **unknown** coefficients a_i

The same trick again − modify the data:

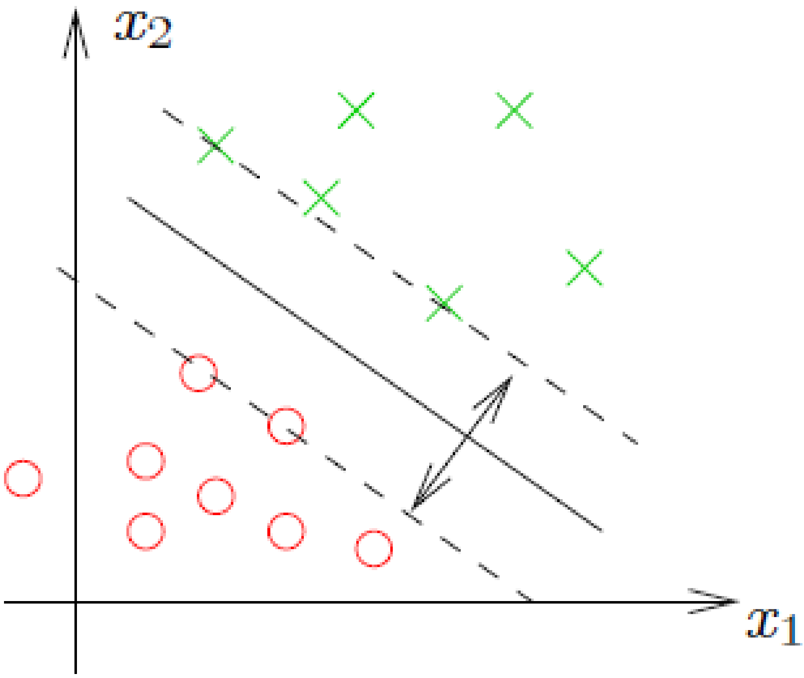
$$w = (a_n, a_{n-1}, \ldots, a_1, a_0)$$
$$\tilde{x} = (x^n, x^{n-1}, \ldots, x, 1)$$

$$\Rightarrow \quad \sum_i a_i x^i = \langle \tilde{x}, w \rangle$$

In general, it is very often possible to learn non-linear decision rules by the Perceptron algorithm using an appropriate transformation of the input space (further extension − SVM).

Kosinec Algorithm

The task:



There are many solutions for the Perceptron in general

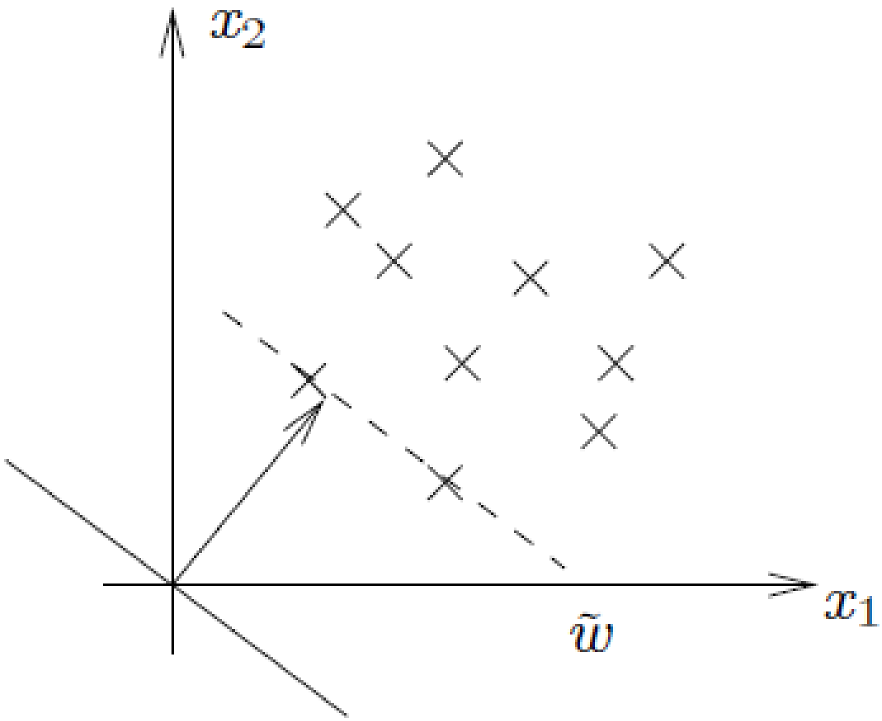
One has to choose one

Idea:
Search for a “stripe of the maximal width” that separates the data

width $\leftrightarrow$ **margin**

“Maximum margin learning”

Kosinec Algorithm



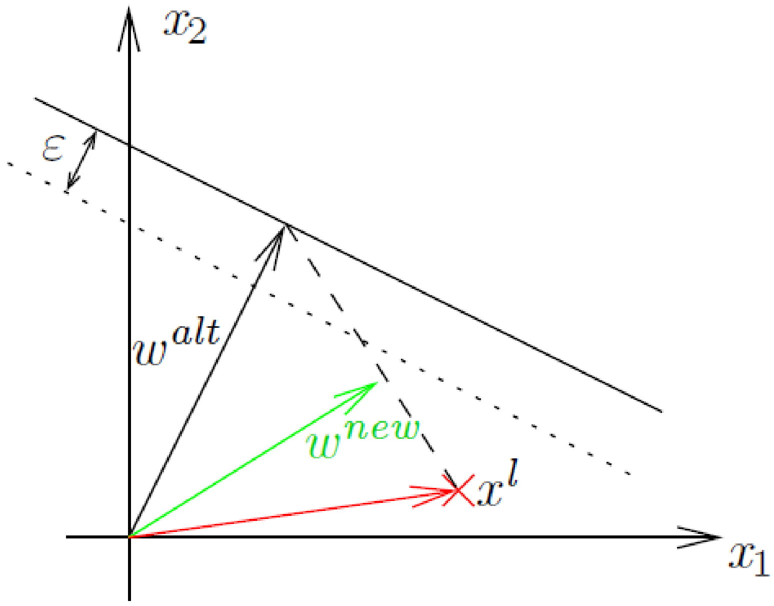
After "Preparation 1 and 2":

$$\min_l \frac{\langle x^l, w \rangle}{\|w\|} \to \max_w$$

(compare with Perceptron)

$$\min_l \frac{\langle x^l, w \rangle}{\|w\|} > 0$$

Kosinec Algorithm (1963?)



ε-precise algorithm:

1. Search for an x^l so that
$$\frac{\langle x^l, w\rangle}{\|w\|} < \|w\| - \varepsilon$$
2. If not found – Stop
3. Search for
$$\gamma^* = \arg\min_\gamma \|w^{alt} + \gamma(x^l - w^{alt})\|^2$$
4. Update
$$w^{neu} = w^{alt} + \gamma(x^l - w^{alt})$$
go to 1.

The algorithm terminates after a finite number of steps, for $\varepsilon > 0$ (proof similar to Perceptron), for $\varepsilon = 0$ does not always terminate ☹