Image Processing

Fourier Transform

Function Spaces

Images are not vectors. Images are mappings: $I : D \subset \mathbb{Z}^2 \to C$
Moreover, images are functions (continuous domain): $I : D \subset \mathbb{R}^2 \to C$
However, functions can be seen as vectors as well:

	Vector $v = (v_1, v_2 \ldots v_n)$	Function $f(x),\ x \in \mathbb{R}$
Domain	$\{1, 2 \ldots n\}$	$\mathbb{R}$
Mapping	$\{1, 2 \ldots n\} \to \mathbb{R}$	$\mathbb{R} \to \mathbb{R}$
Space	$\mathbb{R}^n$	$\mathbb{R}^\infty$
Scalar product	$\langle u, v \rangle = \sum_i u_i v_i$	$\int f(x) g(x) dx$
Length	$\sum_i v_i^2 = \langle v, v \rangle$	$\int f(x)^2 dx$

→ Images are not vectors, they are more, but they are vectors too

Base in vector spaces

Task: decompose a vector $x \in \mathbb{R}^n$ into its "components" in a base

$$x = \sum_i v_i \cdot \lambda_i$$

with the base vectors $v_i \in \mathbb{R}^n$ and coefficients $\lambda_i \in \mathbb{R}$

Properties of base vectors:

1. Vectors should span the space → decomposition exists for all x
2. Vectors should be independent (no vector v_i can be represented as a linear combination of v_j) → decomposition is unique

Special case – orthonormal base:

- All v_i are orthogonal to each other, i.e. $\langle v_i, v_j \rangle = 0$ for all $i \neq j$
- All v_i have the same length (=1), i.e. $\langle v_i, v_i \rangle = 1$

→ $\lambda_i = \langle x, v_i \rangle$

Base in function spaces

The space has infinite dimension →

- Infinite number of base functions $v_i(x)$, i.e. $v(x,y)$, y replaces i
- A continuous function $\lambda(y)$

The task is to decompose a given function $f(x)$ into the base ones:

$$f(x)=\int_y v(x,y)\lambda(y)\,dy$$

Orthonormal base means:

- orthogonal $\quad\int_x v(x,y')v(x,y'')\,dx=0$
- normalized $\quad\int_x v(x,y)v(x,y)\,dx=const$

Then $\quad\lambda(y)=\text{„}\langle\rangle\text{“}=\int_x f(x)v(x,y)\,dx$

Fourier Series

Space:
all periodic functions with the period 2π, i.e. $f(x) = f(x + k \cdot 2\pi),\ k \in \mathbb{Z}$

Base functions: $\sin(kx)$ and $\cos(kx),\ k = 0, \ldots, \infty$

Properties:

1. Orthonormal

$$\int_0^{2\pi} \sin(k_1 x)\cos(k_2 x)dx = \ldots =$$
$$= 1/2 \int_0^{2\pi} \Big[\sin\big((k_1 + k_2)x\big) - \sin\big((k_2 - k_1)x\big)\Big] dx = 0 \quad \forall k_1 \neq k_2$$

2. Span the function space (Jean Baptiste Joseph Fourier, 1822)

Fourier Series

Decomposition:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos(kx) + b_k \sin(kx) \right]$$

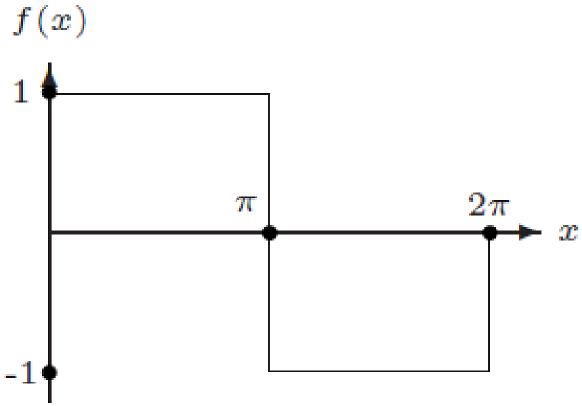
with

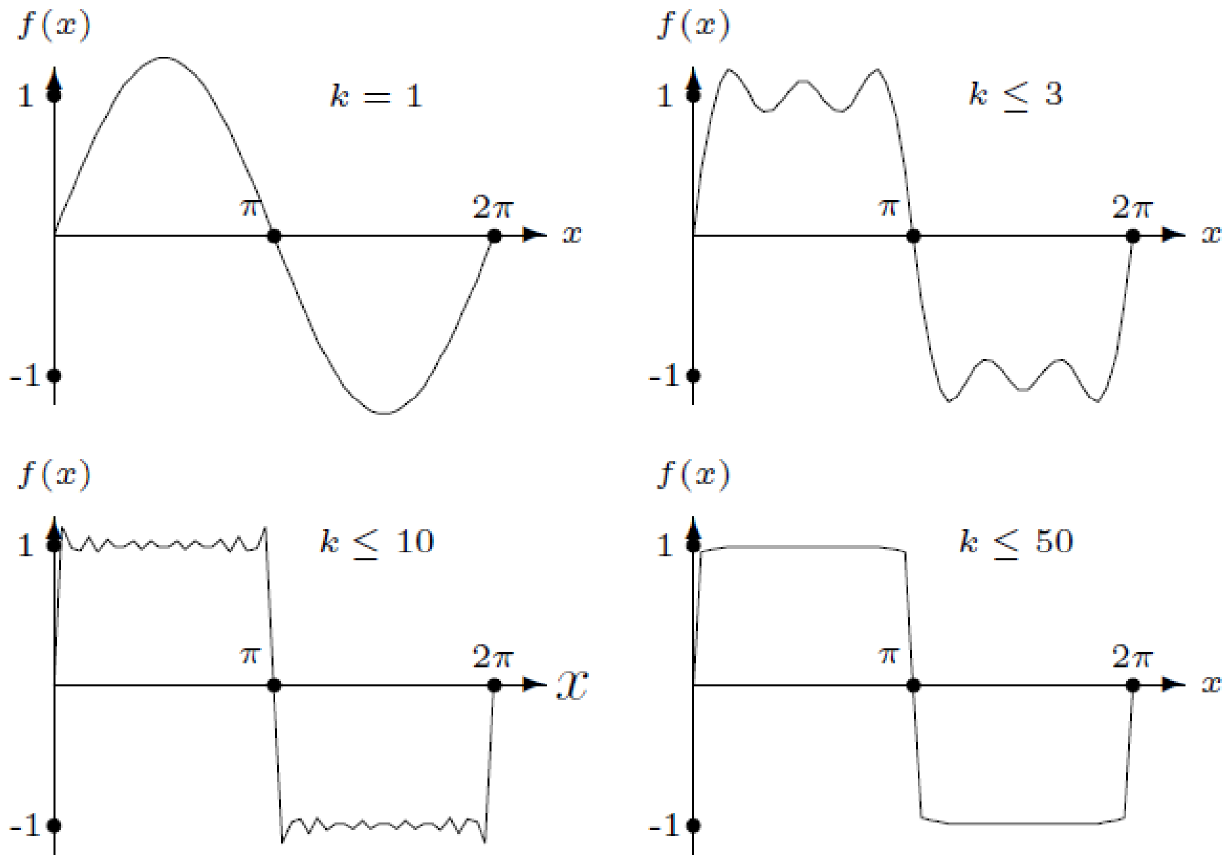
$$a_0 = 1/\pi \int_0^{2\pi} f(x)\,dx$$

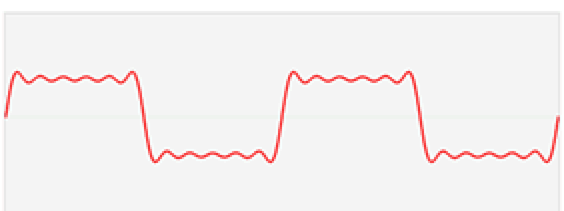
$$a_k = 1/\pi \int_0^{2\pi} f(x) \cos(kx)\,dx$$

$$b_k = 1/\pi \int_0^{2\pi} f(x) \sin(kx)\,dx$$

Fourier Series







Complex numbers

Euler's Formula:

$$\begin{aligned} e^{ikx} &= \cos(kx) + i \cdot \sin(kx) \\ e^{-ikx} &= \cos(kx) - i \cdot \sin(kx) \end{aligned}$$

Decomposition:

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

Coefficients:

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} \, dx = \begin{cases} a_0/2 & k = 0 \\ 1/2 \cdot (a_k - ib_k) & k > 0 \\ 1/2 \cdot (a_{-k} + ib_{-k}) & k < 0 \end{cases}$$

General functions

Arbitrary periodic functions – transition $\cos(kx) \rightarrow \cos(\frac{2\pi kx}{T})$

Arbitrary non-periodic functions – limit $T \rightarrow \infty$

1. Coefficients become continuous
2. The sequence $c_0, c_1, \ldots$ becomes a complex function of a real-valued argument

$$F(u) = R(u) + I(u)$$

The summands are "not interesting" by themselves, but rather:

- **amplitude-spectrum** and

$$|F(u)| = \sqrt{R^2(u) + I^2(u)}$$

- **phase-spectrum**

$$\phi(u) = \tan^{-1} \frac{I(u)}{R(u)}$$

2D Discrete Fourier Transform

Two primary arguments: x and y

Two frequencies: horizontal u and vertical v

Transform:

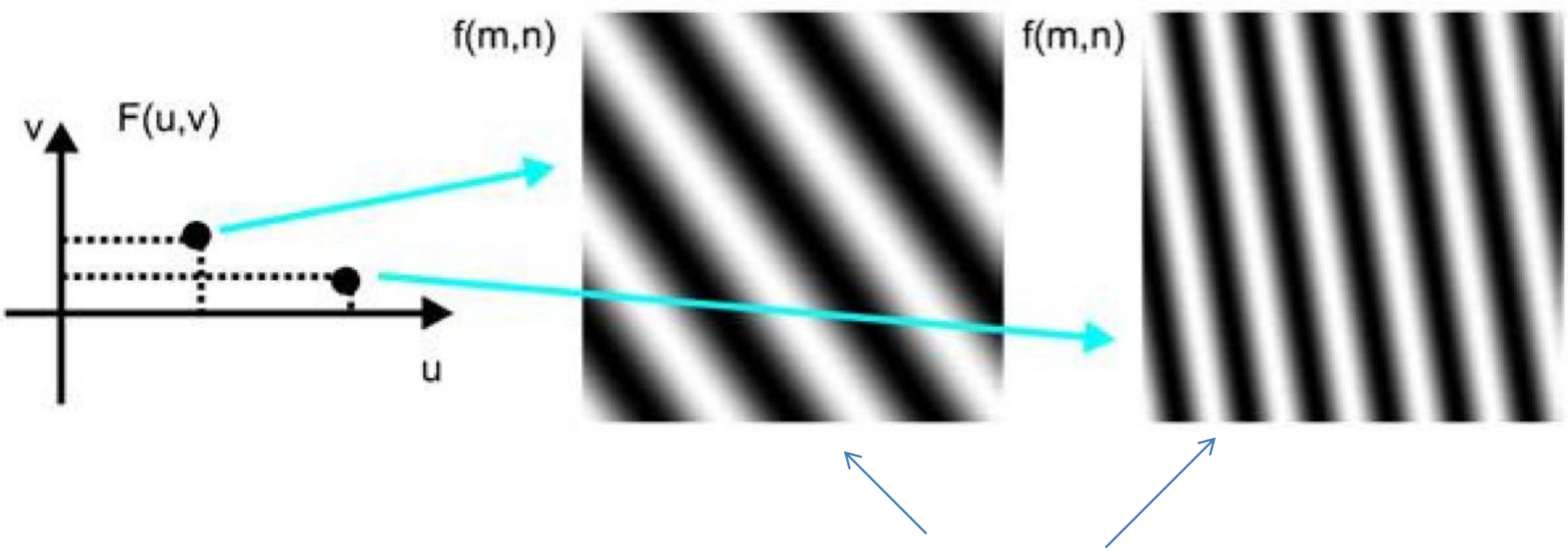
$$F(u, v) = \frac{1}{MN} \cdot \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-i2\pi(xu/M+yv/N)}$$

Inverse:

$$f(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{i2\pi(xu/M+yv/N)}$$

2D Discrete Fourier Transform

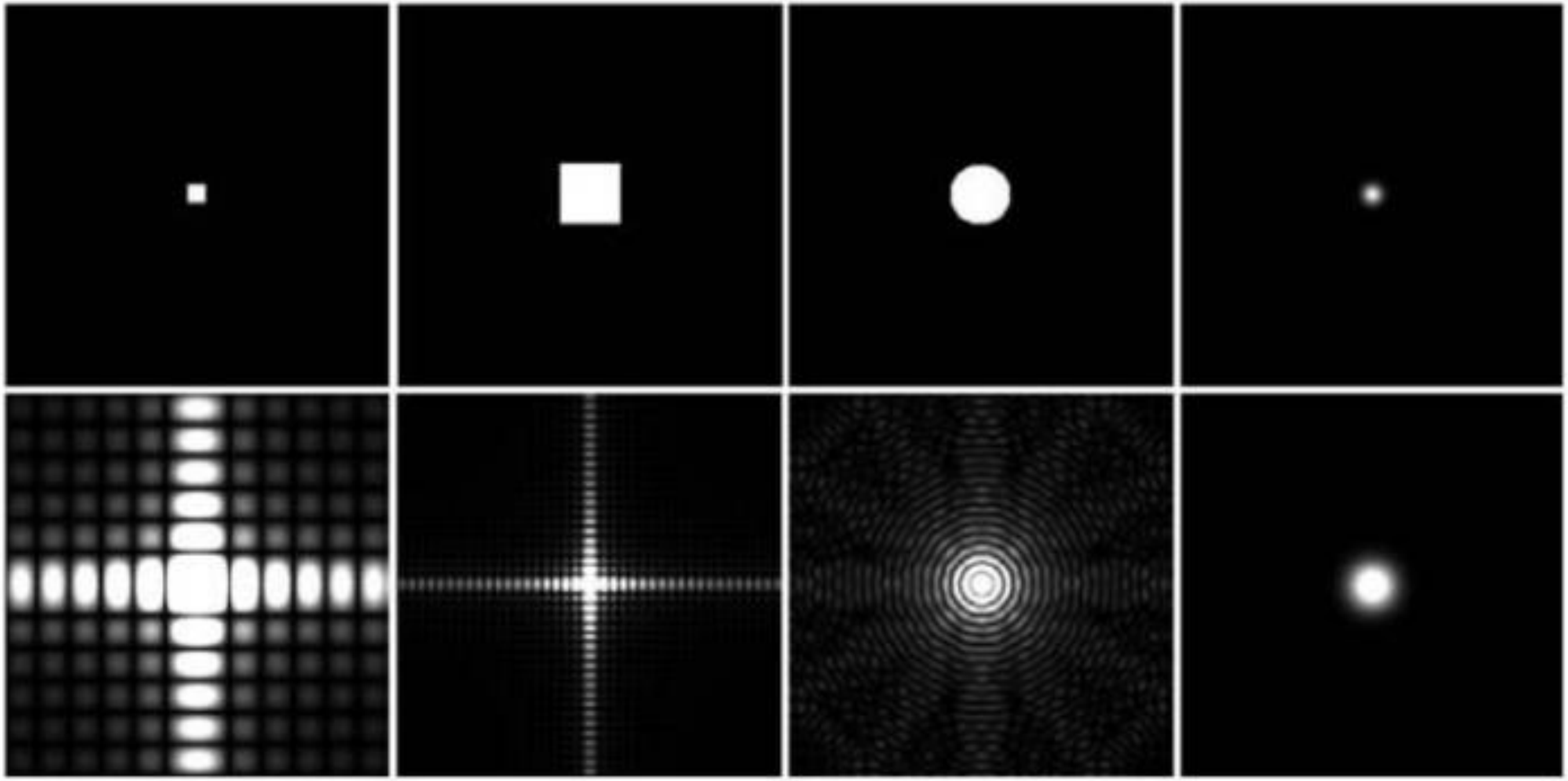
$$F(u,v) = \frac{1}{MN} \cdot \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-i2\pi(xu/M + yv/N)}$$



Convolution masks for different frequencies

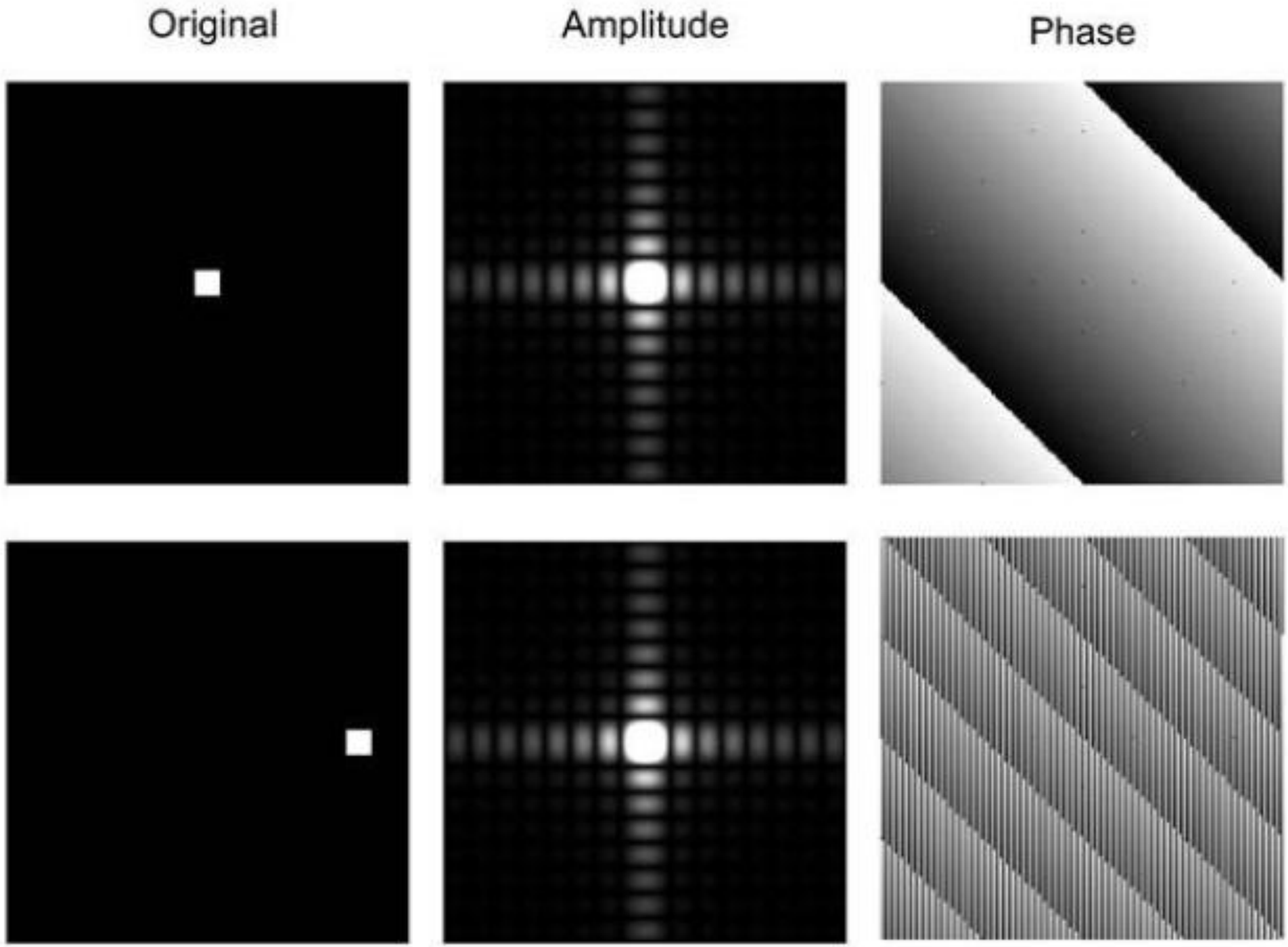
Amplitude-spectrums

Images

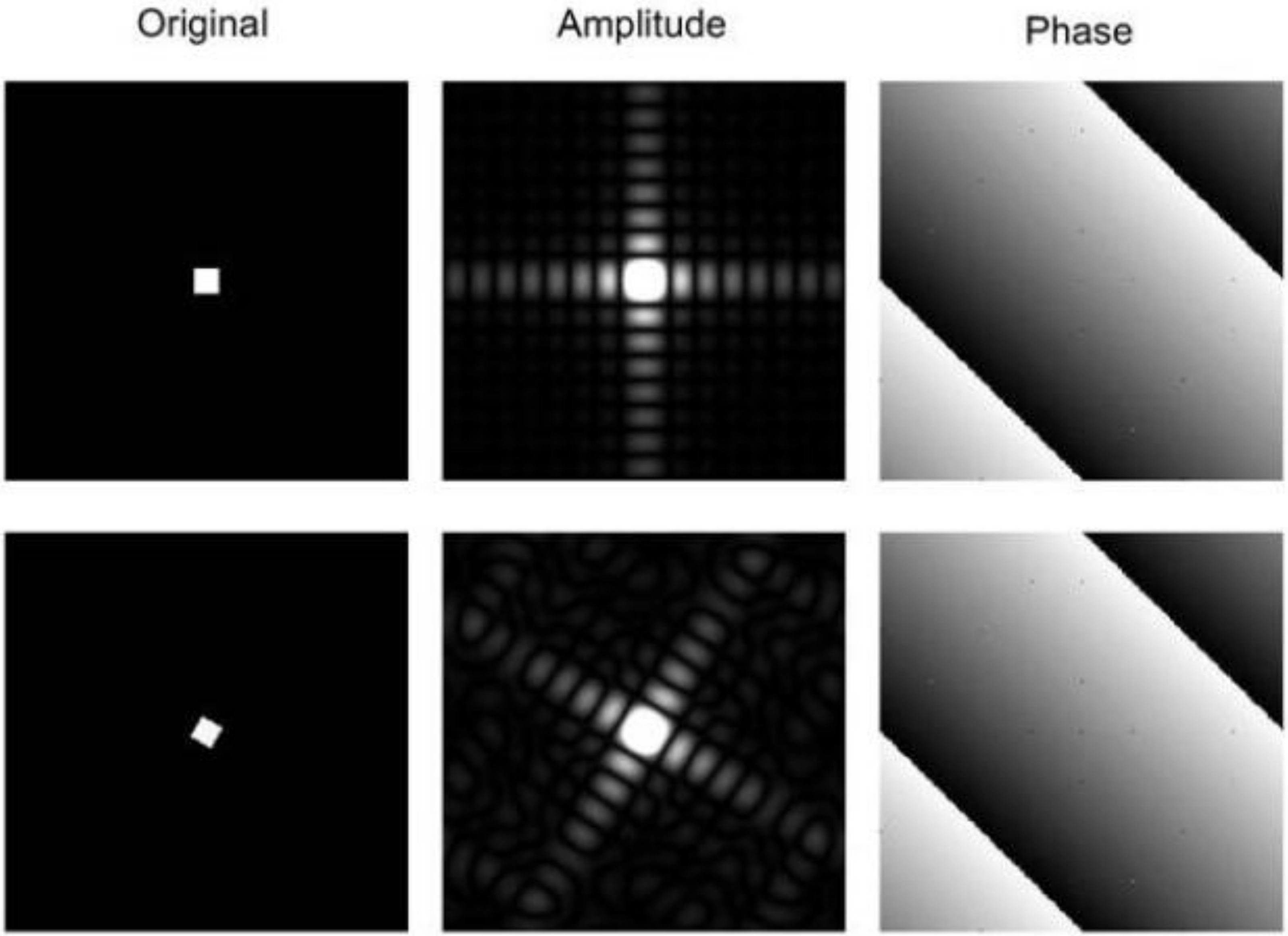


Fourier Transforms

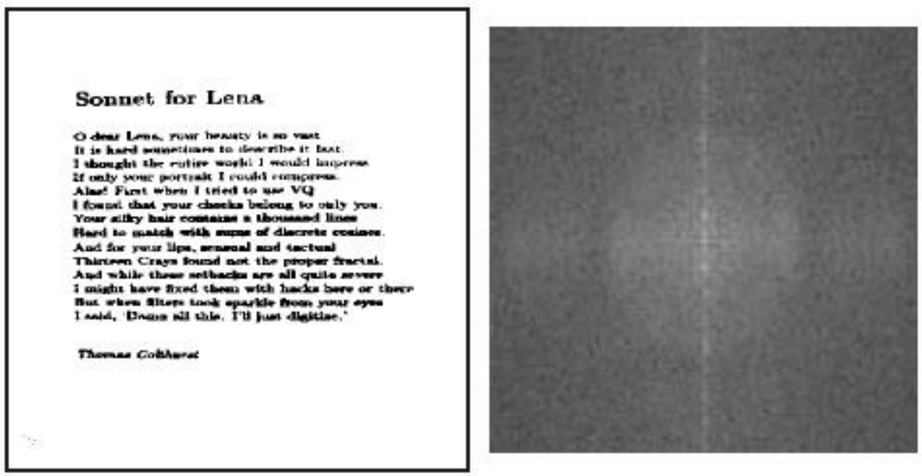
Amplitude vs. Phase

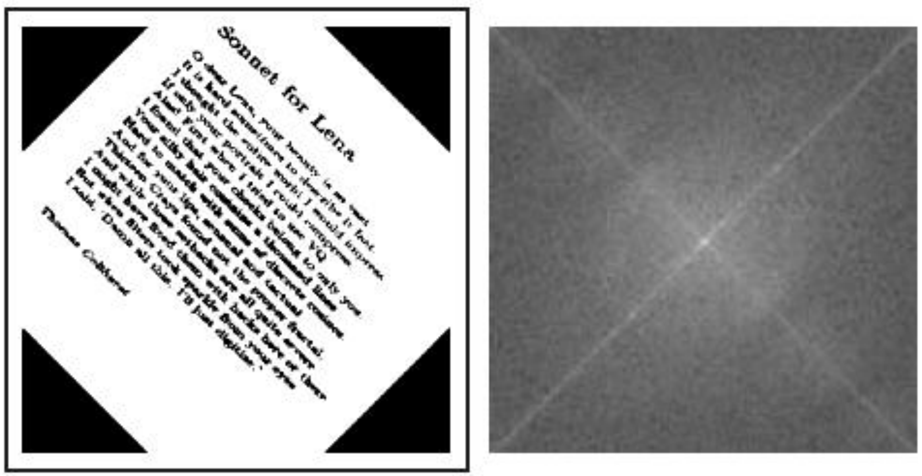


Example – Directions



Example – Directions





Convolution Theorem

$$\mathcal{F}[f * g] = \mathcal{F}[f] \cdot \mathcal{F}[g]$$

$\mathcal{F}$: operator (Fourier Transform)

$F(v) = \mathcal{F}[f]$: the image of a function f in the frequency space

Proof: $f(x) = \mathcal{F}^{-1}[F(v)] = \int_{-\infty}^{\infty} F(v) e^{2\pi i v x} dv,\ g(x) =$ ... analogously

$$f * g = \int_{-\infty}^{\infty} g(x') f(x - x') dx' =$$

$$= \int_{-\infty}^{\infty} g(x') \cdot \left[\int_{-\infty}^{\infty} F(v) e^{2\pi i v (x - x')} dv \right] dx' =$$

$$= \int_{-\infty}^{\infty} F(v) \cdot \left[\int_{-\infty}^{\infty} g(x') e^{-2\pi i v x'} dx' \right] e^{2\pi i v x} dv =$$

$$= \int_{-\infty}^{\infty} F(v) \cdot G(v) e^{2\pi i v x} dv = \mathcal{F}^{-1}[F(v) \cdot G(v)]$$

$$\Rightarrow \mathcal{F}[f * g] = \mathcal{F}[f] \cdot \mathcal{F}[g]$$

Convolution Theorem

Corollary 1:
a convolution $f * g$ can be performed in the frequency space by

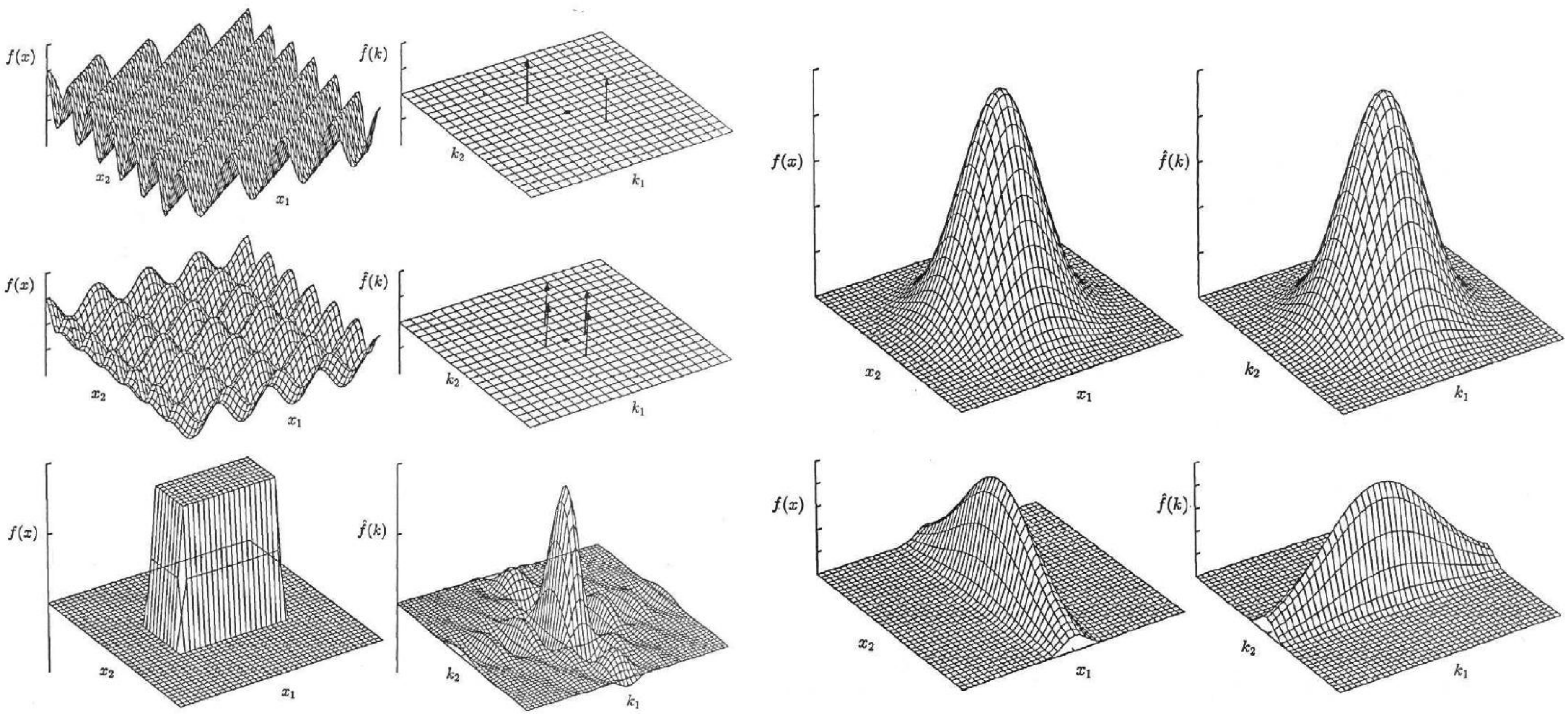
$$f * g = \mathcal{F}^{-1}[F(v) \cdot G(v)]$$

Time complexity:
Fourier Transform can be done with $O(n \log n)$
Component-by-component multiplication: $O(n)$
→ all together $O(n \log n)$
instead of $O(nW)$ by the direct implementation

Corollary 2:
each filter has its spectral characteristics in the frequency space.

1. It is possible to analyze filter characteristics
2. It is possible to design filters with the necessary properties

Some filters and their spectrums



Further themes:

Image say “where” but not “what”.
Spectrums say “what” but not “where”.

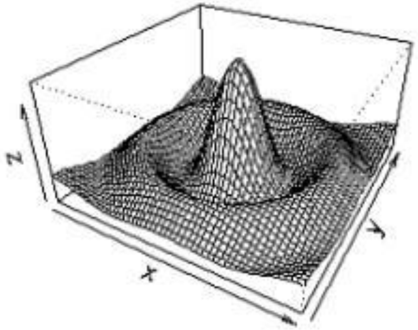
Windowed Fourier Transform – spectrums for (small) windows at each position.

Cosine Transform (1D, discrete, DCT-II – JPEG):

$$F(u) = \sum_{x=0}^{N-1} f(x) \cdot \cos\left[\frac{\pi}{N}\left(x + \frac{1}{2}\right)u\right]$$

Wavelet Transform:

$$F(a, b) \sim \int_{-\infty}^{\infty} f(x) \cdot \psi\left(\frac{x - b}{a}\right) dx$$



$\psi(\cdot)$ – a “mother function”, e.g. Complex Mexican hat wavelet