

## Uniqueness of integrable solutions to $\nabla\zeta = G\zeta$ , $\zeta|_{\Gamma} = 0$ for integrable tensor coefficients $G$ and applications to elasticity

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**Abstract.** Let  $\Omega \subset \mathbb{R}^N$  be a Lipschitz domain and  $\Gamma$  be a relatively open and non-empty subset of its boundary  $\partial\Omega$ . We show that the solution to the linear first-order system

$$\nabla\zeta = G\zeta, \quad \zeta|_{\Gamma} = 0 \tag{1}$$

is unique if  $G \in L^1(\Omega; \mathbb{R}^{(N \times N) \times N})$  and  $\zeta \in W^{1,1}(\Omega; \mathbb{R}^N)$ . As a consequence, we prove

$$\|\cdot\| : C_0^\infty(\Omega; \mathbb{R}^3) \rightarrow [0, \infty), \quad u \mapsto \|\text{sym}(\nabla u P^{-1})\|_{L^2(\Omega)}$$

to be a norm for  $P \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$  with  $\text{Curl } P \in L^p(\Omega; \mathbb{R}^{3 \times 3})$ ,  $\text{Curl } P^{-1} \in L^q(\Omega; \mathbb{R}^{3 \times 3})$  for some  $p, q > 1$  with  $1/p + 1/q = 1$  as well as  $\det P \geq c^+ > 0$ . We also give a new and different proof for the so-called ‘infinitesimal rigid displacement lemma’ in curvilinear coordinates: Let  $\Phi \in H^1(\Omega; \mathbb{R}^3)$  satisfy  $\text{sym}(\nabla\Phi^T \nabla\Psi) = 0$  for some  $\Psi \in W^{1,\infty}(\Omega; \mathbb{R}^3) \cap H^2(\Omega; \mathbb{R}^3)$  with  $\det \nabla\Psi \geq c^+ > 0$ . Then, there exist a constant translation vector  $a \in \mathbb{R}^3$  and a constant skew-symmetric matrix  $A \in \mathfrak{so}(3)$ , such that  $\Phi = A\Psi + a$ .

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### 1. Introduction

Consider the linear first-order system of partial differential equations

$$\nabla\zeta = G\zeta, \quad \zeta|_{\Gamma} = 0. \tag{2}$$

Obviously, one solution is  $\zeta = 0$ . But is this solution unique? The answer is not as obvious as it may seem; consider, for example, in dimension  $N := 1$ ,  $G(t) := 1/t$  in the domain  $\Omega := (0, 1)$  with  $\Gamma := \{0\} \subset \partial\Omega$ . Then,  $\zeta := \text{id} \neq 0$  solves (2). However, in the latter example, the solution becomes unique if  $G \in L^1(\Omega)$ , which is easily deduced from Gronwall’s lemma. Here, we can see that we will need integrability conditions on the coefficient  $G$ ; for a precise formulation of the result, see Sect. 2. The uniqueness of the solution to (2) makes

$$\|u\| := \|\text{sym}(\nabla u P^{-1})\|_{L^2(\Omega)} \tag{3}$$

a norm on

$$C_0^\infty(\Omega; \mathbb{R}^3) := \{u \in C^\infty(\overline{\Omega}; \mathbb{R}^3) : \text{dist}(\text{supp } u, \Gamma) > 0\},$$

where  $C^\infty(\overline{\Omega}; \mathbb{R}^3) := \{u|_{\Omega} : u \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^3)\},$

for  $P \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$  with  $\det P \geq c^+ > 0$  if  $\text{Curl } P \in L^p(\Omega; \mathbb{R}^{3 \times 3})$ ,  $\text{Curl } P^{-1} \in L^q(\Omega; \mathbb{R}^{3 \times 3})$  for some  $p, q > 1$  and  $1/q + 1/p = 1$ . Here the Curl of a matrix field is defined as the row-wise standard curl in  $\mathbb{R}^3$ .

The question whether an expression of the form (3) is a norm arises when trying to generalize Korn's first inequality to hold for non-constant coefficients, that is,

$$\exists c > 0 \quad \forall u \in H_0^1(\Omega, \Gamma; \mathbb{R}^3) \quad \|\text{sym}(\nabla u P^{-1})\|_{L^2(\Omega)} \geq c \|u\|_{H^1(\Omega)}, \quad (4)$$

which was first done for  $P, P^{-1}, \text{Curl } P \in C^1(\bar{\Omega}; \mathbb{R}^{3 \times 3})$  by Neff in [7], cf. [17]. Here,  $H_0^1(\Omega, \Gamma; \mathbb{R}^3)$  denotes the closure of  $C_0^\infty(\Omega, \Gamma; \mathbb{R}^3)$  in  $H^1(\Omega; \mathbb{R}^3)$ . The classical Korn's first inequality is obtained for  $P$  being the identity matrix, see [3, 6, 7, 13–15]. The inequality (4) has been proved in [17] to hold for continuous  $P^{-1}$ , whereas it can be violated for  $P^{-1} \in L^\infty(\Omega)$  or  $P^{-1} \in SO(3)$  a.e. The counterexamples, given by Pompe in [17] and [18], see also [16], each use the fact that for such  $P$  an expression of the form of  $\|\cdot\|$  is not a norm (it has a non-trivial kernel) on the spaces of functions considered. Quadratic forms of the type (4) arise in applications to geometrically exact models of shells, plates, and membranes, in micromorphic and Cosserat type models and in plasticity, [5, 8–11].

The so-called ‘infinitesimal rigid displacement lemma in curvilinear coordinates’, a version of which can be found in [1] and which is important for linear elasticity in curvilinear coordinates (see also [2, 4]), states the following: If  $\Omega \subset \mathbb{R}^N$  is a bounded domain,  $\Psi \in W^{1,\infty}(\Omega; \mathbb{R}^N)$  satisfying  $\det \nabla \Psi \geq c^+ > 0$  a.e. and  $\Phi \in H^1(\Omega; \mathbb{R}^N)$  with  $\text{sym}(\nabla \Phi^\top \nabla \Psi) = 0$  a.e., then on a dense open subset of  $\Omega$ , there exist locally constant mappings  $a : \Omega \rightarrow \mathbb{R}^N$  and  $A : \Omega \rightarrow \mathfrak{so}(N)$  such that locally  $\Phi = A\Psi + a$ . If  $\Omega$  is Lipschitz, then the term ‘locally’ can be dropped. In their proof [1], the authors apply the chain rule to  $\Theta := \Phi \circ \Psi^{-1}$  and use the observation that the conditions  $\text{sym}(\nabla \Phi^\top \nabla \Psi) = 0$  and  $\text{sym}(\nabla \Phi (\nabla \Psi)^{-1}) = 0$  are equivalent by a clever conjugation with  $(\nabla \Psi)^{-1}$ , this is

$$(\nabla \Psi)^{-\top} \text{sym}(\nabla \Phi^\top \nabla \Psi) (\nabla \Psi)^{-1} = \text{sym}(\nabla \Phi (\nabla \Psi)^{-1}) = \text{sym}(\nabla (\Phi \circ \Psi^{-1})) \circ \Psi, \quad (5)$$

together with the classical infinitesimal rigid displacement lemma applied on  $\Theta$ , defined on the domain  $\Psi(\Omega)$ . If to this lemma a boundary condition  $\Phi = 0$  on a relatively open subset of the boundary is added, one obtains  $\Phi = 0$  (cf. [2, 1.7-3(b)]).

The main part of our proof for  $\|\cdot\|$  being a norm is also concerned with obtaining  $u = 0$  from  $\text{sym}(\nabla u P^{-1}) = 0$ . By taking  $P = \nabla \Psi$  to be a gradient, we present another proof of the infinitesimal rigid displacement lemma in dimension  $N = 3$  which yields  $\Phi = A\Psi + a$  with  $A \in \mathfrak{so}(N)$ ,  $a \in \mathbb{R}^N$ . We need slightly more regularity but do not use the chain rule for  $\Theta$ .

The key tool for obtaining our results is Neff's formula for the Curl of the product of two matrices, the first of which is skew-symmetric. We state a generalization of this formula in Sect. 4.1.

This paper is organized as follows: The next section states the main results that will be proven in the subsequent chapters. Section 3.1 provides a tool that gives  $\zeta = 0$  on lines and is used in Sect. 3.2 where this is extended to cubes. Section 3.3 then takes care of the whole domain if a ‘( $\zeta = 0$ )-cube’ is given as starting point. In Sect. 3.4, the uniqueness theorem of Sect. 2 is proven, mainly by putting together the results of the previous sections. After that and before applying the theorem, we have a closer look at the formula for the Curl of a product of matrices (Sect. 4.1). Finally, in Sects. 4.2 and 5, respectively, we prove that  $\|\cdot\|$  is a norm and we present our new proof of the infinitesimal rigid displacement lemma.

## 2. Results

Let us first note that by  $\nabla$  we denote not only the gradient of a scalar-valued function, but also (as an usual gradient row wise) the derivative or Jacobian of a vector field. The Curl of a matrix is to be taken row wise as usual curl for vector fields.

**Theorem 2.1 (Unique Continuation).** *Let  $\Omega \subset \mathbb{R}^N, N \in \mathbb{N}$ , be a Lipschitz domain, and  $\Gamma$  be a relatively open and non-empty subset of  $\partial\Omega$  as well as  $G \in L^1(\Omega; \mathbb{R}^{(N \times N) \times N})$ . If  $\zeta \in W^{1,1}(\Omega; \mathbb{R}^N)$  solves*

$$\nabla\zeta = G\zeta, \quad \zeta|_{\Gamma} = 0,$$

then  $\zeta = 0$ .

From the differential equation itself, it is not a priori clear that  $\zeta$  belongs to  $W^{1,1}(\Omega)$ . But this can be ensured by requiring higher integrability of  $G$  and  $\zeta$ , since for bounded domains, for example, the conditions  $G \in L^2(\Omega)$  and  $\zeta \in L^2(\Omega)$  imply  $\nabla\zeta \in L^1(\Omega)$  and hence  $\zeta \in W^{1,1}(\Omega)$ , where an application of the theorem ensures  $\zeta = 0$ . Thus, we have obtained the uniqueness of  $L^2(\Omega)$  solutions on bounded domains if the coefficient  $G$  is square integrable. Of course, the same holds if  $\zeta \in L^p(\Omega)$  for arbitrary  $p \geq 1$ . Then,  $G$  at least needs to be an  $L^q(\Omega)$  function, where  $1/p + 1/q = 1$ .

**Theorem 2.2 (Norm).** *Let  $\Omega \subset \mathbb{R}^3$  be a Lipschitz domain,  $\emptyset \neq \Gamma \subset \partial\Omega$  be relatively open,  $P \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$  with  $\det P \geq c^+ > 0$ ,  $\text{Curl } P \in L^p(\Omega; \mathbb{R}^{3 \times 3}), \text{Curl } P^{-1} \in L^q(\Omega; \mathbb{R}^{3 \times 3})$  for some  $p, q > 1$  with  $1/p + 1/q = 1$ . Then,*

$$\|\cdot\| : C^\infty(\Omega, \Gamma; \mathbb{R}^3) \rightarrow [0, \infty), \quad u \mapsto \|\text{sym}(\nabla u P^{-1})\|_{L^2(\Omega)} \tag{6}$$

defines a norm.

**Remark 2.3.** *In the case of  $p = q = 2$  and for  $P \in \text{SO}(3)$  a.e.,  $\text{Curl } P^{-1} \in L^2(\Omega)$  is no additional condition, since then  $\text{Curl } P \in L^2(\Omega) \Leftrightarrow \text{Curl } P^{-1} \in L^2(\Omega)$ . (Note that for  $P \in \text{SO}(3)$  a.e., generally  $P, \text{Curl } P \in L^p(\Omega)$  is equivalent to  $P \in W^{1,p}(\Omega)$ , cf. [12].)*

**Conjecture 2.4.** *Theorem 2.2 holds for  $P \in L^\infty(\Omega)$  with  $\text{Curl } P \in L^p(\Omega)$  and  $\det P \geq c^+ > 0$  for some  $p > 1$  or even  $p \geq 1$ .*

**Remark 2.5.** *Since the norms  $\|\cdot\|$  and  $\|\cdot\|_{H^1(\Omega)}$  are not shown to be equivalent, it is not clear whether the spaces  $H^1_\circ(\Omega, \Gamma) = \overline{C^\infty(\Omega, \Gamma)}^{\|\cdot\|_{H^1(\Omega)}}$  and  $\overline{C^\infty(\Omega, \Gamma)}^{\|\cdot\|}$  coincide. However, by [17], these norms are equivalent if  $P \in C^0(\overline{\Omega})$  with  $\det P \geq c^+ > 0$ .*

**Conjecture 2.6.** *The norms are equivalent if  $P \in L^\infty(\Omega)$  with  $\text{Curl } P \in L^p(\Omega)$  and  $\det P \geq c^+ > 0$  for some  $p > 1$  or even  $p \geq 1$ .*

**Theorem 2.7 (Infinitesimal Rigid Displacement Lemma).** *Let  $\Omega \subset \mathbb{R}^3$  be a Lipschitz domain. Moreover, let  $\Phi \in W^{1,p}(\Omega; \mathbb{R}^3)$  and  $\Psi \in W^{1,\infty}(\Omega; \mathbb{R}^3) \cap W^{2,q}(\Omega; \mathbb{R}^3)$  with  $\det \nabla\Psi \geq c^+ > 0$  a.e. and  $p, q > 1, 1/p + 1/q = 1$ . If*

$$\text{sym}(\nabla\Phi^\top \nabla\Psi) = 0,$$

then there exist  $a \in \mathbb{R}^3$  and a constant skew-symmetric matrix  $A \in \mathfrak{so}(3)$ , such that  $\Phi = A\Psi + a$ .

**Remark 2.8.** *When comparing two nearby configurations of an elastic body, namely  $\hat{\Psi} : \Omega \rightarrow \mathbb{R}^3$  and  $\Psi : \Omega \rightarrow \mathbb{R}^3$ , following Ciarlet [4], we may always write  $\hat{\Psi} = \Psi + \Phi$ , where  $\Phi : \Omega \rightarrow \mathbb{R}^3$  is the displacement from  $\Psi$  to  $\hat{\Psi}$ . The respective metric tensors of the two configurations are  $\nabla\hat{\Psi}^\top \nabla\hat{\Psi}$  and  $\nabla\Psi^\top \nabla\Psi$ . In terms of the displacement  $\Phi$  to lowest order, we have for the  $\Phi$ -linearized change of the metric*

$$\begin{aligned} [\nabla\hat{\Psi}^\top \nabla\hat{\Psi} - \nabla\Psi^\top \nabla\Psi]_{\text{lin}, \Phi} &= [(\nabla\Psi + \nabla\Phi)^\top (\nabla\Psi + \nabla\Phi) - \nabla\Psi^\top \nabla\Psi]_{\text{lin}, \Phi} \\ &= \nabla\Phi^\top \nabla\Psi + \nabla\Psi^\top \nabla\Phi = 2 \text{sym}(\nabla\Phi^\top \nabla\Psi). \end{aligned}$$

Therefore, the infinitesimal rigid displacement lemma expresses the fact that if the linearized change of the metric is zero, then the displacement must be (the linearized part of) some rigid displacement.

### 3. Proof of the uniqueness theorem

We start with some preliminaries.

#### 3.1. Vanishing in intervals

Let  $-\infty < a < b < \infty$  and  $I := (a, b)$ .

**Lemma 3.1.** *Let  $G \in L^1(I; \mathbb{R}^{N \times N})$ ,  $\zeta \in W^{1,1}(I; \mathbb{R}^N)$  with  $\zeta' = G\zeta$  and  $\zeta(a) = 0$ . Then,  $\zeta = 0$ .*

*Proof.* We use Gronwall's inequality. Because  $\zeta \in W^{1,1}(I)$ ,  $\zeta$  is absolutely continuous and hence it can be written as an integral over its derivative:

$$\zeta(x) = \zeta(a) + \int_a^x \zeta'(t) dt = \zeta(a) + \int_a^x G(t)\zeta(t) dt \quad \Rightarrow \quad |\zeta(x)| \leq |\zeta(a)| + \int_a^x \|G(t)\|\|\zeta(t)\| dt.$$

An application of Gronwall's inequality leads to

$$|\zeta(x)| \leq |\zeta(a)| \exp\left(\int_a^x \|G(t)\| dt\right) = 0, \quad x \in I,$$

which concludes the proof. □

#### 3.2. Vanishing in cubes

Let  $Q$  be a cuboid in  $\mathbb{R}^N$  and let  $\Gamma$  be a face of  $Q$ , that is,  $Q = \Gamma \times I$  with  $I$  from the previous section. By [19, Th. 2.1.4], we have that for  $u \in L^1(Q)$ , the following is equivalent:  $u \in W^{1,1}(Q)$ , if and only if  $u$  has a representative which is absolutely continuous on almost all line segments in  $Q$  parallel to the coordinate axes and whose (classical a.e.) partial derivatives belong to  $L^1(Q)$ . These classical partial derivatives coincide with the weak derivatives almost everywhere. In particular, if  $u \in W^{1,1}(Q)$ , then  $u_\gamma := u(\gamma, \cdot) \in W^{1,1}(I)$  f.a.a.  $\gamma \in \Gamma$ . Of course, the same holds for  $u \in W^{1,p}(Q)$  with  $p \geq 1$ .

**Lemma 3.2.** *Let  $G \in L^1(Q; \mathbb{R}^{(N \times N) \times N})$  and  $\zeta \in W^{1,1}(Q; \mathbb{R}^N)$  with  $\nabla \zeta = G\zeta$  and  $\zeta|_\Gamma = 0$ . Then,  $\zeta = 0$ .*

*Proof.* Since

$$\|\zeta\|_{L^1(\Omega)} = \int_\Gamma \int_a^b |\zeta(\gamma, x)| dx d\gamma = \int_\Gamma \int_a^b |\zeta_\gamma(x)| dx d\gamma$$

we only have to show  $\zeta_\gamma = 0$  a.e. As  $\zeta \in W^{1,1}(Q)$ ,  $\zeta_\gamma \in W^{1,1}(I)$  f.a.a.  $\gamma \in \Gamma$  by [19, Th. 2.1.4], as mentioned before. Since  $\zeta'_\gamma$  is the last column  $\nabla \zeta e^N$  of  $\nabla \zeta$ , we have

$$\zeta'_\gamma = \nabla \zeta(\gamma, \cdot) e^N = G(\gamma, \cdot) \zeta(\gamma, \cdot) e^N = G(\gamma, \cdot) \zeta_\gamma e^N =: G_\gamma \zeta_\gamma.$$

For fixed  $(\gamma, x) \in Q$ ,  $G(\gamma, x)$  is a linear mapping from  $\mathbb{R}^N$  to  $\mathbb{R}^{N \times N}$ , its product with  $\zeta_\gamma(x) \in \mathbb{R}^N$  is an element of  $\mathbb{R}^{N \times N}$ , and multiplication by  $e^N$  gives an element of  $\mathbb{R}^N$  depending linearly on  $\zeta_\gamma(x)$ . Hence,  $G_\gamma(x)$  is a linear mapping from  $\mathbb{R}^N$  to  $\mathbb{R}^N$  a.e. Even  $G_\gamma \in L^1(I; \mathbb{R}^{N \times N})$  holds, since  $G \in L^1(Q)$ . Also,  $\zeta|_\Gamma = 0$  implies  $\zeta_\gamma(a) = 0$  f.a.a.  $\gamma \in \Gamma$ . By Lemma 3.1, we obtain  $\zeta_\gamma = 0$ . □

### 3.3. Unique continuation

**Lemma 3.3.** *Let  $\Omega \subset \mathbb{R}^N$  be a domain. Let  $G \in L^1(\Omega; \mathbb{R}^{(N \times N) \times N})$  and  $\zeta \in W^{1,1}(\Omega; \mathbb{R}^N)$  with  $\nabla \zeta = G\zeta$ . Moreover, let  $\zeta$  vanish in an open ball  $B \subset \Omega$ . Then,  $\zeta = 0$ .*

*Proof.* Let  $\Omega$  be convex and pick some  $x_1 \in B$ . Then, we can take a straight line between  $x_1$  and some other point  $x_2 \in \Omega$  and a cuboid  $Q$  containing this line and having one face being entirely located in  $B$ . Then, by Lemma 3.2,  $\zeta = 0$  in  $Q$  and hence in a whole neighborhood of  $x_2$ . Since  $x_2$  was arbitrary, we have  $\zeta = 0$  in  $\Omega$ . By induction, this can be carried over to connected unions of finitely many convex sets and hence works for path-connected sets, because every path between two points can be covered by such a finite union. Since domains are path connected, we finally achieve  $\zeta = 0$  in  $\Omega$ .  $\square$

**Remark 3.4.** *Lemma 3.3 can also be stated as follows: The equation  $\nabla \zeta = G\zeta$ , that is, the operator  $\nabla - G$  has the unique continuation property. Moreover, it is enough that  $\zeta$  vanishes on a small part of some  $(N - 1)$ -dimensional hyperplane.*

### 3.4. Proof of Theorem 2.1

Let  $\zeta$  be as in Theorem 2.1. If we can show that  $\zeta$  vanishes on an open set, we can apply Lemma 3.3, and hence,  $\zeta$  must vanish in the whole of  $\Omega$ . To make  $\zeta = 0$  on an open set, we transform a part of  $\Gamma$ , where we know  $\zeta$  to be zero, and a neighborhood  $U$  onto a cuboid  $Q$ , where we can use Lemma 3.2 and the transformed function is forced to vanish, hence also  $\zeta$  must vanish on  $U$ . Let us pick a point on  $\Gamma$  and a corresponding open neighborhood  $U$  as well as a bijective bi-Lipschitz transformation  $\varphi : \hat{Q} := (-1, 1)^N \rightarrow U$ , mapping the cuboid  $Q := (-1, 1)^{N-1} \times (0, 1)$  onto  $U \cap \Omega$  and  $(-1, 1)^{N-1} \times \{0\}$  onto  $U \cap \Gamma$ . Now,  $\tilde{G} := G \circ \varphi \in L^1(Q; \mathbb{R}^{(N \times N) \times N})$  and, see again, for example, [19, Th. 2.2.2],  $\tilde{\zeta} := \zeta \circ \varphi \in W^{1,1}(Q; \mathbb{R}^N)$ . By the chain rule, we have

$$\nabla \tilde{\zeta} = ((\nabla \zeta) \circ \varphi) \nabla \varphi = ((G\zeta) \circ \varphi) \nabla \varphi = \tilde{G} \tilde{\zeta} \nabla \varphi =: \hat{G} \tilde{\zeta}.$$

Since  $\nabla \varphi$  is uniformly bounded, we get

$$\forall z \in Q, y \in \mathbb{R}^N \quad \|\hat{G}(z)y\| \leq \|\tilde{G}(z)\| |y| \|\nabla \varphi(z)\| \leq c \|\tilde{G}(z)\| |y| \quad \Rightarrow \quad \|\hat{G}(z)\| \leq c \|\tilde{G}(z)\|$$

and hence

$$\int_Q \|\hat{G}(z)\| dz \leq c \int_Q \|\tilde{G}(z)\| dz = c \int_{\varphi(Q)} \|G(x)\| |\det \nabla \varphi^{-1}(x)| dx \leq c \int_{\Omega \cap U} \|G(x)\| dx < \infty$$

since  $\det \nabla \varphi^{-1}$  is uniformly bounded as well. Thus,  $\hat{G} \in L^1(Q; \mathbb{R}^{(N \times N) \times N})$ . Because  $\zeta$  vanishes on  $U \cap \Gamma$ ,  $\tilde{\zeta}$  vanishes on  $F := \varphi^{-1}(U \cap \Gamma) = (-1, 1)^{N-1} \times \{0\}$ . Hence,  $\tilde{\zeta} \in W^{1,1}(Q; \mathbb{R}^N)$  solves

$$\nabla \tilde{\zeta} = \hat{G} \tilde{\zeta}, \quad \tilde{\zeta}|_F = 0.$$

Lemma 3.2 implies  $\tilde{\zeta} = 0$  in  $Q$ . Thus,  $\zeta = \tilde{\zeta} \circ \varphi^{-1} = 0$  in  $U \cap \Omega$ , which contains an open ball.

### 4. Proof of the norm property

#### 4.1. Curl of matrix products

We identify  $\mathbb{R}^{3 \times 3}$  and  $\mathbb{R}^9$  by the following isomorphisms:

$$\text{mat} : \mathbb{R}^9 \rightarrow \mathbb{R}^{3 \times 3}, \quad \begin{bmatrix} a_1 \\ \vdots \\ a_9 \end{bmatrix} \mapsto \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}, \quad \text{vec} := \text{mat}^{-1} : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^9$$

We also use the following canonical isomorphism to identify  $\mathbb{R}^3$  and  $\mathfrak{so}(3)$ :

$$\text{axl} : \mathfrak{so}(3) \rightarrow \mathbb{R}^3, \quad \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Moreover, we define

$$\begin{aligned} \text{diagvec} : \mathbb{R}^{3 \times 3} &\rightarrow \mathbb{R}^3, \quad \begin{bmatrix} \textcircled{a_1} & a_2 & a_3 \\ a_4 & \textcircled{a_5} & a_6 \\ a_7 & a_8 & \textcircled{a_9} \end{bmatrix} \mapsto \begin{bmatrix} a_1 \\ a_5 \\ a_9 \end{bmatrix}, \\ \text{skewvec} : \mathbb{R}^{3 \times 3} &\rightarrow \mathbb{R}^3, \quad \begin{bmatrix} a_1 & \textcircled{a_2} & \textcircled{a_3} \\ a_4 & a_5 & \textcircled{a_6} \\ a_7 & a_8 & a_9 \end{bmatrix} \mapsto \begin{bmatrix} -a_6 \\ a_3 \\ a_2 \end{bmatrix}, \\ \text{symvec} : \mathbb{R}^{3 \times 3} &\rightarrow \mathbb{R}^3, \quad \begin{bmatrix} a_1 & a_2 & a_3 \\ \textcircled{a_4} & a_5 & a_6 \\ \textcircled{a_7} & \textcircled{a_8} & a_9 \end{bmatrix} \mapsto \begin{bmatrix} a_8 \\ a_7 \\ a_4 \end{bmatrix}. \end{aligned}$$

We note  $\text{skewvec} = \text{symvec} = \text{axl}$ ,  $\text{diagvec} = 0$  on  $\mathfrak{so}(3)$ . Furthermore, denoting by  $\times$  the cross product,  $Ax = \text{axl}(A) \times x$  and  $\text{axl}^{-1}(a)x = a \times x$  hold for all  $A \in \mathfrak{so}(3)$  and all  $a, x \in \mathbb{R}^3$ .

For a matrix  $Y \in \mathbb{R}^{3 \times 3}$  with  $Y^\top = [y_1 \ y_2 \ y_3]$  and vectors  $y_n \in \mathbb{R}^3$ , we define

$$\begin{aligned} L_{\text{diag}, Y} &= - \begin{bmatrix} \text{axl}^{-1} y_1 & 0 & 0 \\ 0 & \text{axl}^{-1} y_2 & 0 \\ 0 & 0 & \text{axl}^{-1} y_3 \end{bmatrix}, \\ L_{\text{skew}, Y} &= \begin{bmatrix} 0 & -\text{axl}^{-1} y_3 & \text{axl}^{-1} y_2 \\ \text{axl}^{-1} y_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ L_{\text{sym}, Y} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\text{axl}^{-1} y_1 \\ -\text{axl}^{-1} y_2 & \text{axl}^{-1} y_1 & 0 \end{bmatrix}, \\ L_Y &:= L_{\text{skew}, Y} + L_{\text{sym}, Y} = \begin{bmatrix} 0 & -\text{axl}^{-1} y_3 & \text{axl}^{-1} y_2 \\ \text{axl}^{-1} y_3 & 0 & -\text{axl}^{-1} y_1 \\ -\text{axl}^{-1} y_2 & \text{axl}^{-1} y_1 & 0 \end{bmatrix} \end{aligned}$$

and note  $L_Y^\top = L_Y$ . Furthermore, for vector fields  $v$  in  $\mathbb{R}^3$ , we set

$$\hat{\nabla} v := \text{vec } \nabla v,$$

denoting the vector field containing the nine partial derivatives of the three components of  $v$ .

Now, we extend Neff’s formula from [7, Lemma 3.7] in two ways, such that it can be applied with weaker differentiability and for general matrices. For this, we define

$$W^s(\text{Curl}, \Omega; \mathbb{R}^{3 \times 3}) := \{Y \in L^s(\Omega; \mathbb{R}^{3 \times 3}) : \text{Curl } Y \in L^s(\Omega; \mathbb{R}^{3 \times 3})\}.$$

**Remark 4.1.** For skew-symmetric matrix fields, we have  $W^s(\text{Curl}, \Omega; \mathbb{R}^{3 \times 3}) = W^{1,s}(\Omega; \mathbb{R}^{3 \times 3})$ , since in this case the Curl controls all the derivatives, see [12].

**Lemma 4.2.** Let  $r, s \in (1, \infty)$  with  $1/r + 1/s = 1$  and  $X \in W^{1,r}(\Omega; \mathbb{R}^{3 \times 3})$  and  $Y \in W^s(\text{Curl}, \Omega; \mathbb{R}^{3 \times 3})$ . Then,  $XY \in W^1(\text{Curl}, \Omega; \mathbb{R}^{3 \times 3})$  and

$$\text{Curl}(XY) = \text{mat}(L_{\text{diag}, Y} \hat{\nabla} \text{diagvec} + L_{\text{skew}, Y} \hat{\nabla} \text{skewvec} + L_{\text{sym}, Y} \hat{\nabla} \text{symvec})X + X \text{Curl } Y \tag{7}$$

with  $L_{\text{diag}, Y}, L_{\text{skew}, Y}, L_{\text{sym}, Y} \in L^s(\Omega; \mathbb{R}^{9 \times 9})$ . For skew-symmetric  $X$  formula (7) turns to

$$\text{Curl}(XY) = \text{mat } L_Y(\hat{\nabla} \text{axl } X) + X \text{Curl } Y, \tag{8}$$

where  $\det L_Y = -2(\det Y)^3$ . Hence, if  $Y$  is invertible, so is  $L_Y$ .

We note that for smooth ( $C^1$ ) matrices  $X, Y$ , where  $X$  is skew-symmetric, formula (8) was already shown in [7, Lemma 3.7].

*Proof.* Since  $C^\infty(\bar{\Omega})$  is dense in both  $W^{1,r}(\Omega)$  and  $W^s(\text{Curl}, \Omega)$ , we have to show (7) only for smooth matrix fields. But this is a straight forward calculation, which we present in the ‘‘Appendix’’. (8) is a simple consequence from (7), and the assertion about the determinants has been proved already in [7, Lemma 3.7].  $\square$

**4.2. Proof of Theorem 2.2**

Let  $u \in C^\infty_0(\Omega, \Gamma; \mathbb{R}^3)$  with  $\|u\| = 0$ . We have to show  $u = 0$ . Note that  $\text{sym}(\nabla u P^{-1}) = 0$  implies

$$\nabla u P^{-1} = A, \tag{9}$$

where  $A$  is some skew-symmetric matrix field. Moreover, since  $AP = \nabla u$ , we have

$$\text{Curl}(AP) = 0. \tag{10}$$

Without loss of generality, we assume  $\Gamma$  to be bounded (otherwise, replace  $\Gamma$  by a bounded open subset of itself) and that the compact set  $\text{supp } u$  and  $\Gamma$  are both contained in some open ball  $B$ . Define  $\tilde{\Omega} := \Omega \cap B$ . Then,  $\nabla u, P, P^{-1}$  and  $A$  belong to  $L^\infty(\tilde{\Omega}, \mathbb{R}^{3 \times 3}) \subset L^r(\tilde{\Omega}, \mathbb{R}^{3 \times 3})$  for all  $r \in [1, \infty]$ .

Since  $\text{Curl } P^{-1} \in L^q(\tilde{\Omega}, \mathbb{R}^{3 \times 3})$ , Lemma 4.2 with  $X := \nabla u$  and  $Y := P^{-1}$  together with Remark 4.1 show  $A \in W^{1,1}(\tilde{\Omega}, \mathbb{R}^{3 \times 3})$  and by (7) even  $A \in W^{1,q}(\tilde{\Omega}, \mathbb{R}^{3 \times 3})$  holds. Another application of Lemma 4.2 with  $X := A$  and  $Y := P$  gives by (8) and (10)

$$\text{mat } L_P(\hat{\nabla} \text{axl } A) + A \text{Curl } P = 0,$$

since  $A$  is skew-symmetric. Thus,  $\zeta := \text{axl } A \in W^{1,q}(\tilde{\Omega}, \mathbb{R}^3) \subset W^{1,1}(\tilde{\Omega}, \mathbb{R}^3)$  solves

$$\nabla \zeta = - \text{mat } L_P^{-1} \text{vec}(\text{axl}^{-1} \zeta \text{Curl } P) =: G_P \zeta. \tag{11}$$

Since  $L_P, L_P^{-1} \in L^\infty(\tilde{\Omega}, \mathbb{R}^{9 \times 9})$  and  $\text{Curl } P \in L^p(\tilde{\Omega}, \mathbb{R}^{3 \times 3}) \subset L^1(\tilde{\Omega}, \mathbb{R}^{3 \times 3})$ , also  $G_P$  belongs to  $L^p(\tilde{\Omega}, \mathbb{R}^{(3 \times 3) \times 3}) \subset L^1(\tilde{\Omega}, \mathbb{R}^{(3 \times 3) \times 3})$ . Additionally,  $A$  and hence  $\zeta$  vanish on  $\Gamma$  by (9) since  $u$  does. By Theorem 2.1,  $\zeta$  and therefore  $A$  and  $\nabla u$  vanish in  $\tilde{\Omega}$ . Thus,  $u = 0$  in  $\tilde{\Omega}$  because  $u$  vanishes on  $\Gamma$ . Since  $\text{supp } u \subset \tilde{\Omega}$ , we finally obtain  $u = 0$  in  $\Omega$ .

### 5. Proof of Theorem 2.7

$A := \nabla\Phi(\nabla\Psi)^{-1} \in L^p(\Omega; \mathbb{R}^{3 \times 3})$  is skew-symmetric by (5). Since the standard mollification preserves skew-symmetry, we can pick a sequence  $(A_n) \subset C^\infty_0(\Omega; \mathbb{R}^{3 \times 3})$  of skew-symmetric smooth matrices approximating  $A$  in  $L^p(\Omega)$ . Applying Lemma 4.2, that is, (8), to  $A_n \nabla\Psi$ , we get

$$\text{Curl}(A_n \nabla\Psi) = \text{mat } L_{\nabla\Psi}(\hat{\nabla} \text{axl } A_n)$$

with invertible  $L_{\nabla\Psi} \in W^{1,q}(\Omega)$  satisfying  $L_{\nabla\Psi}^{-1} \in W^{1,q}(\Omega)$  by assumption on the regularity of  $\Psi$ . Pick  $\Theta \in C^\infty_0(\Omega; \mathbb{R}^{3 \times 3})$ . Then,  $L_{\nabla\Psi}^{-\top} \text{vec } \Theta \in W^{1,q}_0(\Omega)$  and since  $A \nabla\Psi = \nabla\Phi \in L^p(\Omega; \mathbb{R}^{3 \times 3})$  with  $\text{Curl}(A \nabla\Psi) = 0$  we have

$$\langle A_n \nabla\Psi, \text{Curl}(\text{mat } L_{\nabla\Psi}^{-\top} \text{vec } \Theta) \rangle_{L^2(\Omega)} \rightarrow \langle A \nabla\Psi, \text{Curl}(\text{mat } L_{\nabla\Psi}^{-\top} \text{vec } \Theta) \rangle_{L^2(\Omega)} = 0.$$

On the other hand, we have for the left-hand side

$$\begin{aligned} \langle A_n \nabla\Psi, \text{Curl}(\text{mat } L_{\nabla\Psi}^{-\top} \text{vec } \Theta) \rangle_{L^2(\Omega)} &= \langle \text{Curl}(A_n \nabla\Psi), \text{mat } L_{\nabla\Psi}^{-\top} \text{vec } \Theta \rangle_{L^2(\Omega)} \\ &= \langle L_{\nabla\Psi}(\hat{\nabla} \text{axl } A_n), L_{\nabla\Psi}^{-\top} \text{vec } \Theta \rangle_{L^2(\Omega)} \\ &= \langle \hat{\nabla} \text{axl } A_n, L_{\nabla\Psi}^\top L_{\nabla\Psi}^{-\top} \text{vec } \Theta \rangle_{L^2(\Omega)} \\ &= \langle \nabla \text{axl } A_n, \Theta \rangle_{L^2(\Omega)} \\ &= \langle \text{axl } A_n, \text{Div } \Theta \rangle_{L^2(\Omega)} \rightarrow \langle \text{axl } A, \text{Div } \Theta \rangle_{L^2(\Omega)}. \end{aligned}$$

Hence,  $\nabla \text{axl } A = 0$ , and therefore,  $A \in \mathfrak{so}(3)$  is constant. Thus,  $\nabla(\Phi - A\Psi) = \nabla\Phi - A \nabla\Psi = 0$  and  $\Phi = A\Psi + a$  with some  $a \in \mathbb{R}^3$ .

### A Appendix

We show (7) for smooth matrix fields  $X = [x_{nm}]_{n,m=1,2,3}$  and  $Y = [y_{nm}]_{n,m=1,2,3}$ . The  $l$ -th row of  $XY$  is the transpose of the vector having the entries  $x_{ln}y_{nk}$  for  $k = 1, 2, 3$ . Thus, the  $l$ -th row of  $\text{Curl}(XY)$  is the transpose of the vector having the entries  $\partial_i(x_{ln}y_{nj}) - \partial_j(x_{ln}y_{ni})$  for  $k = 1, 2, 3$ , where the curl of a vector field  $v$  is written as

$$\text{curl } v = \begin{bmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{bmatrix} = [\partial_i v_j - \partial_j v_i]_{k=1,2,3}.$$

Therefore,

$$\begin{aligned} [\text{Curl}(XY)]_{lk} &= \partial_i x_{ln} y_{nj} - \partial_j x_{ln} y_{ni} + x_{ln} \underbrace{(\partial_i y_{nj} - \partial_j y_{ni})}_{=[\text{Curl } Y]_{nk}} \\ &= \partial_i x_{ln} y_{nj} - \partial_j x_{ln} y_{ni} + [X \text{Curl } Y]_{lk}. \end{aligned}$$

With the transpose of the  $n$ -th row of  $Y$  denoted by  $[y_n]_j := y_{nj}$ , we get

$$[\text{Curl}(XY)]_{lk} - [X \text{Curl } Y]_{lk} = [\nabla x_{ln} \times y_n]_k$$

and hence for the  $l$ -th row  $[\text{Curl}(XY) - X \text{Curl } Y]_l = [(\nabla x_{ln} \times y_n)^\top]_l$ . Finally, we obtain:

$$\begin{aligned} \text{Curl}(XY) - X \text{Curl } Y &= \begin{bmatrix} (\nabla x_{1n} \times y_n)^\top \\ (\nabla x_{2n} \times y_n)^\top \\ (\nabla x_{3n} \times y_n)^\top \end{bmatrix} \\ &= \begin{bmatrix} (\nabla x_{11} \times y_1)^\top \\ (\nabla x_{22} \times y_2)^\top \\ (\nabla x_{33} \times y_3)^\top \end{bmatrix} + \begin{bmatrix} (\nabla x_{12} \times y_2)^\top + (\nabla x_{13} \times y_3)^\top \\ (\nabla x_{23} \times y_3)^\top \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ (\nabla x_{21} \times y_1)^\top \\ (\nabla x_{31} \times y_1)^\top + (\nabla x_{32} \times y_2)^\top \end{bmatrix} \end{aligned}$$



$$\begin{aligned}
&= - \begin{bmatrix} (\text{axl}^{-1} y_1 \nabla x_{11})^\top \\ (\text{axl}^{-1} y_2 \nabla x_{22})^\top \\ (\text{axl}^{-1} y_3 \nabla x_{33})^\top \end{bmatrix} - \begin{bmatrix} (\text{axl}^{-1} y_2 \nabla x_{12})^\top + (\text{axl}^{-1} y_3 \nabla x_{13})^\top \\ (\text{axl}^{-1} y_3 \nabla x_{23})^\top \\ 0 \end{bmatrix} \\
&\quad - \begin{bmatrix} 0 \\ (\text{axl}^{-1} y_1 \nabla x_{21})^\top \\ (\text{axl}^{-1} y_1 \nabla x_{31})^\top + (\text{axl}^{-1} y_2 \nabla x_{32})^\top \end{bmatrix} \\
&= - \begin{bmatrix} \text{axl}^{-1} y_1 & 0 & 0 \\ \text{mat } 0 & \text{axl}^{-1} y_2 & 0 \\ 0 & 0 & \text{axl}^{-1} y_3 \end{bmatrix} \begin{bmatrix} \nabla x_{11} \\ \nabla x_{22} \\ \nabla x_{33} \end{bmatrix} \\
&\quad - \text{mat} \begin{bmatrix} 0 & \text{axl}^{-1} y_3 - \text{axl}^{-1} y_2 \\ -\text{axl}^{-1} y_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\nabla x_{23} \\ \nabla x_{13} \\ -\nabla x_{12} \end{bmatrix} \\
&\quad - \text{mat} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \text{axl}^{-1} y_1 \\ \text{axl}^{-1} y_2 - \text{axl}^{-1} y_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \nabla x_{32} \\ -\nabla x_{31} \\ \nabla x_{21} \end{bmatrix} \\
&= \text{mat}(L_{\text{diag}, Y} \hat{\nabla} \text{diagvec } X + L_{\text{skew}, Y} \hat{\nabla} \text{skewvec } X + L_{\text{sym}, Y} \hat{\nabla} \text{symvec } X)
\end{aligned}$$

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