## Linear Elliptic Boundary Value Problems of Second Order

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Dieses Skriptum ist für eine summer-school entstanden, die im August 1993 an der Universität Jyväskylä, Finnland, stattgefunden hat. Es wird in die Theorie der Randwertprobleme für lineare elliptische partielle Differentialgleichungen zweiter Ordnung eingeführt, und die hierzu erforderliche Funktionalanalysis wird bereitgestellt. Hier in Essen wird üblicherweise die Vorlesung "Partielle Differentialgleichungen I" im Sommer als Vorlesung für das 6. Semester angeboten, und es wird empfohlen, im vorangehenden Wintersemester die Vorlesung "Funktionalanalysis I" zu hören. Dieses Skriptum faßt Teile einer Vorlesung "Funktionalanalysis I" und den Hauptteil einer Vorlesung "Partielle Differentialgleichungen I" zusammen.

## Chapter -1

# Introduction

The aim of this course is to introduce the basic methods for the treatment of boundary value problems for second order elliptic partial differential equations. Many physical applications are modelled by equations of the type

$$\Delta u(x) = f(x) \qquad (\Delta u := \sum_{i=1}^{3} \partial_i^2 u = \frac{\partial^2}{\partial x_1^2} u + \frac{\partial^2}{\partial x_2^2} u + \frac{\partial^2}{\partial x_3^2} u)$$
(-1.1)

(the *potential equation*),

$$\frac{\partial}{\partial t}u(x,t) - \Delta u(x,t) = f(x,t) \tag{-1.2}$$

(the heat equation),

$$\frac{\partial^2}{\partial t^2} u(x,t) - \Delta u(x,t) = f(x,t)$$
(-1.3)

(the wave equation),

$$i \frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) + p(x)u(x,t) = 0$$
(-1.4)

(the Schrödinger equation with a given potential p).

Here u is a function defined in some domain  $\Omega$  of Euclidean space  $\mathbb{R}^3$ , say, or in some "space-time" cylinder  $\Omega \times I$ , I an interval on the real line. u should be considered as unknown while f is assumed to be a given function defined in the same domain as u. For example equation (-1.1)

might be used to find a field  $\nabla u = \begin{bmatrix} \partial_1 u \\ \partial_2 u \\ \partial_3 u \end{bmatrix}$  of force from its sources f.

Equation (-1.2) might describe the distribution of temperature u(x, t) at the point x and at time t from heat sources f(x, t). Equation (-1.3) might describe acoustic waves, and the Schrödinger equation describes the evolution of a quantum mechanical particle in a potential. In case of the Schrödinger equation,  $\Omega$  is the whole space.

Of course, the spatial domain  $\Omega$  might also be of one, two and more than three dimensions.

As in the theory of ordinary differential equations neither of these three equations alone are sufficient to describe its solutions uniquely: one has to add boundary conditions and initial conditions. A boundary condition is an equation for u and its derivatives on the boundary of the spatial domain, e.g.

$$u(x) = g(x) \quad \text{for } x \in \partial\Omega \tag{-1.5}$$

(in case of equ. (-1.1)) or

$$u(x,t) = g(x,t) \quad \text{for } x \in \partial\Omega , \ t \in I$$
 (-1.6)

(in case of equ. (-1.2) or (-1.3)). (-1.5) as well as (-1.6) is the so called "Dirichlet (boundary) condition", and due to lack of time we shall concentrate on this condition. This Dirichlet condition says that the unknown function is known on the spatial boundary  $\partial\Omega$  (in case of the potential equation) at any time t (in case of the heat or wave equation). In the case of the two latter equations one also needs *initial conditions* for the determination of u, i.e. one has to describe u and — in case of the wave equation — also  $\frac{\partial u}{\partial t}$  at time t = 0.

Thus the typical problems with respect to potential-, heat-, and wave-equation are:

(EBVP) For given functions  $f: \Omega \to \mathbf{R}, g: \partial \Omega \to \mathbf{R}$  find a function  $u: \overline{\Omega} \to \mathbf{R}$  such that

$$\Delta u(x) = f(x) \quad \text{for } x \in \Omega \tag{-1.7}$$

$$u(x) = g(x) \quad \text{for } x \in \partial \Omega \quad .$$
 (-1.8)

(PIBVP) For given functions  $f : \Omega \times (0,T) \to \mathbf{R}$  (here T > 0),  $g : \partial \Omega \times (0,T) \to \mathbf{R}$  and  $u_0 : \Omega \to \mathbf{R}$ find  $u : \overline{\Omega} \times [0,T) \to \mathbf{R}$  such that

$$\frac{\partial}{\partial t}u(x,t) - \Delta u(x,t) = f(x,t) \quad \text{for } (x,t) \in \Omega \times (0,T)$$
(-1.9)

$$u(x,t) = g(x,t) \quad \text{for } (x,t) \in \partial\Omega \times (0,T)$$
(-1.10)

$$u(x,0) = u_0(x)$$
 for  $x \in \Omega$ . (-1.11)

(HIBVP) For given functions  $f : \Omega \times (0,T) \to \mathbf{R}$ ,  $g : \partial\Omega \times (0,T) \to \mathbf{R}$ ,  $u_0, u_1 : \Omega \to \mathbf{R}$  find  $u : \overline{\Omega} \times [0,T) \to \mathbf{R}$  such that

$$\frac{\partial^2}{\partial t^2} u(x,t) - \Delta u(x,t) = f(x,t)$$
(-1.12)

$$u|_{\partial\Omega} \times (0,T) = g ; \qquad (-1.13)$$

$$u(x,0) = u_0(x)$$
 for  $x \in \Omega$ ; (-1.14)

$$\frac{\partial}{\partial t}u(x,0) = u_1(x) \quad . \tag{-1.15}$$

(f,g) resp.  $(f,g,u_0)$  resp.  $(f,g,u_0,u_1)$  are called the data of the respective problem.

All three problems are linear problems: if u resp. v are solutions with respect to the data U resp. V, then  $\alpha u + \beta v$  is a solution with respect to the data  $\alpha U + \beta V$ ,  $\alpha, \beta \in \mathbf{R}$ .

(EBVP) is a typical *elliptic* problem, while the two others are typical *parabolic* resp. *hyperbolic* problems. The definition of an elliptic problem will be given later. The following example hopefully explains why we concentrate on elliptic problems.

**Example -1.1** (The vibrating string) Consider a string which is fixed at its two end points and after having been extended vibrates in an (x, y)-plane. At any time t the string may be modelled by the graph of a real valued function  $u(\cdot, t)$  on  $\overline{\Omega}$ , the closure of the open intervall  $\Omega = (0, L)$  on the real line. u(x, t) describes the displacement of the point x at time t in the vertical direction.

Under some idealizing assumptions (small amplitudes e.g.) one is lead to a hyperbolic initial boundary value problem (HIBVP):

$$\frac{\partial^2}{\partial t^2} u(x,t) - \frac{\partial^2}{\partial x^2} u(x,t) = 0 \quad \text{for } (x,t) \in \Omega \times I , \qquad I := (0,\infty) , \qquad (-1.16)$$

$$u(0,t) = u(L,t) = 0$$
 for  $t \in \overline{I}$  (-1.17)

$$u(x,0) = u_0(x) \quad \text{for } x \in \Omega \tag{-1.18}$$

$$\frac{\partial u}{\partial t}(x,0) = u_1(x) \quad \text{for } x \in \Omega \quad .$$
 (-1.19)

In order to make the formulation precise we have to say that we wish to find u as an element of

$$u \in C^2(\Omega \times I) \cap C^1(\Omega \times \overline{I}) \cap C^0(\overline{\Omega \times I}) ,$$

which means: u is continuous in  $\Omega \times I$  and has continuous partial derivatives in  $\Omega \times I$  up to the second order, u and its first order partial derivatives can be extended to  $\Omega \times \overline{I}$  as continuous functions, u can be extended as a continuous function to  $\overline{\Omega \times I}$ .

We first of all look for "standing wave solutions". These are nontrivial solutions of (-1.16), (-1.17) of the special form

$$u(x,t) = v(x) w(t)$$

with  $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ ,  $w \in C^2(I) \cap C^1(\overline{I})$ . Insertion into (-1.16) gives for all  $x \in \Omega$ ,  $t \in I$ :

$$v(x) w''(t) - v''(x) w(t) = 0 , \qquad (-1.20)$$

and by (-1.17)

$$v(0) = v(L) = 0 \tag{-1.21}$$

since w is assumed not a vanish identically. Fixing t in such a way that  $w(t) \neq 0$  and putting  $\alpha := \frac{w''(t)}{w(t)}$  we find for all  $x \in \Omega$ 

$$v''(x) = \alpha v(x)$$
 . (-1.22)

Together with (-1.21) and remembering  $v \not\equiv 0$ , we conclude that

$$\alpha \in \{-(\frac{n\pi}{L})^2: n \in \mathbf{N}\}, \quad \mathbf{N} := \{1; 2; 3\cdots\}$$
 (-1.23)

$$u \in \{v_n: n \in \mathbf{N}\}, \quad v_n(x) := \sin(\omega_n x), \quad \omega_n := \frac{n\pi}{L}$$
 (-1.24)

We then obtain that w is of the form

$$w(t) = A_n \cos(\omega_n t) + B_n \sin(\omega_n t)$$

with  $A_n, B_n \in \mathbf{R}$ ,  $(A_n, B_n) \neq (0, 0)$ .

Thus standing waves are of the form

$$u(x,t) = [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)] v_n(x) \quad .$$
(-1.25)

Problem (-1.16) - (-1.19) may now be solved by the ansatz

$$u(x,t) = \sum_{n=1}^{\infty} \left[ A_n \cos(w_n t) + B_n \sin(w_n t) \right] v_n(x)$$
(-1.26)

i.e. a superposition of standing waves. Disregarding any questions about convergence, one expects that (-1.26) solves (-1.16) and (-1.17). To fulfill (-1.18) and (-1.19) we have to choose  $A_n$  and  $B_n$  such that

$$u_0(x) = \sum_{n=1}^{\infty} A_n v_n(x)$$
(-1.27)

$$u_1(x) = \sum_{n=1}^{\infty} (\omega_n B_n) v_n(x) \quad . \tag{-1.28}$$

(-1.27), (-1.28) is just a Fourier-series expansion for  $u_0$  and  $u_1$ : continue  $u_0, u_1$  as odd functions into the intervall [-L, L], then the Fourier series does not contain any cosine terms.

One can make this procedure precise if one assumes some regularity of the data  $u_0, u_1$ . By a similar procedure one may treat the one-dimensional heat equation.

Returning to the problems (-1.12) - (-1.15) resp. (-1.9) - (-1.11) one might ask if they can be solved by a similar procedure as sketched above. The key result was the existence of the sequences  $(-\omega_n^2)_{n \in \mathbb{N}}$  and  $(v_n)_{n \in \mathbb{N}}$  (c.f. (-1.23), (-1.24)) of *eigenvalues* and *eigenfunctions* such that the initial data  $u_0, u_1$  resp. could be expanded into a series of the form  $\sum a_n v_n$ .

We are hence lead to the question whether there exist sequences  $(\lambda_n)_{n \in \mathbb{N}}$  and  $(v_n)_{n \in \mathbb{N}}$  of numbers and functions resp. such that for any n:

$$v_n \neq 0 \tag{-1.29}$$

$$\Delta v_n = \lambda_n v_n \quad \text{in } \Omega \tag{-1.30}$$

$$v_n = 0 \quad \text{on } \partial \Omega \quad .$$
 (-1.31)

 $\lambda_n$  and  $v_n$  are then called an eigenvalue and a corresponding eigenfunction of the Laplace-operator  $\Delta$  under the Dirichlet condition. Having found eigenvalues and eigenfunctions one has to check if there are enough eigenfunctions to expand any given initial data as a series of the form  $\sum_{n \in \mathbf{N}} a_n v_n$ .

As in the theory of Fourier series it is convenient to work in  $L^2(\Omega)$ , the space of (equivalence classes of) square integrable functions on  $\Omega$ . In these lectures we shall consider elliptic boundary value problems as e.g. (-1.7), (-1.8) and the elliptic eigenvalue problem as e.g. (-1.29), (-1.30), (-1.31). We shall introduce suitable Hilbert spaces of square integrable functions and generalize the notion of partial derivation for such functions. It will be shown that the eigenfunctions form a complete orthonormal subset of  $L^2(\Omega)$ , which means that expansion theorems are valid.

The solutions will be considered rather as points in a space than as functions on some domain. This point of view has lead to the development of Functional Analysis, and we shall use some of the methods of this large field of mathematical research. We shall restrict ourselves on those results of functional analysis which are effectively needed for the treatment of our boundary value problem.

## Chapter 0

## Prerequisites

From now on,  $\Omega$  will *always* denote some domain in Euclidean *N*-space  $\mathbf{R}^N$ , i.e. a connected open subset of  $\mathbf{R}^N$ . Moreover we introduce the following spaces of complex valued functions on  $\Omega$ :

- $C^k(\Omega)$ :  $(k \in \mathbf{N}_0 := \{0\} \cup \mathbf{N})$  consists of all continuous functions on  $\Omega$  which in case k > 0 have continuous partial derivatives up to the k-th order.
- $C^k(\overline{\Omega})$ : consists of all functions in  $C^k(\Omega)$  which together with its partial derivatives up to order k can be continued to the closure  $\overline{\Omega}$  of  $\Omega$  as continuous functions.

$$\begin{split} C^{\infty}(\Omega) &:= \bigcap_{k \in \mathbf{N}} C^{k}(\Omega) \\ C^{\infty}_{0}(\Omega) &:= \left\{ \varphi \in C^{\infty}(\Omega) \colon \text{supp } \varphi \subset \subset \Omega \right\}. \end{split}$$

Here

$$\operatorname{supp} \varphi := \overline{\{x \in \Omega \colon \varphi(x) \neq 0\}}$$
(0.1)

is the support of  $\varphi$  and for two subsets  $A, B \subset \mathbf{R}^N$  we write

 $A \subset \subset B : \iff \overline{A}$  is compact and  $\overline{A} \subset B$ .

supp  $\varphi \subset \subset \Omega$  then means that  $\varphi$  vanishes in a neighbourhood of the boundary  $\partial \Omega$  of  $\Omega$  and in case that  $\Omega$  is unbounded — for sufficiently large argument. The elements of  $C_0^{\infty}(\Omega)$  are called test-functions and are always assumed to be continued by 0 into the whole of  $\mathbf{R}^N$ .

The elements of

 $L^{p}(\Omega), p \in [1, \infty)$ , are equivalence classes of (Lebesgue)-measurable functions u, such that  $|u|^{p}$  is integrable on  $\Omega$ . Two functions  $u_{1}, u_{2}$  are equivalent if  $u_{1}$  and  $u_{2}$  differ only on a Lebesgue null-set. The Lebesgue measure in  $\mathbb{R}^{N}$  will be denoted by  $\mu$   $L^{\infty}(\Omega)$  consists of all equivalence classes of measurable functions u such that

$$\mathrm{ess} \sup_{\Omega} |u| := \inf_{M} \sup_{\Omega \setminus M} |u| < \infty$$

Here the inf is to be taken over all Lebesgue null-sets M.

We shall frequently speak somewhat losely about functions in  $L^p(\Omega)$  hereby identifying an equivalence class with one of its representants. For example: if we say that an element  $u \in L^p(\Omega)$  belongs to  $C^k(\Omega)$  we mean that the equivalence class u contains a (unique) function in  $C^k(\Omega)$ , and this function in  $C^k(\Omega)$  is meant by u in further calculations. Thus we may in future write  $L^p(\Omega) \cap C^k(\Omega)$  for example, and mean the space of all functions u in  $C^k(\Omega)$  with  $\int_{\Omega} |u|^p d\mu < \infty$  as well as the space of all equivalence classes in  $L^p(\Omega)$  which contain an element of  $C^k(\Omega)$ .

The following result should be known from the Calculus course:

**Lemma 0.1** For  $p \in [1, \infty)$  the space  $C_0^{\infty}(\Omega)$  is dense in  $L^p(\Omega)$ , i.e. for any  $u \in L^p(\Omega)$  and  $\varepsilon > 0$ there exists a  $\varphi \in C_0^{\infty}(\Omega)$  with supp  $\varphi \subset \subset \Omega$  such that

$$\left(\int_{\Omega} |u-\varphi|^p \ d\mu\right)^{1/p} < \varepsilon \quad . \tag{0.2}$$

The spaces  $L^p(\Omega)$ ,  $p \in [1, \infty]$ , are Banach-spaces when they are equipped with the norms

$$||u||(L^p(\Omega)) := \left(\int_{\Omega} |u|^p \ d\mu\right)^{1/p} \quad \text{(for } p < \infty)$$

respectively

$$||u||(L^{\infty}(\Omega)) = \operatorname{ess sup}_{\Omega} |u| \quad (\text{for } p = \infty)$$

A norm is a real valued function on a real or complex linear space X:

$$\|\cdot\|: X \to \mathbf{R}$$

with the properties

$$||x|| > 0 \quad \text{for all} \ x \in X \setminus \{0\} \tag{0.3}$$

$$\|\lambda x\| = |\lambda| \|x\| \quad \text{for all } x \in X, \ \lambda \in \mathbf{K}$$
(0.4)

where  $\mathbf{K} = \mathbf{R}$  or  $\mathbf{K} = \mathbf{C}$ . Particularly: ||0|| = 0.

$$||x+y|| \le ||x|| + ||y|| \quad \text{for all } x, y \in X \quad . \tag{0.5}$$

The last inequality is called the triangle inequality. A linear space in which a norm is defined is called a *normed space*. In a normed space X one may consider convergence and Cauchy-convergence: A sequence  $(u_n)_{n \in \mathbb{N}}$  in X converges to some element  $u \in X$  if

$$\lim_{n \to \infty} \|u_n - u\| = 0 \quad . \tag{0.6}$$

We shall write  $u_n \to u$  or more precisely  $u_n \stackrel{X}{\to} u$  in this case. A sequence  $(u_n)_{n \in \mathbb{N}}$  in X is called *Cauchy convergent* if there exists a null-sequence  $(\varepsilon_n)$  of positive real numbers such that

$$\forall_{n \in \mathbf{N}} \ \forall_{m \ge n} \ \|u_n - u_m\| \le \varepsilon_n \ . \tag{0.7}$$

Convergent sequences are always Cauchy-convergent. However the converse might be wrong. A normed space X where any Cauchy-convergent sequence is convergent (to some element of X) is called a complete normed space or a *Banach space*.

It is assumed that you are familiar with the very basic notions of convergence in a Banach-space. For example, Lemma 0.1 expresses that  $C_0^{\infty}(\Omega)$  is dense in  $L^p(\Omega)$ , if  $1 \le p < \infty$ .

The proof that  $L^p(\Omega)$  is a Banach space may be found in many textbooks in analysis, e.g. in [5]. In the special case of  $L^p(\Omega)$ ,  $p \in (1, \infty)$ , the triangle inequality is called *Minkowski's inequality* and follows from *Hölder's inequality*: If  $f \in L^p(\Omega)$ ,  $g \in L^q(\Omega)$  and  $\frac{1}{p} + \frac{1}{q} = 1$  then  $fg \in L^1(\Omega)$ and

$$\int_{\Omega} |fg| \, d\mu \, \leq \left(\int_{\Omega} |f|^p \, d\mu\right)^{1/p} \left(\int_{\Omega} |g|^q \, d\mu\right)^{1/q} \, . \tag{0.8}$$

We shall frequently use a family of operators, called *Friedrichs'- mollifiers*: With the notation U(z,r) for an open ball in  $\mathbf{R}^N$  of radius r and with centre z we choose some  $j \in C_0^{\infty}(U(0,1))$  such that  $\int j(x)d\mu(x) = 1$  and  $j \ge 0$ . For example with some R < 1 and a suited c > 0 one might choose

$$j(x) := \begin{cases} c \exp \left[ 1/(|x|^2 - R^2) \right] & \text{if } |x| < R \\ 0 & \text{if } |x| \ge R \end{cases}$$

For  $\varepsilon>0\;,\;x\in {\bf R}^N$  put

$$j_{arepsilon}(x):=arepsilon^{-N}\,j(arepsilon^{-1}x)$$
 .

Then

$$j_{\varepsilon} \in C_0^{\infty}(U(0,\varepsilon)) \tag{0.9}$$

$$\int_{\mathbf{R}^N} j_{\varepsilon}(x) \, dx = 1 \tag{0.10}$$

$$j_{\varepsilon} \ge 0 \quad \text{in } \mathbf{R}^N \tag{0.11}$$

hold. The Friedrichs' mollifiers assign to any  $f \in L^p(\Omega)$ ,  $p \in [1, \infty)$  a family  $(j_{\varepsilon} * f)_{0 < \varepsilon < \varepsilon_0}$  of functions given by

$$(j_{\varepsilon} * f)(x) = \int_{\Omega} j_{\varepsilon}(x - y) f(y) d\mu(y) , \qquad (0.12)$$

the convolution of  $j_{\varepsilon}$  with f. Its result  $j_{\varepsilon} * f$  is in  $C^{\infty}(\mathbf{R}^N)$  since any differentiation may be carried out under the integral by Lebesgue's theorem:

$$[\partial^{\alpha} (j_{\varepsilon} * f)](x) = \int_{\Omega} (\partial^{\alpha} j_{\varepsilon})(x - y) f(y) d\mu(y) \quad . \tag{0.13}$$

Here  $\alpha$  denotes a multi-index  $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbf{N}_0^N$  and

$$\partial^{\alpha} := \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N} := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} , \quad |\alpha| := \alpha_1 + \cdots + \alpha_N .$$
(0.14)

 $|\alpha|$  is called the order of  $\alpha$ .

Friedrichs' mollifier may be used to approximate  $L^p$ -functions by  $C^{\infty}$  functions:

**Lemma 0.2** Let  $p \in [1, \infty)$ ,  $u \in L^p(\Omega)$ . Then

$$\lim_{\varepsilon \searrow 0} \|j_{\varepsilon} * u - u\|(L^p(\Omega)) = 0$$

**Proof:** It suffices to consider the case where  $\Omega = \mathbf{R}^N$ , since any element of  $L^p(\Omega)$  may be identified with an element in  $L^p(\mathbf{R}^N)$  by continuation by 0. Hence if the lemma is proven for  $\mathbf{R}^N$  then it is obtained for  $\Omega$  by the estimate

$$\|j_{\varepsilon} * u - u\|_{\Omega} \le \|j_{\varepsilon} * u - u\|_{\mathbf{R}^{N}}$$

In this proof norms will now be norms in  $L^{p}(\mathbf{R}^{N})$  Later we will see that the proof follows a scheme which can be sketched as: "Stability and consistency yields convergence". In the case of Lemma 0.2 stability means:

$$\forall_{u \in L^p(\mathbf{R}^N)} \ \forall_{\varepsilon > 0} \ \|j_{\varepsilon} * u\| \le \|u\| \ . \tag{0.15}$$

This can be seen with the help of Hölders inequality (H) and the Fubini-Tonelli Theorem (F):

$$\begin{aligned} |j_{\varepsilon} * u(x)|^{p} &= \left| \int \left| j_{\varepsilon}(x-y) u(y) d\mu(y) \right|^{p} \\ &\leq \left( \int \left| j_{\varepsilon}(x-y)^{1/p} |u(y)| \right| \cdot j_{\varepsilon}(x-y)^{1-1/p} d\mu(y) \right)^{p} \\ &\stackrel{(H)}{\leq} \left[ \left( \int \left| j_{\varepsilon}(x-y) |u(y)|^{p} d\mu(y) \right|^{1/p} (\int \left| j_{\varepsilon}(x-y) d\mu(y) \right|^{1-1/p} \right]^{p} \right] \\ &= \int \left| j_{\varepsilon}(x-y) |u(y)|^{p} d\mu(y) \cdot 1 \right|. \end{aligned}$$

Now integration with respect to x yields

$$\int |(j_{\varepsilon} * u((x))|^{p} d\mu(x) \leq \int \int j_{\varepsilon}(x-y) |u(y)|^{p} d\mu(y) d\mu(x)$$

$$\stackrel{(F)}{=} \int (\int j_{\varepsilon}(x-y) d\mu(x)) |u(y)|^{p} d\mu(y)$$

$$= \int |u(y)|^{p} d\mu(y) ,$$

for  $\int j_{\varepsilon}(x-y) d\mu(x) = 1$ .

Consistency means here that one can exhibit a dense subset of  $L^p(\mathbf{R}^N)$  — namely  $C_0^{\infty}(\mathbf{R}^N)^{-1}$  for whose elements the assertion of the lemma holds:

Notice that for any continuous function  $\varphi$  the value  $j_{\varepsilon} * \varphi(x)$  is equal to some value of  $\varphi$  at some point in the  $\varepsilon$ -neighbourhood  $U(x, \varepsilon)$  of x. Since the elements of  $C_0^{\infty}(\mathbf{R}^N)$  (or  $C_0^0(\mathbf{R}^N)$ ) are

<sup>&</sup>lt;sup>1</sup> If suffices to take  $C_0^0(\mathbf{R}^N) := \{ u \in C^0(\mathbf{R}^N) \colon \text{supp } u \subset \mathbf{R}^N \}$  instead

uniformly continuous functions,  $j_{\varepsilon} * \varphi$  tends to  $\varphi$  uniformly, thus in  $L^p(\mathbf{R}^N)$ . For the supports of the functions  $j_{\varepsilon} * \varphi$  are contained in a compact subset K of  $\mathbf{R}^N$  for all  $\varepsilon < \varepsilon_0$  — namely

$$\begin{aligned} \sup p (j_{\varepsilon} * \varphi) &\subset \bigcup_{x \in \operatorname{supp} \varphi} U(x, \varepsilon_0) \\ &\subset \{x \in \mathbf{R}^N : \operatorname{dist} (x, \operatorname{supp} \varphi) \leq \varepsilon_0\} =: K \end{aligned}$$

Now for any  $\varepsilon$  the application  $f \mapsto j_{\varepsilon} * f$  is linear. So let  $u \in L^p(\mathbf{R}^N)$  and  $\delta > 0$  given. Find  $\varphi \in C_0^{\infty}(\mathbf{R}^N)$  (or  $C_0^0(\mathbf{R}^N)$ ) such that

$$\|u - \varphi\| < \delta/3$$

Then

$$egin{aligned} \|j_arepsilonst u-u\| \ &\leq & \|j_arepsilonst (u-arphi)\|+\|j_arepsilonst arphi-arphi)\|+\|u-arphi\| \ &\leq & 2\|u-arphi\|+\|j_arepsilonst arphi-arphi)\| \end{aligned}$$

For sufficiently small  $\varepsilon$  we have  $\|j_{\varepsilon} * \varphi - \varphi\| < \delta/3$ , thus  $\|j_{\varepsilon} * u - u\| < \delta$ . q.e.d.

Friedrichs' mollifiers can also be used to show that for any  $\Omega \subset \mathbf{R}^N$  and any compact subset  $K \subset \subset \Omega$  there exists  $\varphi \in C_0^{\infty}(\Omega)$  with  $\varphi \equiv 1$  in K: Let  $\Omega' \subset \subset \Omega$  such that  $K \subset \subset \Omega'$ , let  $\chi$  denote the characteristic function of  $\Omega'$ , i.e.  $\chi \equiv 1$  on  $\Omega'$ ,  $\chi \equiv 0$  in  $\mathbf{R}^N \setminus \Omega'$  and put  $\varphi := j_{\varepsilon} * \chi$ . When  $\varepsilon$  is sufficiently small this  $\varphi$  will do.

To formulate the Fundamental Lemma of the Calculus of Variations we introduce for  $1 \le p \le \infty$ , the space  $L^p_{\text{loc}}(\Omega)$  of (equivalence classes of) functions u such that  $|u|^p$  is integrable over any compact subset  $K \subset \subset \Omega$ .

Evidently for all  $p \in [1, \infty]$ 

$$L^{p}(\Omega) \subset L^{p}_{loc}(\Omega) \subset L^{1}_{loc}(\Omega)$$

$$(0.16)$$

holds.

**Lemma 0.3** Let  $u \in L^1_{loc}(\Omega)$  and suppose that

$$\int_{\Omega} \ u \cdot \varphi d\mu = 0$$

for any  $\varphi \in C_0^{\infty}(\Omega)$ . Then u = 0 or - if you rather look at u as a function than as an element of  $L^1_{loc}(\Omega) - u(x) = 0$  for almost every  $x \in \Omega$ .

**Proof:** We identify u with one of its representatives and have to show that u vanishes a.e. in some neighbourhood of any point  $z \in \Omega$ . Thus let  $z \in \Omega$  and R > 0 such that  $U(z, 2R) \subset \Omega$ .

For any  $\varepsilon < R$  and  $x \in V := U(z, R)$  we have

$$j_{\varepsilon} * u(x) = \int u(y) j_{\varepsilon}(x-y) d\mu(y) = 0$$
,

since  $y \mapsto j_{\varepsilon}(x-y)$  is in  $C_0^{\infty}(\Omega)$ . Moreover, putting

$$v := \begin{cases} u & \text{in} \quad U(z, 2R) \\ 0 & \text{in} \quad \mathbf{R}^N \setminus U(z, 2R) \end{cases}$$

 $v\in L^1(\mathbf{R}^N) \text{ and } j_\varepsilon\ast v\to v \ \text{ in } \ L^1(\mathbf{R}^N) \text{ as } \ \varepsilon\to 0 \,. \text{ Notice that } j_\varepsilon\ast v=j_\varepsilon\ast u \ \text{ in } \ V.$ 

Thus

$$\begin{aligned} \|u\|(L^{1}(V)) &= \|u - j_{\varepsilon} * u\|(L^{1}(V)) &= \|v - j_{\varepsilon} * v\|(L^{1}(V)) \\ &\leq \|v - j_{\varepsilon} * v\|(L^{1}(\mathbf{R}^{N})) \to 0 \end{aligned}$$

Whence  $||u||(L^1(V)) = 0$  which proves the lemma.

Several times we shall use a method, called *partition of unity*. By this we mean:

#### Lemma 0.4 and Definition

Let K denote some compact subset of  $\mathbf{R}^N$  and  $U_1, \ldots, U_L$  finitely many open sets which cover  $K: K \subset \bigcup_{l=1}^L U_l$ . Then there exist L functions  $\zeta_1 \cdots \zeta_L \in C_0^{\infty}(\mathbf{R}^N)$  such that for all  $l = 1, \ldots, L$ 

- (i) supp  $\zeta_l \subset \subset U_l$
- (ii)  $0 \leq \zeta_l \leq 1$
- (iii)  $\sum_{l=1}^{L} \zeta_l(x) = 1$  for  $x \in K$ .

The family  $\{\zeta_1, \ldots, \zeta_L\}$  is called a partition of unity<sup>2</sup> on K subordinate to the covering  $\{U_1, \ldots, U_L\}$ .

**Proof:** We first construct an open covering  $\{V_1, \ldots, V_L\}$  of K with  $V_l \subset \subset U_l$  for any  $l = 1, \ldots, L$ . Fix  $n \in \{1, \ldots, L\}$  and assume that we have found n-1 open sets  $V_1 \subset \subset U_1, \ldots, V_{n-1} \subset \subset U_{n-1}$  such that  $\{V_1, \ldots, V_{n-1}, U_n, \ldots, U_L\}$  covers K. The case n-1=0 is included and in the following  $\bigcup_{l=k}^m \ldots := \emptyset$  if m < k. Now  $K_n := K \setminus \left(\bigcup_{l=1}^{n-1} V_l \cup \bigcup_{l=n+1}^L U_l\right)$  is compact and contained in  $U_n$ . There exists an open set  $V_n$  such that  $K_n \subset V_n \subset \subset U_n$ . Hence  $\{V_1, \ldots, V_n, U_{n+1}, \ldots, U_L\}$  covers K, and having reached n := L we are done.

Let us define  $U_0 := \bigcup_{l=1}^{L} V_l$ . For  $l := 0 \dots, L$  we now chose nonnegative functions  $\varphi_l \in C_0^{\infty}(\mathbf{R}^N)$  with supports in  $U_l$  resp., which are equal to 1 in K (for l = 0) resp. in  $\overline{V}_l$  (for l > 0). Then

$$\psi := (1 - \varphi_0) + \sum_{l=1}^L \varphi_l$$

is a positive  $C^{\infty}$ –function on  $\mathbf{R}^N$  and the functions

$$\zeta_l := \frac{\varphi_l}{\psi} \quad , \ l = 1, \dots, L$$

<sup>&</sup>lt;sup>2</sup>Some authors do not require (ii) for the notion of a partition of unity.

yield a partition of unity on K subordinate to  $\{U_1, \ldots, U_L\}$ .

We conclude this chapter with some remarks on the multiindex notation, defined in (0.14). We define for  $x \in \mathbf{R}^N$ ,  $\alpha \in \mathbf{N}_0^N$ ,  $\beta \in \mathbf{N}_0^N$ 

$$x^{\alpha} := \prod_{n=1}^{N} x_n^{\alpha_n} \tag{0.17}$$

$$\beta \le \alpha :\iff \beta_n \le \alpha_n \text{ for all } n = 1, \dots, N$$
 (0.18)

$$\beta < \alpha :\iff \beta \le \alpha \text{ and } \beta \ne \alpha$$
 (0.19)

$$\alpha ! := \prod_{n=1}^{N} (\alpha_n !) \tag{0.20}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} := \frac{\alpha!}{\beta!(\alpha - \beta)!} = \prod_{n=1}^{N} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} \quad \text{for } \beta \le \alpha \quad .$$
 (0.21)

Then in generalization of the binomial we have the polynomial formula

$$\left(\sum_{n=1}^{N} x_n\right)^k = \sum_{|\alpha| \le k} \frac{k!}{\alpha!} x^{\alpha} . \tag{0.22}$$

For a function  $u \in C^{m}(\Omega)$  Taylor's formula at some point  $y \in \Omega$  reads

$$u(x) = \sum_{|x| \le m} \frac{1}{\alpha!} \left( \partial^{\alpha} u \right)(y) \left( x - y \right)^{\alpha} + R(x, y) , \quad \lim_{x \to y} |x - y|^{-m} R(x, y) = 0 .$$
 (0.23)

If  $u \in C^{m+1}(\Omega)$  we have for example the representation

$$R(x,y) = \sum_{|x|=m+1} \frac{1}{\alpha!} \partial^{\alpha} u(z)(x-y)^{\alpha} , \qquad (0.24)$$

where z is some point on the line segment between y and x, provided that this segment belongs to  $\Omega$ .

If u and v belong to  $C^m(\Omega)$ , we have Leibniz' rule:

$$\partial^{\alpha}(u \cdot v) = \sum_{\beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \partial^{\beta} u \, \partial^{\alpha - \beta} v . \qquad (0.25)$$

Moreover we shall use the identity

$$\sum_{0 \le \beta \le \alpha} (-1)^{|\beta|} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 .$$
 (0.26)

(0.22) - (0.26) may be proved by induction on the dimension N.

q.e.d.

### Chapter 1

# The Calculus of Weak Derivatives

The aim of this chapter is to introduce the notion of the weak (or distributional) derivative in  $L^2(\Omega)$  which is an essential tool for the treatment of elliptic boundary value problems. We study spaces of "weakly" differentiable functions and give some density results. For a detailed representation the reader is referred to the book [1] by R. A. Adams.

For  $u \in C^1(\Omega)$  and  $\varphi \in C_0^{\infty}(\Omega)$  partial integration (Gauß-theorem) yields

$$\int_{\Omega} \ u \, \partial_i \varphi \, d\mu \ = \ - \ \int \left( \partial_i u \right) \varphi \, d\mu$$

On the other hand: If there exists some  $u_i \in L^1_{loc}(\Omega)$  such that

$$\int u \,\partial_i \varphi \,d\mu \ = \ - \ \int \ u_i \varphi \,d\mu$$

holds for all  $\varphi \in C_0^{\infty}(\Omega)$  then

$$\int_{\Omega} (\partial_{i} u - u_{i}) \varphi \, d\mu = 0 \quad \forall_{\varphi \in C_{0}^{\infty}(\Omega)}$$

and whence  $u_i = \partial_i u$  a.e. in  $\Omega$  by Lemma 0.3

This motivates the following generalization of the notion of a derivative:

**Definition 1.1** Let  $u \in L^1_{loc}(\Omega)$  and  $\alpha \in \mathbf{N}_0^N$  a multiindex of order  $|\alpha| > 0$ . Suppose that there exists a  $u_{\alpha} \in L^1_{loc}(\Omega)$  such that

$$\forall_{\varphi \in C_0^{\infty}(\Omega)} \quad \int_{\Omega} u \partial^{\alpha} \varphi \, d\mu \; (-1)^{|\alpha|} = \int_{\Omega} u_{\alpha} \varphi \, d\mu \; . \tag{1.1}$$

Then we say that  $\partial^{\alpha} u$  exists weakly in  $L^{1}_{loc}(\Omega)$  and we define

$$\partial^{\alpha} u := u_{\alpha}$$

 $(\partial^{\alpha} u \text{ is well defined: if (1.1) holds with } u_{\alpha} \text{ and some } v_{\alpha} \text{ instead of } u_{\alpha}, \text{ then } \int_{\Omega} (v_{\alpha} - u_{\alpha}) \varphi \ d\mu = 0$ for all  $\varphi \in C_{0}^{\infty}(\Omega)$  and whence  $v_{\alpha} = u_{\alpha}$  a.e. in  $\Omega$  by Lemma 0.3)

If it happens that  $\partial^{\alpha} u \in L^{p}(\Omega)$  resp.  $L^{p}_{loc}(\Omega)$ ,  $p \in [1, \infty)$ , we say that  $\partial^{\alpha} u$  exists weakly in  $L^{p}(\Omega)$  resp.  $L^{p}_{loc}(\Omega)$ .

In this case we shall simply write

 $\partial^{\alpha} u \in L^{p}(\Omega)$  resp.  $\partial^{\alpha} u \in L^{p}_{loc}(\Omega)$ .

If  $\partial^{\alpha} u$  exists in  $L^{p}(\Omega)$  resp.  $L^{p}_{loc}(\Omega)$  for any  $\alpha$  with  $|\alpha| \leq m$ ,  $m \in \mathbb{N}$ , then we say that u has weak derivatives up to order m in  $L^{p}(\Omega)$  resp.  $L^{p}_{loc}(\Omega)$ . We shall write  $\partial_{i}u$  instead of  $\partial^{e^{(i)}}u$  where  $e^{(i)}$  denotes the *i*-th unit vector.

An example will be given after we will have stated some properties of the weak derivative

### Lemma 1.1

(i) Let  $u \in L^{1}_{loc}(\Omega)$  and suppose that for some multiindex  $\partial^{\alpha} u$  exists weakly in  $L^{1}_{loc}(\Omega)$ . Then the restriction  $u|_{\Omega'}$  of u onto any subdomain  $\Omega' \subset \Omega$  has weak derivative  $\partial^{\alpha}(u|_{\Omega'})$  in  $L^{1}_{loc}(\Omega')$ , namely the restriction of  $(\partial^{\alpha} u)$  onto  $\Omega'$ :

$$\partial^{\alpha}(u|_{\Omega'}) = (\partial^{\alpha}u)|_{\Omega'}$$

- (ii) If  $u \in C^m(\Omega)$ , then u has weak derivatives in  $L^p_{\text{loc}}(\Omega)$  up to order m for any  $p \ge 1$ , and the weak derivative  $\partial^{\alpha} u$  coincides a.e. with the classical derivative  $\frac{\partial^{|\alpha|} u}{(\partial x_1)^{|\alpha|} \dots (\partial x_N)^{|\alpha|_N}}$ .
- (iii) If  $u \in L^{1}_{loc}(\Omega)$  has weak derivative  $\partial^{\alpha} u \in L^{1}_{loc}(\Omega)$  and if  $\partial^{\alpha} u$  has weak derivative  $\partial^{\beta}(\partial^{\alpha} u)$ in  $L^{1}_{loc}(\Omega)$ ,  $\alpha, \beta$ : multiindices, then u has weak derivative  $\partial^{\alpha+\beta} u$  in  $L^{1}_{loc}(\Omega)$ , and  $\partial^{\alpha+\beta} u = \partial^{\beta}(\partial^{\alpha} u)$ .

We give a proof of (iii) only: Notice that for  $\varphi \in C_0^{\infty}(\Omega)$ ,  $\partial^{\alpha+\beta}\varphi = \partial^{\alpha}\partial^{\beta}\varphi$ , and  $\partial^{\beta}\varphi \in C_0^{\infty}(\Omega)$ . Whence

$$\int_{\Omega} u(\partial^{\alpha+\beta}\varphi) d\mu = \int u \partial^{\alpha}(\partial^{\beta}\varphi) d\mu = (-1)^{|\alpha|} \int (\partial^{\alpha}u) (\partial^{\beta}\varphi) d\mu$$
$$= (-1)^{|\alpha|+|\beta|} \int \partial^{\beta}(\partial^{\alpha}u) \varphi d\mu .$$

The second equality is due to the fact that  $\partial^{\alpha} u \in L^{1}_{\text{loc}}(\Omega)$  and the last equality follows from  $\partial^{\beta}(\partial^{\alpha} u) \in L^{1}_{\text{loc}}(\Omega)$ . The definition of the weak derivative now shows  $\partial^{\alpha+\beta} u \in L^{1}_{\text{loc}}(\Omega)$  and  $\partial^{\alpha+\beta} u = \partial^{\beta}(\partial^{\alpha} u)$ .

**Example 1.1** Let  $\Omega := (-1,1) \subset \mathbf{R}^1$  and u the Heavyside function:

$$u(x) = \begin{cases} 0 & \text{für} \quad x < 0\\ 1 & \text{für} \quad x \ge 0 \end{cases}$$

Then  $u \in L^p(\Omega)$  for any p, and from Lemma 1.1 we get: if  $\partial_1 u$  exists weakly in  $L^1_{\mathsf{loc}}(\Omega)$  then  $\partial_1 u = 0$  in  $(-1,0) \cup (0,1)$ , whence  $\partial_1 u = 0$ , since  $\{0\}$  is of measure 0. However, for  $\varphi \in C_0^{\infty}(-1,1)$ 

$$\int_{(-1,1)} u \varphi' d\mu = \int_0^1 \varphi'(t) dt = -\varphi(0) \neq -\int 0 \cdot \varphi d\mu ,$$

provided that  $\varphi(0) \neq 0$  which may certainly happen. We conclude that u does *not* have weak derivatives in  $L^1_{loc}(\Omega)$ .

**Example 1.2** Let  $\Omega = U(0,1)$  and  $u(x) = \ln |x|$ .

u defines an element in  $L^p(\Omega)$  for any  $p \in [1,\infty)$ . According to the previous lemma we must have

$$\partial_k u(x) = \frac{x_k}{|x|^2} \quad , \tag{1.2}$$

provided that  $\partial_k u$  exists weakly in  $L^1_{\text{loc}}(\Omega)$ .

In the case of N = 1, the function defined by (1.2) does not represent an element in  $L^1_{loc}(\Omega)$ . Whence in this case  $\partial_k u$  does not exist weakly in  $L^1_{loc}(\Omega)$ .

For  $N \ge 2$ , the function defined by (1.2) represents an element in  $L^1(\Omega)$ , more precisely an element of  $L^p(\Omega)$  with  $1 \le p < N$ . Thus  $\partial_k u$  exists weakly in  $L^p(\Omega)$ ,  $1 \le p < N$ , provided that

$$orall_{arphi\in C_0^\infty(\Omega)} \int_\Omega \ln |x| \; \partial_k \; arphi(x) \; d\mu(x) = -\int_\Omega \; rac{x_k}{\left|x
ight|^2} \, arphi(x) \; d\mu(x) \; .$$

Now with  $S(0,R) := \{x: |x| = R\}$  and by the Gauß's Theorem:

$$\begin{split} &\int_{\Omega} \ln |x| \,\,\partial_k \varphi(x) \,d\mu(x) \\ &= \lim_{R \to 0} \,\, \int_{\Omega \setminus U(0,R)} \,\ln |x| \,\,\partial_k \,\,\varphi(x) \,d\mu(x) \\ &= \lim_{R \to 0} \,\, \left[ -\int_{\Omega \setminus U(0,R)} \,\frac{x_k}{|x|^2} \,\varphi(x) \,d\mu(x) - \int_{S(0,R)} \ln R \cdot \frac{x_k}{R} \,\varphi(x) \,d\sigma(x) \right] \\ &= -\int_{\Omega} \,\, \frac{x_k}{|x|^2} \,\varphi(x) \,d\mu(x) \,, \end{split}$$

since the surface integral may be estimated by

$$\int_{S(0,R)} \cdots d\sigma(x) \le (\max |\varphi|) \omega_N \ R^{N-1} \ln R \to 0 \text{ as } R \to 0$$

Here,  $\omega_N$  denotes the surface of the unit sphere in  $\mathbf{R}^N$ .

**Remark 1.1** We immediately used the multiindex notation. Thus we did not attempt to define for example  $\partial_1 \partial_2 u$  in a different way from  $\partial_2 \partial_1 u$ . This is justified by the fact that for  $\varphi \in C_0^{\infty}(\Omega)$ :  $\partial_1 \partial_2 \varphi = \partial_2 \partial_1 \varphi$  and whence  $\int_{\Omega} u \partial_1 \partial_2 \varphi d\mu = \int u \partial_2 \partial_1 \varphi d\mu$ .

We shall however use the notation

$$\partial_i \partial_j u := \partial^{e^{(i)} + e^{(j)}} u$$

Also we shall use

$$\partial^0 u := u$$
 .

We are now in a position to introduce the "Sobolev-spaces". These are  $L^p$ -spaces of functions with weak derivatives up to some order. We will however only consider the case p = 2. This is due to the fact that we only deal with linear problems.

**Definition 1.2** Let  $m \in \mathbf{N}$ . By  $H^m(\Omega)$  we denote the space those elements in  $L^2(\Omega)$  which have weak derivatives in  $L^2(\Omega)$  up to order m:

$$H^m(\Omega) := \{ u \in L^2(\Omega) : \partial^{\alpha} u \in L^2(\Omega) \text{ for } |\alpha| \le m \}$$

For  $u,v\in H^m(\Omega)\,,\ m\in {\bf N}$  , we introduce

$$\langle u, v \rangle_{\Omega} := \langle u, v \rangle_{0,\Omega} := \int_{\Omega} u \,\overline{v} \, d\mu \,, \quad \|u\|_{\Omega} := (\langle u, u \rangle_{0,\Omega})^{1/2} \tag{1.3}$$

$$\langle u, v \rangle_{m,\Omega} := \sum_{0 \le |\alpha| \le m} \langle \partial^{\alpha} u, \partial^{\alpha} v \rangle_{\Omega}, \quad ||u||_{m,\Omega} := (\langle u, u \rangle_{m,\Omega})^{1/2}$$
(1.4)

We shall sometimes write  $\langle u, v \rangle_{0,\Omega}$  resp.  $||u||_{0,\Omega}$  instead of  $\langle u, v \rangle_{\Omega}$  resp.  $||u||_{\Omega}$ , and when no confusion can arise, we omit the indication of the domain  $\Omega$ .

**Theorem 1.1** The mapping

$$\langle \cdot , \cdot \rangle_{m,\Omega} : H^m(\Omega) \times H^m(\Omega) \longrightarrow \mathbf{C}, (u,v) \longmapsto \langle u, v \rangle_{m,\Omega}$$

is a scalar product in  $H^m(\Omega)$ , and with this scalar product  $H^m(\Omega)$  is a Hilbert-space.

**Remark 1.2** A scalar product on a real or complex vector space X is a mapping of  $X \times X$  into the real or complex field **K**,

$$[\cdot,\cdot] : X \times X \longrightarrow \mathbf{K} , \ (\mathbf{x},\mathbf{y}) \longmapsto [\mathbf{x},\mathbf{y}]$$

such that for all  $x,y,z \in X \;,\;\; \alpha,\beta \in \mathbf{K}$ 

$$[x, x] = \overline{[y, x]} \tag{1.5}$$

$$[\alpha x + \beta z, y] = \alpha [x, y] + \beta [z, y]$$
(1.6)

$$[x, x] > 0$$
 for  $x \neq 0$ . (1.7)

Notice that (1.5) and (1.6) imply

$$[x, \alpha y + \beta z] = \overline{\alpha}[x, y] + \overline{\beta}[x, z] .$$
(1.8)

Also [x, x] must be real by (1.5). Thus a scalar product is a hermitian, positive definite sesquilinear form on X.

Then

$$||x|| := [x, x]^{1/2}$$

 $\|\cdot\|$  defines a norm in X.

One easily verifies the properties (0.3) and (0.4) of a norm. To show that the triangle inequality (0.5) holds one needs the *Cauchy* – *Schwarz inequality* 

$$\forall_{x,y\in X} | [x,y] | \leq ||x|| ||y|| .$$
(1.9)

To prove it one may assume  $y \neq 0$ . Then with

$$\alpha := -\frac{[x,y]}{\|y\|^2}$$

one obtains (1.9) from the inequality

$$0 \le [x + \alpha y, x + \alpha y] = ||x||^2 + 2\operatorname{Re} \overline{\alpha}[x, y] + |\alpha|^2 ||y||^2$$
$$= ||x||^2 - \frac{|[x, y]|^2}{||y||^2} .$$

Using (1.9) one obtains for  $x, y \in X$ :

$$\begin{aligned} \|x+y\| &= (\|x\|^2 + 2\operatorname{Re} \langle x, y \rangle + \|y\|^2)^{1/2} \le (\|x\|^2 + 2 \|x\| \|y\| + \|y\|^2)^{1/2} \\ &= \|x\| + \|y\| . \end{aligned}$$

A linear space with a scalar product is called an inner product space. Such an inner product space is always considered a normed space with the norm, which is defined by the scalar product. If it happens that with this norm X is a Banach space, then it is called a *Hilbert space*.

Finally we mention that the scalar product is continuous, i.e. if  $x_n \to x$ ,  $y_n \to y$  with respect to the norm in X, then  $[x_n, y_n] \to [x, y]$ .

#### **Proof of Theorem 1.1:**

It is easy to show that  $\langle \cdot, \cdot \rangle_m$  defines a scalar product on  $H^m(\Omega)$ . So we concentrate on the completeness of  $H^m(\Omega)$ . We already mentioned in chapter **1.** that  $L^2(\Omega) = H^0(\Omega)$  is a Banach-space when equipped with the  $L^2$ -norm. But this norm is exactly the norm which comes from the scalar product  $\langle \cdot, \cdot \rangle_0$  in  $H^0(\Omega)$ . Thus it remains to consider the case  $m \geq 1$ .

Let  $(u_k)_{k \in \mathbb{N}}$  denote a Cauchy sequence in  $H^m(\Omega)$ . Thus there exists a sequence  $\varepsilon_k$  of positive numbers tending to 0 with

$$\forall_{k \in \mathbf{N}} \ \forall_{l \ge k} \|u_k - u_l\|_m = \left(\sum_{0 \le |\alpha| \le m} \|\partial_{u_k}^{\alpha} - \partial_{u_l}^{\alpha}\|^2\right)^{1/2} \le \varepsilon_k .$$

Now for any  $\alpha \in \mathbf{N}_0^N$ ,  $|\alpha| \leq m$ 

$$\|\partial^{\alpha} u_k - \partial^{\alpha} u_l\| \leq \|u_k - u_l\|_{m}$$

holds. Putting  $\alpha = 0$  we find that  $(u_k)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $L^2(\Omega)$  and whence converges in  $L^2(\Omega)$  to some  $u \in L^2(\Omega)$ . Moreover, for any  $0 < |\alpha| \le m$ ,  $(\partial^{\alpha} u_k)$  converges in  $L^2(\Omega)$  to some  $u_{\alpha} \in L^2(\Omega)$ .

But then u has weak derivatives in  $L^2(\Omega)$  up to order m, namely  $\partial^{\alpha} u = u_{\alpha}$ : For any  $\varphi \in C_0^{\infty}(\Omega)$ ,  $\alpha \in \mathbf{N}_0^N$ ,  $|\alpha| \leq m$ 

$$\begin{split} \int_{\Omega} u \,\partial^{\alpha} \varphi \,d\mu &= \langle u, \,\partial^{\alpha} \,\overline{\varphi} \rangle = \lim_{k \to \infty} \langle u_k \,, \,\partial^{\alpha} \overline{\varphi} \rangle \\ &= (-1)^{|\alpha|} \lim_{k \to \infty} \langle \partial^{\alpha} u_k \,, \,\overline{\varphi} \rangle = (-1)^{|\alpha|} \langle u_{\alpha} \,, \,\overline{\varphi} \rangle \\ &= (-1)^{|\alpha|} \int u_{\alpha} \,\varphi \,d\mu \,. \end{split}$$

Thus  $u \in H^m(\Omega)$  and  $||u_k - u||_m \to 0$  as  $k \to \infty$ .

**Remark 1.3** In order to prove  $u_{\alpha} = \partial^{\alpha} u$ , one may prove  $\langle u, \partial^{\alpha} \varphi \rangle = \langle u_{\alpha}, \varphi \rangle$  for any  $\varphi \in C_{0}^{\infty}(\Omega)$ , since  $\varphi \in C_{0}^{\infty}(\Omega)$ , if and only if  $\overline{\varphi} \in C_{0}^{\infty}(\Omega)$ .

For functions in  $H^m(\Omega)$  we have the following version of Leibniz' rule

**Theorem 1.2** Let  $u \in H^m(\Omega)$  and  $a \in C^m(\Omega)$  such that a and its derivation up to order m are bounded. Then  $a \ u \in H^m(\Omega)$ , and for any multiindex  $\alpha$ ,  $|\alpha| \leq m$ :

$$\partial^{\alpha}(a u) = \sum_{\beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (\partial^{\alpha-\beta}a)(\partial^{\beta}u) .$$
 (1.10)

**Proof:** Notice that the right hand side of (1.10) belongs to  $L^2(\Omega)$  since  $\partial^{\alpha-\beta}a$  is bounded and  $\partial^{\beta}u \in L^2(\Omega)$  for any  $|\alpha| \leq m$ ,  $\beta \leq \alpha$ . Whence we have to prove the equality

$$\langle au, \partial^{\alpha}\varphi \rangle = (-1)^{|\alpha|} \langle \left(\sum_{\beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right) \partial^{\alpha-\beta} a \partial^{\beta} u \rangle, \varphi \rangle$$
(1.11)

for any  $\varphi \in C_0^{\infty}(\Omega)$  and  $|\alpha| \leq m$ .

We proceed by induction on m and start the induction with the trivial case m = 0 where (1.11) reduces to  $\langle au, \varphi \rangle = \langle au, \varphi \rangle$ .

Thus assume now that (1.11) holds for functions in  $H^{m'}(\Omega)$  resp.  $C^{m'}(\Omega)$  and any  $|\alpha| \leq m'$ provided that m' < m,  $m \geq 1$ . Let  $u \in H^m(\Omega)$ ,  $a \in C^m$  and  $|\alpha| = m$ . Then for  $\varphi \in C_0^{\infty}(\Omega)$  and

q.e.d.

by (0.25)

$$\begin{array}{ll} \langle au \,,\,\partial^{\alpha}\varphi\rangle &=& \langle u \,,\,\overline{a}\,\partial^{\alpha}\varphi\rangle \\ &=& \langle u \,,\,\partial^{\alpha}(\overline{a}\varphi)\rangle - \sum_{\gamma<\alpha} \,\begin{pmatrix}\alpha\\\gamma\end{pmatrix} \langle u \,,\,\partial^{\alpha-\gamma}\,\overline{a}\cdot\partial^{\gamma}\,\varphi\rangle \\ &=& \langle u \,,\,\partial^{\alpha}(\overline{a}\varphi)\rangle - \sum_{\gamma<\alpha} \,\begin{pmatrix}\alpha\\\gamma\end{pmatrix} \langle (\partial^{\alpha-\gamma}\,a)u \,,\,\partial^{\gamma}\,\varphi\rangle \\ &=& \langle u \,,\,\partial^{\alpha}(\overline{a}\varphi)\rangle - \sum_{\gamma<\alpha} \,\begin{pmatrix}\alpha\\\gamma\end{pmatrix} (-1)^{|\gamma|} \,\langle \partial^{\gamma}((\partial^{\alpha-\gamma}\,a)u) \,,\,\varphi\rangle \ . \end{array}$$

This last equality follows from the induction hypothesis by which one may even write

$$\partial^{\gamma} \left( \left( \partial^{\alpha - \gamma} a \right) u \right) = \sum_{\beta \le \gamma} \begin{pmatrix} \gamma \\ \beta \end{pmatrix} \partial^{\alpha - \beta} a \, \partial^{\beta} u \quad . \tag{1.13}$$

The function  $\overline{a}\varphi$  has compact support in  $\Omega$  and is of class  $C^m(\Omega)$ . Using mollifiers we see that there exists a sequence  $\varphi_n \in C_0^{\infty}(\Omega)$ , such that  $\varphi_n$  and its derivatives up to order m tend to  $\overline{a}\varphi$ and its derivatives uniformly with respect to x.

Whence

$$\langle u, \partial^{\alpha}(\overline{a}\varphi) \rangle = \lim_{n \to \infty} \langle u, \partial^{\alpha}\varphi_n \rangle = (-1)^{|\alpha|} \lim_{n \to \infty} \langle \partial^{\alpha}u, \varphi_n \rangle = (-1)^{|\alpha|} \langle \partial^{\alpha}u, \overline{a}\varphi \rangle = (-1)^{|\alpha|} \langle a\partial^{\alpha}u, \varphi \rangle .$$

$$(1.14)$$

We insert (1.13) and (1.14) into (1.12) and then interchange the order of summation. This yields

$$\langle au, \partial^{\alpha} \varphi \rangle$$

$$= (-1)^{|\alpha|} \langle a\partial^{\alpha} u, \varphi \rangle - \sum_{\gamma < \alpha} \sum_{\beta \le \gamma} {\alpha \choose \gamma} {\gamma \choose \beta} (-1)^{|\gamma|} \langle \partial^{\alpha - \beta} a \partial^{\beta} u, \varphi \rangle$$

$$= (-1)^{|\alpha|} \langle a\partial^{\alpha} u, \varphi \rangle - \sum_{\beta < \alpha} \left( \sum_{\beta \le \gamma < \alpha} {\alpha \choose \gamma} {\gamma \choose \beta} (-1)^{|\gamma|} \right) \langle \partial^{\alpha - \beta} a \partial^{\beta} u, \varphi \rangle$$

$$(1.15)$$

To calculate the sum in the parenthesis we write  $\gamma := \beta + \sigma$  and obtain

$$\begin{split} \sum_{\beta \leq \gamma < \alpha} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \begin{pmatrix} \gamma \\ \beta \end{pmatrix} (-1)^{|\gamma|} \\ &= \sum_{\beta \leq \gamma < \alpha} \frac{\alpha! \, \gamma!}{\gamma! \, (\alpha - \gamma)! \, (\gamma - \beta)! \, \beta!} (-1)^{|\gamma|} \\ &= (-1)^{|\beta|} \frac{\alpha!}{\beta! \, (\alpha - \beta)!} \sum_{\sigma < \alpha - \beta} \begin{pmatrix} \alpha - \beta \\ \sigma \end{pmatrix} (-1)^{|\sigma|} \\ &= (-1)^{|\beta|} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \left[ \left( \sum_{\sigma \leq \alpha - \beta} \begin{pmatrix} \alpha - \beta \\ \sigma \end{pmatrix} (-1)^{|\sigma|} \right) - (-1)^{|\alpha - \beta|} \right] \\ &= -(-1)^{|\alpha|} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} , \end{split}$$

the last equality following by (0.26). Insertion into (1.15) yields

$$\begin{split} \langle \, au \,, \, \partial^{\alpha} \varphi \,\rangle &= (-1)^{|\alpha|} \, \left( \langle \, a \partial^{\alpha} u \,, \, \varphi \, \rangle + \sum_{\beta < \alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \langle \, \partial^{\alpha - \beta} a \, \partial^{\beta} u \,, \, \varphi \, \rangle \right) \\ &= (-1)^{\alpha} \, \langle \, (\sum_{\beta \leq \alpha} \, \partial^{\alpha - \beta} a \, \partial^{\beta} u) \,, \, \varphi \, \rangle \ . \end{split}$$
q.e.d.

We introduce a subspace of  $H^m(\Omega)$ :

$$H_b^m(\Omega) := \{ u \in H^m(\Omega) : \text{ supp } u \text{ is bounded.} \}$$
(1.16)

There are two remarks to be made in connection with the definition of  $H_b^m(\Omega)$ . At first, the notion of the support is not yet defined for elements  $u \in L^2(\Omega)$ . Notice that (0.1) does not yield a correct definition since it is not clear which representant of u should be chosen. On the other hand for smooth functions  $\varphi$ , defined on  $\Omega$ , a point  $x \in \mathbf{R}^N$  does *not* belong to the support of  $\varphi$  if and only if either x does not belong to  $\overline{\Omega}$  or there exists an open neighbourhood U of x, such that  $\varphi$  vanishes in  $U \cap \Omega$ . This can be carried over to define the support of a measurable function  $u : \Omega \to \mathbf{C}$ : We define the complement of supp u by

$$\overline{\Omega} \setminus \text{supp } u := \{ x \in \overline{\Omega} \colon u = 0 \text{ almost everywhere in } \Omega \cap U(x, \rho) \text{ for some } \rho > 0 \}$$
(1.17)

and then put

$$\operatorname{supp} \, u := \overline{\Omega} \setminus (\overline{\Omega} \setminus \operatorname{supp} \, u) \quad . \tag{1.18}$$

Thus — and this is the second remark —

$$H_b^m(\Omega) = H^m(\Omega)$$

if  $\Omega$  is bounded.

Therefore the following Corollary to the Leibniz' rule is nontrivial only for unbounded domains  $\Omega$ :

**Corollary 1.1**  $H_b^m(\Omega)$  is dense in  $H^m(\Omega)$ .

Sketch of the proof: Let  $u \in H^m(\Omega)$ . One has to show that a sequence  $(u_n)_{n \in \mathbb{N}}$  exists with  $u_n \in H_b^m(\Omega)$  for all n and

$$\lim_{n \to \infty} \|u - u_n\|_{m,\Omega} = 0 \quad . \tag{1.19}$$

For  $\varphi \in C_0^{\infty}(U(0,2))$  with  $\varphi \equiv 1$  in U(0,1) we put

$$\varphi_n(x) := \varphi(\frac{1}{n} x) , \quad n \in \mathbf{N} , \quad x \in \mathbf{R}^N$$
.

Then with the help of Lebesgue's theorem and Leibniz' rule one obtains (1.19) for  $u_n := \varphi_n u$ .

Using Corollary 1.1 one obtains

**Theorem 1.3**  $C_0^{\infty}(\mathbf{R}^N)$  is dense in  $H^m(\mathbf{R}^N)$ .

**Proof:** Let  $u \in H^m(\mathbf{R}^N)$  and  $\delta > 0$  be given. One has to show the existence of some  $\varphi \in C_0^{\infty}(\mathbf{R}^N)$  such that

$$\|u-\varphi\|_m \mathbf{R}^N < \delta .$$

Corollary 1.1 guarantees the existence of some  $v \in H^m_b(\mathbf{R}^N)$  with

$$\|u - v\|_{m, \mathbf{R}^N} < \delta/2$$
 (1.20)

Then for any  $\varepsilon > 0$ ,

$$j_{\varepsilon} * v := v_{\varepsilon}$$

belongs to  $C_0^{\infty}(\mathbf{R}^N)$ . Moreover for  $|\alpha| \leq m$ 

$$\partial^{\alpha} v_{\varepsilon}(x) = \int \partial_{x}^{\alpha} j_{\varepsilon}(x-y) v(y) d\mu(y)$$
  
=  $(-1)^{|\alpha|} \int \partial_{y}^{\alpha} j_{\varepsilon}(x-y) v(y) d\mu(y)$  (1.21)  
=  $\int j_{\varepsilon}(x-y) \partial^{\alpha} v(y) d\mu(y)$ .

The last equality is due to the facts that for any fixed x the function  $j_{\varepsilon}(x-\cdot)$  belongs to  $C_0^{\infty}(\mathbf{R}^N)$ and that  $\partial^{\alpha} v$  exists weakly.

Formula (1.21) may be expressed in the form

$$\partial^{\alpha} v_{\varepsilon} = \partial^{\alpha} (j_{\varepsilon} * v) = j_{\varepsilon} * (\partial^{\alpha} v) . \qquad (1.22)$$

Now by Lemma 0.2 for  $\varepsilon \to 0$ 

$$\|j_{\varepsilon} * (\partial^{\alpha} v) - \partial^{\alpha} v\|_{0} \mathbf{R}^{N} \rightarrow 0$$
.

This combined with (1.22) yields

$$\|v_{\varepsilon} - v\|_{m, \mathbf{R}^{N}} \to 0$$

as  $\varepsilon \to 0$ . Thus for sufficiently small  $\varepsilon > 0$  we have

$$\|v_{\varepsilon} - v\|_m \mathbf{R}^N < \delta/2$$

and the assertion follows by (1.20).

In particular the proof of Theorem 1.3 shows:

**Lemma 1.2** If  $u \in H^m(\mathbf{R}^N)$  has compact support then  $j_{\varepsilon} * u \in C_0^{\infty}(\mathbf{R}^N)$  and tends to u in  $H^m(\mathbf{R}^N)$  as  $\varepsilon \to 0$ .

q.e.d.

The proof of the following density result is due to N. Meyers and J. Serrin [3]:

**Theorem 1.4**  $H^m(\Omega) \cap C^{\infty}(\Omega)$  is dense in  $H^m(\Omega)$ .

**Proof:** We introduce a special partition of unity on  $\Omega$ . Notice however, that  $\Omega$  is not compact so that Lemma 0.4 does not hold. With

$$\Omega_n := \{ x \in \Omega : \text{ dist } (x, \partial \Omega) > \frac{\eta}{n}, |x| < \frac{n}{\eta} \}$$

we have for some sufficiently small  $\eta > 0$  and for all  $n \in \mathbf{N}$ 

$$\Omega_n \neq \emptyset$$
  
$$\Omega_n \subset \subset \Omega_{n+1} \subset \subset \Omega$$
$$\bigcup_{n \in \mathbf{N}} \Omega_n = \Omega .$$

Moreover let  $\Omega_0 := \Omega_{-1} := \emptyset$ . Define

$$K_n := \overline{\Omega}_n \setminus \Omega_{n-1}, \quad Z_n := \Omega_{n+1} \setminus \overline{\Omega_{n-2}}$$

Then any  $K_n$  is compact and each  $x \in \Omega$  belongs to at least one of the sets  $K_n$ . Moreover x has a neighbourhood  $U_x$  which has nonempty intersections with at most three of the sets  $Z_n$ .

Let now  $\psi_n \in C_0^{\infty}(Z_n)$  such that

$$\psi_n \ge 0$$
,  $\psi_n = 1$  in  $K_n$ .

Then we may define

$$\psi(x):=\sum_{n=1}^\infty \; \psi_n(x) \quad ext{for any } x\in \Omega \;\;.$$

Notice that in  $U_x$  (defined as above) at most three terms in the sum do not vanish. Thus  $\psi \in C^{\infty}(\Omega)$ . Moreover  $\psi > 0$  in  $\Omega$ , since x belongs to at least one  $K_n$ .

Then

$$\zeta_n := \frac{\psi_n}{\psi} \in C_0^\infty(Z_n)$$

and

$$\sum_{n=1}^{\infty} \zeta_n(x) = 1$$

for any  $x \in \Omega$ . The family  $(\zeta_n)_{n \in \mathbb{N}}$  is the "partition of unity" which we wanted to construct.

For  $u \in H^m(\Omega)$  let

$$u_n := \zeta_n u \in H^m(\Omega)$$

Notice that

$$\operatorname{supp} u_n \subset \subset Z_n \subset \subset \Omega$$

For given  $\varepsilon > 0$  we may find a sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  with

$$\varepsilon_n > 0$$
,  $j_{\varepsilon_n} * u_n \in C_0^\infty(Z_n)$ ,  $\|j_{\varepsilon_n} * u_n - u_n\|_{m,\Omega} < \frac{\varepsilon}{2^n}$ .

Then

$$u_{\varepsilon}(x) := \sum_{n=1}^{\infty} \ (j_{\varepsilon_n} \, \ast \, u_n)(x)$$

defines a function in  $C^{\infty}(\Omega)$ , and

$$||u-u_{\varepsilon}||_{m,\Omega} \leq \sum_{n=1}^{\infty} ||j_{\varepsilon_n} * u_n - u_n||_{m,\Omega} < \varepsilon \sum_{n=1}^{\infty} 2^{-n} = \varepsilon$$
.

This proves the density of  $C^{\infty}(\Omega) \cap H^{m}(\Omega)$  in  $H^{m}(\Omega)$ .

Until now nothing is assumed about the domain  $\Omega$ . If one assumes some regularity about  $\Omega$  then one can prove that even

$$C_0^{\infty}(\overline{\Omega}) := \{\varphi|_{\Omega} : \varphi \in C_0^{\infty}(\mathbf{R}^N)\}$$
(1.23)

is dense in  $H^m(\Omega)$ . For this the segment property is sufficient:

**Definition 1.3** A domain  $\Omega \subset \mathbf{R}^N$  is said to have the segment property iff for any  $\hat{x} \in \partial \Omega$  there exist a neighbourhood V of  $\hat{x}$  and a non-zero vector  $y \in \mathbf{R}^N$ , such that for any  $x \in \overline{\Omega} \cap U$  the open line segment

$$(x, x+y) := \{x+ty: t \in (0, 1)\}$$
(1.24)

is contained in  $\ \Omega$  .

Notice that points in  $\partial \Omega$  do *not* belong to  $\Omega$ !

Another assumption on  $\Omega$  is the assumption of smoothness:

**Definition 1.4** A domain  $\Omega \subset \mathbf{R}^N$  is said to be of class  $C^m$ ,  $m \in \mathbf{N}$ , iff for any  $z \in \partial \Omega$  there exist a neighbourhood V of z and a  $C^m$ - Diffeomorphism  $\Phi$  of V onto the unit ball  $U(0,1) := U \subset \mathbf{R}^N$  with the following properties:

$$\Phi(V \cap \Omega) = U^+ := \{ x \in U : x_N > 0 \}$$
(1.25)

$$\Phi(V \cap \partial \Omega) = U^o := \{ x \in U : x_N = 0 \}$$

$$(1.26)$$

q.e.d.

Then

$$\Phi(V \setminus \overline{\Omega}) = U_{-} := \{ x \in U : x_{N} < 0 \} .$$
(1.27)

Moreover it is assumed that the components of  $\Phi$  and of its inverse  $\Phi^{-1}$  have bounded derivatives up to order m.

One may prove that the boundary of a domain of class  $C^m$  (resp. with segment property) may locally be written as the graph of a  $C^m$ -function (resp. a  $C^0$ -function) of (N-1) variables.

Moreover, if  $\Omega$  is of class  $C^m$  then there exists a vector field  $\nu$  on  $\overline{\Omega}$  with components in  $C^{m-1}(\overline{\Omega})$ such that for any  $x \in \partial \Omega$ ,  $\nu(x)$  is normal to  $\partial \Omega$  at x, has norm 1, and "points into exterior of  $\Omega$ ", i.e.  $x + t\nu(x) \notin \overline{\Omega}$  for sufficiently small positive t. On  $\partial \Omega$ , these three conditions define  $\nu(x)$ uniquely, and  $\nu|_{\partial\Omega}$  is called "the outward unit normal field on  $\partial \Omega$ ".

It is then not difficult to show that any domain of class  $C^m$ ,  $m \in \mathbf{N}$ , has the segment property: In the Definition 1.3 choose y as some negative multiple of  $\nu(\hat{x})$ . In the book by J. Wloka [6] the various conditions on the domains are discussed very carefully.

**Theorem 1.5**  $C_0^{\infty}(\overline{\Omega})$  is dense in  $H^m(\Omega)$  if  $\Omega$  has the segment property.

For the proof we use the following lemma which should be known from the Calculus course

**Lemma 1.3** For  $f \in L^p(\mathbf{R}^N)$ ,  $p \in [1, \infty)$ , and  $z \in \mathbf{R}^N$  let  $f_z$  be given by

$$f_z(x) = f(x-z)$$
 for almost all  $x \in \mathbf{R}^N$ 

Then  $f_z$  tends to f in  $L^p(\mathbf{R}^N)$  when z tends to 0 in  $\mathbf{R}^N$ .

**Proof of Theorem 1.5:** By Corollary 1.1 and Theorem 1.4 it suffices to show that for any  $u \in H_b^m(\Omega) \cap C^\infty(\Omega)$  with bounded support and any  $\varepsilon > 0$  one may find some  $\varphi \in C_0^\infty(\mathbf{R}^N)$  satisfying

$$\|u - \varphi\|_{m,\Omega} < \varepsilon \quad . \tag{1.28}$$

Let U(0, R) denote a ball containing the support of u. For any  $z \in \partial \Omega \cap K(0, R)$  the segment property guarantees the existence of some neighbourhood  $V_z$  and some non-zero vector  $y_z \in \mathbf{R}^N$ such that

$$\forall_{x \in V_z \cap \overline{\Omega}} (x, \ x + y_z) \subset \Omega$$

From the collection of the  $V_z$  choose finitely many neighbourhoods  $V_1 := V_{z^1}, \dots, V_K := V_{z^K}$ , say, which cover the compact set  $\partial \Omega \cap K(0, R)$ .  $y^1, \dots, y^K$  denote the corresponding non-zero vectors,  $y^k := y_{z^k}$ . Put  $V_0 := \Omega$  and choose a partition  $\zeta_0, \zeta_1, \dots, \zeta_K$  of unity on  $\overline{\Omega} \cap K(0, R)$  subordinate to the covering  $V_0, \ldots, V_K$  of  $\overline{\Omega} \cap K(0, R)$ . Since  $\zeta_0 u \in C_0^{\infty}(\Omega)$ , (1.28) will follow if for any  $k = 1, \ldots, K$  one can find  $\varphi \in C_0^{\infty}(\mathbf{R}^N)$  such that

$$\|\zeta_k u - \varphi\|_{m,\Omega} < \varepsilon/K . \tag{1.29}$$

Thus we fix a  $k \in \{1, \ldots, K\}$  and write  $\zeta := \zeta_k$ ,  $V := V_k$ ,  $y := y^{(k)}$ , and (with  $\alpha \in \mathbf{N}_0^N$ )

$$v := \begin{cases} \zeta u & \text{in } V \cap \Omega \\ 0 & \text{elsewhere} \end{cases}, v_{\alpha} := \begin{cases} \partial^{\alpha} v & \text{in } V \cap \Omega \\ 0 & \text{elsewhere} \end{cases}$$

For 0 < h < 1 we introduce  $v_h$  and  $v_{\alpha,h}$  by

$$v_h(x) := v(x + hy), \ v_{\alpha,h} := v_\alpha(x + hy)$$

i.e. we translate v and  $v_{\alpha}$  by the vector -hy. Then  $v_h \in C^{\infty}(\mathbf{R}^N \setminus \Gamma_h)$ , where:

$$\Gamma_h := (\partial \Omega \cap \operatorname{supp} \zeta) - hy = \{x: x + hy \in \partial \Omega \cap \operatorname{supp} \zeta\},\$$

and  $\partial^{\alpha} v_h = v_{\alpha,h}$  in  $\mathbf{R}^N \setminus \Gamma_h$ . Notice that by the segment property  $\Gamma_h \subset \mathbf{R}^N \setminus \overline{\Omega}$ .

Then with  $u_h := v_h \mid_{\Omega}$  and with the help of Lemma 1.3 we obtain as h tends to 0:

$$\begin{aligned} \|u_{h} - \zeta u\|_{m,\Omega}^{2} &= \|u_{h} - \zeta u\|_{0,\Omega}^{2} + \sum_{1 \le |\alpha| \le m} \|\partial^{\alpha} u_{h} - \partial^{\alpha}(\zeta u)\|^{2} \\ &\leq \|v_{h} - v\|_{0,\mathbf{R}^{N}}^{2} + \sum_{1 \le |\alpha| \le m} \|v_{\alpha,h} - v_{\alpha}\|_{0,\mathbf{R}^{N}}^{2} \to 0 \end{aligned}$$

Whence with sufficiently small h > 0 the function  $\varphi$  in (1.29) may be chosen as

$$\varphi = \chi_h u_h$$

where  $\chi_h \in C_0^{\infty}(\mathbf{R}^N \setminus \Gamma_h)$  is such that  $\chi_h \equiv 1$  in  $\Omega \cap U(0, R)$ : For then  $\varphi \in C_0^{\infty}(\mathbf{R}^N)$  and  $\varphi|_{\Omega} = U_h$ . q.e.d.

The Sobolev spaces  $H^m(\Omega)$  are the candidates where the solutions to problems like (-1.7), (-1.8) should be located. But these spaces contain classes of functions which equal to each other outside of sets of measure zero. However, the boundary of a domain usually is a set of measure zero. So, what does it mean for an element of  $H^m(\Omega)$  to satisfy (-1.8), i.e. to attain certain values on  $\partial\Omega$ ? The following theorem gives an answer to this question in the case of m = 1:

**Theorem 1.6** Let  $\Omega$  denote a bounded domain of class  $C^2$ . Then there exists a unique continuous linear operator  $\gamma_0$  — the trace operator — mapping from  $H^1(\Omega)$  into  $L^2(\partial\Omega)$  such that  $\gamma_0\varphi = \varphi|_{\partial\Omega}$  for any  $\varphi \in C_0^{\infty}(\overline{\Omega})$ .

**Remark 1.4** If X and Y are two Banach spaces<sup>1</sup> over the same field  $\mathbf{K} := \mathbf{R}$  or  $\mathbf{K} := \mathbf{C}$  then a linear operator T from X to Y by definition is a linear mapping from a linear subspace D(T) of X

<sup>&</sup>lt;sup>1</sup>One also could assume X and Y to be normed linear spaces only. However in our applications we only have to consider Banach spaces, and there are some results on linear operators which do not hold if X or Y are not complete.

— the "domain of T" — into Y. We express this by the following notation:

$$T: D(T) \subset X \to Y \quad , \tag{1.30}$$

which means that T is a linear operator from X to Y with domain D(T).

Since X and whence D(T) and Y are normed it is clear what it means that T is continuous: if  $(x_n) \subset D(T)$  tends to any  $x \in D(T)$  (with respect to the norm in X), then this implies that  $Tx_n$  tends to Tx with respect to the norm of Y.

For a linear operator  $T: D(T) \subset X \to Y$  the following properties are equivalent

- (i) T is continuous at some point  $x \in D(T)$ , i.e.  $x_n \to x$  implies  $Tx_n \to Tx$  (if  $x_n \in D(T)$ ).
- (ii) T is continuous.
- (iii)  $\sup\{\|Tx\|_Y: x \in D(T), \|x\|_X = 1\} < \infty$ .

Here  $\|\cdot\|_X$  resp.  $\|\cdot\|_Y$  are the norms in X resp. Y.

Because of (iii) "continuous linear operators" usually also are called "bounded linear operators". Due to linearity condition (iii) may equivalently be expressed by

(iv) There exists c > 0 such that  $||Tx||_Y \le c ||x||_X$  for all  $x \in D(T)$ .

The smallest possible constant c in (iv) is just the supremum appearing in (iii), and is called "the norm of T":

$$|T||_{X,Y} := \sup\{||Tx||_Y \colon x \in D(T), ||x||_X = 1\}$$
(1.31)

In the sequel we leave the subscripts X and Y when no confusion may arrise.

We are now in the strange position that we have defined a "norm" without having defined a space in which this "norm" is a norm. So let us introduce some agreements: Whenever we talk about linear operators we will tacitly assume that

$$D(T)$$
 is dense in  $X$ , (1.32)

unless not stated otherwise.

lf then

$$T:D(T)\subset X\to Y$$

is a bounded linear operator then there exists a unique bounded linear operator  $\tilde{T}$  from X to Y with domain  $D(\tilde{T}) = X$  and such that  $Tx = \tilde{T}x$  for  $x \in D(T)$ : If  $x \in X \setminus D(T)$ , then a sequence  $(x_n)_{n \in \mathbb{N}}$  exists in D(T) tending to x (with respect to the norm in X). By (iv) we find for  $n \in \mathbb{N}$ ,  $m \leq n$ :

$$||Tx_n - Tx_m|| = ||T(x_n - x_m)|| \le c ||x_n - x_m|| \le c \varepsilon_n$$
(1.33)

with some sequence  $\varepsilon_n \to 0$ , since  $(x_n)$  is a Cauchy sequence. Whence  $(Tx_n)$  is a Cauchy sequence, and since Y is a Banach space, it converges to some  $y \in Y$ . It is now easy to see that y does not depend on the choice of the special sequence  $(x_n)$ , that necessarily  $\tilde{T}x$  must be equal to y, and that this definition in fact leads to a bounded linear operator  $\tilde{T}$  with the asserted properties.

We may now tacitly assume that continuous linear operators are defined on the whole space X unless explicitly stated otherwise, and accordingly write  $T: X \to Y$  instead of (1.30).

Now for two bounded linear operators  $T_1, T_2$  from X to Y and numbers  $\alpha, \beta \in \mathbf{K}$  we may define the linear combination

$$\alpha T_1 + \beta T_2 : X \longrightarrow Y, x \longmapsto \alpha(T_1x) + \beta(T_2x)$$

yielding another bounded linear operator. Thus the set B(X, Y) of bounded linear operators from X to Y can be considered a linear space, and in fact (1.31) defines a norm in this space, i.e. B(X, Y) is a normed linear space.

Accordingly we also have the notion of convergence resp. Cauchy-convergence of a sequence of bounded linear operators  $(T_n)$ . Notice that if a sequence of bounded operators is Cauchy-convergent with respect to the norm in B(X, Y) then because of

$$\forall x \in X, \ n, m \in \mathbf{N} \ \|T_n x - T_m x\| = \|(T_n - T_m)x\| \le \|T_n - T_m\| \|x\|$$
(1.34)

the sequence  $(T_n x)$  is Cauchy convergent in Y. But Y is a Banach space, whence  $T_n x$  converges to some  $y \in Y$ .

It may then be shown that

$$T : X \longrightarrow Y, x \longmapsto \lim_{n \to \infty} T_n x$$

defines a continuous linear operator and that  $T_n$  actually tends to T in B(X,Y). Thus B(X,Y) is a Banach-space.

The preceding consideration suggests that exept from the convergence of  $(T_n)$  in B(X, Y) one might also consider another notion of convergence of linear operators: it may happen that for any  $x \in X$ the sequence  $T_n x$  converges in Y to some Tx. In this case the sequence  $T_n$  is said to converge strongly to T. Convergence in B(X, Y) is also called *uniform* convergence, and clearly a uniformly convergent sequence of linear operators is strongly convergent. However the conversion is wrong: there exist strongly convergent sequences of operators which do not converge uniformly.

For example, let  $\psi$  denote a function in  $C^0(\mathbf{R}^N)$  with  $\psi(0) = 1$ ,  $\lim_{x \to \infty} \psi(x) = 0$ . Let for  $n \in \mathbf{N}$  the function  $\psi_n$  be defined by

$$\psi_n(x) = \psi(\frac{x}{n})$$
 ,  $x \in \mathbf{R}^N$  .

Then

$$T_n : L^2(\mathbf{R}^N) \longrightarrow L^2(\mathbf{R}^N), f \longmapsto \psi_n f$$

defines a sequence of bounded linear operators. With the help of Lebesgue's Theorem we conclude that for any  $f \in L^2(\mathbf{R}^N)$ 

$$\int_{\mathbf{R}^{N}} |\psi_{n}f - f|^{2} d\mu \rightarrow 0$$

as  $n \to \infty$ . Thus  $T_n$  converges to the identity I strongly. However  $T_n$  does not converge uniformly to I: Choose some  $f \in C_0^{\infty}(\mathbf{R}^N)$  with  $||f||_0 = 1$ . For any given n one may find a translate  $f_n$  of f, given by  $f_n(x) = f(x + \tau_n e^1)$ .  $\tau_n \in \mathbf{R}$ . say, such that the support of  $f_n$  is contained in the subset  $\Omega_n \subset \mathbf{R}^N$ , where  $|\psi_n|$  is less than 1/2. Then

$$\begin{aligned} \|T_n - I\|^2 &\ge \|T_n f_n - f_n\|^2 &= \int_{\mathbf{R}^N} |1 - \psi_n|^2 |f_n|^2 d\mu \\ &\ge \frac{3}{4} \int |f_n|^2 d\mu = \frac{3}{4} \end{aligned}$$

Using the Fourier transform and the above argument one can show that the sequence

$$J_n : L^2(\mathbf{R}^N) \longrightarrow L^2(\mathbf{R}^N), f \longmapsto j_{1/n} * f$$

of mollifiers converges strongly but not unifornmly to the identity.

**Remark 1.5** In Theorem 1.6 the space  $L^2(\partial\Omega)$  appears. For the reader who is not familiar with Lebesgue spaces on manifolds, embedded into  $\mathbb{R}^N$ , we just recall the definition, taylored to fit our situation. From Definition 1.4 and since  $\partial\Omega$  is compact, it may be covered by finitely many open neighbourhoods  $V_1, \ldots, V_K$ , and we have corresponding  $C^m$ -Diffeomorphisms  $\Phi^{(1)}, \ldots, \Phi^{(K)}$ ,  $\Phi^{(k)}$ mapping  $V_k$  onto U(0, 1) =: U with the properties (1.25) - (1.27). Now

$$\Psi^{(k)}: D := \{y \in \mathbf{R}^{N-1}: |y| < 1\} \to \mathbf{R}^N$$

$$\Psi^{(k)}(y) = (\Phi^{(k)})^{-1} ((y, 0))$$

is a parametrization of  $\partial \Omega \cap V_k$ . We call a function  $f : \partial \Omega \to \mathbf{C}$  measurable resp. integrable iff  $f \circ \Psi^{(k)}$ is measurable resp. integrable for any  $k \in \{1, \ldots, K\}$ . Letting  $(\zeta_k)_{k=1,\ldots,K}$  denote a partition of unity on  $\partial \Omega$  subordinate to  $V_1, \ldots, V_K$ , the integral of an integrable f is defined by

$$\int_{\partial\Omega} f \, d\sigma := \sum_{k=1}^{K} \int_{D} \left( \zeta_{k} f \right) \circ \Psi^{(k)}(y) \, \Gamma(\Psi^{(k)}(y)) \, d\mu_{N-1}(y)$$

where  $\mu_{N-1}$  is the Lebesgue measure in  $\mathbf{R}^{N-1}$ , and

$$\Gamma(\Psi^{(k)}(y)) = (\det(\partial_i \Psi^{(k)} \cdot \partial_j \Psi^{(k)})_{i,j=1,...,N-1})^{1/2} ,$$

 $a \cdot b$  denoting the scalar product in  $\mathbf{R}^N$ .

It can be shown that the notions of measurebility, integrability and the value of the integral do not depend on the special choice of the covering  $V_1, \ldots, V_K$ , the Diffeomorphisms  $\Phi^{(1)}, \ldots, \Phi^{(k)}$ , and the partition of unity so that these notions are well defined.

In particular, a subset M of  $\partial\Omega$  is a null-set with respect to the surface measure  $\sigma$  (a  $\sigma$ -null-set) iff  $(\Psi^{(k)})^{-1}(M)$  is a null-set in  $\mathbb{R}^{N-1}$  (with respect to  $\mu_{N-1}$ ) for all k. Then  $L^2(\partial\Omega)$  consists of all equivalence-classes of complex valued measurable functions f on  $\partial\Omega$ , for which  $|f|^2$  is integrable on  $\partial\Omega$ . Of course two functions  $f_1, f_2$  are equivalent iff  $f_1$  and  $f_2$  differ from each other nowhere but on a  $\sigma$ -null-set. With the scalar product

$$\langle f \, , \, g \, 
angle_{\partial\Omega} := \int_{\partial\Omega} \, f \, \overline{g} \, d\sigma$$

 $L^2(\partial \Omega)$  becomes a Hilbert space.

Of course any continuous function on  $\partial\Omega$  is integrable, and the reader should remember Gauß' Theorem:

$$\int_{\Omega} \partial_n u \, d\mu = \int_{\partial \Omega} \nu_n u \, d\sigma \quad , \quad n = 1, \dots, N$$
(1.35)

holds for any  $u \in C^1(\overline{\Omega})$ , if  $\Omega$  is of class  $C^1$  and bounded,  $\nu_n$  denoting the *n*-th component of the outward unit normal field on  $\partial \Omega$ .

We are now ready to prove Theorem 1.6:

**Proof of Theorem 1.6:** All which we have to show is that the restriction of a function  $\varphi \in C_0^{\infty}(\overline{\Omega})$  to the boundary  $\partial\Omega$ , which is a linear operator from  $H^1(\Omega)$  into  $L^2(\partial\Omega)$  with domain  $C_0^{\infty}(\overline{\Omega})$ , in fact is a bounded linear operator. Thus we have to prove the existence of a constant c > 0 such that

$$\|\varphi\|^2_{\partial\Omega} := \int_{\partial\Omega} |\varphi|^2 \ d\sigma \ \leq \ c \, \|\varphi\|^2_{1,\Omega}$$

holds for any  $\varphi \in C_0^{\infty}(\overline{\Omega})$ .

Let  $\nu$  denote a  $C^1$ -extension into  $\overline{\Omega}$  of the outward unit normal on  $\partial\Omega$ . Then by (1.35)

$$\begin{split} \|\varphi\|_{\partial\Omega}^{2} &= \sum_{n=1}^{N} \int_{\partial\Omega} \nu_{n} (\nu_{n} |\varphi|^{2}) d\sigma \\ &= \sum_{n=1}^{N} \int_{\Omega} \partial_{n} (\nu_{n} |\varphi|^{2}) d\mu \\ &= \int_{\Omega} \left[ (\sum_{n=1}^{N} \partial_{n} \nu_{n}) |\varphi|^{2} + \nu_{n} 2 \operatorname{Re} (\partial_{n} \varphi) \overline{\varphi} \right] d\mu \\ &\leq c \|\varphi\|_{1,\Omega}^{2} , \end{split}$$

where c depends on bounds for the  $\nu_n$  and its derivative. Now according to Remark 1.4 for  $u \in H^1(\Omega), \gamma_0 u$  is the limit in  $L^2(\partial \Omega)$  of  $\varphi_n |_{\partial \Omega}$  where  $\varphi_n \in C_0^{\infty}(\overline{\Omega})$  is such that  $||u - \varphi_n||_1 \to 0$ . q.e.d.

We call Theorem 1.6 the weak trace theorem. The attribute 'weak' is due to the fact that Theorem 1.6 does not characterize the range of the trace operator  $\gamma_0$ . Indeed, this is not the whole of  $L^2(\partial\Omega)$  but a dense subspace, called  $H^{1/2}(\partial\Omega)$ .

As a corollary we obtain the rule of partial integration:

**Corollary 1.2** If  $\Omega$  is a bounded domain of class  $C^2$  and  $\gamma_0$  denotes the trace operator, then for all  $u, v \in H^1(\Omega)$ , n = 1, ..., N

$$\langle \partial_n u, v \rangle_{0,\Omega} = - \langle u, \partial_n v \rangle_{0,\Omega} + \langle \nu_n \gamma_0 u, \gamma_0 v \rangle_{\partial\Omega}$$

#### **Proof:**

For  $u, v \in C_0^{\infty}(\overline{\Omega})$  the above formula is just (1.35) applied to  $u\overline{v}$  instead of u. For  $u, v \in H^1(\Omega)$  let  $(u_n), (v_n)$  denote sequences in  $C_0^{\infty}(\overline{\Omega})$  which tend to u resp. v in  $H^1(\Omega)$ . Then  $(\partial_n u_k), (v_k), (u_k), (\partial_n v_k), (\gamma_0 u_k)$  and  $(\gamma_0 v_k)$  tend to  $\partial_n u, v, u, \partial_n v, \gamma_0 u$  and  $\gamma_0 v$  respectively in  $L^2(\Omega)$  resp.  $L^2(\partial \Omega)$ . The assertion follows by the continuity of the scalar products in  $L^2(\Omega)$  resp.  $L^2(\partial \Omega)$ .

#### **Remark 1.6** For a linear operator

 $T: D(T) \subset X \to Y$ , X, Y Banach spaces

the kernel of T is denoted by N(T):

$$N(T) := \{x \in X: Tx = 0\}$$

N(T) is always a linear subspace of X. It need not necessarily be a closed subspace, since  $x_n \to x$  in X and  $Tx_n = 0$  need neither imply  $x \in D(T)$  nor Tx = 0. However if  $T \in B(X, Y)$  then N(T) is a closed subspace of X.

We now want to characterize the kernel of the trace operator:

**Theorem 1.7** Let  $\Omega$  denote a bounded domain of class  $C^2$  and  $\gamma_0$  denote the trace operator. Then  $u \in N(\gamma_0)$  if and only if there exists a sequence  $(\varphi_n)$  in  $C_0^{\infty}(\Omega)$  such that

$$\|u - \varphi_n\|_1 \to 0 \quad . \tag{1.36}$$

**Proof:** Since  $\gamma_0(\varphi_n) = \varphi_n|_{\partial\Omega} = 0$  for  $\varphi_n \in C_0^{\infty}(\Omega)$ , (1.36) implies that u belongs to  $N(\gamma_0)$ .

On the other hand, let  $u \in N(\gamma_0)$ . We let v denote the continuation of u by 0 into the whole of  $\mathbf{R}^N$ . Then  $v \in L^2(\mathbf{R}^N)$  and we claim that  $v \in H^1(\mathbf{R}^N)$ . Indeed, for any  $\varphi \in C_0^{\infty}(\mathbf{R}^N)$  we have by Corollary 1.2 for any  $u \in \{1, \ldots, N\}$ :

$$\langle v, \partial_n \varphi \rangle_{0,\mathbf{R}^N} = \langle u, \partial_n \varphi \rangle_{0,\Omega} = - \langle \partial_n u, \varphi \rangle_{0,\Omega}$$

In the last equation we used  $\gamma_0 u = 0$ . Thus the continuation by 0 of  $\partial_n u$  is the weak derivative of v.

We now introduce the finite covering  $V_1, \ldots, V_K$  of  $\partial\Omega$  and the vectors  $y^1, \ldots, y^K \in \mathbf{R}^N \setminus \{0\}$ guaranteed by the segment property and the compactness of  $\partial\Omega$ . With  $V_0 := \Omega$  let  $\zeta_0, \ldots, \zeta_K$ denote a partition of unity on  $\overline{\Omega}$  subordinate to the covering  $V_0, \ldots, V_K$  of  $\overline{\Omega}$  It suffices to show that  $\zeta_k v$ ,  $k \in \{0, \ldots, K\}$ , can be approximated in  $H^1(\mathbf{R}^N)$ , by  $C_0^{\infty}(\Omega)$ -functions. This is clear for  $\zeta_0 v$ : Since supp  $(\zeta_0 v) \subset \subset \Omega$  we have  $j_{\varepsilon} * (\zeta_0 v) \in C_0^{\infty}(\Omega)$  if  $\varepsilon$  is sufficiently small and by Lemma 1.2  $j_{\varepsilon} * (\zeta_0 v)$  tends to  $\zeta_0 v$  in  $H^m(\Omega)$  as  $\varepsilon \to 0$ .

Thus let  $k \in \{1, \ldots, K\}$ , let us omit the index k from now on, and put  $\tilde{v} := \zeta v$ . For any h > 0 the translate  $\tilde{v}_h$  given by

$$\tilde{v}_h(x) = \tilde{v}(x - hy)$$

belongs to  $H^1(\mathbf{R}^N)$ : By coordinate transformation one obtains for all  $\varphi \in C_0^{\infty}(\mathbf{R}^N)$  with  $\varphi_{-h}(x) := \varphi(x + hy)$ :

$$\begin{split} \int_{\mathbf{R}^N} \tilde{v}_h \partial_n \varphi \, d\mu &= \int_{\Omega} \tilde{v} \partial_n \varphi_{-h} \, d\mu = - \int_{\Omega} \partial_n \tilde{v} \varphi_{-h} \, d\mu \\ &= - \int_{\mathbf{R}^N} (\partial_n \tilde{v})_h \varphi \, d\mu \ , \end{split}$$

where  $(\partial_n \tilde{v})_h$  denotes the translate of  $\partial_n \tilde{v}$ . Thus  $\tilde{v}_h \in H^1(\mathbf{R}^N)$ ,  $\partial_n \tilde{v}_h = (\partial_n \tilde{v})_h$ . It follows that  $\tilde{v}_n$  tends to  $\tilde{v}$  in  $H^1(\mathbf{R}^N)$  as  $h \to 0$ . On the other hand, by the segment property,

supp 
$$\tilde{v}_h \subset \subset \Omega$$
 if  $h < 1$ .

Thus, given  $\varepsilon > 0$ , fix 1 > h > 0 such that  $\|\tilde{v}_h - \tilde{v}\|_{1,\mathbf{R}^N} < \varepsilon/2$ . Then for sufficiently small  $\delta > 0$ ,  $j_{\delta} * \tilde{v}_h \in C_0^{\infty}(\Omega)$  and  $\|j_{\delta} * \tilde{v}_h - \tilde{v}_h\|_{\mathbf{R}^N} < \varepsilon/2$ . An application of the triangle inequality concludes the proof. q.e.d.

If X is a Banach space and Z is some (linear) subspace of X, the closure  $\overline{Z}$  of Z in X, i.e.

 $\overline{Z} := \overline{Z}^X := \{ z \in X \colon ||z_n - z||_X \to 0 \text{ for some sequence } (z_n) \text{ in } Z \} ,$ 

is a closed linear subspace. Forgetting about X,  $\overline{Z}$  itself is a Banach space or even a Hilbert space if X is a Hilbert space. Therefore the space  $(m \in \mathbf{N})$ 

$$H_0^m(\Omega) := \overline{C_0^\infty(\Omega)}^{H^m(\Omega)} , \qquad (1.37)$$

i.e. the closure of the linear subspace  $C_0^{\infty}(\Omega)$  in  $H^m(\Omega)$ , is a closed linear subspace of  $H^m(\Omega)$ , and may be also considered a Hilbert space itself with  $\langle \cdot, \cdot \rangle_m$  as a scalar product. Evidently,  $au \in H_0^m(\Omega)$  if  $a \in C_0^{\infty}(\Omega)$  and  $u \in H^m(\Omega)$  or if  $a \in C_0^{\infty}(\overline{\Omega})$  and  $u \in H_0^m(\Omega)$ .

What we just proved can now be expressed as

$$N(\gamma_0) = H_0^1(\Omega) ,$$
 (1.38)

provided that  $\Omega$  and  $\gamma_0$  are as in Theorem 1.7. We take (1.38) as a motivation to define

**Definition 1.5** Let  $\Omega$  denote any domain in  $\mathbb{R}^N$ . An element  $u \in H^1(\Omega)$  is said to vanish at the boundary  $\partial \Omega$ , iff  $u \in H^1_0(\Omega)$ .

We conclude this chapter by a discription of the behavior of  $H^m(\Omega)$  under diffeomorphisms

**Definition 1.6** Let  $m \in \mathbf{N}$ . By a regular  $C^m$ -diffeomorphism of some domain  $\Omega \subset \mathbf{R}^N$  onto some domain  $\Omega^* \subset \mathbf{R}^N$  we mean an injective mapping

$$\Phi: \Omega \rightarrow \mathbf{R}^N$$

with image  $\Phi(\Omega) = \Omega^*$ , the components of which have bounded and continuous partial derivatives  $\partial^{\alpha} \Phi_n$  up to order m. Moreover the determinants of the Jacobian's  $\Phi'(x) = (\partial_1 \Phi, \dots, \partial_N \Phi)(x)$ of  $\Phi$  must be bounded away from 0, independent of x:

$$\exists_{c>0} \forall_{x\in\Omega} |\det \Phi'(x)| \ge c.$$

If  $\Phi$  is a regular  $C^m$ -diffeomorphism of  $\Omega$  onto  $\Omega^*$  then its inverse  $\Phi^{-1}$  is a regular  $C^m$ diffeomorphism of  $\Omega^*$  onto  $\Omega$ . With  $\Phi$  a pullback operator  $\Phi^*$  is associated which maps the linear space  $C^m(\Omega^*)$  onto  $C^m(\Omega)$  by

$$\Phi^* : C^m(\Omega^*) \longrightarrow C^m(\Omega) , \ u \longmapsto u \circ \Phi , \qquad (1.39)$$

i.e.  $\Phi^* u$  is just the composition of u with  $\Phi$ . One may show by induction on the order  $|\alpha|$  of  $\alpha$ : Let  $A_k$  denote the number of multiindices of order less or equal to k. Then for all multiindices  $\alpha, \beta$  with  $1 \leq |\beta| \leq |\alpha| \leq m$  there exist polynomials  $p_{\alpha,\beta}$  in  $N \cdot A_{|\alpha|}$  variables such that for all  $u \in C^m(\Omega^*)$ 

$$\partial^{\alpha}(u \circ \Phi) = \sum_{1 \le |\beta| \le |\alpha|} p_{\alpha\beta} \left( \{ \partial^{\beta} \Phi_n \}_{1 \le |\beta| \le |\alpha|, 1 \le n \le N} \right) (\partial^{\beta} u) \circ \Phi \quad . \tag{1.40}$$

This shows that in fact  $\Phi^*$  maps  $C^m(\Omega^*)$  into  $C^m(\Omega)$ . This mapping is bijective. Its inverse is  $(\Phi^{-1})^*$ , the pullback of  $\Phi^{-1}$ . Finally  $\Phi^*$  is linear.

From (1.40) we obtain for  $u \in C^m(\Omega) \cap H^m(\Omega)$ 

$$\begin{aligned} \|\partial^{\alpha}(u\circ\Phi)\|_{\Omega}^{2} &= \int_{\Omega} \left| \sum_{1\leq |\beta|\leq |\alpha|} p_{\alpha\beta}(\{\partial^{\alpha}\Phi_{n}\})(\partial^{\beta}u)\circ\Phi \right|^{2} d\mu \\ &= \int_{\Omega^{*}} \left| \left[ \frac{1}{|\det\Phi'|^{1/2}} \sum p_{\alpha\beta}(\partial^{\beta}\Phi_{n})) \right]\circ\Phi^{-1}(\partial^{\beta}u) \right|^{2} d\mu \end{aligned}$$
(1.41)

$$\leq c \|u\|_{m,\Omega^*}^2$$

with some constant c depending on the bounds for the derivatives of  $\Phi$  and on the lower bound for  $|\det \Phi'|$ .

(1.41) implies that

$$\Phi^*: C^m(\Omega^*) \cap H^m(\Omega^*) \subset H^m(\Omega^*) \to H^m(\Omega)$$

is a continuous linear operator from  $H^m(\Omega^*)$  into  $H^m(\Omega)$  and whence can be continued to the whole of  $H^m(\Omega)$  as a continuous linear operator. If  $\tilde{u}$  denotes a representant of some element uin  $H^m(\Omega^*)$ , then  $\tilde{u} \circ \Phi$  is a representant of  $\Phi^* u \in H^m(\Omega)$ .

Altogether we have proved:

**Theorem 1.8** Let  $m \in \mathbb{N}$  and  $\Phi$  denote a regular  $C^m$ -diffeomorphism of the domain  $\Omega$  onto the domain  $\Omega^*$ . Then

$$\Phi^* : H^m(\Omega^*) \longrightarrow H^m(\Omega), \ u \longmapsto u \circ \Phi$$

defines a bounded linear operator from  $H^m(\Omega^*)$  into  $H^m(\Omega)$ . This operator is bijective and its inverse operator is

$$(\Phi^{-1})^*$$
:  $H^m(\Omega) \longrightarrow H^m(\Omega^*)$ ,  $v \longmapsto v \circ \Phi^{-1}$ ,

whence  $(\Phi^*)^{-1}$  is a bounded linear operator, too.

**Remark 1.7** If  $T: D(T) \subset X \to Y$  is an *injective* linear operator from the Banach space X into the Banach space Y then the inverse mapping

$$T^{-1}$$
:  $R(T) := T(D(T)) \subset Y \longrightarrow X$ ,  $Tx \longmapsto x$ 

is a linear operator from Y to X with domain R(T), which need not be dense in  $Y cdot T^{-1}$  is called the inverse of T. Notice that a linear operator T is injective iff  $N(T) = \{0\}$ , i.e. x = 0 is the only solution of the equation Tx = 0.

A fundamental theorem of functional Analysis, the "Bounded Inverse Theorem" states, that the inverse of an injective *bounded* linear operator  $T: X \to Y$  is itself bounded, *provided that the range* R(T) *of* T *is closed.* For the proof of Theorem 1.8, however, the "Bounded Inverse Theorem" was not needed.

### Chapter 2

# Solutions of the Dirichlet problem

From now on we will consider boundary value problems of the type

$$Lu := \sum_{n,m=1}^{N} a_{nm} \partial_n \partial_m u + \sum_{n=1}^{N} b_n \partial_n u + cu = f \text{ in } \Omega$$
(2.1)

$$u|_{\partial\Omega} = g \quad . \tag{2.2}$$

Here  $a_{nm}$ ,  $b_n$ , c, f and g denote (given) measurable complex valued functions on some domain  $\Omega \subset \mathbf{R}^N$  and on  $\partial\Omega$  respectively. f and g are the data of the problem,  $a_{nm}$ ,  $b_n$ , and c (n, m = 1, ..., N) are called the coefficients of the differential expression L. Such a differential expression may be considered as a linear mapping of  $C^2(\Omega)$  into the space of measurable functions on  $\Omega$ : The value of Lu at the point x is the specific linear combination  $\sum a_{nm}(x) \partial_n \partial_m u(x) + \sum b_n(x) \partial_n u(x) + c(x) u(x)$  of u and its derivatives up to the second order in x.

When in Linear Algebra systems Ax = b of linear equations are discussed, one first of all thinks of A as a fixed given matrix and is interested in the dependence of the solutions x on the right hand side b. You may compare the differential expression with such a matrix and the data f, gwith the right hand side.

It is crucial to be very precise about the notion of a solution of (2.1) and (2.2). In order to define the various notions, it is important to make several assumptions on the coefficients and on the data: otherwise certain notions cannot be written down.

We start with the definition of a classical solution:

**Definition 2.1** Suppose that the coefficients  $a_{nm}$ ,  $b_n$ , c belong to  $C^0(\Omega)$ , and let the data f resp. g belong to  $C^0(\Omega)$  resp.  $C^0(\partial\Omega)$ . By a classical solution of (2.1), (2.2) we mean some function  $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$  for which (2.1) and (2.2) hold.
In the notion of a classical solution the derivatives  $\partial_n \partial_m u$ ,  $\partial_n u$  are (classical) partial derivatives as they had been defined in the Calculus course.

We might as well assume that the solutions belong to some Sobolev space and the most natural assumption is to assume them in  $H^2(\Omega)$ . But then some difficulties arrise concerning the equation (2.2): It is obvious that the trace operator (c.f. Theorem 1.6) might be used to express (2.2). But we did not characterize the image of  $H^1(\Omega)$  under  $\gamma_0$ . In fact, for solutions in  $H^2(\Omega)$  one would have to characterize the image of  $H^2(\Omega)$  under  $\gamma_0$  as well, but one might express this within an existence theorem. To avoid this difficulty one replaces in (2.2) g by the trace of some known function from  $H^1(\Omega)$ , again denoted by g:

$$\gamma_0 u = \gamma_0 g \quad . \tag{2.3}$$

In order to write down (2.3) one has to assume that  $\Omega$  is a bounded domain of class  $C^2$ . This can be avoided by the diction from Definition 1.5.

**Definition 2.2** Suppose that the coefficients  $a_{nm}$ ,  $b_n$ , c belong to  $L^{\infty}(\Omega)$ , and let f resp. g belong to  $L^2(\Omega)$  resp.  $H^1(\Omega)$ . By a strong solution of the problem

$$Lu = f , \quad u|_{\partial\Omega} = g|_{\partial\Omega} \tag{2.4}$$

we mean an element  $u \in H^2(\Omega)$  for which (2.1) holds and u - g vanishes at the boundary  $\partial \Omega$ .

Thus the boundary condition is expressed by

$$u - g \in H_0^1(\Omega) \quad , \tag{2.5}$$

and in (2.1) the derivatives must be interpreted as the weak derivatives of u.

Strong solutions have the disadvantage that because of (2.5) one can expect their existence only if  $g \in H^2(\Omega)$ , and even then one needs additional assumptions on the smoothness of the coefficients and of the boundary  $\partial\Omega$ .

The proof of an existence theorem is the easier the less properties of a solution one has to prove. Suppose therefore that u is a strong solution of the problem 'Lu = f,  $u \mid_{\partial\Omega} = g \mid_{\partial\Omega}$ ' for some  $f \in L^2(\Omega)$ ,  $g \in H^1(\Omega)$ . Now 'Lu = f' is equivalent with

$$\langle Lu, \varphi \rangle = \langle f, \varphi \rangle \tag{2.6}$$

for all  $\varphi \in C_0^{\infty}(\Omega)$  by Lemma 0.3.

If  $a_{nm} \in C^1(\Omega)$  for n, m = 1, ..., N and if  $a_{nm}$  and its derivatives are bounded then any of the functions  $a_{nm} \partial_m u$  belong to  $H^1(\Omega)$  by Leibniz' rule. Then

$$\int_{\Omega} a_{nm} \,\partial_n \partial_m \, u \cdot \overline{\varphi} \,d\mu \tag{2.7}$$

$$= \int_{\Omega} \partial_n (a_{nm} \partial_m u) \overline{\varphi} \, d\mu - \int_{\Omega} (\partial_n a_{nm}) \partial_m u \cdot \overline{\varphi} \, d\mu$$
$$= -\int_{\Omega} (a_{nm} \partial_m u) \cdot (\partial_n \overline{\varphi}) \, d\mu - \int_{\Omega} (\partial_n a_{nm}) \partial_n u \, \overline{\varphi} \, d\mu$$

The second equality follows (by partial integration) from the definition of the weak derivatives of  $a_{nm} \partial_m u$ . Plugging (2.7) into the left hand side of (2.6) this formula may be written as

$$B(u,\varphi) := \sum_{n,m=1}^{N} \langle a_{nm}\partial_{n}u, \partial_{n}\varphi \rangle - \sum_{m=1}^{N} \langle a_{m}\partial_{m}u, \varphi \rangle - \langle cu, \varphi \rangle = -\langle f, \varphi \rangle , \qquad (2.8)$$

where we have put

$$a_m := b_m - \sum_{n=1}^N (\partial_n a_{nm})$$
 (2.9)

Thus for a strong solution u the identity (2.8) must hold for all  $\varphi \in C_0^{\infty}(\Omega)$ . Equation (2.8) can be used to define a notion of a solution without talking about its second derivatives:

**Definition 2.3** Suppose that the coefficients  $a_{nm}$  belong to  $C^1(\Omega)$ . Assume further that  $a_{nm}$ , its derivatives,  $b_n$  and c are bounded (n, m = 1, ..., N), and that the data f, g belong to  $L^2(\Omega)$  and  $H^1(\Omega)$  respectively. By a variational solution of the problem (2.4) we mean an element u of  $H^1(\Omega)$  for which u - g vanishes at the boundary, and (2.8) holds for any  $\varphi \in C_0^{\infty}(\Omega)$ . B is called a Dirichlet form for L.

An approximation argument yields:

**Lemma 2.1** Under the assumptions of Definition 2.3 the validity of (2.8) for all  $\varphi \in C_0^{\infty}(\Omega)$  is equivalent with the validity of (2.8) for all  $\varphi \in H_0^1(\Omega)$ .

Let  $L_1, L_2$  denote two differential expressions

$$L_j = \sum_{n,m=1}^N a_{nm}^{(j)} \partial_n \partial_m + \sum_{n=1}^N b_n \partial_n + c ,$$

such that  $L_1 \varphi = L_2 \varphi$  for any  $\varphi \in C^{\infty}(\Omega)$ . This means that we must have  $a_{nm}^{(1)} + a_{mn}^{(1)} = a_{nm}^{(2)} + a_{mn}^{(2)}$ . The notion of a variational solution seems to depend on whether one deals with  $L_1$  or with  $L_2$ . However, this is not true: According to (2.8) we formally have two different forms  $B_1$  resp.  $B_2$  for  $L_1$  resp.  $L_2$ . But for any  $\psi \in C^{\infty}(\Omega)$  and  $\varphi \in C_0^{\infty}(\Omega)$  we have

$$B_j(\psi,\varphi) = -\langle L_j\psi,\varphi\rangle$$
 .

Thus  $B_1(\psi,\varphi) = B_2(\psi,\varphi)$  for all  $\psi \in C^{\infty}(\Omega)$  and  $\varphi \in C_0^{\infty}(\Omega)$ . A density argument then yields  $B_1(u,\varphi) = B_2(u,\varphi)$  for all  $u \in H^1(\Omega)$  and  $\varphi \in C_0^{\infty}(\Omega)$ .

One can try to weaken the notion of a variational solution: if  $a_{nm} \in C^2(\Omega)$ ,  $b_n \in C^1(\Omega)$  then in (2.8) another partial integration can be arried out which yields

$$\langle u, L^* \varphi \rangle = \langle f, \varphi \rangle$$
 (2.10)

with

$$L^*\varphi := \sum_{n,m=1}^N a_{nm} \,\partial_n \partial_m \,\varphi + \sum_{n=1}^N \,b_n^* \,\partial_n \,\varphi + c^*\varphi \tag{2.11}$$

$$b_n^* := \left(\sum_{m=1}^N \partial_m (a_{nm} + a_{mn})\right) - b_n$$
 (2.12)

$$c^* := c - \sum_{m=1}^{N} \partial_m b_m + \sum_{n,m=1}^{N} \partial_n \partial_m a_{nm} . \qquad (2.13)$$

 $L^*$  is called the formal adjoint of L, and we introduce:

**Definition 2.4**  $u \in L^2_{loc}(\Omega)$  is called a weak solution of Lu = f, if (2.10) holds for all  $\varphi \in C_0^{\infty}(\Omega)$ . Here the coefficients of L are assumed to satisfy  $a_{nm} \in C^2(\Omega)$ ,  $b_n \in C^1(\Omega)$ ,  $c \in L^{\infty}_l oc(\Omega)$ , and  $L^*$  is given by (2.11), (2.12), (2.13).

For our boundary value problem the notion of a weak solution will not be advantageous: In order to express the boundary condition we have to assume a possible solution to belong to  $H^1(\Omega)$ . But then the partial integration which led from (2.8) to (2.10) can be reversed and we end up with a variational solution.

The name 'variational solution' is due to the fact that in certain cases such a solution can be characterized as the minimizer of some function from a subset of  $H^1(\Omega)$  into **R**. Such functions are called functionals. For example in the case of the Laplace operator, i.e.  $a_{mn} = \delta_{mn}$ ,  $b_n = c = 0$ , the problem to find a variational solution for the data  $f \in L^2(\Omega)$ ,  $g \in H^1(\Omega)$  is equivalent with the problem:

**(DP)** Find the minimizer  $u \in H^1(\Omega)$  of the functional

$$D_f : H^1(\Omega) \longrightarrow \mathbf{R}, v \longmapsto \sum_{m=1}^N \langle \partial_n v, \partial_n v \rangle + 2 \operatorname{Re} \langle f, v \rangle$$
 (2.14)

among all elements v of  $H^1(\Omega)$  for which  $v - g \in H^1_0(\Omega)$ .

To see this, assume first that u is a variational solution, then for any  $v \in H^1(\Omega)$  with  $v - g \in H^1_0(\Omega)$ :

$$D_{f}(v) = \sum_{n=1}^{N} \langle \partial_{n} u + \partial_{n} (v - u), \partial_{n} u + \partial_{n} (v - u) \rangle + 2 \operatorname{Re} \langle f, u + (v - u) \rangle$$
  
$$= D_{f}(u) + \sum_{n=1}^{N} \langle \partial_{n} (v - u), \partial_{n} (v - u) \rangle$$
  
$$+ 2 \operatorname{Re} \left[ \sum_{n=1}^{N} \langle \partial_{n} u, \partial_{n} (v - u) \rangle + \langle f, v - u \rangle \right]$$
  
$$\geq D_{f}(u) .$$

For the second term is nonnegative and the third term vanishes by Lemma 2.1 because  $v - u = (v - g) - (u - g) \in H_0^1(\Omega)$ .

On the other hand assume that u minimizes  $D_f$  among all  $v \in H^1(\Omega)$  for which  $v - g \in H^1_0(\Omega)$ . Choose any  $\varphi \in C_0^{\infty}(\Omega)$ . Then for any  $z := re^{i\theta} \in \mathbb{C}$ ,  $\theta \in [0, 2\pi)$ ,  $r \ge 0$ :

$$0 \leq D_f(u + z\varphi) - D_f(u)$$
  
=  $\left(\sum_{n=1}^N \langle \partial_n \varphi, \partial_n \varphi \rangle\right) \cdot r^2 + 2r \operatorname{Re}\left[e^{-i\theta} \left(\sum_{n=1}^N \langle \partial_n u, \partial_n \varphi \rangle + \langle f, \varphi \rangle\right)\right]$ .

We now fix  $\theta$  such that the expression in the brackets is real, but not positive. However any polynomial in r of the form  $Ar^2 + Br$  with  $A, B \in \mathbf{R}$ ,  $B < 0 \leq A$  attains negative values for some positive r. Whence

$$\sum_{n=1}^{N} \langle \partial_n u, \partial_n \varphi \rangle + \langle f, \varphi \rangle = 0$$

 $\varphi$  was an arbitrary test-function. Therefore u is a variational solution.

(**DP**) is called 'Dirichlet's principle'. This name was introduced by B. Riemann (1826 - 1866) who — as a student — learnt to know it for f = 0 in Dirichlet's lectures. In those days, of course, the space  $H^1(\Omega)$  was not yet known and classically differentiable functions were used instead. Nevertheless it was commonly accepted to use it in an existence proof for the Dirichlet problem ' $\Delta u = 0$  in  $\Omega$ ,  $u|_{\partial\Omega} = g$ ': In this case the Dirichlet functional is nonnegative and the mathematicians did not hesitate to conclude that it must have a minimum. However Weierstrass pointed out that in this conclusion the notions of a minimum and an infimum were interchanged. It lasted until 1900 that Hilbert was able to use Dirichlet's principle for an existence proof (under suitable restrictions). His techniques were extended by R. Courant and his pupils to the direct methods of the Calculus of Variations.

After what was said above it is easy to guess that our first aim is to look for variational solutions. We will then show that for  $g \in H^2(\Omega)$  and smooth boundary  $\partial\Omega$  these variational solutions in fact are strong solutions. For still smoother f, g and  $\partial\Omega$  they will even be classical solutions. Theorems of this kind are called "regularity theorems".

There is a crucial necessary condition for the existence proofs for the Dirichlet problem: the differential operator L must be elliptic. The ellipticity allows to estimate  $\sum_{n,m=1}^{N} \langle a_{nm} \partial_n v, \partial_m v \rangle$  from below by  $\sum_{n=1}^{N} ||\partial_n v||^2$  (times some constant). There are several notions of ellipticity. For our approach we have to use a condition which in the literature is called *strong*, *uniform ellipticity*. Since we will not deal with any other concept of ellipticity we will not use these two adjectives.

**Definition 2.5** A differential expression  $L = \sum_{n,m=1}^{N} a_{nm} \partial_n \partial_m + b_n \partial_n + c$  in some domain  $\Omega$  is called (uniformly strongly) elliptic in  $\Omega$  iff there exists a constant E > 0 such that for all  $x \in \Omega$  and all  $\xi \in \mathbf{R}^N$ 

$$Re \sum_{m,n=1}^{N} a_{mn}(x)\xi_{m}\xi_{n} \ge E |\xi|^{2} .$$
(2.15)

The largest constant E which fits into (2.15) is called the ellipticity constant (of L in  $\Omega$ ).

The condition says that any eigenvalue of the matrices A(x) with entries  $\frac{1}{2} \operatorname{Re} (a_{nm}(x) + a_{mn}(x))$ is not less than E. Whence the surfaces

$$S(x) = \{\xi \in \mathbf{R}^N \colon \operatorname{Re} \sum_{m,n=1}^N a_{mn}(x)\xi_m\xi_n = 1\}$$

are ellipsoids and lie within the circle of radius 1/E.

The notions of hyperbolicity and parabolicity are also defined in terms of the eigenvalues of the matrix A(x): One speaks of hyperbolicity at some x if A(x) has as well positive as negative eigenvalues but not the eigenvalue 0. Parabolicity means that all eigenvalues are nonnegative but at least one is 0. However we shall not need these notions and so we kept them vague.

### Chapter 3

# Some Functional Analysis in Hilbert Space

In this chapter we introduce some facts on Hilbert spaces which are needed for the solution of the Dirichlet problem in the so called strongly coercive case. This is the case, in which

$$\left|\sum_{n,m=1}^{N} \left\langle a_{nm} \partial_n v , \partial_m v \right\rangle + \sum_{n=1}^{N} \left\langle a_n \partial_n v , v \right\rangle + \left\langle cv , v \right\rangle \right| \ge c_+ \|v\|_1^2 \tag{3.1}$$

holds for all  $v \in H_0^1(\Omega)$  with some constant  $c_+ > 0$ , independent of v.

For this purpose we shall need the Riesz–representation theorem, the approximation theorem, the projection theorem and the Lax–Milgram theorem. To formulate the Riesz–representation theorem we need

**Definition 3.1** Let X be a Banach space over the field  $\mathbf{K} = \mathbf{C}$  or  $\mathbf{K} = \mathbf{R}$ . A linear functional F on X is a linear operator from X into  $\mathbf{K}$ :

$$F: D(F) \subset X \rightarrow \mathbf{K}$$

D(F) does not need to be dense or closed. However if we speak of a continuous linear functional we will assume that D(F) = X, unless stated otherwise. The space  $B(X, \mathbf{K})$  of continuous linear functionals on X is denoted by X' and is called the dual space of X. If F is a linear functional and  $\mathbf{K} = \mathbf{C}$ , then  $\overline{F}$ :  $D(F) \subset X \longrightarrow \mathbf{C}$ ,  $x \longmapsto \overline{F(x)}$  is called an antilinear functional.  $\overline{F}$  is continuous if and only if F is continuous.

For example for any fixed  $u \in H^1(\Omega)$  the mapping

$$\bar{F} : H_0^1(\Omega) \longrightarrow \mathbf{C} 
v \longmapsto \sum_{n,m=1}^N \langle a_{nm} \partial_n u, \partial_m v \rangle + \sum_{n=1}^N \langle a_n \partial_n u, v \rangle + \langle cu, v \rangle$$
(3.2)

is a continuous antilinear functional on  $H_0^1(\Omega)$  if  $a_{nm}$ ,  $a_n$ ,  $c \in L^{\infty}(\Omega)$ .

The representation theorem by F. Riesz (1880–1956) states that any linear functional on a Hilbert space can be written as the scalar product with a fixed vector:

**Theorem 3.1** If H is a Hilbert space then for any linear functional  $F : H \to \mathbf{K}$  there exists a unique vector  $v \in H$  such that

$$Fx = \langle x, v \rangle \tag{3.3}$$

for all  $x \in H$ . Moreover,

$$||v|| = ||F||$$

Here  $\langle \cdot, \cdot \rangle$  resp.  $\|\cdot\|$  denote scalar product and norm in H. The norm in  $H' = B(H, \mathbf{K})$  is denoted by  $\|\cdot\|$  as well.

**Proof:** The proof is carried out in a way which shows the relationship with variational problems.

 $\operatorname{Let}$ 

$$\Phi : H \longrightarrow \mathbf{R}, x \longmapsto ||x||^2 - 2 \operatorname{Re} F(x) .$$
(3.4)

Then  $\Phi$  is bounded from below:

$$\Phi(x) \ge ||x||^2 - 2||F|| ||x|| \ge \min_{t \in \mathbf{R}} (t^2 - (2||F||) t) = -||F||^2 .$$

We prove that  $\Phi$  attains its minimum at some  $v \in H$  and that with this v the representation (3.3) is valid.

We make use of the parallelogram equality

$$||v + w||^{2} + ||v - w||^{2} = 2||v||^{2} + 2||w||^{2}$$
(3.5)

which is valid for all  $v, w \in H$ , as is proved by direct calculation. In fact the validity of (3.5) for all elements v, w from a normed space V characterizes V as an inner product space.

Since  $\Phi$  is bounded from below it has an infimum,  $\mu \in \mathbf{R}$  say. There exists a sequence  $(v_n)$  in H such that  $\Phi(v_n)$  tends to  $\mu$ . Assume that  $\varepsilon > 0$  is given and  $m, n \in \mathbf{N}$  are so large that

$$\Phi(v_n) < \mu + \varepsilon/4, \quad \Phi(v_m) < \mu + \varepsilon/4$$

Then

$$\begin{split} \|v_n - v_m\|^2 &= 2\|v_n\|^2 + 2\|v_m\|^2 - \|v_n + v_m\|^2 \\ &= 2\Phi(v_n) + 2\Phi(v_m) - 4\left[\|\frac{1}{2}(v_n + v_m)\|^2 - 2\operatorname{Re} F(\frac{1}{2}(v_n + v_m))\right] \\ &< 4\mu + \varepsilon - 4\Phi(\frac{1}{2}(v_n + v_m)) \le \varepsilon \end{split}$$

since  $\Phi(y) \ge \mu$  for any  $y \in H$ .

Thus  $(v_n)$  is a Cauchy sequence in H, and by the completeness of H, has a limit  $v \in H$ . Since  $\Phi$  is continuous we conclude  $\Phi(v) = \lim_{n \to \infty} \Phi(v_n) = \mu$ , whence v is a minimizer of  $\Phi$ .

Let now  $x \in H$  be arbitrary and  $z := re^{i\theta} \in \mathbf{C}$ , r > 0,  $\theta \in [0, 2\pi)$ . (In case  $\mathbf{K} = \mathbf{R}$  we choose  $\theta = 0$  or  $\theta = \pi$ ).

Then (c.f. the reasoning below the introduction of Dirichlet's principle):

$$0 \le \Phi(v + zx) - \Phi(v) = r^2 ||x||^2 + 2r \operatorname{Re} e^{+i\theta} (\langle x, v \rangle - F(x)) \quad .$$
(3.6)

Choose  $\theta$  such that

Re 
$$\left(e^{i\theta}(\langle x, v \rangle - F(x))\right) =: \left|\langle x, v \rangle - F(x)\right|$$

If  $\langle x, v \rangle - F(x) \neq 0$  then there exists r > 0 such that the right side of (3.6) is less than 0. Since x was arbitrary we have just shown that for all  $x \in H$ 

$$\langle x, v \rangle = F(x)$$

By the Cauchy Schwarz inequality then

$$|Fx| \le ||v|| ||x|| ,$$

whence  $||F|| \le ||v||$ . On the other hand with x := v:

$$|v||^2 = \langle v, v \rangle = F(v) \le ||F|| ||v||$$
.

Thus  $||F|| \ge ||v||$  and we conclude ||F|| = ||v||.

If there is a second element  $w \in H$  such that for all  $x \in H$ 

$$\langle x, w \rangle = F(x),$$

then

$$||v - w||^{2} = \langle v - w, v \rangle - \langle v - w, w \rangle = F(v - w) - F(v - w) = 0$$

i.e. v = w. This proves the representation theorem.

**Corollary 3.1** Let  $\Omega$  denote any domain in  $\mathbb{R}^N$ ,  $f \in L^2(\Omega)$  and  $g \in H^1(\Omega)$ . Then there exists a unique variational solution u of ' $\Delta u - u = f$ ,  $u|_{\partial\Omega} = g|_{\partial\Omega}$  '.

**Proof:** We have to show that there exists a unique  $u \in H^1(\Omega)$  with  $u - g \in H^1_0(\Omega)$  and

$$\sum_{n=1}^{N} \langle \partial_n u, \partial_n \varphi \rangle + \langle u, \varphi \rangle = - \langle f, \varphi \rangle$$
(3.7)

for all  $\varphi \in H_0^1(\Omega)$ . Putting u - g =: v this is equivalent with the unique existence of  $v \in H_0^1(\Omega)$ such that for all  $\varphi \in H_0^1(\Omega)$ 

$$\langle v, \varphi \rangle_1 = -\langle f, \varphi \rangle - \langle g, \varphi \rangle_1 .$$
 (3.8)

Instead of (3.8) we might as well write

$$\langle \varphi, v \rangle_1 = - \langle \varphi, f \rangle - \langle \varphi, g \rangle_1 .$$
 (3.9)

But

$$F : H_0^1(\Omega) \longrightarrow \mathbf{C}, \ \varphi \longmapsto -\langle \varphi, f \rangle - \langle \varphi, g \rangle_1$$

is a continuous linear functional on  $H_0^1(\Omega)$ : Linearity is clear, continuity follows by the estimate

$$|F(\varphi)| \leq |\langle \varphi, f \rangle| + |\langle \varphi, g \rangle_{1}|$$

$$\leq ||f|| ||\varphi|| + ||g||_{1} ||\varphi||_{1}$$

$$\leq (||f|| + ||g||_{1}) ||\varphi||_{1} ,$$
(3.10)

since  $\|\varphi\| \leq \|\varphi\|_1$ . By the Riesz representation theorem there exists a unique  $v \in H_0^1(\Omega)$  such that

$$\langle \varphi, v \rangle_1 = F(\varphi)$$

for all  $\varphi \in H_0^1(\Omega)$ .

**Remark 3.1** We use the notations from the proof above. Notice, that by (3.10)

$$||v||_1 = ||F|| \le ||f|| + ||g||_1$$

hence with u = v + g:

$$||u||_1 \leq ||v||_1 + ||g||_1 \leq ||f|| + 2 ||g||_1 .$$
(3.11)

The Cartesian product of two Hilbert spaces  $\tilde{H}$ ,  $\hat{H}$  is a Hilbert space  $\tilde{H} \times \hat{H}$  itself, if one introduces the scalar product

$$\langle \, ( ilde{x}, \hat{x}) \, , \, ( ilde{y}, \hat{y}) \, 
angle_{ar{H} imes \hat{H}} \; := \; \langle \, ilde{x} \, , \, ilde{y} \, 
angle_{ar{H}} + \langle \, \hat{x} \, , \, \hat{y} \, 
angle_{\hat{H}}$$
 .

The data (f,g) of the Dirichlet problem are elements of the Cartesian product  $L^2(\Omega) \times H^1(\Omega)$  and we may define

$$G : L^2(\Omega) \times H^1(\Omega) \longrightarrow H^1(\Omega) , (f,g) \longmapsto u$$

where u is the variational solution of ' $\Delta u + u = f$ ,  $u \mid_{\partial\Omega} = g \mid_{\partial\Omega}$ '. G is a linear operator as is easily checked. (3.11) implies that G is continuous since the right hand side may be estimated further by  $5^{1/2} \cdot (||f||^2 + ||g||_1^2)^{1/2}$ , which is  $5^{1/2}$  times the norm of (f,g) in  $L^2(\Omega) \times H^1(\Omega)$ .

We now use the representation theorem for a proof of the approximation theorem:

**Theorem 3.2** Let H be a Hilbert space and M some closed subspace of H. For any  $x \in H$  there exists a unique  $v \in M$  such that

$$||x - v|| = d = \min_{y \in M} ||x - y|| .$$
(3.12)

This v is characterized by

$$\langle x - v, y \rangle = 0 \quad for \ all \ y \in M \quad .$$

$$(3.13)$$

**Proof:** If  $v \in M$  satisfies (3.13), then for any  $y \in M$ :

$$||x - y||^{2} = ||x - v + (v - y)||^{2} = ||x - v||^{2} + 2 \operatorname{Re} \langle x - v, v - y \rangle + ||v - y||^{2}$$
$$= ||x - v||^{2} + ||v - y||^{2}.$$

The last equality follows from (3.13) and the fact that  $v - y \in M$ . From the equation above we see that ||v - y|| will be larger than d unless y = v. This means that if v satisfies (3.13) it is the unique minimizer of (3.12).

Thus we have to prove the existence of some v for which (3.13) or equivalently

$$\langle y, v \rangle = \langle y, x \rangle \tag{3.14}$$

holds for all  $y \in M$ . But

$$F : M \longrightarrow \mathbf{K}, y \longmapsto \langle y, x \rangle$$

is a continuous linear functional on M and by Theorem 3.1 one may find  $v \in M$  such that (3.14) holds for all  $y \in M$ . Notice, that we used here that a closed subspace of a Hilbert space can be considered a Hilbert space itself. q.e.d.

Theorem 3.2 was proved with the help of Theorem 3.1 One may as well do this the other way round.

From the approximation theorem we obtain the projection theorem. For this we introduce some notations. For some subset S of a Hilbert space H let

$$S^{\perp} := \{ y \in H \colon \langle y, s \rangle = 0 \text{ for all } s \in S \} .$$

 $S^{\perp}$  is called the orthogonal complement of S (in H) and it is easy to see that  $S^{\perp}$  is always a closed linear subspace of H. Moreover, if S is a subspace of H then its orthogonal complement is the same as that of its closure:

$$S^{\perp} = (\overline{S})^{\perp}$$

The projection theorem now reads

**Theorem 3.3** Let M denote a closed subspace of the Hilbert space H. Then for any  $x \in H$  there exist a unique  $y \in M$  and a unique  $z \in M^{\perp}$  such that

$$x = y + z \; .$$

Moreover, Pythagoras' theorem

$$||x||^2 = ||y||^2 + ||z||^2$$

is valid.

**Proof:** For any  $x \in H$  we can find some  $y \in M$  such that

$$||x - y|| = \min_{m \in M} ||x - m||$$

Putting z := x - y we find  $z \in M^{\perp}$  by (3.13), and whence the existence of  $y \in M$ ,  $z \in M^{\perp}$  with x = y + z is proved.

Pythagoras' theorem is a direct consequence of the orthogonality of y and z:

$$||y + z||^2 = ||y||^2 + ||z||^2 + 2 \operatorname{Re} \langle y, z \rangle = ||y||^2 + ||z||^2$$

Now assume that there is a second pair  $\hat{y}inM$ ,  $\hat{z} \in \hat{M}$  with  $x = \hat{y} + \hat{z}$ . Then  $y - \hat{y} \in M$ ,  $z - \hat{z} \in M^{\perp}$ and  $(y - \hat{y}) + (z + \hat{z}) = 0$ . Consequently by Pythagoras' theorem  $||y - \hat{y}||^2 + ||z - \hat{z}||^2 = 0$ , i.e.  $y = \hat{y}$  and  $z = \hat{z}$ . This shows that the decomposition of x is unique. q.e.d.

As a corollary we note

Corollary 3.2 For any subspace M of a Hilbert space H

$$(M^{\perp})^{\perp} = \overline{M}$$

**Proof:** Of course  $M \subset (M^{\perp})^{\perp}$  and since orthogonal complements are closed,  $\overline{M} \subset (M^{\perp})^{\perp}$ . Since  $(\overline{M})^{\perp} = M^{\perp}$ , any  $x \in (M^{\perp})^{\perp}$  can be written as

$$x = y + z$$
, thus  $0 = z + (y - x)$ 

where  $y \in \overline{M}$  and  $z \in M^{\perp}$ . Since  $y - x \in (M^{\perp})^{\perp}$  we have by Pythagoras' theorem  $||z||^2 + ||y - x||^2 = 0$  which implies z = 0 and  $x = y \in \overline{M}$ . q.e.d.

Theorem 3.3 can be written in the form

$$H = M \oplus M^{\perp} \tag{3.15}$$

for any closed subspace M of a Hilbert space H. (3.15) should be read: H is the direct sum of M and  $M^{\perp}$ .

If  $M_1$  and  $M_2$  are subspaces of H, one defines

$$M_1 + M_2 := \{ y + z \colon y \in M_1 , z \in M_2 \}$$
.

The notation  $M_1 \oplus M_2$  means  $M_1 + M_2$  and additionally expresses that  $M_1 \cap M_2 = \{0\}$ . Thus

$$M_1 + M_2 = M_1 \oplus M_2 : \Leftrightarrow M_1 \cap M_2 = \{0\}$$
.

The sum  $M_1 + M_2$  always is a subspace of H.

If  $M_1 + M_2$  is direct, then the representation y + z of any  $x \in M_1 + M_2$  is unique.

Moreover if  $x_1 = y_1 + z_1$ ,  $x_2 = y_2 + z_2$ ,  $y_1, y_2 \in M_2$ ,  $z_1, z_2 \in M_2$  then for any  $\alpha, \beta \in \mathbf{K}$ :

$$\alpha x_1 + \beta x_2 = (\alpha y_1 + \beta y_2) + (\alpha z_1 + \beta z_2) ,$$

and  $\alpha y_1 + \beta y_2 \in M_1$ ,  $\alpha z_1 + \beta z_2 \in M_2$ . Thus, if H is a direct sum of two subspaces  $M_1$  and  $M_2$ , then there exist two linear operators

$$P_1, P_2 : H \rightarrow H$$

such that

$$P_1(y+z) = y$$
,  $P_2(y+z) = z$ 

for any  $y \in M_1$ ,  $z \in M_2$ .  $P_1$  and  $P_2$  are called projections of H onto  $M_1$  resp.  $M_2$  along  $M_2$  resp.  $M_1$ . Of course,

$$P_2 = I - P_1 , \quad I: \text{ the identity }, \qquad (3.16)$$

$$P_1 P_1 = P_1, \quad P_2 P_2 = P_2 \text{ and}$$
 (3.17)

$$P_1 P_2 = 0 (3.18)$$

In the case of  $M_1 := M$ ,  $M_2 := M^{\perp}$  these projections are called orthogonal projections onto M resp. onto  $M^{\perp}$ . Denoting the orthogonal projection onto M by P and putting Q := I - P, we find by Pythagoras theorem:

$$||Px||^2 + ||Qx||^2 = ||x||^2$$
.

Therefore

$$||Px|| \le ||x|| , \quad ||Qx|| \le ||x||$$
(3.19)

for all  $x \in H$  which implies that P and Q are bounded linear operators H into H and

$$||P|| = ||Q|| = 1$$

provided that neither M nor  $M^{\perp}$  are the whole of H. Of course (3.19) only implies ||P||,  $||Q|| \le 1$ . However if M resp.  $M^{\perp}$  contain a nonzero element y then Py = y resp. Qy = y which implies that  $||P|| \ge 1$  resp.  $||Q|| \ge 1$ .

**Example 3.1** Let  $\Omega$  denote some domain in  $\mathbb{R}^N$ .  $H_0^1(\Omega)$  can be considered as a closed linear subspace of  $H^1(\Omega)$ . Thus by the projection theorem

$$H^1(\Omega) = H^1_0(\Omega) \oplus (H^1_0(\Omega))^{\perp}$$

and there exist orthogonal projections P, Q of  $H^1(\Omega)$  onto  $H^1_0(\Omega)$  resp.  $(H^1_0(\Omega))^{\perp}$ . The space  $(H^1_0(\Omega))^{\perp}$  consists of those  $u \in H^1(\Omega)$  for which

$$\langle u, v \rangle_1 = 0$$
 for all  $v \in H^1_0(\Omega)$ 

holds. This is nothing but

$$\sum_{n=1}^N \left< \partial_n u \,,\, \partial_n \varphi \right> \,+\, \left< u \,,\, \varphi \right> \,=\, 0 \quad \text{for all } \varphi \in H^1_0(\Omega) \;.$$

Thus  $(H_0^1(\Omega))^{\perp}$  is the space of all variational solutions of the Dirichlet problems ' $\Delta u - u = 0$ ,  $u|_{\partial\Omega} = g|_{\partial\Omega}$ ', when g varies over all of  $H^1(\Omega)$ , and the orthogonal projection Q assigns to any  $g \in H^1(\Omega)$  the variational solution of the Dirichlet problem above.

To handle more general Dirichlet problems we need a slight modification of the Riesz representation theorem, the Lax–Milgram theorem. For this we need

**Definition 3.2** A sesquilinear form on a Hilbert space H over the field  $\mathbf{K}$ ,  $\mathbf{K} = \mathbf{R}$  or  $\mathbf{K} = \mathbf{C}$ , is a mapping

$$B: H \times H \rightarrow \mathbf{K}$$

such that for all  $u, v, w \in H$ ,  $\alpha, \beta \in \mathbf{K}$ 

$$\begin{split} B(\alpha u + \beta v, w) &= \alpha \, B(u, w) + \beta \, B(v, w) \\ B(u, \alpha v + \beta w) &= \overline{\alpha} \, B(u, v) + \overline{\beta} \, B(u, w) \end{split}$$

hold. B is called "bounded" if with some c > 0:

$$|B(u,v)| \le c ||u|| ||v|| \tag{3.20}$$

holds for all  $u, v \in H$ .

As with linear operators, the bounded sesquilinear forms are exactly the continuous sesquilinear forms.

**Theorem 3.4** (P.D. Lax, A.N. Milgram, 1954) Let B denote a bounded sesquilinear form on a Hilbert space  $\mathbf{K}$ . Suppose that there exists some  $c_+ > 0$  such that for any  $u \in H$ 

$$|B(u,u)| \ge c_+ ||u||^2 . (3.21)$$

Then for any continuous linear functional F on H there exist a unique  $u \in H$  and a unique  $v \in H$  such that for all  $w \in H$ 

$$B(w,v) = F(w) \tag{3.22}$$

$$B(u,w) = \overline{F(w)} \quad . \tag{3.23}$$

#### **Proof:**

If B is a bounded sesquilinear form, which satisfies (3.21), then the same is true for  $\tilde{B}$ , given by

$$\tilde{B}$$
 :  $H \times H \longrightarrow \mathbf{K}$ ,  $(u,v) \longmapsto \overline{B(v,u)}$ .

Therefore it suffices to prove the existence of a unique v, such that (3.22) is valid for all  $w \in H$ .

For any  $g \in H$  the mapping

$$B(\cdot,g) : H \longrightarrow \mathbf{K}, \ w \longmapsto B(w,g)$$
 (3.24)

is a linear functional which is continuous since

$$|B(w,g)| \le c \|w\| \|g\| = c_g \|w\| \tag{3.25}$$

holds for all  $w \in H$  with  $c_g := c ||g||$ , c the constant from Definition 3.2. Hence by the Riesz representation theorem, there exists a unique  $\tilde{g} \in H$  such that for all  $w \in H$ 

$$B(w,g) = \langle w, \tilde{g} \rangle . \tag{3.26}$$

Thus we may define an operator  $T: H \to H$  by this procedure

$$T \; : \; H \; \longrightarrow \; H \; , \; g \; \longmapsto \; \widetilde{g} \; \; ,$$

where  $\tilde{g}$  is given by the validity of (3.26) for all  $w \in H$ . T can easily be seen to be a linear operator. For example, if  $\tilde{g} = Tg$  then for all  $w \in H$   $B(w, \alpha g) = \overline{\alpha}B(w, g) = \overline{\alpha}\langle w, \tilde{g} \rangle = \langle w, \alpha \tilde{g} \rangle$ , i.e.  $T(\alpha g) = \alpha Tg$ .

T is a continuous operator: Putting w := Tg in (3.26) we find from (3.20)

$$||Tg||^2 = \langle Tg, Tg \rangle = B(Tg, g) \le c ||Tg|| ||g||$$

by which  $||Tg|| \le c ||g||$  follows.

Suppose now that  $T^{-1}$  exists and is defined on whole H. Then if F is a linear functional, we find  $\tilde{v} \in H$  such that  $\langle w, \tilde{v} \rangle = F(w)$  for all  $w \in H$ . Hence with  $v := T^{-1} \tilde{v}$  we get  $B(w, v) = \langle w, \tilde{v} \rangle = F(w)$ .

It remains to show that  $T^{-1}$  exists, is defined on whole H, and that the v obtained above is unique. By (3.21) and the definition of T we obtain for any  $g \in H$ 

$$|c_+||g||^2 \le |B(g,g)| = |\langle g, Tg \rangle| \le ||g|| ||Tg||$$
,

 $_{\mathrm{thus}}$ 

$$c_{+} \|g\| \leq \|Tg\| \quad . \tag{3.27}$$

This estimate shows

- (i) T is injective: For ||Tg|| = 0 implies ||g|| = 0.
- (ii) R(T) is closed.
- (iii)  $T^{-1}: R(T) \subset H \to H$  is continuous.

To see (ii) let  $(Tx_n)$  tend to some  $y \in H$ . Then

$$||x_n - x_m|| \le \frac{1}{c_+} ||Tx_n - Tx_m||$$

Since  $(Tx_n)$  is a Cauchy sequence so is  $(x_n)$ , and there exists  $x \in H$  with

$$\lim_{n \to \infty} x_n = x .$$

But then  $Tx = \lim_{n \to \infty} Tx_n = y$ , i.e.  $y \in R(T)$ .

To see (iii) let  $y \in R(T)$ ,  $g = T^{-1}y$ . Then

$$||T^{-1}y|| = ||g|| \le \frac{1}{c_+} ||Tg|| = \frac{1}{c_+} ||y||$$
.

Thus  $T^{-1}$  is bounded and defined on the closed subspace R(T) of H.

To see that R(T) = H it suffices to show  $R(T)^{\perp} = \{0\}$  by Corollary 3.2. Let  $z \in R(T)^{\perp}$ , i.e.

$$\langle z, Tg \rangle = 0$$
 for all  $g \in H$ .

Then especially  $\langle z, Tz \rangle = 0$ , and we get from the definition of T and by (3.21):

$$0 = |\langle z, Tz \rangle| = |B(z, z)| \ge c_+ ||z||^2$$

thus z = 0.

It remains to show the uniqueness of v. Assume that  $v_1, v_2 \in H$  are such that  $B(w, v_i) = F(w)$  for all  $w \in H$  and  $i \in \{1, 2\}$ . Then

$$0 = F(v_2 - v_1) - F(v_2 - v_1) = B(v_2 - v_1, v_2) - B(v_2 - v_1, v_1)$$
$$= B(v_2 - v_1, v_2 - v_1) .$$

By (3.21) then  $v_2 = v_1$  follows.

### Chapter 4

# Solution of Strongly Coercive Dirichlet Problems

To find a variational solution of the Dirichlet problem

$$Lu = f, \quad u|_{\partial\Omega} = g|_{\partial\Omega} \tag{4.1}$$

we may use the Lax–Milgram lemma in special cases.

We shall from now on assume<sup>1</sup> that  $\Omega \subset \mathbf{R}^N$  is some domain and

$$L(x,\partial) = \sum_{n,m=1}^{N} a_{nm} \partial_n \partial_m + \sum_{n=1}^{N} b_n \partial_n + c$$
(4.2)

where

$$a_{nm} = a_{mn} \in C^1(\Omega) \tag{4.3}$$

$$a_{nm}, \ \partial_k a_{nm}, \ b_n, \ c \in L^{\infty}(\Omega)$$
, (4.4)

and L is elliptic, i.e. there exist some E>0 such that for all  $x\in\Omega$ ,  $\xi\in {\bf R}^N$ 

Re 
$$\sum_{n,m=1}^{N} a_{nm}(x) \xi_n \xi_m \ge E |\xi|^2$$
 . (4.5)

Moreover we fix a Dirichet form for L, namely

$$B(u,v) := \sum_{n,m=1}^{N} \langle a_{nm} \partial_m u, \partial_n v \rangle + \sum_{m=1}^{N} \langle a_m \partial_m u, v \rangle + \langle au, v \rangle ,$$

where

$$a_m := (\sum_{n=1}^N \partial_n a_{nm}) - b_m , \quad a := -c .$$
 (4.6)

<sup>&</sup>lt;sup>1</sup>After Corollary 4.1 we shall introduce another assumption

Notice that (4.5) and (4.3) imply

Re 
$$\sum_{n,m=1}^{N} a_{nm}(x) \zeta_n \overline{\zeta}_m \ge E |\zeta|^2 := E \left(\sum_{n=1}^{N} |\zeta_n|^2\right)$$
 (4.7)

for all  $x \in \Omega$ ,  $\zeta \in \mathbf{C}^N$ .

For the following lemma we introduce the notation

$$|u|_{m} := |u|_{m,\Omega} := \left(\sum_{|\alpha|=m} \|\partial^{\alpha} u\|_{0,\Omega}\right)^{1/2}$$
(4.8)

for  $u \in H^m(\Omega)$ .

Lemma 4.1 Let

$$\beta := \operatorname{ess} \sup_{x \in \Omega} \left( \sum_{m=1}^{N} |a_m(x)|^2 \right)^{1/2}$$
(4.9)

$$\alpha := \operatorname{ess inf}_{x \in \Omega} (\operatorname{Re} a(x)) = -\operatorname{ess sup}(\operatorname{Re} c(x)) . \tag{4.10}$$

Then for all  $u \in H^1(\Omega)$ 

Re 
$$B(u, u) \ge E |u|_1^2 - \beta |u|_1 ||u||_0 + \alpha ||u||_0^2$$
. (4.11)

Moreover there exists a constant  $\gamma > 0$  such that for  $u, v \in H^1(\Omega)$ 

$$|B(u,v)| \leq \gamma ||u||_1 ||v||_1 .$$
(4.12)

From Lemma 4.1 one concludes that for certain differential expressions L the Dirichlet form B satisfies

Re 
$$B(u, u) \ge c_+ ||u||_1^2$$
 (4.13)

with some  $c_+ > 0$  for all  $u \in H^1(\Omega)$ .

However, we only need with some  $c_+ > 0$ 

$$|B(\varphi,\varphi)| \ge c_+ \|\varphi\|_1^2 \quad \text{for all } \varphi \in C_0^\infty(\Omega) \quad . \tag{4.14}$$

If (4.13) or (4.14) hold for all  $\varphi \in C_0^{\infty}(\Omega)$  then it also holds for all  $u \in H_0^1(\Omega)$  since (4.12) implies the continuity of B.

Hence for the sesquilinear form B — restricted to  $H_0^1(\Omega)$  — the assumptions of the Lax–Milgram theorem hold, and we can use it to prove the unique existence of a solution of the Dirichlet problem:

**Theorem 4.1** Assume that the coefficients of L are such that the Dirichlet form

$$B: H^1(\Omega) \times H^1(\Omega) \to \mathbf{C}$$

satisfies (4.14) for all  $\varphi \in C_0^{\infty}(\Omega)$ . Then for any  $f \in L^2(\Omega)$  and  $g \in H^1(\Omega)$  there exists a unique variational solution  $u \in H^1(\Omega)$  of the Dirichlet problem (4.1)

$$Lu = f in \Omega$$
,  $u|_{\partial\Omega} = g|_{\partial\Omega}$ .

**Proof:** u solves (4.1) if and only if  $(u-g) =: \tilde{u} \in H_0^1(\Omega)$  and  $B(u, v) = -\langle f, v \rangle$  for all  $v \in H_0^1(\Omega)$ , the latter being equivalent with

$$B(\tilde{u}, v) = -\langle f, v \rangle - B(g, v) \quad \text{for all } v \in H_0^1(\Omega) \quad .$$

$$(4.15)$$

Now

$$F \ : \ H^1_0(\Omega) \ \longrightarrow \ \mathbf{C} \ , \ v \ \longmapsto \ \overline{[- \ \langle \ f \ , \ v \ \rangle - B(g,v)]}$$

is a continuous linear functional in  $H_0^1(\Omega)$ : Linearity is clear, and to prove continuity we estimate

$$Fv| \leq \|f\|_0 \|v\|_0 + \gamma \|g\|_1 \|v\|_1 \leq (\|f\|_0 + \gamma \|g\|_1) \|v\|_1$$

for all  $v \in H_0^1(\Omega)$ .

By the Lax–Milgram theorem there exists a unique  $\tilde{u}$  such that  $B(\tilde{u}, v) = \overline{F}(v)$  for all  $v \in H_0^1(\Omega)$ which is (4.15). q.e.d.

From Lemma 4.1 we see that the conditions of Theorem 4.1 are satisfied if for example  $\beta = 0$  and  $\alpha > 0$ , i.e. if

$$b_m = \sum_{n=1}^N \partial_n a_{nm}$$

and if Re c is bounded from above by a negative constant.

Notice however that (4.14) needs to be satisfied for  $\varphi \in C_0^{\infty}(\Omega)$  only. So we may play around with partial integrations. For example, for  $\varphi \in C_0^{\infty}(\Omega)$ 

$$\operatorname{Re} B(\varphi, \varphi) = \sum_{n,m=1}^{N} \operatorname{Re} \langle a_{nm} \partial_n \varphi, \partial_m \varphi \rangle + \frac{1}{2} \sum_{n=1}^{N} \operatorname{Re} \langle a_n \partial_n \varphi, \varphi \rangle$$
$$- \frac{1}{2} \sum_{n=1}^{N} \operatorname{Re} \langle a_n \varphi, \partial_n \varphi \rangle + \operatorname{Re} \langle a \frac{1}{2} (\sum_{n=1}^{N} \partial_n a_n) \varphi, \varphi \rangle.$$

If therefore the coefficients  $a_n$  are real valued  $C^1(\Omega)$ -functions then the second and the third term cancel each other, and (4.13) will hold if

ess 
$$\inf_{\Omega} (a - \frac{1}{2} \sum_{n=1}^{N} (\partial_n a_n)) > 0$$
 . (4.16)

From Lemma 4.1 we also get for any  $\varepsilon > 0$ :

$$\operatorname{Re} B(u, u) \geq E |u|_{1}^{2} - \frac{1}{2} \varepsilon \beta |u|_{1}^{2} - \frac{1}{2\varepsilon} \beta ||u|_{0}^{2} + \alpha ||u||_{0}^{2} \qquad (4.17)$$
$$= (E - \frac{1}{2} \varepsilon \beta) |u|_{1}^{2} + (\alpha - \frac{\beta}{2\varepsilon}) ||u||_{0}^{2}.$$

Hence, if  $\beta^2 < 4\alpha E$  one may find  $\varepsilon > 0$  such that both  $E - \frac{1}{2} \varepsilon \beta$  as well as  $\alpha - \frac{\beta}{2\varepsilon}$  are positive, and (4.14) holds.

**Definition 4.1** A sesquilinear form B on a closed subspace V of  $H^1(\Omega)$  is called coercive over V if there exist constants  $c_+ > 0$  and  $c_- \ge 0$  such that for all  $v \in V$ 

Re 
$$B(v,v) \ge c_+ ||v||_1^2 - c_- ||v||_0^2$$
. (4.18)

B is called strongly coercive over V if there exists  $c_+ > 0$  such that

Re 
$$B(v,v) \ge c_+ ||v||_1^2$$
 (4.19)

holds for all  $v \in V$ .

Thus the essential condition (4.14) for Theorem 4.1 is satisfied if B is strongly coercive over  $H_0^1(\Omega)$ . Notice however that strong coerciveness is sufficient but not necessary for (4.14). Lemma 4.1 together with the calculation which led to (4.17) shows that the Dirichlet form B is coercive on  $H^1(\Omega)$  and whence on  $H_0^1(\Omega)$ .

If B is coercive on some closed subspace V of  $H^1(\Omega)$ , and if  $c_+$ ,  $c_-$  denote the constants, appearing in (4.18), then for any  $\lambda > c_- - c_+$  the form  $B_{\lambda}$ , given by

$$B_{\lambda}(u,v) := B(u,v) + \lambda \langle u, v \rangle_{0,\Omega}$$

$$(4.20)$$

is strongly coercive. This is the reason why one rather deals with Re B than with |B|. With respect to our differential expression L this means:

**Corollary 4.1** For any  $\lambda \in \mathbf{C}$  let  $L_{\lambda}$  denote the differential expression

$$L_{\lambda}(x,\partial) = \sum_{n,m=1}^{N} a_{nm}(x)\partial_n\partial_m + \sum_{m=1}^{N} b_m(x)\partial_m + (c(x) - \lambda) \quad .$$
(4.21)

There exists  $\lambda_0 \in \mathbf{R}$  such that for any  $\lambda$  with  $\operatorname{Re} \lambda \geq \lambda_0$  and any  $f \in L^2(\Omega)$ ,  $g \in H^1(\Omega)$  the Dirichlet problem " $L_{\lambda}u = f$  in  $\Omega$ ,  $u|_{\partial\Omega} = g|_{\partial\Omega}$ " has a unique variational solution.

**Proof:** Choose  $\lambda_0$  such that  $B_{\lambda_0}$  is strongly coercive ! q.e.d.

Up to now we are not yet able to solve the Dirichlet problem for the Laplace operator  $L = \Delta$ . In fact the Dirichlet form for  $\Delta$  is not strongly coercive for certain unbounded domains, including the whole space. For a counterexample in  $\Omega := \mathbf{R}^N$  choose some nonzero  $\varphi \in C_0^{\infty}(\mathbf{R}^N)$  and, with  $n \in \mathbf{N}$ , define

$$\varphi_n(x) := \varphi(\frac{1}{n}x)$$
.

Then

$$\varphi_n \|_0^2 = n^N \|\varphi\|_0^2, \quad |\varphi_n|_1^2 = n^{N-2} |\varphi|_1^2.$$
 (4.22)

The Dirichlet form for  $\Delta$  is given by  $B(u, v) := \sum \langle \partial_n u, \partial_n v \rangle$  such that Re  $B(u, u) = B(u, u) = |u|_1^2$ . Inserting  $\varphi_n$  for V in (4.19) would yield

$$n^{N-2} |\varphi|_1^2 \ge c_+ n^N ||\varphi||_0^2$$

with some  $c_+ > 0$ . But this is impossible since  $\|\varphi\|_0^2 \neq 0$ .

In bounded domains, however, we can do better. This is due to the Poincaré estimate:

**Lemma 4.2** Suppose that the domain  $\Omega$  lies between two parallel hyperplanes of distance d. Then for all  $v \in H_0^1(\Omega)$ 

$$\|v\|_0 \le d \|v\|_1 \quad . \tag{4.23}$$

**Remark 4.1** Estimate (4.23) may be sharpened to  $\|\varphi\|_0 \leq \frac{d}{\pi} \|\varphi\|_1$ .

**Proof of Lemma** 4.2: Without loss of generality we may assume that  $\Omega$  lies between the planes  $\{x : x_1 = 0\}$  and  $\{x : x_1 = d\}$ , i.e.

$$0 < x_1 < d$$

holds for any  $x \in \Omega$ . Moreover, it suffices to prove (4.23) for each  $\varphi \in C_0^{\infty}(\Omega)$ . Notice that we assumed elements of  $C_0^{\infty}(\Omega)$  to be defined in the whole of  $\mathbf{R}^N$ , namely by  $\varphi = 0$  outside of  $\Omega$ , and that they are in  $C^{\infty}(\mathbf{R}^N)$  then.

An integration yields for  $x \in \Omega$ 

$$\varphi(x) = \varphi(x_1, x_2, \dots, x_N) = \int_0^{x_1} \partial_1 \varphi(t, x_2, \dots, x_N) dt$$

From this we obtain

$$\left|\varphi(x)\right|^{2} \leq \left(\int_{0}^{d} 1 \cdot \left|\partial_{1}\varphi(t, x_{2}, \dots, x_{N})\right| dt\right)^{2} \leq d \int_{0}^{d} \left|\partial_{1}\varphi(t, x_{2}, \dots, x_{N})\right|^{2} dt$$

The last estimate follows by the Cauchy Schwarz inequality.

Integrating this inequality with respect to  $x_2, \ldots, x_N$  we obtain

$$\int_{\mathbf{R}^{N-1}} |\varphi(x_1, x_2, \dots, x_N)|^2 d(x_2, \dots, x_N) \le d |\varphi|_1^2 .$$

Another integration with respect to  $x_1$  from 0 to d yields the desired estimate. q.e.d.

With Poincaré's estimate we obtain unique solvability for  $L = \Delta$ , too:

**Theorem 4.2** Let  $\Omega$  denote a domain in  $\mathbf{R}^N$  which lies between two parallel hyperplanes of distance d > 0, i.e. for each  $x \in \Omega$  and some  $\gamma \in \mathbf{R}$ ,  $\xi \in \mathbf{R}^N$ ,  $|\xi| = 1$ ,

$$\gamma < \xi \cdot x < \gamma + d$$

holds. Assume that the coefficients of L satisfy  $\beta = 0$  and  $\alpha > -d^{-2}E$ , where  $\alpha, \beta$  and E are given by (4.9), (4.10) and (4.5) respectively. Then the Dirichlet form is strongly coercive over  $H_0^1(\Omega)$ .

**Proof:** For all  $v \in H_0^1(\Omega)$  and with  $\varepsilon = E^{-1}d^2\alpha + 1 > 0$  Lemma 4.1 yields

$$\begin{aligned} \operatorname{Re} \ B(v,v) &\geq E \ |v|_1^2 + \alpha \ \|v\|_0^2 \\ &= E \ \left( |v|_1^2 - d^{-2}(1 - \varepsilon/2) \ \|v\|_0^2 + d^{-2} \varepsilon/2 \ \|v\|_0^2 \right) \\ &\geq E \left( |v|_1^2 - (1 - \varepsilon/2) \ |v|_1^2 + d^{-2} \varepsilon/2 \ \|v\|_0^2 \right) \ , \end{aligned}$$

the last estimate following from Poincaré's inequality. Thus with  $c_{+} = \min\{\frac{E\varepsilon}{2}, \frac{E\varepsilon}{2d}\}$  inequality (4.19) follows. q.e.d.

**Remark 4.2** Differential expressions with  $\beta = 0$  satisfy

$$b_m = \sum_{n=1}^N \partial_n a_{nm}$$
.

They are usually written in the form

$$\sum_{n,m=1}^N \,\partial_n(a_{nm}\partial_m) + c$$

and are said to be in 'divergence form'.

#### Chapter 5

## The Fredholm Alternative

If Ax = y denotes a system of linear equations in  $\mathbf{R}^N$  we know from Linear Algebra that the following alternative is true:

- Either the system Ax = y is uniquely solvable for all  $y \in \mathbf{R}^N$ .
- Or the homogeneous system Ax = 0 has nontrivial solutions. In this case the adjoint problem<sup>1</sup>  $A^*x = 0$  has nontrivial solutions, too, and the dimensions of the kernels of A and  $A^*$  are equal (and finite). The problem Ax = y can be solved if and only if y is orthogonal to any vector in the kernel of  $A^*$ .

This alternative can be reformulated as (5.1), (5.2):

$$\dim N(A) = \dim N(A^*) < \infty \tag{5.1}$$

$$R(A) = N(A^*)^{\perp} . (5.2)$$

Here N(A) and R(A) denote the kernel and the range of A respectively. Also we did distinguish the matrix A from the linear operator which is described by A with respect to the standard basis in  $\mathbf{R}^{N}$ .

In the case of infinite dimension the above alternative is usually wrong. The aim of this section is to exhibit a class of linear operators A for which it still holds. The above alternative carries the name of the Norwegian mathematician I. Fredholm (1866 - 1927) who was the first to obtain such a result for an infinitely dimensional problem.

We will restrict ourselves to the case of operators in a Hilbert space H which makes some notions and results easier. At first we have to define the notion of the adjoint of a linear operator. Here

 $<sup>{}^{1}</sup>A^{*}$  is the adjoint matrix of matrix A.

we restrict ourselves to the case of a bounded operator  $A: H_1 \to H_2$ , where  $H_1, H_2$  are Hilbert spaces with scalar products  $\langle \cdot, \cdot \rangle_1$  resp.  $\langle \cdot, \cdot \rangle_2$ . These should not be confused with the scalar products in some Sobolev spaces. In the sequel  $H_1, H_2, H_3, H$  always denote Hilbert spaces over the same field **K**, **K** = **R** or **K** = **C**.

**Theorem 5.1 (and Definition)** Let  $A \in B(H_1, H_2)$  (defined on whole  $H_1$ ). Then there exists a unique linear operator  $A^* \in B(H_2, H_1)$  (defined on whole  $H_2$ ) such that

$$\langle Au, v \rangle_2 = \langle u, A^*v \rangle_1 \tag{5.3}$$

for all  $u \in H_1$ ,  $v \in H_2$ . This operator  $A^*$  is called the adjoint of A.

**Proof:** For any  $v \in H_2$  the mapping

$$F_v : H_1 \longrightarrow \mathbf{K}, \ u \longmapsto \langle Au, v \rangle$$

is a continuous linear functional on  $H_1$ . The continuity follows from the estimate

$$|F_v u| \leq ||Au||_2 ||v||_2 \leq ||A|| ||u||_1 ||v||_2$$

whence

$$||F_v|| \leq ||A|| \, ||v|| . \tag{5.4}$$

By the Riesz representation theorem (Theorem 3.1) there exists a unique  $f \in H_1$  such that for all  $u \in H_1$ 

$$\langle Au, v \rangle = F_v u = \langle u, f \rangle$$
.

Thus necessarily  $A^*v := f$ , and it remains to prove that  $A^*$  is linear and continuous.

For the linearity let  $w = \alpha v_1 + \beta v_2$ ,  $v_1, v_2 \in H_2$ ,  $\alpha, \beta \in \mathbf{K}$ . Then for any  $u \in H_1$ 

$$F_{w}u = \langle Au, \alpha v_{1} + \beta v_{2} \rangle = \overline{\alpha} \langle Au, v_{1} \rangle + \overline{\beta} \langle Au, v_{2} \rangle = \overline{\alpha} F_{v_{1}} + \overline{\beta} F_{v_{2}}.$$

Hence for all  $u \in H_1$ 

$$\langle u, \alpha A^* v_1 + \beta A^* v_2 \rangle = \overline{\alpha} \langle u, A^* v_1 \rangle + \overline{\beta} \langle u, A^* v_2 \rangle = F_w u$$

On the other hand by the definition of  $A^*$ 

$$F_w u = \langle u, A^* w \rangle = \langle u, A^* (\alpha v_1 + \beta v_2) \rangle$$

for all  $u \in H_1$ . Since the representation of  $F_w$  is unique we conclude

$$\alpha A^* v_1 + \beta A^* v_2 = A^* (\alpha v_1 + \beta v_2) .$$

For the continuity notice that by Theorem 3.1 and by (5.4)

$$||A^*v||_1 = ||F_v|| \le ||A|| ||v||$$

from which the continuity of  $A^*$  and moreover

$$||A^*|| \leq ||A|| \tag{5.5}$$

follow.

We did not yet consider the composition of two linear operators  $A \in B(H_1, H_2)$ ,  $B \in B(H_2, H_3)$ . This yields a linear operator

$$BA : H_1 \longrightarrow H_2, u \longmapsto B(Au)$$

which is continuous and for which

$$||BA|| \leq ||B|| \, ||A|| \tag{5.6}$$

holds. If A is onto and has a bounded inverse  $A^{-1}$  then  $A^{-1}A = I_1$  and  $AA^{-1} = I_2$  where  $I_1, I_2$ denote the identities in  $H_1$  resp.  $H_2$ . Of course  $I_2A = AI_1 = A$  and we have distributive laws

$$B(\alpha A_1 + \beta A_2) = \alpha(BA_1) + \beta(BA_2) \tag{5.7}$$

$$(\alpha B_1 + \beta B_2)A = \alpha(B_1A) + \beta(B_2A) \tag{5.8}$$

for  $A, A_1, A_2 \in B(H_1, H_2)$ ,  $B, B_1, B_2 \in B(H_2, H_3)$  and  $\alpha, \beta \in \mathbb{C}$ . Because of (5.7) and (5.8) one usually omits the parenthesis on the right hand side of (5.7), (5.8).

We may now gather some properties of the adjoints:

**Lemma 5.1** Let  $A, A_1, A_2 \in B(H_1, H_2)$ ,  $B \in B(H_2, H_3)$  and  $\alpha, \beta \in \mathbf{K}$ . Then the following assertions hold

- (i)  $A^{**} := (A^*)^* = A$
- (ii)  $(\alpha A_1 + \beta A_2)^* = \overline{\alpha} A_1^* + \overline{\beta} A_2^*$
- (iii)  $(BA)^* = A^*B^*$
- (iv)  $I^* = I$  if I denotes the identity in H
- $(\mathbf{v}) \ (A^{-1})^* = (A^*)^{-1}$ .

Assertion  $(\mathbf{v})$  is meant in the sense that whenever one of these two inverses exists and is defined on the whole space then so does the other and equation  $(\mathbf{v})$  holds.

**Remark 5.1** Assertion (i) and (5.5) imply  $||A|| = ||A^*||$  for any  $A \in B(H_1, H_2)$ .

**Proof of Lemma 5.1:** We only prove the fifth assertion, using (i), (ii) and (iv). If  $A^{-1}$  exists and is defined on the whole of  $H_2$  then

$$I_1^* = I_1^* = (A^{-1}A)^* = A^*(A^{-1})^*$$
  
 $I_2 = I_2^* = (AA^{-1})^* = (A^{-1})^*A^*$ .

The first equation implies that  $A^*$  is surjective, and the second equation yields that  $A^*$  is injective. Whence  $(A^*)^{-1}$  exists and is defined on the whole of  $H_1$ . Then either of the two equations yields

$$(A^{-1})^* = (A^*)^{-1}$$
.

On the other hand, if  $(A^*)^{-1}$  exists and is defined on the whole of  $H_1$ , then the above argument shows that  $(A^{**})^{-1}$  exists. But this is  $A^{-1}$  by (i). q.e.d.

There is a fundamental relation between the range of A and the kernel of its adjoint:

**Theorem 5.2** For  $A \in B(H_1, H_2)$ 

$$R(A)^{\perp} = N(A^*)$$

holds for the range  $R(A) := \{Ax: x \in H_1\}$  and the kernel  $N(A^*) = \{y \in H_2: A^*y = 0\}$ .

**Proof:**  $y \in H_2$  belongs to  $R(A)^{\perp}$  if and only if

$$\langle Ax, y \rangle = 0 \tag{5.9}$$

for all  $x \in H_1$ . By the definition of  $A^*$  this is equivalent with

$$\langle x, A^*y \rangle = 0 \tag{5.10}$$

for all  $x \in H_1$ . Of course,  $A^*y = 0$  is sufficient for (5.10). That it is also necessary can be seen by specializing  $x := A^*y$ . q.e.d.

It is usefull to abstract the above argument a little bit:

**Lemma 5.2** The vector  $z \in H$  is 0 if and only if

$$\langle x, z \rangle = 0$$

for all  $x \in H$  (or for all x from a dense subspace of H).

Notice that by Corollary 3.2 and Theorem 5.2:

$$N(A^*)^{\perp} = \overline{R(A)} . \tag{5.11}$$

In order to obtain (5.2) for a continuous linear operator in an infinitely dimensional Hilbert space it is therefore necessary and sufficient to prove that R(A) is closed!

The operators  $A \in B(H, H)$  for which we wish to prove Fredholm's alternative are of the type A = I - K where I is the identity in the Hilbert space H and K is a compact operator, the latter being defined as follows:

**Definition 5.1** A linear operator  $C \in B(H_1, H_2)$  is compact iff any bounded sequence  $(u_n)_{n \in \mathbb{N}}$ in  $H_1$  contains a subsequence  $(u_{n'})_{n \in \mathbb{N}}$  such that  $(Cu_{n'})_{n \in \mathbb{N}}$  converges.

This means that the unit ball in  $H_1$  is mapped onto a precompact subset of  $H_2$ , i.e. a subset the closure of which is compact.<sup>2</sup> It is well known that compact subsets of metric spaces are closed and bounded. Hence any compact operator is a bounded linear operator. Moreover in  $\mathbf{C}^N$  or  $\mathbf{R}^N$ ,  $N \in \mathbf{N}$ , the bounded closed subsets are exactly the compact subsets. From this we may conclude that any bounded linear operator with a finite dimensional range is compact.

More important, however, is the following property of compact operators.

**Lemma 5.3** Let  $A, C_1 \in B(H_1, H_2)$ ,  $B, C_2 \in B(H_2, H_3)$  and suppose that  $C_1, C_2$  are compact. Then also  $BC_1$  and  $C_2A$  are compact.

**Proof:** Let  $(u_n)_{n \in \mathbb{N}}$  denote some bounded sequence in  $H_1$ . Since  $C_1$  is compact,  $(u_n)$  contains a subsequence  $(u_{n'})$  for which  $C_1(u_{n'})$  converges to some  $v \in H_2$ , and the continuity of B implies  $\lim_{n \to \infty} BC_1 u_{n'} = Bv$ . Thus  $BC_1$  is compact.

To see that  $C_2A$  is compact notice that  $(Au_n)$  is bounded in  $H_2$ , whence  $(u_n)$  contains a subsequence  $(u_{n'})$  such that  $C_2(Au_{n'}) = C_2Au_{n'}$  converges. **q.e.d.** 

The adjoint of a compact operator is a compact operator, too:

**Lemma 5.4** If  $C \in B(H_1, H_2)$  is compact then so is  $C^*$ .

**Proof:** By Lemma 5.3 the operator  $CC^*$  is compact. So let  $(u_n)_{n \in \mathbb{N}}$  denote a bounded sequence in  $H_2$ ,  $||u_n||_2 \leq \gamma < \infty$  say, and consider a subsequence  $(u_{n'})$  such that  $(CC^*u_{n'})$  converges. Then from

$$\|C^*(u_{n'}-u_{m'})\|_1^2 = \langle CC^*(u_{n'}-u_{m'}), u_{n'}-u_{m'} \rangle_2$$

<sup>&</sup>lt;sup>2</sup>Compactness of a subset S in a metric space may be defined by several equivalent properties of which one is the property that any sequence  $(s_n)$  in S contains a subsequence converging towards some element of S.

$$\leq \|CC^*(u_{n'} - u_{m'})\|_2 \|u_{n'} - u_{m'}\|_2$$
  
$$\leq 2\gamma \|CC^*(u_{n'} - u_{m'})\|_2$$

we conclude that  $(C^*u_{n'})_{n \in \mathbb{N}}$  is a Cauchy sequence and whence convergent. **q.e.d.** 

We are now ready to prove the main result of this section, namely the Fredholm alternative for the solvability of the equation (I - K)x = y for compact operators K:

**Theorem 5.3** For a compact operator  $K \in B(H, H)$ 

$$\dim N(I-K) = \dim N(I-K^*) < \infty$$
(5.12)

and

$$R(I - K) = N(I - K^*)^{\perp}$$
(5.13)

hold.

The proof is divided into several lemmata.

**Lemma 5.5** For a compact operator  $K \in B(H, H)$  the range R(I - K) of I - K is closed.

**Lemma 5.6** For a compact operator  $K \in B(H, H)$  the kernel N(I - K) has finite dimension.

**Lemma 5.7** For a compact operator  $K \in B(H, H)$ 

$$\dim N(I-K) \leq \dim N(I-K^*)$$

Assume that these three lemmata are proven. Then (5.13) follows by (5.11) and Lemma 5.5. From Lemma 5.6 and Lemma 5.4 one obtains that dim N(I - K) and dim  $N(I - K^*)$  are finite, and Lemma 5.7 together with Lemma 5.1 and Lemma 5.4 yield

dim 
$$N(I - K) \le \dim N(I - K^*) \le \dim N(I - K^{**}) = \dim N(I - K)$$

and whence (5.12). It remains to prove the lemmata.

**Proof of Lemma** 5.5 Let y denote an accumulation point of R(I - K). Then there exists a sequence  $(\tilde{x}_n)_{n \in \mathbb{N}}$  with  $\lim_{n \to \infty} (I - K)\tilde{x}_n = y$ . We have to show that some  $x \in H$  exists with

$$y = (I - K) x$$
 . (5.14)

According to Theorem 3.3 we decompose any  $\tilde{x}_n$  as

$$\tilde{x}_n = x_n + \hat{x}_n$$
,  $x_n \in (N(I-K))^{\perp}$ ,  $\hat{x}_n \in N(I-K)$ .

Notice here that the kernel of a linear operator  $A \in B(H_1, H_2)$  is always a closed subspace! Since  $(I - K)\hat{x}_n = 0$  the sequence  $(x_n)$  satisfies

$$\lim_{n \to \infty} (I - K) x_n = y$$
$$x_n \in N(I - K)^{\perp} \quad \text{for all } n \in \mathbb{N} .$$

Assume first that  $(x_n)$  is a bounded sequence. Then it contains a subsequence  $(x_{n'})$  such that  $Kx_{n'}$  and whence  $x_{n'} = Kx_{x'} - (I - K)x_{n'}$  converge. Let  $x := \lim_{n \to \infty} x_{n'}$ . Then

$$(I-K) x = \lim_{n \to \infty} (I-K) x_{n'} = y$$

by the continuity of I - K.

It remains to lead the assumption, that  $(x_n)$  is unbounded, to a contradiction. After an eventual passage to a subsequence we may assume that  $||x_n||$  tends to infinity. Consider

$$u_n := ||x_n||^{-1} x_n, \quad n \in \mathbf{N}$$

Then for each  $n \in \mathbf{N}$ 

$$||u_n|| = 1 , (5.15)$$

$$u_n \in [N(I-K)]^{\perp} \quad , \tag{5.16}$$

$$\lim_{n \to \infty} \left( I - K \right) u_n = 0 \quad . \tag{5.17}$$

As previously there exists a subsequence  $u_{n'}$  converging to some  $u \in H$ , and (I - K)u = 0, i.e.  $u \in N(I - K)$ . By (5.16)

$$u \in N(I - K) \cap N(I - K)^{\perp} = \{0\}$$

which contradicts (5.15).

**Proof of Lemma 5.6:** If dim  $N(I-K) = \infty$  then there exists a sequence  $(u_n)$  of by pairs distinct elements  $u_n \in N(I-K)$  such that  $\{u_n : n \in \mathbb{N}\}$  is linear independent<sup>3</sup>. Employing E. Schmidt's orthogonalization process

$$\left. \begin{array}{ll} x_{1} & := & \|u_{1}\|^{-1} u_{1} \\ x_{n+1} & := & \|\tilde{x}_{n+1}\|^{-1} \tilde{x}_{n+1} , \text{ where} \\ \tilde{x}_{n+1} & := & u_{n+1} - \sum_{m=1}^{n} \langle u_{n+1} , x_{m} \rangle x_{m} \end{array} \right\}$$
(5.18)

one obtains a sequence  $(x_n)_{n \in \mathbb{N}}$  in N(I-K) of normalized vectors which are by pairs orthogonal

$$\langle x_n, x_m \rangle = \delta_{nm} := \begin{cases} 1 & \text{for } n = m \\ 0 & \text{for } n \neq m \end{cases}$$
 (5.19)

The set  $\{x_n : n \in \mathbf{N}\}$  resp. the sequence  $(x_n)_{n \in \mathbf{N}}$  are called an orthonormal system resp. an orthonormal sequence. By the way, for any  $n_0 \in \mathbf{N} \cup \{\infty\}$ 

$$\operatorname{span} \{ u_n \colon 1 \le n < n_0 \} = \operatorname{span} \{ x_n \colon 1 \le n < n_0 \} , \qquad (5.20)$$

<sup>&</sup>lt;sup>3</sup> this means that any nontrivial linear combination of *finitely* many distinct vectors  $u_n$  is nonzero.

where the span of a subset S of a vector space X is the subspace of all (finite) linear combinations of elements of S.

Since  $x_n \in N(I - K)$  we have  $x_n = Kx_n$  and whence  $(x_n)_{n \in \mathbb{N}}$  must contain a convergent subsequence. But this is impossible by Pythagoras' theorem:

$$||x_n - x_m||^2 = ||x_n||^2 + ||x_m||^2 = 2 \text{ for } n \neq m .$$
 (5.21)

Thus N(I-K) must be finite.

**Proof of Lemma 5.7:** The proof is in two steps. In a first step we show that surjectivety of I - K implies injectivety. By Theorem 5.2 this means that the assertion of the lemma is true if dim  $N(I - K^*) = 0$ .

Thus assume that I - K is onto and consider the sequence of spaces

$$N_n := N((I-K)^n) , \quad n \in \mathbf{N}_0 .$$
 (5.22)

Of course,

$$(I-K)^0 := I$$
,  $(I-K)^n := (I-K)^{n-1} (I-K) = (I-K)(I-K)^{n-1}$ 

These spaces  $N_n$  are closed subspaces of H and are nested as follows:

$$N_n \supset N_{n-1} \quad \text{for } n \in \mathbf{N} \quad .$$
 (5.23)

Suppose now that I - K is not injective. We prove by induction that then

$$N_n \neq N_{n-1} \tag{5.24}$$

holds for all  $n \in \mathbf{N}$ . The assumption that I - K is not injective yields (5.24) for n = 1. Suppose that (5.24) holds for some n, then there exists  $y \in N_n \setminus N_{n-1}$  and by the surjectivity one finds some x such that

$$(I-K) x = y$$

But then

$$(I - K)^{n+1} x = (I - K)^n y = 0 \quad (\text{since } y \in N_n)$$
$$(I - K)^n x = (I - K)^{n-1} y \neq 0 \quad (\text{since } y \notin N_{n-1}),$$

whence  $x \in N_{n+1} \setminus N_n$ , and (5.24) is proven.

We may now construct a sequence  $(x_n)$  such that for all  $n \in \mathbf{N}$ 

$$||x_n|| = 1 (5.25)$$

$$x_n \in N_n \cap N_{n-1}^{\perp} . \tag{5.26}$$

Choose  $u_n \in N_n \setminus N_{n-1}$  and according to Theorem 3.3 decompose  $u_n$  as

$$u_n = \tilde{x}_n + y_n, \quad \tilde{x}_n \in N_{n-1}^{\perp}, \quad y_n \in N_{n-1}$$

Then

$$(I-K)^n \tilde{x}_n = (I-K)^n u_n - (I-K) ((I-K)^{n-1} y_n) = 0$$

and whence  $\tilde{x}_n \in N_n \cap N_{n-1}^{\perp}$ . Moreover  $\tilde{x}_n \neq 0$  since  $u_n \notin N_{n-1}$ . Thus we may choose

$$x_n := \|\tilde{x}_n\|^{-1} \tilde{x}_n$$
 .

We now show that  $(Kx_n)$  cannot contain a convergent subsequence, since for  $n \neq m$ 

$$||K x_n - K x_m|| \ge 1 . (5.27)$$

To show (5.27) assume that n > m. Then

$$K x_n - K x_m = x_n - w$$

where

$$w := (I - K) x_n + (I - K) x_m + x_m \in N_{n-1}$$

From Pythagoras' theorem we obtain

$$||K x_n - K x_m||^2 = ||x_n - w|| = ||x_n||^2 + ||w||^2 \ge 1$$
.

Hence the assumption that I - K is not injective has been lead to a contradiction.

In the second step the case dim  $N(I - K^*) > 0$  is reduced to the first step. Assume that

$$\dim N(I - K^*) < \dim N(I - K) < \infty , \qquad (5.28)$$

the second inequality following from Lemma 5.6 Then there exists a (bounded) linear operator

$$E : N(I-K) \to N(I-K^*)$$

which is surjective. Since E maps into a finite dimensional space it is compact. Let P denote the orthogonal projection onto N(I - K).

We consider

$$\tilde{K} : H \longrightarrow H, x \longmapsto Kx - E(Px)$$

and show that  $\tilde{K}$  is compact and  $I - \tilde{K}$  is surjective but not injective. This will yield a contradiction to what was proven in the first step.

The compactness of  $\tilde{K}$  follows from the fact that K is compact and  $\tilde{K} - K$  is a bounded linear operator with finite dimensional range.

For any  $y \in H$  a solution of  $(I - \tilde{K})x = y$  can be found as follows: decompose y according to Theorem 3.3 as

$$y = y_1 + y_2$$
,  $y_1 \in N(I - K^*)^{\perp}$ ,  $y_2 \in N(I - K^*)$ .

Since  $N(I - K^*)^{\perp} = R(I - K)$  there exists  $x_1 \in H$  such that  $(I - K)x_1 = y_1$ . The ansatz

$$x := x_1 + x_2, \quad x_2 \in N(I - K)$$

then leads to

$$(I - \tilde{K})x = y \iff E x_2 = y_2 - E(Px_1)$$

which by the surjectivity of E is solvable. Thus  $I - \tilde{K}$  is surjective.

But  $I - \tilde{K}$  is not injective: Because of (5.28) E cannot be injective. Hence there exists  $x \in N(I - K) \setminus \{0\}$  with Ex = 0. For this x

$$(I - \tilde{K})x = (I - K)x - Ex = 0$$

This proves the lemma.

I. Fredholm obtained his results for integral equations of the second kind. These are equations of the type (I - K)u = f in  $L^2(\Omega)$ , say, where K is given by

$$(Ku)(x) := \int_{\Omega} k(x,y)u(y) d\mu(y) ,$$

and k is some 'kernel'-function. The conditions which one imposes on k guarantee that K is a compact operator. For example  $k \in L^2(\Omega \times \Omega)$  is such a condition, if  $\Omega$  is bounded.

By an integral equation of the first kind one means an equation of the type Ku = f, where K is given as above. These are much more difficult to handle. The following theorem contains a negative result in this direction.

**Theorem 5.4** Let K denote a compact operator in B(H, H).

(i) If  $N(I-K) = \{0\}$  then  $(I-K)^{-1} \in B(H,H)$ .

(ii) If  $N(K) = \{0\}$  and if dim  $H = \infty$  then  $K^{-1}$  is not continuous.

#### **Proof:**

(i) By Theorem 5.3  $(I - K)^{-1}$  is defined on the whole of H. Assume that  $(I - K)^{-1}$  is not bounded. Then a sequence  $(y_n)_{n \in \mathbb{N}}$  in H exists with

$$\|y_n\| = 1$$
 for all  $n$ ,  
 $\|(I-K)^{-1}y_n\| \to \infty$  as  $n \to \infty$ .

Putting

$$x_n := \|(I-K)^{-1}y_n\|^{-1} (I-K)^{-1}y_n$$
(5.29)

we obtain

$$||x_n|| = 1 \quad \text{for all} \ n \in \mathbf{N} \quad (5.30)$$

 $\lim_{n \to \infty} (I - K) x_n = 0 .$ 

Choose a subsequence  $(x_{n'})$  for which  $Kx_{n'}$  converges to some  $x \in H$ . Then also

$$x_{n'} = K x_{n'} + (I - K) x_{n'} \rightarrow x \text{ as } n \rightarrow \infty$$
.

Thus

$$(I-K)x = \lim_{n \to \infty} (I-K) x_{n'} = 0$$

and whence x = 0. However, by (5.29) we obtain ||x|| = 1 — a contradiction.

(ii) If N(K) = {0} and dim H = ∞, the range R(K) cannot have finite dimension: No injective linear operator exists from an infinite dimensional space into a finite dimensional space. As in the proof of Lemma 5.6 we may construct an infinite orthonormal sequence (y<sub>n</sub>)<sub>n∈N</sub> with y<sub>n</sub> ∈ R(K), and this sequence cannot contain a converging subsequence. However, the continuity of K<sup>-1</sup> implies that the sequence (x<sub>n</sub>), given by

$$x_n = K^{-1} y_n ,$$

is bounded. This contradicts the compactness of K.

#### Chapter 6

## Solution of the Dirichlet Problem

In Chapter 5 the Dirichlet problem was solved under the assumption that the Dirichlet form B of L was strongly coercive on  $H_0^1(\Omega)$ . We remind the assumptions made on L at the beginning of that chapter. Moreover we always assume:

Assumption: The coefficients of L are such that the Dirichlet form B is strongly coercive on  $H_0^1(\Omega)$ .

In this chapter we do not want to treat problems with strongly coercive<sup>1</sup> Dirichlet forms only. But if we have some differential expression,  $\tilde{L}$  say, which does not have a strongly coercive form we will write it as  $\tilde{L} = L_{-\lambda}$  where  $L := \tilde{L}_{\lambda}$  is strongly coercive. So we still assume that the Dirichlet form B is strongly coercive. But now we want to solve

$$L_{\lambda} u = f , \quad u|_{\partial\Omega} = g|_{\partial\Omega}$$
(6.1)

for arbitrary  $\lambda \in \mathbf{C}$ .

From Theorem 4.1 we get

**Theorem 6.1** There exists a linear operator  $G_0 \in B(L^2(\Omega), H_0^1(\Omega))$  which assigns to any  $f \in L^2(\Omega)$  the variational solution u of the Dirichlet problem 'Lu = f,  $u|_{\partial\Omega} = 0$ '.  $G_0$  is called the 'solution operator'.

**Remark 6.1** Notice that for given  $f \in L^2(\Omega)$  the value  $G_0 f$  is uniquely defined by the relation

 $G_0 f \in H^1_0(\Omega) \quad \text{and} \quad B(G_0 f, v) = -\langle f, v \rangle \text{ for all } v \in H^1_0(\Omega) \quad .$  (6.2)

<sup>&</sup>lt;sup>1</sup> If the subspace V, on which a form is (strongly) coercive, is not mentioned explicitly we assume it to be  $H_0^1(\Omega)$ .

**Proof of Theorem 6.1** From Theorem 4.1 it is clear that  $G_0 f$  is well defined for each  $f \in L^2(\Omega)$ . Moreover  $G_0$  is linear, since  $u := \alpha u_1 + \beta u_2$  is the variational solution of ' $Lu = \alpha f_1 + \beta f_2$ ,  $u|_{\partial\Omega} = 0$ ' if the  $u_j$ , j = 1, 2, are the variational solutions of ' $Lu_j = f_j$ ,  $u_j|_{\partial\Omega} = 0$ '.

The continuity follows from the strong coercivity of B (c.f. (4.19)), the boundedness of B (c.f. (4.12)) and from (6.2):

$$c_{+} \|G_{0}f\|_{1}^{2} \leq \operatorname{Re} \ B(G_{0}f, G_{0}f) = -\operatorname{Re} \ \langle f, G_{0}f \rangle \leq \gamma \|f\|_{0} \|G_{0}f\|_{0} \leq \gamma \|f\|_{0} \|G_{0}f\|_{1}.$$

Hence

$$||G_0 f||_1 \leq (c_+^{-1} \gamma) ||f||_0$$
(6.3)

for any  $f \in L^2(\Omega)$ .

Now let  $\lambda \in \mathbb{C}$  be fixed and  $f \in L^2(\Omega)$ ,  $g \in H^1_0(\Omega)$ . By Theorem 4.1 there exists an  $h \in H^1(\Omega)$  such that

$$g - h \in H_0^1(\Omega) \quad , \tag{6.4}$$

$$B(h, v) = 0 \quad \text{for all } v \in H_0^1(\Omega) . \tag{6.5}$$

By the ansatz

 $u \ := \ \widetilde{u} + h$ 

problem (6.1) is equivalent with the problem to find  $\tilde{u}$  satisfying

$$\tilde{u} \in H_0^1(\Omega) \tag{6.6}$$

$$B(\tilde{u}, v) + \lambda \langle \tilde{u}, v \rangle = - \langle f + \lambda h, v \rangle \text{ for all } v \in H^1_0(\Omega) , \qquad (6.7)$$

and (6.7) is equivalent with

$$\tilde{u} = \lambda G_0 \tilde{u} + G_0 (f + \lambda h) .$$
(6.8)

Equation (6.8) may be considered as an equation in  $H_0^1(\Omega)$  or in  $L^2(\Omega)$ . The latter consideration is more usual because in this case it is easier to describe 'the adjoint problem'.

 $\operatorname{Let}$ 

$$J : H_0^1(\Omega) \longrightarrow L^2(\Omega), \ u \longmapsto u \tag{6.9}$$

describe the canonic inclusion of  $H_0^1(\Omega)$ , into  $L^2(\Omega)$ . J is a bounded linear operator defined on the whole of  $H_0^1(\Omega)$ . The boundedness of J follows directly from

$$||u||_1 \leq ||u||_0$$
 for all  $u \in H_0^1(\Omega)$ .

We then define

$$G := J G_0 , (6.10)$$

i.e. we consider the solution operator as an operator into  $L^2(\Omega)$  instead of  $H^1_0(\Omega)$ . We then obtain

**Lemma 6.1** A function  $u \in H^1(\Omega)$  solves (6.1) for given  $f \in L^2(\Omega)$ ,  $g \in H^1(\Omega)$  if and only if

$$u = \tilde{u} + h \tag{6.11}$$

where h is given by (6.4), (6.5), and  $\tilde{u} \in L^2(\Omega)$  solves

$$\tilde{u} - \lambda G \tilde{u} = G(f + \lambda h) . \tag{6.12}$$

**Proof:** The only thing which remains to be shown is that (6.12) is equivalent with (6.8). Clearly (6.8) implies (6.12): Apply J on both sides. On the other hand (6.12) implies  $\tilde{u} \in R(G) \subset H_0^1(\Omega)$  and thus (6.8). q.e.d.

We want to apply the theory, provided in the previous section. For this we have to calculate the adjoint of G and to show that G is compact. The latter is not true for arbitrary  $\Omega$ . However for bounded domains  $\Omega$  it follows from Rellich's compactness theorem (F. Rellich (1906 - 1955)):

**Theorem 6.2** If  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ , then the operator J given by (6.9) is compact.

**Proof:** We must show that any sequence  $(u_n)$  in  $H_0^1(\Omega)$ , for which

$$\sup_{n \in \mathbf{N}} \|u_n\|_1 < \infty$$

contains a subsequence which converges in  $L^2(\Omega)$ .

For each  $n \in \mathbf{N}$  let  $\varphi_n \in C_0^{\infty}(\Omega)$  with

 $\|\varphi_n - u_n\|_1 \leq 1/n .$ 

If it is possible to extract a  $L^2(\Omega)$ -convergent subsequence  $(\varphi_{n'})$  from  $(\varphi_n)$  then the corresponding subsequence  $(u_{n'})$  converges, too. Whence it suffices to show that any sequence  $(\varphi_n)$  in  $C_0^{\infty}(\Omega)$ , for which

$$\sup_{n \in \mathbf{N}} \|\varphi_n\|_1 =: \gamma < \infty , \qquad (6.13)$$

contains a convergent subsequence.

To do so we use the mollifier and put for  $n, k \in \mathbf{N}$ 

$$\varphi_{n,k} := j_{1/k} * \varphi_n$$

For any fixed k the sequence  $(\varphi_{n,k})_{n\in\mathbb{N}}$  is equicontinuous: For  $h\in\mathbb{R}^N$  we have (with  $\eta:=j_{1/k}$ )

$$\begin{aligned} |\varphi_{n,k}(x+h) - \varphi_{n,k}(x)| &= \int \left[\eta(x+h-y) - \eta(x-y)\right] \varphi_n(y) \, d\mu(y) \\ &\leq \left[\int |\eta(x+h-y) - \eta(x-y)|^2 \, d\mu(y)\right]^{1/2} \|\varphi_n\|_0 \\ &\leq \left[\int |\eta(z+h) - \eta(z)|^2 \, d\mu(z)\right]^{1/2} \, \cdot \, \gamma \end{aligned}$$

The last term is independent of n and tends to zero as h tends to zero by Lebesgue's theorem.

There exists a compact subset B of  $\mathbf{R}^N$  which contains the support of all the functions  $\varphi_{n,k}$ . Whence we may apply Arzela's theorem and select a subsequence  $(\varphi_{n_1})$  of  $(\varphi_n)$  such that  $\varphi_{n_1,1}$  converges uniformly. From  $(\varphi_{n_1})$  we select a subsequence  $(\varphi_{n_2})$  such that  $(\varphi_{n_2,2})$  converges uniformly. Proceeding in this manner we find for any  $k \in \mathbf{N}$  a subsequence  $(\varphi_{n_{k+1}})$  of  $(\varphi_{n_k})$  such that  $(\varphi_{n_{k+1},k+1})$  converges uniformly. Then the "diagonal" sequence  $(\varphi_{n'})$ , given by

$$\varphi_{n'} := \varphi_{n_n} ,$$

has the property that  $(\varphi_{n',k})$  converges uniformly for any  $k \in \mathbf{N}$ . Moreover, the estimate

$$\|\varphi_{n',k} - \varphi_{m',k}\|_0 \leq (\mu(B))^{1/2} \sup |\varphi_{n',k} - \varphi_{m',k}|$$

 $(\mu(B) :$  the Lebesgue–measure of B) shows that  $\varphi_{n',k}$  converges in  $L^2(\mathbf{R}^N)$  for any  $k \in \mathbf{N}$ .

In a moment we shall prove that for all  $\varphi \in C_0^\infty(\mathbf{R}^N)$ ,  $\delta > 0$ 

$$\|j_{\delta} * \varphi - \varphi\|_{0} \leq \delta |\varphi|_{1} \quad . \tag{6.14}$$

Assuming (6.14) it follows that  $(\varphi_{n'})$  converges in  $L^2(\Omega)$ . For all  $k \in \mathbb{N}$ 

$$\begin{aligned} \|\varphi_{n'} - \varphi_{m'}\|_{0,\Omega} &\leq \|\varphi_{n'} - \varphi_{n',k}\| + \|\varphi_{n',k} - \varphi_{m',k}\| + \|\varphi_{m',k} - \varphi_{m'}\| \\ &\leq \|\varphi_{n',k} - \varphi_{m',k}\| + 2\gamma/k \end{aligned}$$

Whence for given  $\varepsilon > 0$  we first choose k such that  $2\gamma/k < \varepsilon/2$ , and then fix  $n_0$  such that  $\|\varphi_{n',k} - \varphi_{m',k}\| < \varepsilon/2$  for  $n, m > n_0$ . For these n, m then  $\|\varphi_{n'} - \varphi_{m'}\|_{0,\Omega} < \varepsilon$ .

It remains to prove (6.14): Since  $j_{\delta} \ge 0$ , since  $\int j_{\delta} = 1$ , and with the help of the Cauchy Schwarz inequality we obtain for any fixed  $x \in \mathbf{R}^N$ 

$$\begin{aligned} |j_{\delta} * \varphi(x) - \varphi(x)|^2 &= \left| \int \left[ j_{\delta}(x-y) \right]^{1/2} \left[ j_{\varepsilon}(x-y) \right]^{1/2} \left( \varphi(y) - \varphi(x) \right) d\mu(y) \right|^2 \\ &\leq \int \left[ j_{\delta}(x-y) d\mu(y) \cdot \int \left[ j_{\delta}(x-y) \left| \varphi(y) - \varphi(x) \right|^2 d\mu(y) \right]^2 d\mu(y) \\ &= \int \left[ j_{\delta}(x-y) \left| \varphi(y) - \varphi(x) \right|^2 d\mu(y) \right]^2 d\mu(y) \end{aligned}$$

$$(6.15)$$

The difference  $\varphi(y) - \varphi(x)$  may be written as an integral

$$\varphi(y) - \varphi(x) = \int_0^1 \frac{d}{dt} \varphi(x + t(y - x)) dt = \sum_{n=1}^N (y_n - x_n) \int_0^1 \partial_n \varphi(x + t(y - x)) dt .$$

Notice that the integrals in (6.15) extend over  $U(x, \delta)$  which contains the support of  $j_{\delta}(x - \cdot)$ . Thus for  $y \in U(x, \delta)$  we may estimate

$$\begin{aligned} |\varphi(y) - \varphi(x)| &\leq |y - x| \left[ \sum_{n=1}^{N} \left| \int_{0}^{1} \partial_{n} \varphi(x + t(y - x)) dt \right|^{2} \right]^{1/2} \\ &\leq |y - x| \left( \int_{0}^{1} \sum_{n=1}^{N} \left| \partial_{n} \varphi(x + t(y - x))^{2} dt \right)^{1/2} \\ &\leq \delta \left( \int_{0}^{1} \left| \nabla \varphi(x + t(y - x))^{2} dt \right)^{1/2} . \end{aligned}$$
The second inequality follows by the Cauchy Schwarz inequality. Insertion into (6.15) yields

$$\begin{aligned} \left| j_{\delta} * \varphi(x) - \varphi(x) \right|^2 &\leq \delta^2 \int_{\mathbf{R}^N} \int_0^1 \left| j_{\delta}(x-y) \left| \nabla \varphi(x+t(y-x)) \right|^2 dt \, d\mu(y) \\ &= \delta^2 \int_0^1 \int_{\mathbf{R}^N} \left| j_{\delta}(x-y) \left| \nabla \varphi(x+t(y-x)) \right|^2 d\mu(y) \, dt \end{aligned}$$

The last equation follows by Fubini's theorem. We introduce new coordinates  $z := t(x-y), d\mu(y) = t^{-N} d\mu(z)$  and obtain

$$\begin{aligned} |j_{\delta} * \varphi(x) - \varphi(x)|^2 &\leq \delta^2 \int_0^1 \int_{\mathbf{R}^N} t^{-N} j_{\delta}(z/t) \left| \nabla \varphi(x-z) \right|^2 \, d\mu(z) \, dt \\ &= \delta^2 \int_0^1 \left[ j_{(t\delta)} * \left( |\nabla \varphi|^2 \right) \right] (x) \, dt \end{aligned}$$

Integration with respect to x and another application of Fubini's theorem yields

$$\|j_{\delta} * \varphi - \varphi\|^2 \le \delta^2 \int_0^1 \left( \int_{\mathbf{R}^N} j_{(t\delta)} * (|\nabla \varphi|)^2(x) \, d\mu(x) \right) \, dt$$

By (0.15) the inner integral can be estimated by

$$\left\| \left| \nabla \varphi \right|^2 \left\| \left( L^1(\mathbf{R}^N) \right) \right\| = \left| \varphi \right|_1^2 .$$

Hence

$$\|j_{\delta} \ast \varphi - \varphi\|^2 \le \delta^2 \int_0^1 |\varphi|_1^2 dt = \delta^2 |\varphi|_1^2 ,$$

which is (6.14).

Rellich's compactness theorem yields the compactness of the operator G (c.f. (6.10), Theorem 6.1 and Lemma 5.3). We now find the adjoint of G. Assume for a moment, that the coefficients of L are so smooth that one can write down the formal adjoint operator  $L^*$  (c.f. (2.10), (2.11)). In some sense L is the inverse of G and we have for  $\varphi, \psi \in C_0^{\infty}(\Omega)$ ,  $\langle L\varphi, \psi \rangle = \langle \varphi, L^*\psi \rangle$ . So it is a good guess that  $G^*$  might be the solution operator for the problem ' $L^*u = f$ ,  $u|_{\partial\Omega} = 0$ '. Notice that

$$B^* : H_0^1(\Omega) \times H_0^1(\Omega) \longrightarrow \mathbf{K} , \ (v,w) \longmapsto \overline{B(w,v)}$$
(6.16)

is the Dirichlet form for  $L^*$  . We obtain

**Theorem 6.3** Let  $G_0^+ \in B(L^2(\Omega), H_0^1(\Omega))$  be defined by

$$B(v, G_0^+ f) = -\langle v, f \rangle \tag{6.17}$$

for all  $v \in H_0^1(\Omega)$ . Then  $G^* = JG_0^+$ .

**Proof:** The Lax-Milgram theorem 3.4 yields that there exists a unique  $G_0^+ f \in H_0^1(\Omega)$  such that (6.17) holds for all  $v \in H_0^1(\Omega)$ . For  $v \mapsto \langle v, f \rangle$  is a continuous linear functional on  $H_0^1(\Omega)$ . Now for any  $f_1, f_2 \in L^2(\Omega)$ 

$$\langle Gf_1, f_2 \rangle = -B(Gf_1, G_0^+ f_2) = \langle f_1, G_0^+ f_2 \rangle = \langle f_1, JG_0^+ f_2 \rangle$$

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q.e.d.

Hence  $G^* f_2 = J G_0^+ f_2$ .

**Remark 6.2** Of course, (6.17) is equivalent with

$$B^*(G_0^+f, v) = -\langle f, v \rangle$$

for all  $v \in H_0^1(\Omega)$ . Thus indeed  $G_0^+$  is the solution operator for the 'adjoint' Dirichlet problem ' $L^*u = f$ ,  $u \mid_{\partial\Omega} = 0$ ', provided that the coefficients  $a_{nm}$  are in  $C^2(\Omega)$  with bounded second derivatives. However,  $G_0^+$  is defined even if  $L^*$  is not.

We now obtain:

**Theorem 6.4** Let  $\lambda \in \mathbf{C}$  be given and  $\Omega \subset \mathbf{R}^N$  be a bounded domain. Then either (i) or (ii) holds:

- (i) For all f ∈ L<sup>2</sup>(Ω), g ∈ H<sup>1</sup>(Ω) there exists a unique variational solution u of the Dirichlet problem 'L<sub>λ</sub>u = f, u|<sub>∂Ω</sub> = g|<sub>∂Ω</sub> '.
- (ii) There exists a nontrivial variational solution of the homogeneous Dirichlet problem 'L<sub>λ</sub>u = 0, u<sub>|∂Ω</sub> = 0'. In this case the space N of all solutions of this homogeneous problem has finite dimension. Then the space<sup>2</sup>

$$\mathcal{N}^* := \{ w \in H^1_0(\Omega) \colon B(v, w) + \lambda \langle v, w \rangle = 0 \text{ for all } v \in H^1_0(\Omega) \}$$

has the same dimension. The problem  $L_{\lambda}u = f$ ,  $u|_{\partial\Omega} = g$  'is solvable if and only if

$$\langle f, w \rangle + B(g, w) + \lambda \langle g, w \rangle = 0$$

for any  $w \in \mathcal{N}^*$ . Then all solutions are obtained as the sum of one special solution and any element from  $\mathcal{N}$ .

**Proof:** Let h denote the variational solution of 'Lh = 0,  $h|_{\partial\Omega} = g|_{\partial\Omega}$ ', i.e. h satisfies (6.4), (6.5). By Lemma 6.1 the solvability of " $L_{\lambda}u = f$ ,  $u|_{\partial\Omega} = g|_{\partial\Omega}$ " is equivalent with the solvability of (6.12):

$$\tilde{u} - \lambda G \tilde{u} = G(f + \lambda h)$$

Assume that the homogeneous problem '  $L_{\lambda}u = 0$ ,  $u \mid_{\partial\Omega} = 0$ ' only admits the trivial solution. Then

$$N(I - \lambda G) = \{0\}$$

<sup>&</sup>lt;sup>2</sup> This is the space of variational solutions to the homogeneous adjoint problem  $L_{\bar{\lambda}}^* w = 0$ ,  $w|_{\partial\Omega} = 0$ , provided that  $L^*$  is defined.

Since  $\lambda G$  is a compact operator we conclude by Theorem 5.3 that  $R(I - \lambda G) = L^2(\Omega)$  and whence (6.12) is solvable for any f and h.

Assume now that the homogeneous problem '  $L_{\lambda}u = 0$  ,  $u \mid_{\partial\Omega} = 0$  ' admits nontrivial solutions. Then

$$\mathcal{N} := \{ u: u \text{ solves } ``L_{\lambda} u = 0, u |_{\partial \Omega} = 0'' \} = N(I - \lambda G)$$

and whence has finite dimension, n say. Then by Theorem 5.3

$$\dim N(I - (\lambda G)^*) = n ,$$

too. Now  $(\lambda G)^* = \overline{\lambda} G^*$ , and of course  $\lambda \neq 0$ . Then

$$w \in N(I - (\lambda G)^*) \Leftrightarrow \frac{1}{\overline{\lambda}} w = G^* w \Leftrightarrow \forall_{v \in H^1_0(\Omega)} \frac{1}{\lambda} B(v, w) = -\langle v, w \rangle.$$

This means that

$$N(I - (\lambda G)^*) = \mathcal{N}^* ,$$

and

$$\dim \mathcal{N}^* = \dim \mathcal{N} \ .$$

Again Theorem 5.3 implies that ' Lu = f ,  $u|_{\partial\Omega} = g|_{\partial\Omega}$  ' is solvable if and only if

$$\langle G(f + \lambda h), w \rangle = 0 \tag{6.18}$$

holds for all  $w \in \mathcal{N}^*$ .

Now

$$\langle G(f + \lambda h), w \rangle = -B(G(f + \lambda h), G^*w) \quad \text{(by Theorem 6.3)}$$
  
$$= -B(G(f + \lambda h), \frac{1}{\lambda}w) \quad \text{(by (6.18))}$$
  
$$= \frac{1}{\lambda} \langle f + \lambda h, w \rangle \qquad \text{(by (6.1))}.$$

Moreover

$$\lambda \langle h , w \rangle = B(h, w) + \lambda \langle h , w \rangle \qquad (by (6.5)) .$$

However  $h \notin H_0^1(\Omega)$  so that we cannot use the relation in the definition of  $\mathcal{N}^*$ . But  $h - g \in H_0^1(\Omega)$ . Hence

$$\lambda \langle h, w \rangle = B(h - g, w) + \lambda \langle h - g, w \rangle + B(g, w) + \lambda \langle g, w \rangle$$
  
=  $B(g, w) + \lambda \langle g, w \rangle$ . (6.20)

Now (6.19) and (6.20) imply that (6.18) is equivalent with

$$\langle f, w \rangle + B(g, w) + \lambda \langle g, w \rangle = 0$$

for all  $w \in \mathcal{N}^*$ .

**Definition 6.1** A complex number  $\lambda$  is called an eigenvalue of L in  $\Omega$  under the Dirichlet condition, if  $L_{\lambda}u = 0$ ,  $u|_{\partial\Omega} = 0$  has a nontrivial variational solution. Any such nontrivial solution u is called a Dirichlet eigenfunction of L in  $\Omega$  with respect to the eigenvalue  $\lambda$ . Since it is clear that we only handle with the Dirichlet problem and since L and  $\Omega$  are fixed, we simply say that u is an eigenfunction with respect to  $\lambda$ .

Our aim is to prove the existence of eigenvalues and to show that the corresponding eigenfunctions span the whole space in the case  $L = L^*$ . Before doing so, we need some results on orthonormal systems in Hilbert spaces.

## Chapter 7

## **Orthonormal Systems**

In chapter 6 we already constructed orthonormal sequences with the help of E. Schmidt's orthogonalization process. Generally it can be shown that any Hilbert space H contains an orthonormal system which spans a dense subspace of H. The proof requires a set theoretical tool, Zorn's lemma, which is equivalent with the axiom of choice. However if one confines oneself to the case of *separable* Hilbert spaces, E. Schmidt's process suffices.

**Definition 7.1** A metric space (and whence a Hilbert space) is separable if it contains a countable (or finite) dense subset.

**Definition 7.2** An orthonormal system in a Hilbert space H is a subset  $S \subset H$  such that for all  $u, v \in S$ 

$$\langle u, v \rangle = \begin{cases} 0 & if \quad u \neq v \\ 1 & if \quad u = v \end{cases}$$

An orthonormal system S is called complete if it is maximal, i.e. if no nonzero vector x exists which is orthogonal to every  $u \in S$ , i.e. if  $S^{\perp} = \{0\}$ . An orthonormal sequence  $(u_n)_{n \in \mathbb{N}}$  in H is a sequence in H such that  $\langle u_n, u_m \rangle = \delta_{nm}$  for all  $n, m \in \mathbb{N}$ .

From now on let H denote a Hilbert space of infinite dimension.

**Theorem 7.1** H is separable if and only if it contains a countable complete orthonormal system.

**Proof:** To prove the 'if' part, assume that H contains a complete orthonormal system S which can be written as

$$S = \{u_n: n \in \mathbf{N}\}$$

with an orthonormal sequence  $(u_n)_{n \in \mathbb{N}}$ . Let M denote a countable subset of  $\mathbf{K}$ , for example  $\mathbf{Q}$  if  $\mathbf{K} = \mathbf{R}$  or  $\mathbf{Q} + i\mathbf{Q}$  if  $\mathbf{K} = \mathbf{C}$ . We show that the set T of all (finite) linear combinations of S with coefficients in M,

$$T := \{ x \in H \colon \exists_{k \in \mathbf{N}} \exists_{\mu_1, \dots, \mu_k \in M} x = \sum_{n=1}^k \mu_n u_n \}$$

is dense in H. As Cantor's diagonal argument shows, T is countable, and whence the 'if' part will follow.

To prove the density of T let  $y \in H$  and  $\varepsilon > 0$  be given. Since S is complete we have  $S^{\perp} = \{0\}$ and whence span S is dense in H. Thus there exists a  $k \in \mathbb{N}$  and numbers  $\kappa_1, \ldots, \kappa_k \in \mathbb{K}$  such that

$$\|y - \sum_{n=1}^k \kappa_n u_n\| < \varepsilon/2$$

Now choose  $\mu_1, \ldots, \mu_k \in M$  such that

$$|\kappa_n - \mu_n| \le \varepsilon \cdot 2^{-n-1}$$

Then  $x := \sum_{n=1}^{k} \mu_n u_n \in T$  and

$$||y - x|| \leq ||y - \sum_{n=1}^{k} \kappa_n u_n|| + ||\sum_{n=1}^{k} (\kappa_n - \mu_n) u_n||$$
  
$$< \frac{\varepsilon}{2} + \sum_{n=1}^{k} |\kappa_n - \mu_n| \leq \varepsilon .$$

Thus T is dense in H.

Suppose on the other hand that H contains a countable dense subset  $T := \{v_n: n \in \mathbf{N}\}$ . There exists a subsequence  $(v_{n'})$  of  $(v_n)$  such that  $T' := \{v_{n'}: n \in \mathbf{N}\}$  is linear independent and  $X := \operatorname{span} T'$  is dense in H: choose 1' as the smallest index such that  $v_{1'} \neq 0$ , then choose 2' as the smallest index for which  $\{v_{1'}, v_{2'}\}$  is linear independent etc. The E. Schmidt orthonormalization process, applied to the sequence  $(v_{n'})_{n\in\mathbf{N}}$ , yields the complete orthonormal system (c.f. Proof of Lemma 5.6). q.e.d.

Bessel's inequality is an essential tool when working with orthonormal systems. In particular it can be used to show that any complete orthonormal system in a separable space must be countable.

**Theorem 7.2** Let S denote some orthonormal system in H and  $x \in H$ . Then the following assertions are true:

(i) For any finite subset  $S' \subset S$ 

$$\sum_{s\in S'} \left| \langle x \, , \, s \, \rangle \right|^2 \; \leq \; \|x\|^2 \; \; .$$

(ii) The subset

$$S_x := \{ s \in S \colon \langle x, s \rangle \neq 0 \}$$

is at most countable.

(iii) Bessel's inequality holds:

$$\sum_{s \in S} \left| \langle x, s \rangle \right|^2 \leq ||x||^2$$

This sum has to be interpreted to run over an enumeration of  $S_x$ . Since the sum contains positive terms only, its value is independent of the choice of the enumeration.

### **Proof:**

(i) Let  $S' = \{s_1, ..., s_J\}$  and put

$$y:=\sum_{i=1}^J\left<\,x\,,\;s_i\,\right>\,s_i~~.$$

Then  $x - y \in (S')^{\perp}$ , since for any  $j \in \{1, \ldots, J\}$ 

$$\langle x - y, s_j \rangle = \langle x, s_j \rangle - \sum_{i=1}^{J} \langle x, s_i \rangle \langle s_i, s_j \rangle = 0$$

Here we need  $\langle s_i, s_j \rangle = \delta_{ij}$ . Since  $(S')^{\perp} = (\text{span } S')^{\perp}$  the vectors y and x-y are orthogonal. Hence by Pythagoras' theorem

$$\begin{aligned} \|x\|^2 &= \|x - y\|^2 + \|y\|^2 \ge \|y\|^2 = \langle \sum_{i=1}^J \langle x, s_i \rangle s_i, \sum_{j=1}^J \langle x, s_j \rangle s_j \rangle \\ &= \sum_{i=1}^J \sum_{j=1}^J \langle x, s_i \rangle \overline{\langle x, s_j \rangle} \langle s_i, s_j \rangle = \sum_{i=1}^J |\langle x, s_i \rangle|^2 . \end{aligned}$$

(ii) For  $n \in \mathbb{N}$  let

$$S_{x,n} := \{ s \in S \colon |\langle x, s \rangle| \ge 1/n . \}$$

Because of (i) any of the sets  $S_{x,n}$  must be finite, whence

$$S_x = \bigcup_{n=1}^{\infty} S_{x,n}$$

is at most countable.

(iii) now follows from (i) and (ii).

From Theorem 7.2 one may prove that with respect to a *complete* orthonormal system S one can develop any given  $x \in H$  into a series of multiples of elements from S. More generally:

q.e.d.

**Theorem 7.3** Let S denote an orthonormal system in H. Then for any  $x \in H$ 

$$\sum_{s \in S} \langle x, s \rangle s = Px , \qquad (7.1)$$

where P is the orthogonal projection of H onto  $(\overline{span S})$ . The sum in (7.1) must be interpreted as follows: Either  $S_x$ , defined in Theorem 7.2, is finite. Then  $\sum_{s \in S} \langle x, s \rangle s := \sum_{s \in S_x} \langle x, s \rangle s$ . Or  $S_x$  is countable. Let  $(s_n)_{n \in \mathbb{N}}$  denote an enumeration of  $S_x$ . Then (7.1) means that

$$\sum_{n=1}^{\infty} \left\langle \, x \, , \, s_n \, \right\rangle s_n := \lim_{m \to \infty} \sum_{n=1}^m \left\langle \, x \, , \, s_n \, \right\rangle s_n$$

exists and equals Px (which implies that it is independent of the choice of the enumeration).

In particular , if S is complete then

$$x = \sum_{s \in S} \langle x, s \rangle s \tag{7.2}$$

$$||x||^2 = \sum_{s \in S} |\langle x, s \rangle|^2$$
 (7.3)

and for any  $y \in H$ 

$$\langle x, y \rangle = \sum_{s \in S} \langle x, s \rangle \langle s, y \rangle .$$
 (7.4)

The equality (7.3) is called 'Parseval's identity'. Its validity for all  $x \in H$  is necessary and sufficient for the completeness of S.

**Example 7.1** It is known from the Calculus course that with  $I := (-\pi, \pi)$  any  $\varphi \in C_0^{\infty}(I)$  can be written as a uniformly convergent Fourier series

$$\varphi(x) = \sum_{n \in \mathbf{Z}} a_n e^{in\varphi}$$

where

$$a_n = \frac{1}{2\pi} \int_I \varphi(x) e^{-inx} d\mu(x)$$
.

The functions  $u_n$  given by

$$u_n(x) = \frac{1}{\sqrt{2\pi}} e^{-inx}, \quad x \in (-\pi, \pi), \quad n \in \mathbf{Z}$$

form an orthonormal system S in  $L^2(I)$ . The starting remark shows that

$$span \ S = \{ u \in L^2(I) \colon \exists_{m=m(n)} \ \exists_{a_{-m}, \dots, a_0, \dots, a_m} \ u = \sum_{n=-m}^m \ a_n u_n \}$$

is dense in  $L^2(I)$ , for  $C_0^{\infty}(I)$  is dense in  $L^2(I)$  and uniform convergence in I implies  $L^2(I)$ convergence. Hence  $S^{\perp} = (\overline{\operatorname{span} S})^{\perp} = \{0\}$ , i.e. S is complete, and Theorem 7.3 just provides
the well known Fourier-expansion of  $L^2(I)$ -functions.

**Proof of Theorem** 7.3 If for  $x \in H$ 

$$S_x := \{ s \in S \colon \langle x, s \rangle \neq 0 \}$$

is finite then we already found in the proof of Theorem 7.2*i* that  $Px = \sum_{x \in S} \langle x, s \rangle s$ .

So let  $S_x$  be countable and  $(s_n)_{n \in \mathbb{N}}$  an enumeration of  $S_x$ . By Bessel's inequality,

$$\sum_{n=1}^{\infty} |\langle x, s_n \rangle|^2 < ||x||^2 , \qquad (7.5)$$

whence the sum on the left hand side of (7.5) is convergent. Put

$$x_m := \sum_{n=1}^m \langle x, s_n \rangle s_n .$$

Then for all  $m \in \mathbf{N}$ ,  $k \in \mathbf{N}$  by Pythagoras' theorem:

$$\|x_{m+k} - x_m\|^2 = \|\sum_{n=m+1}^{m+k} \langle x, s_n \rangle s_n\|^2 = \sum_{n=m+1}^{m+k} |\langle x, s_n \rangle|^2 \quad , \tag{7.6}$$

which by (7.5) implies that  $(x_m)$  is a Cauchy sequence. Denote its limit by y. Then for any  $m \in \mathbb{N}$ 

$$\begin{array}{ll} \langle \, x - y \, , \, s_m \, \rangle & = & \lim_{k \to \infty} \, \langle \, x - \sum_{j=1}^k \langle \, x \, , \, s_j \, \rangle s_j \, , \, s_m \, \rangle \\ \\ & = & \langle \, x \, , \, s_m \, \rangle - \langle \, x \, , \, s_m \, \rangle = 0 \ . \end{array}$$

Moreover for any  $s \in S \setminus S_x$ 

$$\langle x - y, s \rangle = \lim_{k \to \infty} \langle x - \sum_{j=1}^{k} \langle x, s_j \rangle s_j, s, \rangle = 0 - 0 = 0$$

Thus  $x - y \in S^{\perp}$ , and y = Px. If S is complete then  $S^{\perp} = \{0\}$  and P is the identity.

Parseval's identity is a consequence of the more general equality

$$||Px||^{2} = \sum_{s \in S} |\langle x, s \rangle|^{2} .$$
(7.7)

Letting  $(s_n)_{n \in \mathbb{N}}$  be any enumeration of  $S_x$ . Then

$$\begin{split} \|Px\|^2 &= \lim_{m \to n} \|\sum_{n=1}^m \langle x, s_n \rangle s_n\|^2 = \lim_{m \to \infty} \sum_{n=1}^m |\langle x, s_n \rangle|^2 \\ &= \sum_{s \in S} |\langle x, s \rangle|^2 \ . \end{split}$$

Thus if S is complete Px = x.

If S is not complete there exists some  $x \in H$  with ||x|| > 0 and Px = 0. For this x Parseval's identity is wrong by (7.7). Hence the validity of Parseval's identity for any  $x \in H$  is necessary and sufficient for the completeness of S.

Finally, to prove (7.4) let  $\{s_n: x \in \mathbf{N}\}$  denote an enumeration of  $S_x \cup S_y$ . Then

$$\begin{array}{ll} \langle x \,, \, y \,\rangle &=& \lim_{n \to \infty} \,\langle \sum_{k=1}^n \langle x \,, \, s_k \,\rangle s_k \,, \, \sum_{l=1}^n \,\langle y \,, \, s_l \,\rangle s_l \,\rangle \\ \\ &=& \lim_{n \to \infty} \, \sum_{k=1}^n \,\sum_{l=1}^n \,\langle x \,, \, s_k \,\rangle \,\overline{\langle y \,, \, s_l \,\rangle} \,\langle s_k \,, \, s_l \,\rangle \\ \\ &=& \lim_{n \to \infty} \, \sum_{k=1}^n \,\langle x \,, \, s_k \,\rangle \,\overline{\langle y \,, \, s_k \,\rangle} = \sum_{k=1}^\infty \,\langle x \,, \, s_k \,\rangle \,\overline{\langle y \,, \, s_k \,\rangle} \,. \end{array}$$

Since  $\overline{\langle y, s_k \rangle} = \langle s_k, y \rangle$  the assertion follows.

q.e.d.

### Chapter 8

## **Eigenvalues and Eigenfunctions**

We start this chapter with the spectral theorem for self adjoint compact operators. To shorten the exposition we shall consider only the case which will be interesting in connection with our boundary value problems.

**Theorem 8.1** Let K denote a compact operator in the infinite dimensional Hilbert space H and assume that K is self adjoint, i.e.  $K^* = K$ , and positive, i.e.

$$\langle Kx, x \rangle > 0 \text{for all } x \in H$$
. (8.1)

Then there exists a non increasing sequence  $(\lambda_n)_{n \in \mathbb{N}}$  (of eigenvalues) and an orthonormal sequence  $(u_n)_{n \in \mathbb{N}}$  (of eigenvectors) such that

- (i)  $0 < \lambda_{n+1} \leq \lambda_n$  for any  $n \in \mathbb{N}$
- (ii)  $\lim_{n\to\infty} \lambda_n = 0$
- (iii)  $Ku_n = \lambda_n u_n$  for any  $n \in \mathbb{N}$
- (iv) The orthonormal system  $\{u_n: n \in \mathbf{N}\}$  is complete.
- (v) For any  $x \in H$  and for any  $\mu \in \mathbf{C} \setminus \{\lambda_n : n \in \mathbf{N}\}$

$$Kx = \sum_{n=1}^{\infty} \lambda_n \langle x, u_n \rangle u_n$$
(8.2)

$$(\mu I - K)^{-1} x = \sum_{n=1}^{\infty} \frac{1}{\mu - \lambda_n} \langle x, u_n \rangle u_n$$
(8.3)

(vi)  $y \in H$  is in the range of K if and only if

$$\sum_{n=1}^{\infty} |\langle y, u_n \rangle|^2 < \infty .$$

Any  $\lambda \neq 0$  which is not a value of the sequence  $(\lambda_n)$  belongs to  $\rho(K)$  (see Definition 8.1 below).

Before we start to prove this theorem we have to fix the notions of eigenvalues and eigenvectors, of self-adjointness and prove some lemmata which are of general interest in connection with eigenvalues and eigenvectors. In this connection it is reasonable to deal with complex Hilbert spaces only.

**Definition 8.1** Let H denote a complex Hilbert space and  $T \in B(H, H)$ . T is called self-adjoint if  $T = T^*$ . An eigenvalue of T is a number  $\lambda \in \mathbf{C}$  for which  $N(\lambda I - T) \neq \{0\}$ . Any nonzero element in  $N(\lambda I - T)$  is called an eigenvector corresponding to the eigenvalue  $\lambda$ .

 $\lambda \in \mathbf{C}$  is called a spectral value of T if one of the three following conditions hold:

- (i)  $\lambda$  is an eigenvalue
- (ii)  $\lambda$  is not an eigenvalue and the range of  $(\lambda I T)$  is not dense in  $H: \overline{R(\lambda I T)} \neq H$
- (iii)  $\lambda$  is not an eigenvalue, the range of T is dense in H, but not equal to  $H: R(\lambda I T) \neq H$ ,  $\overline{R(\lambda I - T)} = H$ .

The set of all spectral values is called the spectrum of T and is denotes by  $\sigma(T)$ . Its complement  $\mathbf{C} \setminus \sigma(T)$  is called the resolvent set of T and is denoted by  $\rho(T)$ . The spectrum decomposes into three disjoint subsets:

 $\sigma_p(T)$ , the point spectrum of T, contains the eigenvalues

 $\sigma_r(T)$ , the residual spectrum of T, contains all spectral values of type (ii)

- $\sigma_c(T)$ , the continuous spectrum, contains all spectral values of type (iii)
- $\sigma(T) = \sigma_p(T) \dot{\cup} \sigma_r(T) \dot{\cup} \sigma_c(T).$

**Remark 8.1** If  $\lambda \in \sigma_r(T)$  then by the basic relation

$$R(\lambda I - T)^{\perp} = N((\lambda I - T)^*) = N(\overline{\lambda}I - T^*)$$

we conclude

$$\overline{\lambda} \in \sigma_p(T^*)$$
.

**Remark 8.2** From a basic theorem of Functional Analysis, the bounded inverse theorem one can conclude that for an operator  $T \in B(H_1, H_2)$ ,  $(H_1, H_2$  Hilbert spaces) with  $\overline{R(T)} = H_2$  and with  $N(T) = \{0\}$  the two following assertions are equivalent

(i) 
$$R(T) = H_2$$

(ii)  $T^{-1}$ :  $R(T) \subset H_2 \longrightarrow H_1$ ,  $Tx \longmapsto x$  is continuous.

Therefore the resolvent set of T is the set of all  $\lambda$  for which the inverse of  $(\lambda I - T)$  exists and belongs to B(H, H). The spectrum is the set of all  $\lambda$  for which 'something goes wrong' with the inverse of  $(\lambda I - T)$ .

The proof of the bounded inverse theorem will not be carried out here. However, with our tools we may at least show that ' $\lambda \in \sigma_c(T)$ ' implies that the inverse  $(\lambda I - T)^{-1}$  is not continuous. Notice that for  $\lambda \in \sigma_c(T)$ 

$$(\lambda I - T)^{-1}$$
 :  $R(\lambda I - T) \subset H \longrightarrow H$ ,  $(\lambda I - T)x \longmapsto x$ 

is a linear operator defined on a dense subspace of  $H : \lambda I - T$  is injective since  $\lambda \notin \sigma_p(T)$ . Let  $y \in H \setminus R(\lambda I - T)$  and  $(x_n)_{n \in \mathbb{N}}$  a sequence in H such that  $(\lambda I - T)x_n$  tends to y. Such a sequence exists since  $R(\lambda I - T)$  is dense in H. If  $(\lambda I - T)^{-1}$  was a bounded operator then  $x_n := (\lambda I - T)^{-1} (\lambda I - T) x_n$  would be a Cauchy sequence and whence tend to some  $x \in H$ . But then  $(\lambda I - T)x = \lim_{n \to \infty} (\lambda I - T)x_n = y$ , i.e.  $y \in R(\lambda I - T)$ , a contradiction.

In the case, which we are interested in, the residual spectrum is empty:

**Lemma 8.1** If  $T \in B(H,H)$  is self adjoint then  $\sigma_r(T) = \emptyset$ . Moreover the spectrum of T is restricted to the real line:  $\sigma(T) \subset \mathbf{R}$ .

**Proof:** If T is self adjoint then for all  $x \in H$ 

$$\langle Tx, x \rangle = \langle x, Tx \rangle , \qquad (8.4)$$

which means that  $\langle Tx, x \rangle$  is real for any  $x \in H$ . Consider the sesquilinear form

$$t : H \times H \longrightarrow \mathbf{C}, (x, y) \longmapsto \langle (\lambda I - T)x, y \rangle$$

t is bounded

$$|t(x,y)| \le |\lambda| ||x|| ||y|| + ||Tx|| ||y|| \le (|\lambda| + ||T||) ||x|| ||y||,$$

and satisfies the assumption (3.21) of the Lax-Milgram Theorem for non-real  $\lambda$ :

$$|t(x,x)| \ge |\text{Im } t(x,x)| = |\text{Im } \lambda| \ ||x||^2 \ . \tag{8.5}$$

Hence for any  $z \in H$  there exists a unique  $x \in H$  such that

$$t(x,y) = \langle z, y \rangle \quad \text{for all } y \in H .$$
(8.6)

But this means that

$$(\lambda I - T) x = z \quad . \tag{8.7}$$

On the other hand, any solution x of (8.7) satisfies (8.6). Whence the solutions of (8.7) are unique.

We have shown that for non real  $\lambda$  the range of  $(\lambda I - T)$  is the whole of H and that  $\lambda$  is no eigenvalue. Whence  $\lambda \in \rho(T)$ . (The continuity of  $(\lambda I - T)$  can easily be deduced from the (8.5).)

Assume now that there exists  $\lambda \in \sigma_r(T)$ . Then  $\lambda$  is real and by Remark 8.1  $\lambda \in \sigma_p(\lambda I - T^*)$ . But  $T^* = T$ , thus  $\lambda \in \sigma_p(T)$ , a contradiction.

**Lemma 8.2** If  $T \in B(H, H)$  is self adjoint and  $\lambda, \mu$  are two different eigenvalues of T with corresponding eigenvectors x resp. y, then x and y are orthogonal:

$$\langle x, y \rangle = 0$$

**Proof:**  $\lambda$  and  $\mu$  are real by Lemma 8.1 Hence because of  $\lambda x = Tx$ ,  $\mu y = Ty$ :

$$(\lambda - \mu) \langle x, y \rangle = \langle \lambda x, y \rangle - \langle x, \mu y \rangle = \langle Tx, y \rangle - \langle x, Ty \rangle = 0$$

**Lemma 8.3** If  $T \in B(H, H)$  is self adjoint and V is a finite dimensional subspace of H which is spanned by eigenvectors of T, then the restriction  $T_1$  of T to

$$H_1 := V^{\perp}$$

maps  $H_1$  into itself and is self adjoint on  $H_1$ .  $\lambda$  is an eigenvalue of  $T_1$  if and only if  $\lambda$  is an eigenvalue of T with an eigenvector  $x \notin V$ .

**Proof:** Notice that T maps V into itself since this is true on a basis of V. Whence for all  $x \in V^{\perp}$  and  $y \in V$ 

$$\langle Tx, y \rangle = \langle x, Ty \rangle = 0$$

i.e.  $Tx \in H_1$  if  $x \in H_1$ . Self adjointness is clear and it is also clear that an eigenvalue  $\lambda$  of  $T_1$  is an eigenvalue of T with an eigenvector in  $H_1$  and whence not in V. To see the converse, let  $\lambda$ denote an eigenvalue of T with an eigenvector  $x \notin V$ . V has a basis consisting of eigenvectors and by Lemma 8.2 x is orthogonal to any basis vector which is eigenvector with respect to an eigenvalue  $\mu \neq \lambda$ . If there exists some basis vectors, which are eigenvectors  $x_1, \ldots, x_k$  with respect to  $\lambda$  one may find a linear combination  $\sum_{i=1}^k \alpha_i x_i$  such that  $\hat{x} := x - \sum_{i=1}^k \alpha_i x_i \in H_1$ , and of course  $\hat{x}$  is an eigenvector with respect to  $\lambda$ . **q.e.d.** 

We now start to consider compact operators. From Theorem 5.3 one obtains that non zero spectral values must be eigenvalues.

**Lemma 8.4** If K is a compact operator in B(H, H) then  $\sigma(K) \setminus \{0\} \subset \sigma_p(K)$  and  $0 \in \sigma(K)$ . In particular: if K is injective and self adjoint then  $0 \in \sigma_c(K)$  and  $K^{-1}$  is not continuous.

**Remark 8.3** With the equivalence in Remark 8.2 one might prove that for any compact operator K

$$0 \in \sigma(K)$$
.

Assuming that  $0 \notin \sigma_p(K) \cup \sigma_r(K)$ , we obtain from Theorem 5.4 and Remark 8.2 i that  $0 \in \sigma_c(K)$ .

We are now ready to start the

Proof of Theorem 8.1 The proof uses Rayleigh quotients.

Notice that for any eigenvalue  $\mu$  of K there exists a normalized eigenvector x for which  $\langle Kx, x \rangle = \langle \mu x, x \rangle = \mu$ . Thus any eigenvalue is nonnegative and is not greater than

$$\lambda_1 := \sup_{x \in H, \|x\|=1} \langle Kx, x \rangle .$$
(8.8)

The supremum exists since for ||x|| = 1 the quadratic form  $\langle Kx, x \rangle$  is real and bounded from above:

$$\langle Kx, x \rangle \le ||Kx|| \, ||x|| \le ||K|| \, ||x||^2 = ||K|| .$$
 (8.9)

Hence  $\lambda_1 \leq ||K||$ .

In fact,

$$\lambda_1 = ||K|| .$$

This can be seen as follows: let k denote the hermetian sesquilinear form

$$k \; : \; \; H \times H \; \longrightarrow \; \mathbf{C} \; , \; \; (u,v) \; \longmapsto \; \langle \, Ku \, , \, v \, \rangle \; \; .$$

An easy calculation shows for all  $u, v \in H$ 

Re 
$$k(u,v) = \frac{1}{4} [k(u+v, u+v) - k(u-v, u-v)]$$
.

With ||x|| = 1 and

$$\alpha := ||Kx||^{1/2} > 0$$

we replace u by  $\alpha x$  and v by  $\frac{1}{\alpha}Kx$ . Since  $\langle Ky, y \rangle \leq \lambda_1 ||y||^2$  holds for any  $y \in H$  we obtain:

$$\begin{split} \|Kx\|^2 &= \operatorname{Re} k(\alpha x, \frac{1}{\alpha} Kx) \\ &\leq \frac{1}{4} k(\alpha x + \frac{1}{\alpha} Kx, \alpha x + \frac{1}{\alpha} Kx) \\ &\leq \frac{\lambda_1}{4} \|\alpha x + \frac{1}{\alpha} Kx\|^2 \\ &\leq \frac{\lambda_1}{4} (\alpha^2 + 2\langle Kx, x \rangle + \frac{1}{\alpha^2} \|Kx\|^2) \\ &\leq \frac{\lambda_1}{4} (\alpha^2 + \frac{1}{\alpha^2} \|Kx\|^2) + \frac{\lambda_1^2}{2} \\ &= \frac{\lambda_1}{2} \|Kx\| + \frac{\lambda_1^2}{2} . \end{split}$$

Hence

$$||Kx||^2 - \frac{\lambda_1}{2} ||Kx|| - \frac{\lambda_1^2}{2} \le 0$$

from which

$$||Kx|| \leq \lambda_1$$

follows. Thus  $||K|| = \sup_{||x||=1} ||Kx|| \le \lambda_1$ .

Suppose now that the supremum is attained at u, say. Then with

$$v := (K - \lambda_1 I)u$$

we define the functions

$$F : \mathbf{R}^2 \longrightarrow \mathbf{R} , (s,t) \longmapsto \langle K(su+tv), su+tv \rangle ,$$
$$G : \mathbf{R}^2 \longrightarrow \mathbf{R} , (s,t) \longmapsto ||su+tv||^2 - 1 .$$

Then under the side condition G(s,t) = 0 the function F attains its maximum at the point (s,t)=(1,0). By the Lagrangian rule there exists some real  $\mu$  such that  $\partial_s F(1,0) = \mu \partial_s G(1,0)$  and  $\partial_t F(1,0) = \mu \partial_t G(1,0)$ , i.e.

$$2\langle Ku, u \rangle = 2\mu ||u||^2, \quad 2\text{Re} \langle Ku, v \rangle = 2\text{Re} (\mu \langle u, v \rangle)$$

Since  $\langle Ku, u \rangle = \lambda_1$  and ||u|| = 1 the first equation yields  $\mu = \lambda_1$ . Then the second equation and the definition of v give

$$||(K - \lambda_1 I)u||^2 = 0$$
,

i.e. u is an eigenvector for the eigenvalue  $\lambda_1$ .

We now prove that in fact the supremum is attained, at  $u_1$ , say. Then it is the largest eigenvalue with normalized eigenvector  $u_1$ .

Choose a sequence  $(x_n)_{n \in \mathbb{N}}$  with  $||x_n|| = 1$  and such that

$$\lim_{n \to \infty} \langle K x_n , x_n \rangle = \lambda_1 .$$

We may without loss of generality assume that  $Kx_n$  converges, otherwise we replace it by a subsequence which has the desired property. Since  $||x_n|| = 1$  we obtain

$$\begin{aligned} \|(\lambda_1 - K)x_n\|^2 &= \lambda_1^2 - 2\lambda_1 \langle Kx_n, x_n \rangle + \|Kx_n\|^2 \\ &= 2\lambda_1 (\lambda_1 - \langle Kx_n, x_n \rangle) + (\|Kx_n\|^2 - \lambda_1^2) \\ &\leq 2\lambda_1 (\lambda_1 - \langle Kx_n, x_n \rangle) + (\|K\|^2 - \lambda_1^2) \\ &= 2\lambda_1 (\lambda_1 - \langle Kx_n, x_n \rangle) . \end{aligned}$$

The right hand side tends to 0. Thus

$$\lim_{n\to\infty} (\lambda_1 x_n - K x_n) = 0 ,$$

and as  $(Kx_n)$  converges so does  $(x_n)$ . But then

$$u_1 := \lim_{n \to \infty} x_n$$

has norm 1 and  $\langle Ku_1, u_1 \rangle = \lambda_1$ .

By Lemma 8.3 any other eigenvalue of K is an eigenvalue of  $K_1$ , the restriction of K to  $H_1 := \{u_1\}^{\perp}$ . So we may construct a sequence  $(\lambda_n)_{n \in \mathbb{N}}$  of eigenvalues and an orthonormal sequence  $(u_n)_{n \in \mathbb{N}}$  of eigenvectors by the recursion

$$\begin{array}{lll} \lambda_{n+1} & := & \max_{x \in \{u_1, \dots, u_n\}^{\perp}, \ \|x\| = 1} \langle Kx, x \rangle \\ \\ u_{n+1} & : & \text{a maximizer of } \langle Kx, x \rangle \text{ in } \{u_1, \dots, u_n\}^{\perp} \cap \{x : \|x\| = 1\} \end{array}$$

Of course the sequence  $\lambda_n$  is non increasing and by Lemma 8.3 any eigenvalue is a value in the sequence. Also, the sequence attains any value at most finitely often: If  $\lambda_n = \lambda > 0$  for all  $u \ge u_0$  then  $N(\lambda I - K) = N(I - \frac{1}{\lambda}K)$  would have infinite dimension which contradicts (5.12).

By construction  $\lambda_n > 0$  for all  $n \in \mathbf{N}$ . We show that

$$\lim_{n \to \infty} \lambda_n = 0 . \tag{8.10}$$

Assume that

$$\lim_{n \to \infty} \lambda_n = \lambda > 0$$

Choose a subsequence  $(\lambda_{n'})_{n \in \mathbb{N}}$  such that  $(Ku_{n'})_{n \in \mathbb{N}}$  converges to v, say. Then  $u_{n'}$  tends to  $\lambda^{-1}v$ : By  $\lambda_{n'}u_{n'} = Ku_{n'}$  we have

$$\|\lambda u_{n'} - v\| = \|(\lambda - \lambda_{n'})u_{n'} + (Ku_{n'} - v)\| \le |\lambda - \lambda_{n'}| + \|Ku_{n'} - v\| \to 0$$

But this contradicts the fact that  $(u_{n'})_{n \in \mathbb{N}}$  form an orthonormal sequence, and (8.10) follows.

To show that  $\{u_n: n \in \mathbb{N}\}$  is complete let  $u \in H$ , ||u|| = 1 and assume  $\langle u, u_n \rangle = 0$  for all  $n \in \mathbb{N}$ . By (8.1)

$$\langle Ku, u \rangle > 0$$

and it exists  $n \in \mathbf{N}$  such that

$$\lambda_{n+1} < \langle Ku, u \rangle \leq \lambda_n$$
 .

But

$$\langle Ku, u \rangle \leq \max_{\|x\|=1, x \in \{u_1, \dots, u_n\}^{\perp}} \langle Kx, x \rangle = \lambda_{n+1} ,$$

a contradiction.

To prove (8.2) notice that

$$Kx = \sum_{n=1}^{\infty} \langle Kx, u_n \rangle u_n .$$

 $\operatorname{But}$ 

$$\langle Kx, u_n \rangle = \langle x, Ku_n \rangle = \lambda_n \langle x, u_n \rangle ,$$

whence (8.2) follows. Similarly

$$(\mu I - K)^{-1}x = \sum_{n=1}^{\infty} \langle (\mu I - K)^{-1}x, u_n \rangle u_n$$
$$= \sum_{n=1}^{\infty} \langle x, (\overline{\mu} I - K)^{-1}u_n \rangle u_n$$

Now

$$(\overline{\mu} I - K) \frac{1}{\overline{\mu} - \lambda_n} u_n = \frac{1}{\overline{\mu} - \lambda_n} (\overline{\mu} u_n - \lambda_n u_n) = u_n .$$

Thus

$$(\overline{\mu} I - K)^{-1} u_n = \frac{1}{\overline{\mu} - \lambda_n} u_n$$

 $\operatorname{and}$ 

$$(\mu I - K)^{-1} x = \sum_{n=1}^{\infty} \langle x, \frac{1}{\overline{\mu} - \lambda_n} u_n \rangle u_n = \sum_{n=1}^{\infty} \frac{1}{\mu - \lambda_n} \langle x, u_n \rangle u_n ,$$

which is (8.3).

It remains to prove (v): If  $y = Kx, x \in H$ , then

$$\langle y, u_n \rangle = \lambda_n \langle x, u_n \rangle$$

Whence

$$\infty > ||x||^2 = \sum_{n=1}^{\infty} \lambda_n^{-2} |\langle y, u_n \rangle|^2$$

If, on the other hand, y is such that

$$\sum_{n=1}^{\infty} |\langle y , u_n \rangle|^2 < \infty$$

then (see the step from (7.5) to (7.6))

$$x := \sum_{n=1}^{\infty} \lambda_n^{-1} \langle y, u_n \rangle u_n$$

defines an element in H for which y = Kx.

Combining Theorems 8.1 iv and 7.1 one obtains

**Corollary 8.1** If H is a Hilbert space such that B(H, H) contains a compact, injective, self adjoint operator K then H is separable.

q.e.d.

**Proof:** In this case  $K^2$  satisfies the assumptions of Theorem 8.1 and whence H contains a countable complete orthonormal system.

The problem to find eigenvalues and eigenfunctions of L in  $\Omega$  under Dirichlet condition (c.f. Definition 6.1) is directly connected with the problem of finding the eigenvalues and eigenvectors of the solution operator G defined in Theorem 6.1 and by (6.10). Since we want to use the compactness of G we assume henceforth that  $\Omega$  is bounded. Also in addition to (4.3), (4.4), (4.5) and the assumption below Corollary 4.1, the coefficients of L are assumed to be such that the Dirichlet form B is hermetian, i.e.

$$B(u,v) = \overline{B(v,u)} \quad \text{for all } u,v \in H_0^1(\Omega) \quad .$$
(8.11)

For example,

$$b_m = \sum_{n=1}^{N} \partial_n a_{nm}$$
 for  $m = 1, ..., N$  (8.12)

$$a_{nm}$$
, c real valued (8.13)

are sufficient for the validity of (8.11). Another sufficient condition is that the coefficients are so smooth that one can write down the formal adjoint  $L^*$  of L and

$$L^* = L$$

holds.

Under the assumption (8.11) it follows from Theorem 6.3 and from Remark 6.1 that the solution operator G is self adjoint. Moreover (6.1) implies  $B(Gu, u) = -||u||^2$  for any  $u \in L^2(\Omega)$ . Thus  $Gu \neq 0$ . Another application of (6.1) shows

$$|-\langle Gu, u \rangle = -\langle u, Gu \rangle = B(Gu, Gu) \ge c_+ ||Gu||_1^2$$

for all  $u \in L^2(\Omega)$ . Since G is injective the far right hand side only vanishes if u = 0. Hence we have shown:

**Lemma 8.5** K := -G is an injective positive, self adjoint compact operator. Hence there exists a non increasing sequence  $(\lambda_n)_{n \in \mathbb{N}}$  of eigenvalues and an orthonormal sequence  $(u_n)_{n \in \mathbb{N}}$  of corresponding eigenvectors which satisfy the assertions of Theorem 8.1

Definition 6.1 and Lemma 6.1 imply that  $\lambda \in \mathbf{C}$  is an eigenvalue of L with eigenfunction u if and only if  $u \in L^2(\Omega)$ ,  $u \neq 0$  and

$$u - \lambda G u = 0 \tag{8.14}$$

holds. But (8.14) is equivalent with

$$(-\frac{1}{\lambda}) u - Ku = 0 . (8.15)$$

Notice, that  $\lambda = 0$  surely is not an eigenvalue of L.

Thus the eigenvalues of L in  $\Omega$  are just  $\mu_n := -\frac{1}{\lambda_n}$ , where  $\lambda_n$  are the eigenvalues of K and the orthonormal sequence  $(u_n)_{n \in \mathbb{N}}$  asserted in Lemma 8.5, provides the sequence of eigenfunctions. So we have partly proven:

**Theorem 8.2** Under the conditions on L and  $\Omega$  stated above there exists a non increasing sequence  $(\mu_n)_{n \in \mathbb{N}}$  of eigenvalues of L in  $\Omega$  and an orthonormal (in  $L^2(\Omega)$ ) sequence  $(u_n)_{n \in \mathbb{N}}$ of corresponding eigenfunctions  $(u_n)_{n \in \mathbb{N}}$  such that

- (i)  $0 > \mu_1 \ge \ldots \ge \mu_n \ge \mu_{n+1} \rightarrow -\infty$  as  $n \rightarrow \infty$ .
- (ii)  $u_n$  is a variational solution of  $L_{\mu_n} u = 0$ ,  $u_n \mid_{\partial \Omega} = 0$ .
- (iii) The orthonormal system  $\{u_n: n \in \mathbf{N}\}$  is complete.
- (iv)  $u \in H_0^1(\Omega)$  iff  $\sum |\mu_n| |\langle u, u_n \rangle|^2 < \infty$ .
- (v) If  $u, v \in H_0^1(\Omega)$  then

$$B(u,v) = \sum_{n=1}^{\infty} |\mu_n| \langle u, u_n \rangle \langle u_n, v \rangle$$

(vi)  $u \in L^2(\Omega)$  is a variational solution of a Dirichlet problem "Lu = f in  $\Omega$ ,  $u|_{\partial\Omega} = 0$ " for some  $f \in L^2(\Omega)$ , iff

$$\sum_{n=1}^{\infty} \ \left| \mu_n \right|^2 \ \left| \left\langle \, u \, , \, u_n \, \right\rangle \right|^2 \ < \ \infty \ .$$

(vii) If  $f \in L^2(\Omega)$  and  $\mu$  is not a value of the the sequence  $(\mu_n)_{n \in \mathbb{N}}$  then the solution u of  $L_{\mu}u = f$ ,  $u|_{\partial\Omega} = 0$  is given by

$$u = \sum_{n=1}^{\infty} \frac{1}{\mu_n - \mu} \langle f, u_n \rangle u_n$$
.

#### **Proof:**

(i) - (iii) is the part of the theorem which is already proven.

To prove (iv) notice that  $B(\cdot, \cdot)$  can be considered as a scalar product in  $H_0^1(\Omega)$ . It then induces a norm  $\| \|_b$  in  $H_0^1(\Omega)$ , and for any  $u \in H_0^1(\Omega)$ 

$$c_{+}^{1/2} \|u\|_{1} \leq \|u\|_{b} \leq \gamma^{1/2} \|u\|_{1}$$

(c.f. (4.19), (4.12)). Thus convergence with respect to the original norm  $\|\cdot\|_1$  is the same as convergence with respect to the new norm  $\|\cdot\|_b$ , and  $H_0^1(\Omega)$  is a Hilbert space with respect to the scalar product  $B(\cdot, \cdot)$ , too.

Now for any  $n, m \in \mathbf{N}$ 

$$B(u_n, u_m) = - \langle \mu_n u_n, u_m \rangle = - \mu_n \langle u_n, u_m \rangle$$
$$= - \mu_n \delta_{nm} = |\mu_n| \delta_{nm} .$$

Hence  $\{|\mu_n|^{-1/2} u_n: n \in \mathbb{N}\}$  is an orthonormal system in  $H_0^1(\Omega)$  (with respect to new the scalar product  $B(\cdot, \cdot)$ ).

This orthonormal system is complete: Let  $u \in H_0^1(\Omega)$  and  $B(u_n, u) = 0$  for all  $n \in \mathbb{N}$ . Since  $u_n$  are the eigenvectors with respect to the nonzero eigenvelaues  $\mu_n$ :

$$0 = B(u_n, u) = -\mu_n \langle u_n, u \rangle \Rightarrow \langle u_n, u \rangle = 0$$

The latter is true for all  $n \in \mathbf{N}$ . Since  $\{u_n: n \in \mathbf{N}\}$  is complete in  $L^2(\Omega)$  we obtain that u must be 0, whence  $\{|\mu_n|^{-1/2} u_n: n \in \mathbf{N}\}$  is complete in  $H_0^1(\Omega)$ .

Before going further we notice:

**Remark 8.4**  $H_0^1(\Omega)$  is separable.

We continue the proof of (iv). If  $u \in H_0^1(\Omega)$  then by Parseval's identity

$$\infty > B(u, u) = \sum_{n=1}^{\infty} \left| B(u, |\mu_n|^{-1/2} u_n) \right|^2 = \sum_{n=1}^{\infty} |\mu_n| |\langle u, u_n \rangle|^2$$

If on the other hand  $u \in L^2(\Omega)$  and

$$\sum_{n=1}^{\infty} |\mu_n| |\langle u, u_n \rangle|^2 < \infty ,$$

then the sequence  $(v_n)_{n \in \mathbb{N}}$ , given by

$$v_n \;=\; \sum_{j=1}^n \, \langle \, u \,, \, u_j \, 
angle \; u_j$$

converges in  $H_0^1(\Omega)$ :

$$\begin{aligned} \|v_n - v_{n+k}\|_b^2 &= \|\sum_{j=n+1}^{n+k} \langle u, u_j \rangle u_j\|_b^2 \\ &\leq \sum_{j=n+1}^{\infty} |\langle u, u_j \rangle|^2 |\mu_n| \to 0 \end{aligned}$$

as  $n \to \infty$ . Whence  $(v_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $H_0^1(\Omega)$ . Consequently it has a limit v in  $H_0^1(\Omega)$ . Then

$$\begin{aligned} \|u - v\|_{0} &\leq \|u - v_{n}\|_{0} + \|v_{n} - v\|_{0} \\ &\leq \|u - v_{n}\|_{0} + \|v_{n} - v\|_{1} \\ &\leq \|u - v_{n}\|_{0} + c_{+}^{-1/2} \|v_{n} - v\|_{b} \to 0 \end{aligned}$$

as  $n \to \infty$ , i.e.  $u = v \in H_0^1(\Omega)$ .

(v) follows from (7.4), applied to the complete *B*-orthonormal system  $\{|\mu_n|^{-1/2} u_n: n \in \mathbf{N}\}$ :

$$B(u,v) = \sum_{n=1}^{\infty} B(u, |\mu_n|^{-1/2} u_n) B(|\mu_n|^{-1/2} u_n, v)$$
$$= \sum_{n=1}^{\infty} |\mu_n| \langle u, u_n \rangle \langle u_n, v \rangle .$$

To prove (vii) let

$$u := \sum_{n=1}^{\infty} \frac{1}{\mu_n - \mu} \langle f, u_n \rangle u_n .$$
 (8.16)

Thus  $\langle u, u_n \rangle = (\mu_n - \mu)^{-1} \langle f, u_n \rangle$ . Then  $u \in H_0^1(\Omega)$  because of **(iv)** and for any  $v \in H_0^1(\Omega)$ 

$$\begin{split} B(u,v) + \mu \left\langle u, v \right\rangle &= \sum_{n=1}^{\infty} -\mu_n \frac{1}{\mu_n - \mu} \left\langle f, u_n \right\rangle \left\langle u_n, v \right\rangle \\ &+ \sum_{n=1}^{\infty} \mu \frac{1}{\mu_n - \mu} \left\langle f, u_n \right\rangle \left\langle u_n, v \right\rangle \\ &= -\sum_{n=1}^{\infty} \left\langle f, u_n \right\rangle \left\langle u_n, v \right\rangle = - \left\langle f, v \right\rangle \;. \end{split}$$

Hence  $\,u\,,$  given by (8.15), is the unique solution of ' $L_{\mu}u\,=f\,,\,u\,|_{\partial\Omega}=$  0'.

Finally (vi) may be easily deduced from (vii).

### Chapter 9

## Regularity

To prove regularity results means to show that the solutions obtained in an existence theorem can be differentiated more often then asserted in that theorem. Generally one distinguishes between local regularity results and global regularity results. Local regularity means that additional differentiability is proven in small neighbourhoods of any point in  $\Omega$ . Global regularity means that additional differentiability is proven up to the boundary.

A typical global result of this kind is

**Theorem 9.1** Let  $k \in \mathbf{N}_0$  and  $\Omega$  be a bounded domain of class  $C^{k+2}$ . Assume that the coefficients  $a_{mn}, b_m$  and c satisfy (m, n = 1, ..., N):

$$a_{mn} \in C^{k+1}(\Omega)$$
$$b_m \in C^k(\Omega)$$
$$c \in C^k(\Omega)$$

and that  $a_{mn}, b_m, c$  and all their derivatives, the existence of which is assumed, are bounded. If u is the variational solution of  $L_{\mu}u = f$ ,  $u|_{\partial\Omega} = 0$  and if  $f \in H^k(\Omega)$  then  $u \in H^{k+2}(\Omega)$  and there exists a constant, independent of f and u such that

$$||u||_{k+2,\Omega} \leq c (||f||_{k,\Omega} + ||u||_{0,\Omega})$$

For example, if k = 0 then under the assumptions of the theorem the variational solution of  $L_{\mu}u = f$ ,  $u|_{\partial\Omega} = 0$  is a strong solution.

There is another result which should be mentioned in this connection, namely Sobolev's embedding theorem. It states roughly that under mild assumptions on  $\Omega$ , any element in  $H^m(\Omega)$  (with m > N/2) has a representative which is in  $C^l(\overline{\Omega})$  where l is the largest integer less then k - N/2. For example, if N = 3,  $f \in H^2(\Omega)$ , then  $f \in C^0(\overline{\Omega})$ , and the solution u of " $L_{\mu}u = f$ ,  $u|_{\partial\Omega} = 0$ " belongs to  $H^4(\Omega)$  and whence to  $C^2(\overline{\Omega})$ , i.e. it is a classical solution.

We wish to prove Theorem 9.1 only in the case k = 0. For this gives an idea how to prove such theorems. We shall use the following result from Functional Analysis:

#### **Theorem 9.2** (Weak compactness theorem)

Let H denote some separable<sup>1</sup> Hilbert space and  $(y_n)_{n \in \mathbb{N}}$  a bounded sequence in H. Then  $(y_n)$  contains a subsequence  $(y_{n'})_{n \in \mathbb{N}}$  which converges weakly to some  $y \in H$ , i.e. there exists some y such that

$$\lim_{n \to \infty} \langle y_{n'}, x \rangle = \langle y, x \rangle \tag{9.1}$$

for all  $x \in H$ .

As to the concept of weak convergence which is introduced by (9.1) we have to note that weak limits are unique. To be precise: A weak limit of a sequence  $(y_n)_{n \in \mathbb{N}}$  in H is an element y in Hsuch that  $\langle y_n, x \rangle$  tends to  $\langle y, x \rangle$  for all  $x \in H$ . If such a weak limit exists the sequence is said to converge weakly (to y). Assume that z is another weak limit of  $(y_n)$ . Then

$$\begin{aligned} |y - z||^2 &= \langle y, y - z \rangle - \langle z, y - z \rangle \\ &= \lim_{n \to \infty} \left[ \langle y_n, y - z \rangle - \langle y_n, y - z \rangle \right] = 0 \end{aligned}$$

Hence weak limits are unique. Moreover, convergent sequences are always weakly convergent.

#### **Proof of Theorem 9.2**

We assume of course dim  $H = \infty$ , since for finite dimension one may even select a convergent subsequence from  $(y_n)$ .

Consider a complete orthonormal system  $\{u_n: n \in \mathbf{N}\}$ . Find a subsequence  $(y_{n_1})$  of  $(y_n)$  for which  $\langle y_{n_1}, u_1 \rangle$  converges. From  $(y_{n_1})$  one may select a subsequence  $(y_{n_2})$  for which  $\langle y_{n_2}, u_2 \rangle$  converges. Proceeding in this manner we find for any  $k \in \mathbf{N}$  subsequences  $(y_{n_k})$ , such that  $(y_{n_k})$  is a subsequence of  $y_{n_{k-1}}$   $(y_{n_0} := y_n)$  and  $\langle y_{n_k}, u_k \rangle$  converges. Put  $y_{n'} := y_{n_n}$ , the diagonal sequence. This has the property that for any  $k \in \mathbf{N}$ ,  $\langle y_n, u_k \rangle$  converges to  $\alpha_k$ , say.

We show that  $\sum_{k=1}^{\infty} \alpha_k u_k$  converges to some element y which is the weak limit of  $(y_{m'})$ .

By construction of  $y_{n'}$  we find that for any  $N \in \mathbf{N}$  the sequence

$$y_{n'}^N := \sum_{k=1}^N \langle y_{n'} , u_k \rangle u_k$$

tends to

$$y^N := \sum_{k=1}^N lpha_k u_k$$

 $<sup>^{1}</sup>H$  needs not to be separable, however we need the result for separable spaces only. To obtain it for non-separable spaces one has to proof it for separable spaces first and then provide an additional argument.

Hence

$$||y^{N}|| = \lim_{n \to \infty} ||y^{N}_{n'}|| \le \sup_{n \in \mathbf{N}} ||y^{N}_{n'}|| \le \sup_{n \in \mathbf{N}} ||y_{n'}|| =: \gamma$$
,

and we conclude

$$\sum_{k=1}^{N} |\alpha_k|^2 = ||y^N||^2 \le \gamma^2 ,$$

from which

$$\sum_{k=1}^{\infty} |\alpha_k|^2 \le \gamma^2 < \infty$$

follows. But then  $\sum_{k=1}^{\infty} \alpha_k u_k$  converges and defines an element  $y \in H$ , and  $\langle y, u_k \rangle = \alpha_k$ .

For any  $w \in \text{span} \{u_n : n \in \mathbb{N}\}$ , i.e. for any w which is a linear combination of finitely many of the  $u_n$ , we obtain

$$\lim_{n\to\infty}\,\langle\,y_{n'}\,,\,w\,\rangle\ =\ \langle\,y\,,\,w\,\rangle\ .$$

Now let  $v \in H$  and  $\varepsilon > 0$  be given: We may find  $w \in \text{span} \{u_n : n \in \mathbb{N}\}$  such that

$$\gamma \|v - w\| < \varepsilon/4$$

Whence

$$\begin{split} |\langle y_{n'}, v \rangle - \langle y, v \rangle| &\leq |\langle y_{n'}, v - w \rangle| + |\langle y, v - w \rangle| + |\langle y_{n'}, w \rangle - \langle y, w \rangle| \\ &\leq 2 \gamma ||v - w|| + |\langle y_{n'}, w \rangle - \langle y, w \rangle| \\ &< \varepsilon/2 + |\langle y_{n'}, w \rangle - \langle y, w \rangle| \quad . \end{split}$$

For sufficiently large n the last term is less than  $\varepsilon/2$ , and hence  $y_{n'}$  tends weakly to y. **q.e.d.** 

From now on let

$$\Xi := \mathbf{R}^N_+ := \{ x \in \mathbf{R}^N \colon x_N > 0 \} \quad \text{or} \quad \Xi := \mathbf{R}^N \quad .$$

$$(9.2)$$

Moreover we assume that

$$j \in \{1, \dots, N-1\}$$
 if  $\Xi = \mathbf{R}_{+}^{N}$  or  $j \in \{1, \dots, N\}$  if  $\Xi := \mathbf{R}^{N}$ , (9.3)

and define with  $h \in \mathbf{R}$ ,  $h \neq 0$ :

$$\tau_h : L^2(\Xi) \to L^2(\Xi) , \ (\tau_h u)(x) := u(x + he^{(j)}) .$$
 (9.4)

Of course, (9.4) has to be understood in the sense that whenever  $\tilde{u}$  is a representant of u then  $x \mapsto \tilde{u}(x + he^{(j)})$  is a representant of  $\tau_h u$ .  $\tau_h$  is a linear operator of  $L^2(\Xi)$  into  $L^2(\Xi)$  and for all  $u \in L^2(\Xi)$ 

$$\|\tau_h u\| = \|u\| . (9.5)$$

Moreover  $\tau_h$  is surjective, since

$$\tau_h \tau_{-h} = I \quad . \tag{9.6}$$

**Remark 9.1** A surjective operator U of a Hilbert space H, into another Hilbert space  $H_2$ , for which  $||Uu||_2 = ||u||_1$  holds for any  $u \in H_1$  is called a unitary operator. Unitary operators can also be characterized by the equation  $U^* = U^{-1}$ .

We now introduce the difference quotionts.

$$\delta_h : L^2(\Xi) \longrightarrow L^2(\Xi) , \ u \longmapsto \frac{1}{h} (\tau_h u - u) , \qquad (9.7)$$

where  $h \in \mathbf{R} \setminus \{0\}$ . The following lemma decovers the connection between the difference quotients and the weak derivative which in some sense is easier than for the classical derivative.

**Lemma 9.1** Let  $u \in L^2(\Xi)$  and  $h_0 > 0$ . If  $\{\delta_h u: 0 < |h| < h_0\}$  is a bounded subset of  $L^2(\Xi)$  then  $\partial_j u$  exists weakly in  $L^2(\Xi)$ , and

$$\|\partial_j u\| \leq \sup_{0 < |h| < h_0} \|\delta_h u\|$$
 .

If  $u \in H^1(\Xi)$  then

$$\|\delta_h u\|_{0,\Xi} \leq \|\partial_j u\|_{0,\Xi} , \qquad (9.8)$$

and  $\delta_h u$  converges in  $L^2(\Omega)$  to  $\partial_j u$ .

**Proof:** Notice that by coordinate transformation for all  $u, v \in L^2(\Xi)$  and  $h \in \mathbf{R} \setminus \{0\}$ 

$$\langle \tau_h u, v \rangle = \int_{\Xi} u(x + he^j) \,\overline{v}(x) \, d\mu(x) = \int u(z) \overline{v}(z - he^j) d\mu(z) = \langle u, \tau_{-h}v \rangle .$$
(9.9)

This implies

$$\langle \delta_h u, v \rangle = - \langle u, \delta_{-h} v \rangle . \qquad (9.10)$$

To prove the first assertion we invoke Theorem 9.2 and select a sequence  $h_n \to 0$  such that  $\delta_{h_n} u$  tends weakly to some  $w \in L^2(\Xi)$ . For  $\varphi \in C_0^{\infty}(\Xi)$  and  $h \in \mathbf{R} \setminus \{0\}$  we have  $\delta_h \varphi \in C_0^{\infty}(\Xi)$  and  $\delta_h \varphi \to \partial_j \varphi$  in  $L^2(\Xi)$  as  $h \to 0$  (by Lebesgue's theorem for example). Hence for any  $\varphi \in C_0^{\infty}$ 

$$\langle w, \varphi \rangle = \lim_{n \to \infty} \langle \delta_{h_n} u, \varphi \rangle = - \lim_{n \to \infty} \langle u, \delta_{-h_n} \varphi \rangle = - \langle u, \partial_j \varphi \rangle.$$

This means that the weak derivative  $\partial_j u$  exists in  $L^2(\Xi)$  (and is equal to w).

We now prove (9.8) for  $u = \varphi|_{\Xi}$ ,  $\varphi \in C_0^{\infty}(\mathbf{R}^N)$ . Notice that

$$\delta_h \varphi(x) = \int_0^1 \, \partial_j \varphi(x + \tau h e^{(j)}) \, d\tau \quad ,$$

and whence by the Cauchy Schwarz inequality

$$\left|\delta_h\varphi(x)\right|^2 \le \left(\int_0^1 1 \cdot \left|\partial_j\varphi(x+\tau h e^{(j)}\right| \, d\tau\right)^2 \le \int_0^1 \left|\partial_j\varphi(x+\tau h e^{(j)}\right|^2 \, d\tau \, d\tau$$

From this estimate we obtain by Fubini's theorem

$$\begin{split} \|\delta_h u\|^2 &= \int_{\Xi} |\delta_h \varphi(x)|^2 d\mu(x) \leq \int_{\Xi} \int_0^1 \left| \partial_j \varphi(x + \tau h e^{(j)} \right|^2 d\tau d\mu(x) \\ &= \int_0^1 \left[ \int_{\Xi} \left| \partial_j \varphi(x + \tau h e^{(j)} \right|^2 d\mu(x) \right] d\tau = \int_0^1 \|\partial_j u\|^2 d\tau = \|\partial_j u\|^2 \,. \end{split}$$

A density argument now gives (9.8) for all  $u \in H^1(\Omega)$ : Let  $\varphi_n \in C_0^{\infty}(\Xi)$  denote a sequence for which  $\varphi_n |_{\Xi} \to u$  in  $H^1(\Omega)$ . Then  $\delta_h(\varphi_n |_{\Xi}) \to \delta_h u$  in  $L^2(\Xi)$ , since  $\delta_h$  is a continuous operator in  $L^2(\Xi)$ , and  $\partial_j \varphi_n |_{\Xi}$  tends to  $\partial_j u$  in  $L^2(\Xi)$ .

We obtain:

$$\|\delta_h u\|_{0,\Xi} = \lim_{n \to \infty} \|\delta_h \varphi_n\|_{0,\Xi} \le \lim_{n \to \infty} \|\partial_j \varphi_n\|_{0,\Xi} = \|\partial_j u\|_{0,\Xi} .$$

The convergence of  $\delta_h u$  to  $\partial_j u$  now follows by a 'stability and consistency yields convergence'argument (see Lemma 0.2). q.e.d.

By a partition of unity and by coordinate transforms according to Theorem 1.8 one can reduce the regularity question for functions in  $\Omega$  to regularity questions in  $\Xi$ . The idea then is to estimate the difference quotients of the first derivatives. This will be done in the next lemma:

**Lemma 9.2** For  $n, m \in \{1, \ldots, N\}$  let  $b_{nm} \in C^1(\Xi)$  satisfy

$$b_{nm} = b_{mn} \quad in \ \Xi , \qquad (9.11)$$

Re 
$$\sum_{n,m=1}^{N} b_{nm}(x) \xi_n \xi_m \ge E |\xi|^2$$
 with some  $E > 0$  for all  $x \in \Xi$ ,  $\xi \in \mathbf{R}^N$ , (9.12)

$$b_{nm}$$
,  $|\nabla b_{nm}|$  are bounded in  $\Xi$ . (9.13)

With some  $\gamma \geq 0$  let  $v \in H_0^1(\Xi)$  satisfy

$$\left|\sum_{n,m=1}^{N} \langle b_{nm} \partial_n v, \partial_m \varphi \rangle\right| \leq \gamma ||\varphi||_{0,\Xi}$$
(9.14)

for all  $\varphi \in C_0^{\infty}(\Xi)$ . Then  $v \in H^2(\Xi)$ , and there exists a constant c > 0 independent of v and  $\gamma$ , such that

$$||v||_{2,\Xi} \leq c \left(\gamma + ||v||_{1,\Xi}\right) . \tag{9.15}$$

**Proof:** We introduced  $\delta_h$  and  $\tau_h$  as operators acting in  $L^2(\Xi)$ . However, we shall write  $\delta_h f$ ,  $\tau_h f$  for any function f defined on  $\Xi$  to denote difference quotient and translation. Then one may easily deduce the following rules:

$$\delta_h(fg) = f(\delta_h g) + (\delta_h f) (\tau_h g) \tag{9.16}$$

for all functions f, g on  $L^2(\Xi)$ , and

$$\delta_h \partial_n u = \partial_n \delta_h u \tag{9.17}$$

for all  $u \in H^1(\Xi)$ . The latter is true for  $\varphi \in C_0^{\infty}(\Xi)$  and hence

$$\langle \delta_h u, \partial_n \varphi \rangle = - \langle u, \delta_{-h} \partial_n \varphi \rangle = - \langle u, \partial_n (\delta_{-h} \varphi) \rangle = - \langle \partial_n u, \delta_{-h} \varphi \rangle = \langle \delta_h \partial_n u, \varphi \rangle .$$

$$(9.18)$$

Here we used that with  $\varphi$  also  $\delta_h \varphi$  belongs to  $C_0^{\infty}(\Xi)$ . Now (9.18) implies that  $\delta_h u \in H^1(\Xi)$  and  $\partial_n \delta_h u = \delta_h \partial_n u$ .

We use these rules to calculate  $\langle b_{nm}\partial_n(\delta_h v), \partial_m \varphi \rangle$  for  $\varphi \in C_0^{\infty}(\Xi)$ :

$$\begin{split} \langle b_{nm}\partial_n(\delta_h v) \,, \, \partial_m \varphi \,\rangle &= \langle b_{nm}\delta_h(\partial_n v) \,, \, \partial_m \varphi \,\rangle \\ &= \langle \delta_h(\partial_n v) \,, \, \overline{b}_{nm}\partial_m \varphi \,\rangle \\ &= -\langle \partial_n v \,, \, \overline{b}_{nm}\partial_m(\delta_{-h}\varphi) \,\rangle - \langle \partial_n v \,, \, (\delta_{-h}\overline{b}_{nm})\tau_h\partial_m \varphi \,\rangle \\ &= -\langle b_{nm}\partial_n v \,, \, \partial_m(\delta_h \varphi) \,\rangle - \langle \partial_n v \,, \, (\delta_{-h}\overline{b}_{nm})\tau_{-h}\partial_m \varphi \,\rangle \,. \end{split}$$

Using (9.14) we may estimate

$$\left|\sum_{n,m=1}^{N} \langle b_{nm} \partial_n(\delta_h v), \partial_m \varphi \rangle \right| \leq \gamma \|\delta_{-h} \varphi\|_{0,\Xi} + c_1 \|v\|_{1,\Xi} \|\varphi\|_{1,\Xi},$$

where  $c_1$  depends on bounds for the derivatives of  $b_{nm}$  only. Notice that by (9.8) the right hand can be estimated further, and one gets

$$\left|\sum_{n,m=1}^{N} \langle b_{nm} \partial_n(\delta_h v), \partial_m \varphi \rangle\right| \le (\gamma + c_1 \|v\|_{1,\Xi}) \|\varphi\|_{1,\Xi} \quad .$$

$$(9.19)$$

Now  $\tau_h$  and whence  $\delta_n$  maps  $H_0^1(\Xi)$  into itself: If the sequence  $(\varphi_n)$  from  $C_0^{\infty}(\Xi)$  approximates some  $w \in H_0^1(\Xi)$  with respect to the  $\|\cdot\|_{1,\Xi}$ -norm then  $(\tau_h\varphi_n)$  is a sequence in  $C_0^{\infty}(\Xi)$  which in  $H^1(\Xi)$  tends to  $\tau_h w$ . An approximation argument shows that in (9.19) one may replace  $\varphi$  by any element of  $H_0^1(\Xi)$ , especially by  $\delta_h v$ . Using (9.12) we obtain

$$E(\|\delta_h v\|_1^2 - \|\delta_h v\|_0^2) \le \operatorname{Re} \sum_{n,m=1}^N \langle b_{nm} \partial_n(\delta_h v), \partial_m(\delta_h v) \rangle \le (\gamma + c_1 \|v\|_{1,\Xi}) \|\delta_h v\|_{1,\Xi}.$$

Hence, using (9.8):

$$\|\delta_h v\|_1^2 \leq \frac{1}{E} \left(\gamma + (c_1 + E^2) \|v\|_{1,\Xi}\right) \|\delta_h v\|_{1,\Xi}$$

from which with some  $c_2$  depending on E and  $c_1$ :

$$\|\delta_h v\|_1 \leq \frac{1}{E} \left(\gamma + (c_1 + E^2) \|v\|_{1,\Xi}\right) \leq c_2(\gamma + \|v\|_{1,\Xi}).$$

But then for any  $n \in \{1, \ldots, N\}$ 

$$\|\delta_h \partial_n v\|_{0,\Xi} \leq \|\delta_h v\|_1 \leq c_2 (\gamma + \|v\|_{1,\Xi})$$

From Lemma 9.1 it follows that  $\partial_j \partial_n v$  exists weakly in  $L^2(\Xi)$  and

$$\|\partial_j \partial_n v\| \le c_2 \left(\gamma + \|v\|_{1,\Xi}\right) . \tag{9.20}$$

In the case  $\Xi = \mathbf{R}^N$  therefore the lemma is proven.

In the case  $\Xi = \mathbf{R}^N_+$  we still have to show that  $\partial_N \partial_N v$  exists weakly in  $L^2(\Xi)$  and can be estimated by the right hand side of (9.20). Notice that (9.14) implies that by

$$\begin{array}{rcl} F & : & C_0^{\infty}(\Xi) \subset L^2(\Xi) & \longrightarrow & \mathbf{C} \\ & \varphi & \longmapsto & \sum_{n,m=1}^N \overline{\langle b_{nm} \partial_n v \,, \, \partial_m \varphi \rangle} \end{array}$$

a continuous linear functional is defined on  $C_0^{\infty}(\Xi)$  which may be continued uniquely as a continuous linear functional on the whole of  $L^2(\Omega)$  with norm not greater than  $\gamma$ . The Riesz representation theorem then yields the existence of a (unique)  $f \in L^2(\Xi)$  such that

$$\sum_{n,m=1}^{N} \left\langle b_{nm} \partial_n v , \, \partial_m \varphi \right\rangle = \left\langle f , \, \varphi \right\rangle \tag{9.21}$$

for all  $\varphi \in C_0^{\infty}(\Omega)$ , and

 $\|f\| \, \leq \, \gamma$  .

A density argument shows that (9.21) is true for all  $\varphi \in H_0^1(\Xi)$ . Equation (9.21) means that v is the variational solution of some Dirichlet problems with respect to the differential expression

$$\sum_{n,m=1}^{N} b_{nm} \partial_n \partial_m + \left(\sum_{m=1}^{N} \partial_m b_{nm}\right) \partial_n \quad .$$

Notice that this expression contains  $\partial_N \partial_N$  with a non-vanishing factor  $b_{NN}$ , since by ellipticity

$$Re \ b_{NN} = Re \ \sum_{n,m=1}^{N} \ b_{nm} \ \delta_{nN} \ \delta_{mN} \ \ge \ E \ .$$

The idea now is that it should be possible to express  $\partial_N \partial_N v$  by f and derivatives of v which are already known to belong to  $L^2(\Xi)$ .

This can be done as follows. For  $\varphi\in C_0^\infty(\Xi)$  and with  $b:=1/\overline{b_{NN}}$  :

$$\begin{array}{ll} \langle v \,, \, \partial_N \partial_N \varphi \rangle &= - \langle \partial_N v \,, \, \partial_N \varphi \rangle = - \langle b_{NN} \partial_N v \,, \, b \partial_N \varphi \rangle \\ \\ &= - \langle b_{NN} \partial_N v \,, \, \partial_N (b\varphi) \rangle + \langle b_{NN} \partial_N v \,, \, (\partial_N b) \cdot \varphi \rangle \\ \\ &= - \sum_{n,m=1}^N \langle b_{nm} \partial_n v \,, \, \partial_m (b\varphi) \rangle + \sum_{n,m=1, \ (n,m) \neq (N,N)} \langle \partial_m (b_{nm} \partial_n v) \,, \, b\varphi \rangle \\ \\ &+ \langle (\partial_N \overline{b}) b_{NN} \partial_N v \,, \, \varphi \rangle \ . \end{array}$$

Notice that  $b\varphi \in H_0^1(\Omega)$ . Hence by (9.21)

$$\langle v , \partial_N \partial_N \varphi \rangle = \langle g , \varphi \rangle$$

where

$$g := -\overline{b}f + \left(\sum_{n,m=1, (n,m)\neq (N,N)} \overline{b} \,\partial_m (b_{nm}\partial_n v)\right) + (\partial_N \overline{b}) b_{NN} \partial_N v$$

belongs to  $L^2(\Xi)$  and can be estimated by

$$\begin{aligned} \|g\| &\leq c_3 \left( \|f\| + c_2 \left( \gamma + \|v\|_1 \right) \right. \\ &\leq c_4 \left( \gamma + \|v\|_1 \right) \end{aligned}$$

where  $c_3$  and  $c_4$  do not depend on v or  $\gamma$ .

Hence  $v \in H^2(\Xi)$  and  $||v||_{2,\Xi}$  can be estimated as asserted. q.e.d.

Remark 9.2 In the preceding Lemma 9.2 one may replace estimate (9.15) by

$$\|v\|_{2,\Xi} \leq c \left(\gamma + \|v\|_{0,\Xi}\right) , \qquad (9.22)$$

where c does not depend von v and  $\gamma$ . This follows from the coercivity of the Dirichlet form  $\sum_{n,m=1}^{N} \langle b_{nm} \partial_n \cdot, \partial_m \cdot \rangle$ :

$$E(\|v\|_1^2 - \|v\|_0^2) \leq \operatorname{Re} \sum_{n,m=1}^N \langle b_{nm} \partial_n v, \partial_m v \rangle \leq \gamma \|v\|_0,$$

from which

$$||v||_1^2 \leq \frac{\gamma}{E} ||v||_0 + ||v_0||^2$$

and hence

$$||v||_1 \leq \frac{\gamma}{2E} + ||v||_0$$

follows.

We are now ready for the

### **Proof of Theorem** 9.1 (in the case k = 0):

Since  $\Omega$  is a bounded domain of class  $C^2$  (c.f. Definition 1.4) there exists a finite open covering  $V_1, \ldots, V_K$  of  $\partial \Omega$  and regular  $C^2$ -diffeomorphism  $\Phi^1, \ldots, \Phi^K$  onto U = U(0,1) such that (for  $k = 1, \ldots, K$ )

$$\Phi^{(k)}(V_k \cap \Omega) = U^+ := \{ x \in U : x_N > 0 \}$$

We introduce  $V_0 := \Omega$  so that  $V_0, \ldots, V_K$  is an open covering of  $\overline{\Omega}$  and select a partition of unity  $(\zeta_k)_{k=0,\ldots,K}$  on  $\overline{\Omega}$  which is subordinate to the covering  $V_0, \ldots, V_K$ , i.e.  $\zeta_k \in C_0^{\infty}(V_k)$ .

If u is a variational solution of  $L_{\mu}u = f$ ,  $u|_{\partial\Omega} = 0$  then  $u \in H_0^1(\Omega)$  and for all  $\varphi \in H_0^1(\Omega)$ 

$$\sum_{n,m=1}^{N} \langle a_{nm} \partial_n u, \partial_m \varphi \rangle = - \langle f, \varphi \rangle - \sum_{n=1}^{N} \langle a_n \partial_n u, \varphi \rangle - \langle (c+\mu)u, \varphi \rangle$$
  
$$\leq c_1 \gamma ||\varphi||_0.$$
(9.23)

 $c_1$  depends on the coefficients of L and  $\mu$  only, and

$$\gamma := \|f\|_{0,\Omega} + \|u\|_{1,\Omega}$$

Using coercivity as in the preceding remark we may redefine  $c_1$  and  $\gamma$  by replacing  $||u||_{1,\Omega}$  by  $||u||_{0,\Omega}$ 

$$\gamma := \|f\|_{0,\Omega} + \|u\|_{0,\Omega} . \tag{9.24}$$

For any  $k = 0, \ldots, K$  the function  $\zeta_k u$  belongs to  $H_0^1(\Omega)$  and we obtain for any  $\varphi \in H_0^1(\Omega)$ :

$$\sum_{n,m=1}^{N} \langle a_{nm} \partial_n(\zeta_k u), \partial_m \varphi \rangle$$
  
= 
$$\sum_{n,m=1}^{N} (\langle a_{nm} \partial_n u, \partial_m(\zeta_k \varphi) \rangle + \langle a_{nm}(\partial_n \zeta_k) u, \partial_m \varphi \rangle - \langle a_{nm} \partial_n u, (\partial_m \zeta_k) \varphi \rangle \}$$

The second term on the right can be integrated by parts. Using  $a_{nm} = a_{mn}$  one gets:

$$\sum_{n,m=1}^{N} \langle a_{nm} \partial_n(\zeta_k u), \partial_m \varphi \rangle$$
  
= 
$$\sum_{n,m=1}^{N} \langle a_{nm} \partial_n u, \partial_m(\zeta_k \varphi) \rangle + \sum_{n,m=1}^{N} \langle (\partial_m(a_{nm} \partial_n \zeta_k)) u, \varphi \rangle - 2 \sum_{n,m=1}^{N} \langle a_{nm} \partial_n \zeta_k \partial_m u, \varphi \rangle.$$

The modulus of the first term on the right hand side can be estimated by  $c_1 \gamma \|\zeta_k \varphi\|$  and whence by  $c_1 \gamma \|\varphi\|_{0, V_{k \cap \Omega}}$ . A similar estimate holds for the moduli of the two other terms. Hence with some  $c_2$  independent of f and u:

$$\left|\sum_{n,m=1}^{N} \left\langle a_{nm} \partial_n(\zeta_k u), \, \partial_m \varphi \right\rangle \right| \leq c_2 \gamma ||\varphi||_{0, V_k \cap \Omega} .$$
(9.25)

For  $\zeta_0 u$  we may apply Lemma 9.2 with  $\Xi = \mathbf{R}^N$  to obtain  $\zeta_0 u \in H^2(\mathbf{R}^N)$  and with some c independent of u, f:

$$\left\|\zeta_0 u\right\|_{2,\mathbf{R}^N} \leq c \gamma \quad . \tag{9.26}$$

Of course, we have continued  $\zeta_0 u$  from  $\Omega$  into  $\mathbf{R}^N$  by zero. It remains to prove  $\zeta_k u \in H^2(V_k \cap \Omega)$ and the validity of (9.26) for  $k \in \{1, \ldots, K\}$  (instead of k = 0). Let us fix k and omit the index k henceforth. Moreover let us put

$$\Psi := \Phi^{-1}, \quad \psi_{ij} := \partial_j \Psi_i, \quad \varphi_{ij} = \partial_j \Phi_i.$$

Then

$$\sum_{j=1}^{N} (\psi_{ij} \circ \Phi) \varphi_{jl} = \delta_{il} \quad .$$

Consider the pull-back operators

$$\Phi^* : H^1(U^+) \longrightarrow H^1(\Omega \cap V) , \quad w \longmapsto w \circ \Phi$$
$$\Psi^* : H^1(\Omega \cap V) \longrightarrow H^1(U^+) , \quad \tilde{w} \longmapsto \tilde{w} \circ \Psi$$

as defined in Theorem 1.8 and put

$$v := \Psi^*(\zeta u) ,$$

ζ

whence

$$u = \Phi^* v \quad .$$

Then

$$\operatorname{supp} v \subset \overline{U^+} \cap U(0,\rho) \tag{9.27}$$

with some  $\rho < 1$ . Moreover,  $v \in H_0^1(U^+)$  since  $\Psi^*$  maps  $C_0^{\infty}(\Omega \cap V)$  into a space of continuously differentiable functions with compact support in  $U_+$ . Therefore we may continue v by 0 into  $\mathbf{R}^N_+$  and obtain an element of  $H_0^1(\mathbf{R}^N_+)$ . An approximation argument shows that the chain rule can be applied to  $v|_{U^+} = (\zeta u) \circ \Phi$ . Hence in  $U^+$  with  $\chi \in C_0^{\infty}(\mathbf{R}^+_N)$ :

$$\sum_{n,m=1}^{N} a_{nm} \partial_n(\zeta u) \cdot \partial_m(\overline{\chi} \circ \Phi) = \sum_{n,m=1}^{N} a_{nm} \partial_n(v \circ \Phi) \partial_m(\overline{\chi} \circ \Phi)$$
$$= \sum_{k,l=1}^{N} \sum_{n,m=1}^{N} a_{nm} \varphi_{kn}(\partial_k v) \circ \Phi \varphi_{lm}(\partial_l \overline{\chi}) \circ \Phi \qquad (9.28)$$
$$= \sum_{k,l=1}^{N} (\tilde{b}_{kl} \partial_k v \partial_l \overline{\chi}) \circ \Phi |\det \Phi'| \quad ,$$

where

$$\Phi' = (\varphi_{ij})_{i,j=1,...,N}$$
,  $\Psi' = (\psi_{ij})_{i,j=1,...,N}$ 

are the Jacobians of  $\Phi$  resp.  $\Psi$ , and

$$\tilde{b}_{kl} = \sum_{n,m=1}^{N} |\det \Psi'| (a_{nm} \varphi_{kn} \varphi_{lm}) \circ \Psi$$

Notice that  $\tilde{b}_{kl}$  is in  $C^1(U^+)$ , is bounded, and has bounded derivatives. Moreover for  $x \in U^+$ ,  $\xi \in \mathbf{R}^N$ , and with  $y = \Psi(x)$ 

Re 
$$\sum_{k,l=1}^{N} \tilde{b}_{kl}(x)\xi_k\xi_l \ge |\det \Psi'| E |\Phi'(y)\xi|^2 \ge \tilde{E} |\xi|^2$$

with some  $\tilde{E} > 0$ .

We now redefine the  $\tilde{b}_{kl}$  in  $U^+ \setminus U(0, \rho)$  such that continuations  $b_{kl}$  into whole of  $\mathbf{R}^N_+$  exist which satisfy the assumptions of Lemma 9.2 with  $\Xi = \mathbf{R}^N_+$ . Let  $\eta \in C_0^\infty(U)$  with  $0 \le \eta \le 1$  and  $\eta = 1$  in  $U(0, \rho)$ . Put

$$b_{kl} = \begin{cases} \eta \, \tilde{b}_{kl} + (1-\eta)\delta_{kl} & \text{in} \quad U^+ \\ \delta_{kl} & \text{in} \quad \mathbf{R}^N_+ \setminus U^+ \end{cases}$$

Then the  $b_{kl}$  belong to  $C^1(\mathbf{R}^N_+)$  and satisfy (9.11), (9.12), (9.13). Since v vanishes where  $b_{kl}$  and  $\tilde{b}_{kl}$  differ from each other we get from (9.28) by a coordinate transformation for  $\chi \in C_0^{\infty}(U^+)$ 

$$\sum_{k,l=1}^{N} \left\langle b_{kl} \partial_{k} v , \partial_{l} \chi \right\rangle = \sum_{n,m=1}^{\infty} \left\langle a_{nm} \partial_{n}(\zeta u) , \partial_{m}(\chi \circ \Phi) \right\rangle,$$

and whence from (9.25) and the continuity of the pull-back operators:

$$\left| \sum \langle b_{kl} \partial_k v, \partial_l \chi \rangle \right| \le c_2 \gamma \| \chi \circ \Phi \|_{\Omega \cap V} \le c_3 \gamma \| \chi \|_{U^+} .$$
(9.29)

Because of (9.27), estimate (9.29) remains to hold for any  $\chi \in C_0^{\infty}(\mathbf{R}^N_+)$ . An application of Lemma 9.2 now proves the desired result.

# Bibliography

- [1] R. Adams, Sobolev spaces, New York, Academic Press 1975
- [2] S. Agmon, Lectures on elliptic boundary value problems, Princeton, Van Nostrand, 1965
- [3] N. Meyers and J. Serrin,  $H \equiv W$ , Proc. Nat. Acad. Sci. USA, 51 (1964), 1055–1056
- [4] M. Renardy, R. C. Rogers, An introduction to partial differential equations, New York, Springer, 1993
- [5] W. Rudin, Principles of mathematical analysis, New York, McGraw-Hill, 1964
- [6] J. Wloka, Partial Differential Equations, Cambridge, Cambridge University Press 1987