ON THE LINEAR TRANSMISSION PROBLEM FOR THE LAPLACIAN: ANALYSIS AND A POSTERIORI ERROR ESTIMATES

DIRK PAULY

Contents

1. Introduction

We continue and extend our studies from [3, 1] to a classical transmission problem for the Laplacian. Throughout this paper, let $\widetilde{\Omega} \subset \mathbb{R}^N$ be a bounded domain, i.e., an open, connected, and bounded set, let $\widehat{\Omega} := \mathbb{R}^N \setminus \overline{\widetilde{\Omega}}$ be its open complement (unbounded domain), and let $\Gamma := \partial\,\widetilde{\Omega} = \partial\,\widehat{\Omega}$ denote their common boundary/interface. We introduce

$$\mathbb{R}^N_\Gamma := \mathbb{R}^N \setminus \Gamma = \widetilde{\Omega} \,\dot{\cup}\, \widehat{\Omega}.$$

For the use of weighted Sobolev spaces we define the polynomial weight $\rho := (1 + r^2)^{1/2}$ where $r(x) = |x|$ for $x \in \mathbb{R}^N$. Until otherwise stated explicitly we assume $N \geq 3$ and *no regularity* for Γ at all.

For these introductory lines let Γ be smooth enough, and let the data

$$\widetilde{f} \in \mathsf{L}^2(\widetilde{\Omega}), \quad \rho\widehat{f} \in \mathsf{L}^2(\widehat{\Omega}), \qquad g_\Gamma \in \mathsf{H}^{1/2}(\Gamma), \quad h_\Gamma \in \mathsf{H}^{-1/2}(\Gamma),$$

and an elliptic $\Theta \in \mathrm{L}^\infty(\mathbb{R}^N_\Gamma, \mathbb{R}^{N\times N}_{\mathsf{sym}})$ be given. In other words, Θ is a reel, symmetric, bounded, and uniformly positive definite tensor/matrix field. Here we utilise standard notations for the Lebesgue and Sobolev spaces. For

$$u = \begin{cases} \widetilde{u} & \text{in } \widetilde{\Omega}, \\ \widehat{u} & \text{in } \widehat{\Omega}, \end{cases} \qquad f = \begin{cases} \widetilde{f} & \text{in } \widetilde{\Omega}, \\ \widehat{f} & \text{in } \widehat{\Omega}, \end{cases} \qquad \Theta = \begin{cases} \widetilde{\Theta} & \text{in } \widetilde{\Omega}, \\ \widehat{\Theta} & \text{in } \widehat{\Omega}, \end{cases}$$

Date: Fall 2025.
Key words and phrases. Laplacian, transmission problem, a posteriori error estimates.

where the $\widetilde{\cdot} = \cdot|_{\widetilde{\Omega}}$ and $\widehat{\cdot} = \cdot|_{\widehat{\Omega}}$ notation shall be used throughout the whole paper, we consider the *transmission problem* (TP)

$$\begin{aligned}
-\operatorname{div}\widetilde{\Theta}\nabla\widetilde{u} &= \widetilde{f} && \text{in } \widetilde{\Omega},\\
-\operatorname{div}\widehat{\Theta}\nabla\widehat{u} &= \widehat{f} && \text{in } \widehat{\Omega},\\
\widetilde{u}|_\Gamma - \widehat{u}|_\Gamma &= g_\Gamma && \text{on } \Gamma,\\
\partial_{\widetilde{\Theta}\nu}\widetilde{u}|_\Gamma - \partial_{\widehat{\Theta}\nu}\widehat{u}|_\Gamma &= h_\Gamma && \text{on } \Gamma,\\
\rho^{-1}u &\in \mathsf{L}^2(\mathbb{R}^N),
\end{aligned} \tag{1}$$

where the Neumann traces are given by

$$\begin{aligned}
\partial_{\widetilde{\Theta}\nu}\widetilde{u}|_\Gamma &= \widetilde{\Theta}\nu\cdot\nabla\widetilde{u}|_\Gamma = \nu\cdot\widetilde{\Theta}\nabla\widetilde{u}|_\Gamma,\\
\partial_{\widehat{\Theta}\nu}\widehat{u}|_\Gamma &= \widehat{\Theta}\nu\cdot\nabla\widehat{u}|_\Gamma = \nu\cdot\widehat{\Theta}\nabla\widehat{u}|_\Gamma
\end{aligned}$$

with outer unit normal ν for $\Gamma = \partial\widetilde{\Omega}$ which is the inner unit normal for $\Gamma = \partial\widehat{\Omega}$.

In future contributions we shall extend or findings also to equations of other Hilbert complexes, such as the de Rham, the elasticity, or the biharmonic complex. Out techniques are flexible and general enough to handle those systems as well. A prominent example is a transmission problem for static Maxwell's equations, such as

$$\begin{aligned}
\operatorname{rot}\widetilde{\Theta}\operatorname{rot}\widetilde{E} &= \widetilde{F} && \text{in } \widetilde{\Omega},\\
\operatorname{rot}\widehat{\Theta}\operatorname{rot}\widehat{E} &= \widehat{F} && \text{in } \widehat{\Omega},\\
\nu\times\widetilde{E}|_\Gamma\times\nu - \nu\times\widehat{E}|_\Gamma\times\nu &= G_\Gamma && \text{on } \Gamma,\\
\nu\times\widetilde{\Theta}\operatorname{rot}\widetilde{E}|_\Gamma - \nu\times\widehat{\Theta}\operatorname{rot}\widehat{E}|_\Gamma &= H_\Gamma && \text{on } \Gamma,\\
\rho^{-1}E &\in \mathsf{L}^2(\mathbb{R}^2).
\end{aligned}$$

2. Analysis

We recall the weight ρ and the geometry $\widetilde{\Omega}$, $\widehat{\Omega}$, Γ, $\mathbb{R}^N_\Gamma$ from the introduction.

2.1. **Preliminaries.** Let $\Omega \subset \mathbb{R}^N$ be open (bounded or unbounded, connected or not). We introduce the standard Lebesgue and Sobolev spaces $\mathsf{L}^2(\Omega)$, $\mathsf{H}^1(\Omega)$, $\mathsf{H}(\operatorname{div},\Omega)$, and $\mathring{\mathsf{H}}^1(\Omega)$, $\mathring{\mathsf{H}}(\operatorname{div},\Omega)$, where the latter are defined as closures of $\mathring{\mathsf{C}}^\infty(\Omega)$ (test functions) in the respective graph norms, as well as the polynomially weighted Sobolev spaces

$$\begin{aligned}
&\mathsf{L}^2_{\pm1}(\Omega) := \big\{\varphi\in\mathsf{L}^2_{\mathsf{loc}}(\Omega)\,\big|\,\rho^{\pm1}\varphi\in\mathsf{L}^2(\Omega)\big\},\\
&\mathsf{H}^1_{-1}(\Omega) := \big\{\varphi\in\mathsf{L}^2_{-1}(\Omega)\,\big|\,\nabla\varphi\in\mathsf{L}^2(\Omega)\big\}, && \mathring{\mathsf{H}}^1_{-1}(\Omega) := \overline{\mathring{\mathsf{C}}^\infty(\Omega)}^{\mathsf{H}^1_{-1}(\Omega)},\\
&\mathsf{H}_0(\operatorname{div},\Omega) := \big\{\Phi\in\mathsf{L}^2(\Omega)\,\big|\,\operatorname{div}\Phi\in\mathsf{L}^2_1(\Omega)\big\}, && \mathring{\mathsf{H}}_0(\operatorname{div},\Omega) := \overline{\mathring{\mathsf{C}}^\infty(\Omega)}^{\mathsf{H}_0(\operatorname{div},\Omega)}.
\end{aligned}$$

Moreover, we shall utilise the standard (inner and outer) scalar and normal traces

$$\begin{aligned}
&\widetilde{\operatorname{tr}}_{\mathsf{s}} : \mathsf{H}^1(\widetilde{\Omega}) \to \mathsf{H}^{1/2}(\Gamma), && \widetilde{\operatorname{tr}}_{\mathsf{n}} : \mathsf{H}(\operatorname{div},\widetilde{\Omega}) \to \mathsf{H}^{-1/2}(\Gamma),\\
&\widehat{\operatorname{tr}}_{\mathsf{s}} : \mathsf{H}^1_{-1}(\widehat{\Omega}) \to \mathsf{H}^{1/2}(\Gamma), && \widehat{\operatorname{tr}}_{\mathsf{n}} : \mathsf{H}_0(\operatorname{div},\widehat{\Omega}) \to \mathsf{H}^{-1/2}(\Gamma),
\end{aligned}$$

provided that Γ is regular enough, e.g., Lipschitz. Here we have the convention that $\widetilde{\operatorname{tr}}_{\mathsf{n}}$ uses the outer normal ν and that $\widehat{\operatorname{tr}}_{\mathsf{n}}$ uses the inner normal $-\nu$. At this point, let us also introduce the duality between $\mathsf{H}^{1/2}(\Gamma)$ and its dual $\mathsf{H}^{-1/2}(\Gamma) = \mathsf{H}^{1/2}(\Gamma)'$ by

$$\forall\,\xi\in\mathsf{H}^{-1/2}(\Gamma)\quad \eta\in\mathsf{H}^{1/2}(\Gamma)\qquad \langle\!\langle\eta,\xi\rangle\!\rangle_\Gamma := \langle\!\langle\eta,\xi\rangle\!\rangle_{\mathsf{H}^{1/2}(\Gamma),\mathsf{H}^{-1/2}(\Gamma)}.$$

As $\mathbb{R}^N_\Gamma$ has the two connected components $\widetilde{\Omega}$ and $\widehat{\Omega}$ we see

$$\begin{aligned}
\mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma) &= \big\{\phi\in\mathsf{L}^2_{-1}(\mathbb{R}^N)\,\big|\,\widetilde{\phi}\in\mathsf{H}^1(\widetilde{\Omega})\ \wedge\ \widehat{\phi}\in\mathsf{H}^1_{-1}(\widehat{\Omega})\big\},\\
\mathsf{H}_0(\operatorname{div},\mathbb{R}^N_\Gamma) &= \big\{\psi\in\mathsf{L}^2(\mathbb{R}^N)\,\big|\,\widetilde{\psi}\in\mathsf{H}(\operatorname{div},\widetilde{\Omega})\ \wedge\ \widehat{\psi}\in\mathsf{H}_0(\operatorname{div},\widehat{\Omega})\big\},
\end{aligned}$$

and hence we have the *broken differential operators*

$$\nabla\phi=\begin{cases}\nabla\widetilde{\phi} & \text{in } \widetilde{\Omega},\\ \nabla\widehat{\phi} & \text{in } \widehat{\Omega},\end{cases}\qquad \operatorname{div}\psi=\begin{cases}\operatorname{div}\widetilde{\psi} & \text{in } \widetilde{\Omega},\\ \operatorname{div}\widehat{\psi} & \text{in } \widehat{\Omega}.\end{cases}$$

Note that $\mathsf{H}^1_{-1}(\mathbb{R}^N)\subsetneq\mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma)$ and $\mathsf{H}_0(\operatorname{div},\mathbb{R}^N)\subsetneq\mathsf{H}_0(\operatorname{div},\mathbb{R}^N_\Gamma)$.

2.2. **Transmission Problem.** We follow the rationale for "a simple transmission problem" from the book of Rolf Leis [4, p. 31], see also [9, p. 85, 93, 98]. As pointed out before we are interested in the transmission problem (1) which reads as

$$\begin{aligned}-\operatorname{div}\Theta\nabla u&=f &&\text{in } \mathbb{R}^N_\Gamma,\\ \widetilde{\operatorname{tr}}_{\mathsf{s}}\widetilde{u}-\widehat{\operatorname{tr}}_{\mathsf{s}}\widehat{u}&=g_\Gamma &&\text{on } \Gamma,\\ \widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\nabla\widetilde{u}-\widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{\Theta}\nabla\widehat{u}&=h_\Gamma &&\text{on } \Gamma,\end{aligned}\tag{2}$$

with $u\in\mathsf{L}^2_{-1}(\mathbb{R}^N)$, more precisely,

$$u\in\mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma),\qquad \Theta\nabla u\in\mathsf{H}_0(\operatorname{div},\mathbb{R}^N_\Gamma).\tag{3}$$

The next remark is well known.

Remark 2.1. *Let $\varphi\in\mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma)$ and $\psi\in\mathsf{H}_0(\operatorname{div},\mathbb{R}^N_\Gamma)$. If Γ is Lipschitz then:*

- **(i)** *$\varphi\in\mathsf{H}^1_{-1}(\mathbb{R}^N)$ if and only if φ satisfies the first transmission condition of (2) with $g_\Gamma=0$, i.e.,*
$$\widetilde{\operatorname{tr}}_{\mathsf{s}}\widetilde{\varphi}=\widehat{\operatorname{tr}}_{\mathsf{s}}\widehat{\varphi}.$$
- **(ii)** *$\psi\in\mathsf{H}_0(\operatorname{div},\mathbb{R}^N)$ if and only if ψ satisfies the second transmission condition of (2) with $h_\Gamma=0$, i.e.,*
$$\widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\psi}=\widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{\psi}.$$
- **(iii)** *$\varphi\in\mathsf{H}^1_{-1}(\mathbb{R}^N)$ and $\Theta\nabla\varphi\in\mathsf{H}_0(\operatorname{div},\mathbb{R}^N)$ if and only if φ and $\Theta\nabla\varphi$ satisfy both transmission conditions of (2) with $g_\Gamma=0$ and $h_\Gamma=0$, i.e.,*
$$\widetilde{\operatorname{tr}}_{\mathsf{s}}\widetilde{\varphi}=\widehat{\operatorname{tr}}_{\mathsf{s}}\widehat{\varphi}\quad\text{and}\quad\widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\nabla\widetilde{\varphi}=\widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{\Theta}\nabla\widehat{\varphi}.$$

We emphasise that all introduced Lebesgue and Sobolev spaces are Hilbert spaces.

2.3. **Weak Formulations.** Generally, let

$$f\in\mathsf{L}^2_1(\mathbb{R}^N).$$

W.l.o.g. let us assume that g_Γ and h_Γ are given by

$$\widetilde{g}\in N(\operatorname{div}\widetilde{\Theta}\nabla,\widetilde{\Omega})\subset\mathsf{H}^1(\widetilde{\Omega}),\qquad \widetilde{h}\in\mathsf{H}(\operatorname{div},\widetilde{\Omega}),$$

i.e., $\operatorname{div}\widetilde{\Theta}\nabla\widetilde{g}=0$ in $\widetilde{\Omega}$ and

$$\widetilde{\operatorname{tr}}_{\mathsf{s}}\widetilde{g}=g_\Gamma,\quad \widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{h}=h_\Gamma,$$

where $N(\mathrm{A})$ notes the kernel of the maximal L^2-realisation of an operator A. $\widetilde{g}$ is often called harmonic or minimal norm extension. The existence of $\widetilde{g}$ and $\widetilde{h}$ is always guaranteed even if no regularity of Γ is assumed (as we do here), cf. [2]. If Γ is regular enough, e.g., Lipschitz, then the existence is well known also by classical techniques. For later purpose we set

$$g:=\begin{cases}\widetilde{g} & \text{in } \widetilde{\Omega},\\ \widehat{g}:=0 & \text{in } \widehat{\Omega},\end{cases}\qquad h:=\begin{cases}\widetilde{h} & \text{in } \widetilde{\Omega},\\ \widehat{h}:=0 & \text{in } \widehat{\Omega}.\end{cases}$$

Remark 2.2. *There is some freedom in the choice of the extensions g and h. For example, we may also pick*

$$g:=\begin{cases}0 & \text{in } \widetilde{\Omega},\\ \widehat{g} & \text{in } \widehat{\Omega},\end{cases}\qquad h:=\begin{cases}0 & \text{in } \widetilde{\Omega},\\ \widehat{h} & \text{in } \widehat{\Omega}\end{cases}$$

with $\widehat{g}\in N(\operatorname{div}\widehat{\Theta}\nabla,\widehat{\Omega})\subset\mathsf{H}^1_{-1}(\widehat{\Omega})$, $\widehat{h}\in\mathsf{H}_0(\operatorname{div},\widehat{\Omega})$ and $\widehat{\operatorname{tr}}_{\mathsf{s}}\widehat{g}=g_\Gamma$, $\widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{h}=h_\Gamma$, or combinations of both.

Now, we seek weak versions of the two Dirichlet and Neumann transmission conditions in (2) and a weak (variational) formulation for the whole system (2) and (3).

• *A weak formulation of the first (Dirichlet) transmission condition* is simply given by saying that $u \in \mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma)$ is given by

$$u = u_0 + g = \begin{cases} \widetilde{u} = u_0 + \widetilde{g} & \text{in } \widetilde{\Omega}, \\ \widehat{u} = u_0 & \text{in } \widehat{\Omega} \end{cases} \tag{4}$$

with some $u_0 \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$ since then, e.g., in case of a Lipschitz interface Γ, we have by Remark 2.1

$$\widetilde{\operatorname{tr}}_{\mathsf{s}}\widetilde{u} - \widehat{\operatorname{tr}}_{\mathsf{s}}\widehat{u} = \underbrace{\widetilde{\operatorname{tr}}_{\mathsf{s}}u_0 - \widehat{\operatorname{tr}}_{\mathsf{s}}u_0}_{=0} + \widetilde{\operatorname{tr}}_{\mathsf{s}}\widetilde{g} - \underbrace{\widehat{\operatorname{tr}}_{\mathsf{s}}\widehat{g}}_{=0} = g_\Gamma. \tag{5}$$

Hence we shall put the first transmission condition as a strong condition included in the solution space which indeed turns out to be $\mathsf{H}^1_{-1}(\mathbb{R}^N) + \{g\} \subset \mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma)$.

• *A weak version of the second (Neumann) transmission condition* can be implemented as a weak (natural) condition into a variational formulation. For this we compute, e.g., in case of a Lipschitz interface Γ, for all $\varphi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$

$$\begin{aligned}
&\langle\!\langle h_\Gamma, \widetilde{\operatorname{tr}}_{\mathsf{s}}\varphi\rangle\!\rangle_\Gamma = \langle\!\langle \widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{h}, \widetilde{\operatorname{tr}}_{\mathsf{s}}\varphi\rangle\!\rangle_\Gamma = \langle \widetilde{h}, \nabla\varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} + \langle \operatorname{div}\widetilde{h}, \varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} \\
&= \langle\!\langle \widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\nabla\widetilde{u} - \widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{\Theta}\nabla\widehat{u}, \widetilde{\operatorname{tr}}_{\mathsf{s}}\varphi\rangle\!\rangle_\Gamma \\
&= \langle\!\langle \widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\nabla\widetilde{u}, \widetilde{\operatorname{tr}}_{\mathsf{s}}\varphi\rangle\!\rangle_\Gamma - \langle\!\langle \widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{\Theta}\nabla\widehat{u}, \widehat{\operatorname{tr}}_{\mathsf{s}}\varphi\rangle\!\rangle_\Gamma \\
&= \langle \widetilde{\Theta}\nabla\widetilde{u}, \nabla\varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} + \langle \underbrace{\operatorname{div}\widetilde{\Theta}\nabla\widetilde{u}}_{=-\widetilde{f}}, \varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} + \langle \widehat{\Theta}\nabla\widehat{u}, \nabla\varphi\rangle_{\mathsf{L}^2(\widehat{\Omega})} + \langle \underbrace{\operatorname{div}\widehat{\Theta}\nabla\widehat{u}}_{=-\widehat{f}}, \varphi\rangle_{\mathsf{L}^2(\widehat{\Omega})} \\
&= \langle \Theta\nabla u, \nabla\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} - \langle f, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N}
\end{aligned}$$

since $\widehat{\operatorname{tr}}_{\mathsf{s}}\varphi = \widetilde{\operatorname{tr}}_{\mathsf{s}}\varphi$ by Remark 2.1. This computation shows that a weak formulation of the second transmission condition $\widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\nabla\widetilde{u} - \widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{\Theta}\nabla\widehat{u} = h_\Gamma$ is given by

$$\forall\,\varphi \in \mathsf{H}^1_{-1}(\mathbb{R}^N) \qquad \langle \Theta\nabla u, \nabla\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} = \langle f, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N} + \langle \widetilde{h}, \nabla\varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} + \langle \operatorname{div}\widetilde{h}, \varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})}, \tag{6}$$

which can be stated without any regularity assumptions on the interface Γ.

• Now we search for *a variational formulation* for u using (6) by the canonical ansatz (4), i.e.,

$$u := u_0 + g \in \mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma) \qquad \text{with} \qquad u_0 \in \mathsf{H}^1_{-1}(\mathbb{R}^N).$$

Then for all $\varphi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$ by (6)

$$\begin{aligned}
\langle f, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} &= \langle \Theta\nabla u, \nabla\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} - \langle \widetilde{h}, \nabla\varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} - \langle \operatorname{div}\widetilde{h}, \varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} \\
&= \langle \Theta\nabla u_0, \nabla\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} + \langle \widetilde{\Theta}\nabla\widetilde{g} - \widetilde{h}, \nabla\varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} - \langle \operatorname{div}\widetilde{h}, \varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})}.
\end{aligned}$$

Thus we are given the bounded bilinear form $B : \mathsf{H}^1_{-1}(\mathbb{R}^N) \times \mathsf{H}^1_{-1}(\mathbb{R}^N) \to \mathbb{R}$ and the bounded linear functional $F : \mathsf{H}^1_{-1}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\begin{aligned}
B(u_0, \varphi) &:= \langle \Theta\nabla u_0, \nabla\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}, \\
F(\varphi) &:= \langle f, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} + \langle \widetilde{h} - \widetilde{\Theta}\nabla\widetilde{g}, \nabla\varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} + \langle \operatorname{div}\widetilde{h}, \varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})}.
\end{aligned} \tag{7}$$

Remark 2.3. *Note that:*

(i) $\mathring{\mathsf{C}}^\infty(\mathbb{R}^N)$ *is dense in* $\mathsf{H}^1_{-1}(\mathbb{R}^N)$.

(ii) *The* $\mathsf{L}^2(\mathbb{R}^N)$*-inner product induces a canonical duality between* $\mathsf{L}^2_1(\mathbb{R}^N)$ *and its dual space* $\mathsf{L}^2_1(\mathbb{R}^N)' \cong \mathsf{L}^2_{-1}(\mathbb{R}^N)$.

Therefore, a term like $\langle f, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}$ *with* $f \in \mathsf{L}^2_1(\mathbb{R}^N)$ *and* $\varphi \in \mathsf{L}^2_{-1}(\mathbb{R}^N)$ *is to be understood as*

$$\langle f, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} = \langle \rho f, \rho^{-1}\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} = \langle\!\langle f, \varphi\rangle\!\rangle_{\mathsf{L}^2_1(\mathbb{R}^N), \mathsf{L}^2_{-1}(\mathbb{R}^N)} =: \langle\!\langle f, \varphi\rangle\!\rangle_{\mathbb{R}^N}.$$

Finally, we have peeled out the following *variational problem* replacing the transmission problem (2) and (3): Find $u_0 \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$ such that

$$\forall\, \varphi \in \mathsf{H}^1_{-1}(\mathbb{R}^N) \qquad B(u_0, \varphi) = F(\varphi). \tag{8}$$

2.4. **Solution by Variational Techniques.** According to (2), (3), (4), and (6) we introduce three solution concepts for the transmission problem (TP), a strong, a weak, and a variational one.

Definition 2.4 (Solutions of the Transmission Problem)**.** *Given u let $u_0 := u - g$.*
• *We call u a variational solution of (TP), if $u_0 \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$ solves (8).*
• *We call u a weak/strong solution of (TP), if $u \in \mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma)$ with $\Theta\nabla u \in \mathsf{H}_0(\mathrm{div}, \mathbb{R}^N_\Gamma)$, cf. (3), solves*

$$-\operatorname{div}\Theta\nabla u = f \qquad \text{in } \mathbb{R}^N_\Gamma$$

together with the two transmission conditions in (2) in the

- **(w)** *weak sense, i.e., $u_0 \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$, cf. (4), and u satisfies (6),*
- **(s)** *strong sense, i.e., $\widetilde{\mathrm{tr}}_{\mathsf{s}}\widetilde{u} - \widehat{\mathrm{tr}}_{\mathsf{s}}\widehat{u} = g_\Gamma$ holds in $\mathsf{H}^{1/2}(\Gamma)$ and $\widetilde{\mathrm{tr}}_{\mathsf{n}}\widetilde{\Theta}\nabla\widetilde{u} - \widehat{\mathrm{tr}}_{\mathsf{n}}\widehat{\Theta}\nabla\widehat{u} = h_\Gamma$ holds in $\mathsf{H}^{-1/2}(\Gamma)$, provided that $\mathsf{H}^{\pm 1/2}(\Gamma)$ are well defined.*

From [4, page 57], see also [8, 6], we get a Poincaré type estimate in the whole space:

Lemma 2.5 (Poincaré's estimate in $\mathbb{R}^N$)**.** *Let $c_N := \dfrac{2}{N-2}$. It holds*

$$\forall\, \varphi \in \mathsf{H}^1_{-1}(\mathbb{R}^N) \qquad |\varphi|_{\mathsf{L}^2_{-1}(\mathbb{R}^N)} \le c_N |\nabla\varphi|_{\mathsf{L}^2(\mathbb{R}^N)}.$$

Theorem 2.6 (V-Solution)**.** *There exists a unique variational solution of (TP). The solution operator mapping $F \mapsto u_0 = u - g$ is an isometric isomorphism between $\mathsf{H}^1_{-1}(\mathbb{R}^N)'$ and $\mathsf{H}^1_{-1}(\mathbb{R}^N)$.*

Proof. Lemma 2.5 shows that B is positive/coercive on $\mathsf{H}^1_{-1}(\mathbb{R}^N)$ since Θ is elliptic. Hence B defines an inner product for $\mathsf{H}^1_{-1}(\mathbb{R}^N)$. Riesz' representation theorem (or Lax-Milgram's lemma) yields a unique solution $u_0 \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$ of (8) with $|u_0|_{\mathsf{H}^1_{-1}(\mathbb{R}^N)} = |F|_{\mathsf{H}^1_{-1}(\mathbb{R}^N)'}$. The definition $u := u_0 + g$ shows the assertion. □

Theorem 2.7 (W-Solution)**.** *There exists a unique weak solution of (TP), and this one coincides with the variational one.*

Proof. Let $u = u_0 + g$ with $u_0 \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$ be the unique variational solution of (TP) from Theorem 2.6. Then $u \in \mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma)$ and

$$\widetilde{u} = u_0 + \widetilde{g} \in \mathsf{H}^1(\widetilde{\Omega}), \qquad \widehat{u} = u_0 \in \mathsf{H}^1_{-1}(\widehat{\Omega}).$$

We consider different test functions.
• Let $\varphi \in \mathring{\mathsf{C}}^\infty(\widehat{\Omega})$. Then

$$\langle\widehat{\Theta}\nabla u_0, \nabla\varphi\rangle_{\mathsf{L}^2(\widehat{\Omega})} = B(u_0, \varphi) = F(\varphi) = \langle\widehat{f}, \varphi\rangle_{\mathsf{L}^2(\widehat{\Omega})},$$

i.e., $\widehat{\Theta}\nabla u_0 \in \mathsf{H}_0(\mathrm{div}, \widehat{\Omega})$ and $-\operatorname{div}\widehat{\Theta}\nabla u_0 = \widehat{f}$. Note that $\widehat{u} = u = u_0$ in $\widehat{\Omega}$.
• Let $\varphi \in \mathring{\mathsf{C}}^\infty(\widetilde{\Omega})$. Then

$$\begin{aligned}\langle\widetilde{\Theta}\nabla u_0, \nabla\varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} = B(u_0, \varphi) = F(\varphi) &= \langle\widetilde{f}, \varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} - \underbrace{\langle\widetilde{\Theta}\nabla\widetilde{g}, \nabla\varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})}}_{=0} \\ &\quad + \underbrace{\langle\widetilde{h}, \nabla\varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} + \langle\operatorname{div}\widetilde{h}, \varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})}}_{=0}\end{aligned}$$

i.e., $\widetilde{\Theta}\nabla u_0 \in \mathsf{H}(\mathrm{div}, \widetilde{\Omega})$ and $-\operatorname{div}\widetilde{\Theta}\nabla u_0 = \widetilde{f}$. Note that $\widetilde{u} = u = u_0 + \widetilde{g}$ in $\widetilde{\Omega}$ and that $\widetilde{\Theta}\nabla\widetilde{g} \in \mathsf{H}(\mathrm{div}, \widetilde{\Omega})$. Hence $\widetilde{\Theta}\nabla\widetilde{u} \in \mathsf{H}(\mathrm{div}, \widetilde{\Omega})$ and

$$-\operatorname{div}\widetilde{\Theta}\nabla\widetilde{u} = -\operatorname{div}\widetilde{\Theta}\nabla u_0 - \underbrace{\operatorname{div}\widetilde{\Theta}\nabla\widetilde{g}}_{=0} = \widetilde{f}.$$

- Let $\varphi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$. Then (6) holds as

$$\begin{aligned}\langle \Theta \nabla u, \nabla \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} &= \underbrace{\langle \Theta \nabla u_0, \nabla \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}}_{= B(u_0,\varphi) = F(\varphi)} + \langle \widetilde{\Theta} \nabla \widetilde{g}, \nabla \varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} \\ &= \langle f, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} + \langle \widetilde{h}, \nabla \varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} + \langle \operatorname{div} \widetilde{h}, \varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})}.\end{aligned}$$

Thus u is a weak solution of (TP).

Now, let u with $u_0 = u - g \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$ be a weak solution of (TP). Then for all $\varphi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$ by (6)

$$\begin{aligned}B(u_0,\varphi) &= \langle \Theta \nabla u_0, \nabla \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} \\ &= \langle \Theta \nabla u, \nabla \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} - \langle \widetilde{\Theta} \nabla \widetilde{g}, \nabla \varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} \\ &= \langle f, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N} + \langle \widetilde{h} - \widetilde{\Theta} \nabla \widetilde{g}, \nabla \varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} + \langle \operatorname{div} \widetilde{h}, \varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} \\ &= F(\varphi).\end{aligned}$$

Therefore, u is the (unique) variational solution of (TP). □

Theorem 2.8 (S-Solution)**.** *Let Γ be Lipschitz. Then there exists a unique strong solution of* (TP)*, and this one coincides with the weak and the variational one.*

Proof. Let $u = u_0 + g$ with $u_0 \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$ be the unique variational/weak solution of (TP) from Theorem 2.6 and Theorem 2.7. (5) shows in $\mathsf{H}^{1/2}(\Gamma)$

$$\widetilde{\operatorname{tr}}_{\mathsf{s}} \widetilde{u} - \widehat{\operatorname{tr}}_{\mathsf{s}} \widehat{u} = g_\Gamma.$$

Moreover, using (6) we see for all $\varphi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$

$$\begin{aligned}\langle\!\langle \widetilde{\operatorname{tr}}_{\mathsf{n}} \widetilde{\Theta} \nabla \widetilde{u} - \widehat{\operatorname{tr}}_{\mathsf{n}} \widehat{\Theta} \nabla \widehat{u}, \widetilde{\operatorname{tr}}_{\mathsf{s}} \varphi\rangle\!\rangle_\Gamma &= \langle\!\langle \widetilde{\operatorname{tr}}_{\mathsf{n}} \widetilde{\Theta} \nabla \widetilde{u}, \widetilde{\operatorname{tr}}_{\mathsf{s}} \varphi\rangle\!\rangle_\Gamma - \langle\!\langle \widehat{\operatorname{tr}}_{\mathsf{n}} \widehat{\Theta} \nabla \widehat{u}, \widehat{\operatorname{tr}}_{\mathsf{s}} \varphi\rangle\!\rangle_\Gamma \\ &= \langle \widetilde{\Theta} \nabla \widetilde{u}, \nabla \varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} + \langle \widehat{\Theta} \nabla \widehat{u}, \nabla \varphi\rangle_{\mathsf{L}^2(\widehat{\Omega})} \\ &\qquad + \langle \underbrace{\operatorname{div} \widetilde{\Theta} \nabla \widetilde{u}}_{= -\widetilde{f}}, \varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} + \langle \underbrace{\operatorname{div} \widehat{\Theta} \nabla \widehat{u}}_{= -\widehat{f}}, \varphi\rangle_{\mathsf{L}^2(\widehat{\Omega})} \\ &= \langle \Theta \nabla u, \nabla \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} - \langle f, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} \\ &= \langle \widetilde{h}, \nabla \varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} + \langle \operatorname{div} \widetilde{h}, \varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} \\ &= \langle\!\langle \widetilde{\operatorname{tr}}_{\mathsf{n}} \widetilde{h}, \widetilde{\operatorname{tr}}_{\mathsf{s}} \varphi\rangle\!\rangle_\Gamma = \langle\!\langle h_\Gamma, \widetilde{\operatorname{tr}}_{\mathsf{s}} \varphi\rangle\!\rangle_\Gamma\end{aligned} \tag{9}$$

as $\widehat{\operatorname{tr}}_{\mathsf{s}} \varphi = \widetilde{\operatorname{tr}}_{\mathsf{s}} \varphi$. Hence we have $\widetilde{\operatorname{tr}}_{\mathsf{n}} \widetilde{\Theta} \nabla \widetilde{u} - \widehat{\operatorname{tr}}_{\mathsf{n}} \widehat{\Theta} \nabla \widehat{u} = h_\Gamma$ in $\mathsf{H}^{-1/2}(\Gamma)$. Therefore, u is a strong solution of (TP).

Now, let u be a strong solution of (TP). We set $u_0 = u - g$. Then as in (5)

$$\widetilde{\operatorname{tr}}_{\mathsf{s}} \widetilde{u}_0 - \widehat{\operatorname{tr}}_{\mathsf{s}} \widehat{u}_0 = \underbrace{\widetilde{\operatorname{tr}}_{\mathsf{s}} \widetilde{u} - \widehat{\operatorname{tr}}_{\mathsf{s}} \widehat{u}}_{= g_\Gamma} - \underbrace{\widetilde{\operatorname{tr}}_{\mathsf{s}} \widetilde{g}}_{= g_\Gamma} + \underbrace{\widehat{\operatorname{tr}}_{\mathsf{s}} \widehat{g}}_{= 0} = 0,$$

i.e., $u_0 \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$ by Remark 2.1. For $\varphi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$ we compute as in (9)

$$\begin{aligned}\langle \Theta \nabla u, \nabla \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} &= \langle f, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N} + \langle\!\langle \underbrace{\widetilde{\operatorname{tr}}_{\mathsf{n}} \widetilde{\Theta} \nabla \widetilde{u} - \widehat{\operatorname{tr}}_{\mathsf{n}} \widehat{\Theta} \nabla \widehat{u}}_{= h_\Gamma = \widetilde{\operatorname{tr}}_{\mathsf{n}} \widetilde{h}}, \widetilde{\operatorname{tr}}_{\mathsf{s}} \varphi\rangle\!\rangle_\Gamma \\ &= \langle f, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N} + \langle \widetilde{h}, \nabla \varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})} + \langle \operatorname{div} \widetilde{h}, \varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})},\end{aligned}$$

which is (6). Thus u is the unique weak solution of (TP). □

2.5. **Operator Theory.** The corresponding linear operator A, i.e.,

$$\begin{aligned} \mathrm{A} : \mathsf{H}^1_{-1}(\mathbb{R}^N) &\to \mathsf{H}^1_{-1}(\mathbb{R}^N)' \\ v &\mapsto \mathrm{A}\, v := B(v,\,\cdot\,) := \langle \Theta\, \nabla\, v, \nabla\,\cdot\,\rangle_{\mathsf{L}^2(\mathbb{R}^N)}, \end{aligned}$$

for the bilinear form B in (7) is an isometric isomorphism, cf. Theorem 2.6, and the variational formulation (8) reads

$$\mathrm{A}\, u_0 = F.$$

For some Hibert space H, let

$$\begin{aligned} &\mathcal{R}_{\mathsf{H}} : \mathsf{H} \to \mathsf{H}', && \phi \mapsto \mathcal{R}_{\mathsf{H}}\, \phi := \langle \phi,\,\cdot\,\rangle_{\mathsf{H}}, && \text{i.e.,} && \mathcal{R}_{\mathsf{H}}\, \phi(\varphi) = \langle \phi, \varphi\rangle_{\mathsf{H}}, \\ &\mathcal{I}_{\mathsf{H}} : \mathsf{H} \to \mathsf{H}'', && \phi \mapsto \mathcal{I}_{\mathsf{H}}\, \phi := \cdot\, \phi, && \text{i.e.,} && \mathcal{I}_{\mathsf{H}}\, \phi(\psi) = \psi(\phi), \end{aligned}$$

denote the Riesz and the reflexivity isometric isomorphisms. Note that

$$\mathcal{I}_{\mathsf{H}}\, \phi(\mathcal{R}_{\mathsf{H}}\, \varphi) = \langle \varphi, \phi\rangle_{\mathsf{H}}.$$

Moreover, let us interpret Θ as bounded and selfadjoint isomorphism $\Theta : \mathsf{L}^2(\mathbb{R}^N) \to \mathsf{L}^2(\mathbb{R}^N)$.

Remark 2.9 (A as bounded or unbounded linear operator)**.** *Note that:*

(i) $\mathrm{A} = \mathcal{R}_{\mathsf{H}^1_{-1}(\mathbb{R}^N)} : \mathsf{H}^1_{-1}(\mathbb{R}^N) \to \mathsf{H}^1_{-1}(\mathbb{R}^N)'$ *is actually the Riesz isometry if* $\mathsf{H} = \mathsf{H}^1_{-1}(\mathbb{R}^N)$ *is equipped with the weighted (half) inner product* $B = \langle \Theta\, \nabla\,\cdot\,, \nabla\,\cdot\,\rangle_{\mathsf{L}^2(\mathbb{R}^N)}$*, cf. Lemma 2.5.*

(ii) *We have* $\mathrm{A} = -\operatorname{div} \Theta\, \nabla : \mathsf{H}^1_{-1}(\mathbb{R}^N) \to \mathsf{H}^1_{-1}(\mathbb{R}^N)'$ *with*

$$\begin{aligned} \nabla : \mathsf{H}^1_{-1}(\mathbb{R}^N) &\to \mathsf{L}^2(\mathbb{R}^N), && \varphi \mapsto \nabla\, \varphi \\ \nabla' : \mathsf{L}^2(\mathbb{R}^N)' &\to \mathsf{H}^1_{-1}(\mathbb{R}^N)', && \psi \mapsto \nabla'\, \psi := \psi\, \nabla, \\ -\operatorname{div} := \nabla'\, \mathcal{R}_{\mathsf{L}^2(\mathbb{R}^N)} : \mathsf{L}^2(\mathbb{R}^N) &\to \mathsf{H}^1_{-1}(\mathbb{R}^N)', && \vartheta \mapsto -\operatorname{div} \vartheta = \nabla'\, \mathcal{R}_{\mathsf{L}^2(\mathbb{R}^N)}\, \vartheta = \langle \vartheta, \nabla\,\cdot\,\rangle_{\mathsf{L}^2(\mathbb{R}^N)}, \end{aligned}$$

where ∇' *denotes the Banach space adjoint of the gradient* ∇ *seen as bounded linear operator.* $R(\nabla)$ *and* $R(\nabla') = R(\operatorname{div}) = R(\mathrm{A})$ *are closed by Lemma 2.5 and the closed range theorem.*

(iii) *Let* $\mathcal{E}_{\mathsf{L}^2_1(\mathbb{R}^N)} : \mathsf{L}^2_1(\mathbb{R}^N) \to \mathsf{H}^1_{-1}(\mathbb{R}^N)'$ *be the bounded embedding defined by*

$$\mathcal{E}_{\mathsf{L}^2_1(\mathbb{R}^N)}\, \phi := \langle\!\langle \phi,\,\cdot\,\rangle\!\rangle_{\mathbb{R}^N} = \langle \rho\phi, \rho^{-1}\,\cdot\,\rangle_{\mathsf{L}^2(\mathbb{R}^N)} = \langle \phi,\,\cdot\,\rangle_{\mathsf{L}^2(\mathbb{R}^N)}.$$

Moreover, let $\mathrm{A}\, v = F_k$ *with* $F_k := \mathcal{E}_{\mathsf{L}^2_1(\mathbb{R}^N)}\, k = \langle k,\,\cdot\,\rangle_{\mathsf{L}^2(\mathbb{R}^N)}$ *given by some* $k \in \mathsf{L}^2_1(\mathbb{R}^N)$*. Then by defintion* $\Theta\, \nabla\, v \in \mathsf{H}_0(\operatorname{div}, \mathbb{R}^N)$ *and* $-\operatorname{div} \Theta\, \nabla\, v = k$*.*

Let us consider the unbounded but densely defined and closed linear operator

$$\nabla : \mathsf{H}^1_{-1}(\mathbb{R}^N) \subset \mathsf{L}^2_{-1}(\mathbb{R}^N) \to \mathsf{L}^2(\mathbb{R}^N), \qquad \varphi \mapsto \nabla\, \varphi.$$

Its Hilbert space adjoint is given by $-\rho^2 \operatorname{div}$*, i.e.,*

$$\begin{aligned} -\rho^2 \operatorname{div} = \nabla^* : \mathsf{H}_0(\operatorname{div}, \mathbb{R}^N) \subset \mathsf{L}^2(\mathbb{R}^N) &\to \mathsf{L}^2_{-1}(\mathbb{R}^N), && \vartheta \mapsto -\rho^2 \operatorname{div} \vartheta, \\ -\operatorname{div} = \rho^{-2}\, \nabla^* : \mathsf{H}_0(\operatorname{div}, \mathbb{R}^N) \subset \mathsf{L}^2(\mathbb{R}^N) &\to \mathsf{L}^2_1(\mathbb{R}^N), && \vartheta \mapsto -\operatorname{div} \vartheta, \end{aligned}$$

which are also densely defined and closed linear operators, since for $\vartheta \in D(\nabla^*)$

$$\forall\, \varphi \in \mathsf{H}^1_{-1}(\mathbb{R}^N) \qquad \langle \nabla\, \varphi, \vartheta\rangle_{\mathsf{L}^2(\mathbb{R}^N)} = \langle \varphi, \nabla^*\, \vartheta\rangle_{\mathsf{L}^2_{-1}(\mathbb{R}^N)} = \langle \varphi, \rho^{-2}\, \nabla^*\, \vartheta\rangle_{\mathsf{L}^2(\mathbb{R}^N)},$$

i.e., $-\operatorname{div} \vartheta = \rho^{-2}\, \nabla^*\, \vartheta$*. Hence in this case we have*

$$\mathrm{A} = -\,\mathcal{E}_{\mathsf{L}^2_1(\mathbb{R}^N)} \operatorname{div} \Theta\, \nabla = \mathcal{E}_{\mathsf{L}^2_1(\mathbb{R}^N)}\, \rho^{-2}\, \nabla^*\, \Theta\, \nabla\,.$$

As before, $R(\nabla)$ *and* $R(\nabla^*)$*,* $R(\operatorname{div})$ *are closed.*

(iv) *Alternatively, we may consider the unbounded but densely defined and closed linear operator*

$$\nabla_\Theta : \mathsf{H}^1_{-1}(\mathbb{R}^N) \subset \mathsf{L}^2_{-1}(\mathbb{R}^N) \to \mathsf{L}^2_\Theta(\mathbb{R}^N), \qquad \varphi \mapsto \nabla\, \varphi,$$

where $\mathsf{L}^2_\Theta(\mathbb{R}^N) := \mathsf{L}^2(\mathbb{R}^N)$ *is equipped with the inner product* $\langle \Theta\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2(\mathbb{R}^N)}$*. Its Hilbert space adjoint is given by* $-\rho^2 \operatorname{div}_\Theta$*, i.e., by the densely defined and closed linear operators*

$$-\rho^2 \operatorname{div}_\Theta = \nabla^*_\Theta : \mathsf{H}_0(\operatorname{div} \Theta, \mathbb{R}^N) \subset \mathsf{L}^2_\Theta(\mathbb{R}^N) \to \mathsf{L}^2_{-1}(\mathbb{R}^N), \qquad \vartheta \mapsto -\rho^2 \operatorname{div} \Theta\vartheta,$$

$$-\operatorname{div}_\Theta = \rho^{-2}\nabla_\Theta^* : \mathsf{H}_0(\operatorname{div}\Theta,\mathbb{R}^N) \subset \mathsf{L}^2_\Theta(\mathbb{R}^N) \to \mathsf{L}^2_1(\mathbb{R}^N), \qquad \vartheta \mapsto -\operatorname{div}\Theta\vartheta,$$

since for $\vartheta \in D(\nabla_\Theta^*)$

$$\forall\,\varphi \in \mathsf{H}^1_{-1}(\mathbb{R}^N) \qquad \langle \nabla\varphi, \Theta\vartheta\rangle_{\mathsf{L}^2(\mathbb{R}^N)} = \langle \nabla_\Theta\varphi, \vartheta\rangle_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}$$
$$= \langle \varphi, \nabla_\Theta^*\vartheta\rangle_{\mathsf{L}^2_{-1}(\mathbb{R}^N)} = \langle \varphi, \rho^{-2}\nabla_\Theta^*\vartheta\rangle_{\mathsf{L}^2(\mathbb{R}^N)},$$

i.e., $-\operatorname{div}\Theta\vartheta = \rho^{-2}\nabla_\Theta^*\vartheta$. *Hence* $\mathrm{A} = -\mathcal{E}_{\mathsf{L}^2_1(\mathbb{R}^N)}\operatorname{div}_\Theta\nabla_\Theta = \mathcal{E}_{\mathsf{L}^2_1(\mathbb{R}^N)}\rho^{-2}\nabla_\Theta^*\nabla_\Theta$. *Again,* $R(\nabla_\Theta)$ *and* $R(\nabla_\Theta^*)$, $R(\operatorname{div}_\Theta)$ *are closed.*

Remark 2.10 (adjoints of A)**.** *Let us compute the adjoints of* A*:*

(i) *We consider the Banach space adjoint* $\mathrm{A}' : \mathsf{H}^1_{-1}(\mathbb{R}^N)'' \to \mathsf{H}^1_{-1}(\mathbb{R}^N)'$. *For* $\varphi,\phi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$ *we have by symmetry of the inner product*

$$\mathrm{A}'\,\mathcal{I}_{\mathsf{H}^1_{-1}(\mathbb{R}^N)}\varphi(\phi) = \mathcal{I}_{\mathsf{H}^1_{-1}(\mathbb{R}^N)}\varphi(\mathrm{A}\,\phi) = \mathrm{A}\,\phi(\varphi),$$
$$\mathrm{A}\,\phi(\varphi) = -\operatorname{div}\Theta\nabla\phi(\varphi) = \nabla'\mathcal{R}_{\mathsf{L}^2(\mathbb{R}^N)}\Theta\nabla\phi(\varphi) = \mathcal{R}_{\mathsf{L}^2(\mathbb{R}^N)}\Theta\nabla\phi(\nabla\varphi)$$
$$= \underbrace{\langle \Theta\nabla\phi, \nabla\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}}_{=\langle \Theta\nabla\varphi, \nabla\phi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}} = \mathrm{A}\,\varphi(\phi),$$

i.e., the Banach space adjoint of A *is given by* $\mathrm{A}'\,\mathcal{I}_{\mathsf{H}^1_{-1}(\mathbb{R}^N)} = \mathrm{A}$, *meaning that up to the isometry* $\mathcal{I}_{\mathsf{H}^1_{-1}(\mathbb{R}^N)}$ *(reflexivity)* A *is "selfadjoint".*

(ii) *Note that* $\rho^{\pm 2}$ *are (Hilbert space) adjoint to each other, this is*

$$\rho^{-2} : \mathsf{L}^2_{-1}(\mathbb{R}^N) \to \mathsf{L}^2_1(\mathbb{R}^N), \qquad \rho^2 = (\rho^{-2})^* : \mathsf{L}^2_1(\mathbb{R}^N) \to \mathsf{L}^2_{-1}(\mathbb{R}^N),$$

as $\langle \rho^{-2}\varphi, \phi\rangle_{\mathsf{L}^2_1(\mathbb{R}^N)} = \langle \varphi, \phi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} = \langle \varphi, \rho^2\phi\rangle_{\mathsf{L}^2_{-1}(\mathbb{R}^N)}$, *i.e.,* $(\rho^{-2})^* = \rho^2$, *and the also* $(\rho^2)^* = (\rho^{-2})^{**} = \rho^{-2}$. *We conclude that*

$$-\rho^2\operatorname{div}\Theta\nabla = \nabla^*\Theta\nabla : D(\rho^2\operatorname{div}\Theta\nabla) \subset \mathsf{L}^2_{-1}(\mathbb{R}^N) \to \mathsf{L}^2_{-1}(\mathbb{R}^N)$$

is selfadjoint, and that the Hilbert space adjoint of

$$-\operatorname{div}\Theta\nabla : D(\operatorname{div}\Theta\nabla) \subset \mathsf{L}^2_{-1}(\mathbb{R}^N) \to \mathsf{L}^2_1(\mathbb{R}^N)$$

is

$$(-\operatorname{div}\Theta\nabla)^* : D\big((\operatorname{div}\Theta\nabla)^*\big) \subset \mathsf{L}^2_1(\mathbb{R}^N) \to \mathsf{L}^2_{-1}(\mathbb{R}^N)$$

given by

$$(-\operatorname{div}\Theta\nabla)^* = (\rho^{-2}\nabla^*\Theta\nabla)^* = (\nabla^*\Theta\nabla)^*(\rho^{-2})^* = \nabla^*\Theta\nabla\rho^2 = -\rho^2\operatorname{div}\Theta\nabla\rho^2.$$

(iii) *We have that*

$$-\rho^2\operatorname{div}_\Theta\nabla_\Theta = \nabla_\Theta^*\nabla_\Theta : D(\rho^2\operatorname{div}_\Theta\nabla_\Theta) \subset \mathsf{L}^2_{-1}(\mathbb{R}^N) \to \mathsf{L}^2_{-1}(\mathbb{R}^N)$$

is selfadjoint, and that the Hilbert space adjoint of

$$-\operatorname{div}_\Theta\nabla_\Theta : D(\operatorname{div}_\Theta\nabla_\Theta) \subset \mathsf{L}^2_{-1}(\mathbb{R}^N) \to \mathsf{L}^2_1(\mathbb{R}^N)$$

is

$$(-\operatorname{div}_\Theta\nabla_\Theta)^* : D\big((\operatorname{div}_\Theta\nabla_\Theta)^*\big) \subset \mathsf{L}^2_1(\mathbb{R}^N) \to \mathsf{L}^2_{-1}(\mathbb{R}^N)$$

given by

$$(-\operatorname{div}_\Theta\nabla_\Theta)^* = (\rho^{-2}\nabla_\Theta^*\nabla_\Theta)^* = \nabla_\Theta^*\nabla_\Theta\rho^2 = -\rho^2\operatorname{div}_\Theta\nabla_\Theta\rho^2.$$

By Theorem 2.6 there is a unique variational solution u of (TP), i.e., (8), with

$$u = u_0 + g \in \mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma), \qquad \Theta\nabla u \in \mathsf{H}_0(\operatorname{div},\mathbb{R}^N_\Gamma), \qquad u_0 = \mathrm{A}^{-1}F \in \mathsf{H}^1_{-1}(\mathbb{R}^N),$$

cf. (3), and u coincides with the weak and strong one (if it exists). From Remark 2.9 we known that

$$\mathrm{A} = \mathcal{R}_{\mathsf{H}^1_{-1}(\mathbb{R}^N)} = -\operatorname{div}\Theta\nabla : \mathsf{H}^1_{-1}(\mathbb{R}^N) \to \mathsf{H}^1_{-1}(\mathbb{R}^N)'$$

is an isometric isomorphism (Riesz isometry). Our problem (TP) splits linearly into four parts, namely

$$\mathrm{A}\,u_0(\varphi)=B(u_0,\varphi)=F(\varphi)=\underbrace{\langle f,\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}}_{=:\,F_f(\varphi)}\underbrace{-\langle\widetilde{\Theta}\,\nabla\,\widetilde{g},\nabla\,\varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})}}_{=:\,F_{\nabla\,\widetilde{g}}(\varphi)}+\underbrace{\langle\widetilde{h},\nabla\,\varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})}}_{=:\,F_{\widetilde{h}}(\varphi)}+\underbrace{\langle\operatorname{div}\widetilde{h},\varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})}}_{=:\,F_{\operatorname{div}\widetilde{h}}(\varphi)}$$

with $F, F_f, F_{\nabla\,\widetilde{g}}, F_{\widetilde{h}}, F_{\operatorname{div}\widetilde{h}}\in\mathsf{H}^1_{-1}(\mathbb{R}^N)'$ and

$$\begin{aligned}
F_f(\varphi)&=\langle f,\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}=\langle\!\langle f,\varphi\rangle\!\rangle_{\mathbb{R}^N}=\mathcal{E}_{\mathsf{L}^2_1(\mathbb{R}^N)}\,f(\varphi),\\
F_{\nabla\,\widetilde{g}}(\varphi)&=-\langle\Theta\,\nabla\,g,\nabla\,\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}=\operatorname{div}\Theta\,\nabla\,g(\varphi),\\
F_{\widetilde{h}}(\varphi)&=\langle h,\nabla\,\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}=-\operatorname{div}h(\varphi),\\
F_{\operatorname{div}\widetilde{h}}(\varphi)&=\langle\operatorname{div}h,\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}=\langle\!\langle\operatorname{div}h,\varphi\rangle\!\rangle_{\mathbb{R}^N}=\mathcal{E}_{\mathsf{L}^2_1(\mathbb{R}^N)}\operatorname{div}h(\varphi).
\end{aligned}$$

We have $f,\ \operatorname{div}h\in\mathsf{L}^2_1(\mathbb{R}^N)$ and

$$\mathcal{E}_{\mathsf{L}^2_1(\mathbb{R}^N)}\,f,\ \mathcal{E}_{\mathsf{L}^2_1(\mathbb{R}^N)}\operatorname{div}h,\ \operatorname{div}\Theta\,\nabla\,g,\ \operatorname{div}h\in\mathsf{H}^1_{-1}(\mathbb{R}^N)'.$$

Let us consider the partial solutions $g\in\mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma)$ and

$$\begin{aligned}
u_{0,f}&:=\mathrm{A}^{-1}F_f\in\mathsf{H}^1_{-1}(\mathbb{R}^N), & u_{0,\nabla\,\widetilde{g}}&:=\mathrm{A}^{-1}F_{\nabla\,\widetilde{g}}\in\mathsf{H}^1_{-1}(\mathbb{R}^N),\\
u_{0,\operatorname{div}\widetilde{h}}&:=\mathrm{A}^{-1}F_{\operatorname{div}\widetilde{h}}\in\mathsf{H}^1_{-1}(\mathbb{R}^N), & u_{0,\widetilde{h}}&:=\mathrm{A}^{-1}F_{\widetilde{h}}\in\mathsf{H}^1_{-1}(\mathbb{R}^N).
\end{aligned}$$

Theorem 2.11 (Splitting of Solution)**.** *The partial solutions sum up to the unique solution*

$$\begin{aligned}
u&=u_0+g\in\mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma),\\
u_0&=u_{0,f}+u_{0,\nabla\,\widetilde{g}}+u_{0,\operatorname{div}\widetilde{h}}+u_{0,\widetilde{h}}\in\mathsf{H}^1_{-1}(\mathbb{R}^N),
\end{aligned}$$

and it holds

$$\begin{aligned}
\mathrm{A}\,u_0=F&=F_f+F_{\nabla\,\widetilde{g}}+F_{\operatorname{div}\widetilde{h}}+F_{\widetilde{h}}\\
&=\mathcal{E}_{\mathsf{L}^2_1(\mathbb{R}^N)}\,f+\operatorname{div}\Theta\,\nabla\,g+\mathcal{E}_{\mathsf{L}^2_1(\mathbb{R}^N)}\operatorname{div}h-\operatorname{div}h
\end{aligned}$$

with

$$F(\varphi)=\langle f,\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}-\langle\widetilde{\Theta}\,\nabla\,\widetilde{g},\nabla\,\varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})}+\langle\operatorname{div}\widetilde{h},\varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})}+\langle\widetilde{h},\nabla\,\varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})},$$

cf. (7)*. Moreover, the partial solutions solve the five transmission problems*

$$\begin{aligned}
-\operatorname{div}\Theta\,\nabla\,g&=0 && \text{in } \mathbb{R}^N_\Gamma,\\
\widetilde{\operatorname{tr}}_{\mathsf{s}}\widetilde{g}-\widehat{\operatorname{tr}}_{\mathsf{s}}\widehat{g}&=g_\Gamma && \text{on } \Gamma,\\
\widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\,\nabla\,\widetilde{g}-\widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{\Theta}\,\nabla\,\widehat{g}&=\widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\,\nabla\,\widetilde{g} && \text{on } \Gamma,
\end{aligned}$$

$$\begin{aligned}
-\operatorname{div}\Theta\,\nabla\,u_{0,f}&=f, & -\operatorname{div}\Theta\,\nabla\,u_{0,\operatorname{div}\widetilde{h}}&=\operatorname{div}h && \text{in } \mathbb{R}^N_\Gamma,\\
\widetilde{\operatorname{tr}}_{\mathsf{s}}\widetilde{u}_{0,f}-\widehat{\operatorname{tr}}_{\mathsf{s}}\widehat{u}_{0,f}&=0, & \widetilde{\operatorname{tr}}_{\mathsf{s}}\widetilde{u}_{0,\operatorname{div}\widetilde{h}}-\widehat{\operatorname{tr}}_{\mathsf{s}}\widehat{u}_{0,\operatorname{div}\widetilde{h}}&=0 && \text{on } \Gamma,\\
\widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\,\nabla\,\widetilde{u}_{0,f}-\widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{\Theta}\,\nabla\,\widehat{u}_{0,f}&=0, & \widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\,\nabla\,\widetilde{u}_{0,\operatorname{div}\widetilde{h}}-\widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{\Theta}\,\nabla\,\widehat{u}_{0,\operatorname{div}\widetilde{h}}&=0 && \text{on } \Gamma,
\end{aligned}$$

$$\begin{aligned}
-\operatorname{div}\Theta\,\nabla\,u_{0,\nabla\,\widetilde{g}}&=0, & -\operatorname{div}\Theta\,\nabla\,u_{0,\widetilde{h}}&=-\operatorname{div}h && \text{in } \mathbb{R}^N_\Gamma,\\
\widetilde{\operatorname{tr}}_{\mathsf{s}}\widetilde{u}_{0,\nabla\,\widetilde{g}}-\widehat{\operatorname{tr}}_{\mathsf{s}}\widehat{u}_{0,\nabla\,\widetilde{g}}&=0, & \widetilde{\operatorname{tr}}_{\mathsf{s}}\widetilde{u}_{0,\widetilde{h}}-\widehat{\operatorname{tr}}_{\mathsf{s}}\widehat{u}_{0,\widetilde{h}}&=0 && \text{on } \Gamma,\\
\widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\,\nabla\,\widetilde{u}_{0,\nabla\,\widetilde{g}}-\widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{\Theta}\,\nabla\,\widehat{u}_{0,\nabla\,\widetilde{g}}&=-\widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\,\nabla\,\widetilde{g}, & \widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\,\nabla\,\widetilde{u}_{0,\widetilde{h}}-\widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{\Theta}\,\nabla\,\widehat{u}_{0,\widetilde{h}}&=h_\Gamma && \text{on } \Gamma.
\end{aligned}$$

For $u_{g_\Gamma}:=u_{0,\nabla\,\widetilde{g}}+g\in\mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma)$ *and* $u_{h_\Gamma}:=u_{0,\widetilde{h}}+u_{0,\operatorname{div}\widetilde{h}}\in\mathsf{H}^1_{-1}(\mathbb{R}^N)$ *it holds*

$$u=u_{0,f}+u_{g_\Gamma}+u_{h_\Gamma}\in\mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma)$$

and

$$\begin{aligned}
-\operatorname{div}\Theta\,\nabla\,u_{g_\Gamma}&=0, & -\operatorname{div}\Theta\,\nabla\,u_{h_\Gamma}&=0 && \text{in } \mathbb{R}^N_\Gamma,\\
\widetilde{\operatorname{tr}}_{\mathsf{s}}\widetilde{u}_{g_\Gamma}-\widehat{\operatorname{tr}}_{\mathsf{s}}\widehat{u}_{g_\Gamma}&=g_\Gamma, & \widetilde{\operatorname{tr}}_{\mathsf{s}}\widetilde{u}_{h_\Gamma}-\widehat{\operatorname{tr}}_{\mathsf{s}}\widehat{u}_{h_\Gamma}&=0 && \text{on } \Gamma,
\end{aligned}$$

$$\widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\nabla\widetilde{u}_{g_\Gamma}-\widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{\Theta}\nabla\widehat{u}_{g_\Gamma}=0,\qquad \widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\nabla\widetilde{u}_{h_\Gamma}-\widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{\Theta}\nabla\widehat{u}_{h_\Gamma}=h_\Gamma\qquad \textit{on } \Gamma.$$

2.6. **Helmholtz Decomposition.** We recall Remark 2.9(iv) and consider the unbounded but densely defined and closed linear operator ∇_Θ together with its Hilbert space adjoint ∇_Θ^* and $-\operatorname{div}_\Theta=\rho^{-2}\nabla_\Theta^*$. By Lemma 2.5 and the projection theorem we have closed ranges and Helmholtz type decompositions in the whole space.

Lemma 2.12 (Closed Range of the Gradient)**.** *The range*

$$R(\nabla_\Theta)=\nabla\,\mathsf{H}^1_{-1}(\mathbb{R}^N)=\overline{\nabla\,\mathsf{H}^1(\mathbb{R}^N)}$$

is closed in $\mathsf{L}^2_\Theta(\mathbb{R}^N)$.

Lemma 2.13 (Helmholtz Decomposition in $\mathbb{R}^N$)**.** *The orthogonal decomposition*

$$\mathsf{L}^2_\Theta(\mathbb{R}^N)=R(\nabla_\Theta)\oplus_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}N(-\rho^2\operatorname{div}_\Theta)=\nabla\,\mathsf{H}^1_{-1}(\mathbb{R}^N)\oplus_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}N(\operatorname{div}_\Theta)$$

holds with corresponding orthonormal projectors

$$\pi_\nabla:\mathsf{L}^2_\Theta(\mathbb{R}^N)\to\nabla\,\mathsf{H}^1_{-1}(\mathbb{R}^N),\qquad \pi_0:\mathsf{L}^2_\Theta(\mathbb{R}^N)\to N(\operatorname{div}_\Theta).$$

2.7. **The Case N=2.** For $\Omega\subset\mathbb{R}^2$ open (bounded or unbounded, connected or not) we introduce the weighted Sobolev spaces

$$\mathsf{L}^2_{\pm\ln}(\Omega):=\Big\{\varphi\in\mathsf{L}^2_{\mathsf{loc}}(\Omega)\;\Big|\;\big(\frac{\rho}{2}\ln(e+r^2)\big)^{\pm1}\varphi\in\mathsf{L}^2(\Omega)\Big\},$$

$$\mathsf{H}^1_{-\ln}(\Omega):=\big\{\varphi\in\mathsf{L}^2_{-\ln}(\Omega)\;\big|\;\nabla\varphi\in\mathsf{L}^2(\Omega)\big\},\qquad \mathring{\mathsf{H}}^1_{-\ln}(\Omega):=\overline{\mathring{\mathsf{C}}^\infty(\Omega)}^{\mathsf{H}^1_{-\ln}(\Omega)},$$

$$\mathsf{H}_0(\operatorname{div},\Omega):=\big\{\Phi\in\mathsf{L}^2(\Omega)\;\big|\;\operatorname{div}\Phi\in\mathsf{L}^2_{\ln}(\Omega)\big\},\qquad \mathring{\mathsf{H}}_0(\operatorname{div},\Omega):=\overline{\mathring{\mathsf{C}}^\infty(\Omega)}^{\mathsf{H}_0(\operatorname{div},\Omega)}.$$

Note that constants belong to $\mathsf{H}^1_{-\ln}(\Omega)\subset\mathsf{L}^2_{-\ln}(\Omega)$ even is Ω is unbounded.

Lemma 2.14 (Poincaré's estimate in $\mathbb{R}^2$)**.** *There exists* $c_2>0$ *such that*

$$\forall\,\varphi\in\mathsf{H}^1_{-\ln}(\mathbb{R}^2)\qquad |\varphi|_{\mathsf{L}^2_{-\ln}(\mathbb{R}^2)}\le c_2\Big(|\nabla\varphi|_{\mathsf{L}^2(\mathbb{R}^2)}+\big|\langle\varphi,1\rangle_{\mathsf{L}^2_{-\ln}(\mathbb{R}^2)}\big|\Big).$$

Proof. From [8, 6] we cite a Poincaré type estimate in the exterior of the unit ball $\Xi:=\mathbb{R}^2\setminus\overline{B_1}$

$$\forall\,\phi\in\mathring{\mathsf{H}}^1_{-\ln}(\Xi)\qquad |\phi|_{\mathsf{L}^2_{-\ln}(\Xi)}\le\Big|\frac{\phi}{r\ln r}\Big|_{\mathsf{L}^2(\Xi)}\le 2|\nabla\phi|_{\mathsf{L}^2(\Xi)}.\tag{10}$$

Let $\varphi\in\mathsf{H}^1_{-\ln}(\mathbb{R}^2)$ and $\xi\in\mathring{\mathsf{C}}^\infty(B_3)$ with $\xi|_{B_2}=1$. Then $(1-\xi)\in\mathring{\mathsf{C}}^\infty(\Xi)$ with $(1-\xi)|_{\Xi\setminus B_3}=1$. Hence $\xi\varphi\in\mathring{\mathsf{H}}^1(B_3)$ and $(1-\xi)\varphi\in\mathring{\mathsf{H}}^1_{-\ln}(\Xi)$. By the standard Friedrichs estimate and (10) we compute

$$\begin{aligned}|\varphi|_{\mathsf{L}^2_{-\ln}(\mathbb{R}^2)}\le c|\xi\varphi|_{\mathsf{L}^2(B_3)}+c\big|(1-\xi)\varphi\big|_{\mathsf{L}^2_{-\ln}(\Xi)}&\le c\big|\nabla(\xi\varphi)\big|_{\mathsf{L}^2(B_3)}+c\Big|\nabla\big((1-\xi)\varphi\big)\Big|_{\mathsf{L}^2(\Xi)}\\&\le c|\nabla\varphi|_{\mathsf{L}^2(\mathbb{R}^2)}+c|\varphi|_{\mathsf{L}^2(\operatorname{supp}\nabla\xi)}.\end{aligned}\tag{11}$$

Now, let us assume that the inequality of the lemma is wrong. Then there exists a sequence $(\varphi_n)\subset\mathsf{H}^1_{-\ln}(\mathbb{R}^2)$ such that $|\varphi_n|_{\mathsf{L}^2_{-\ln}(\mathbb{R}^2)}=1$ and $|\nabla\varphi_n|_{\mathsf{L}^2(\mathbb{R}^2)}+\big|\langle\varphi_n,1\rangle_{\mathsf{L}^2_{-\ln}(\mathbb{R}^2)}\big|<1/n$. Rellich's selection theorem yields a subsequence (again denoted by) (φ_n) being a Cauchy sequence in $\mathsf{L}^2(\operatorname{supp}\nabla\xi)$. By (11) (φ_n) is a Cauchy sequence in $\mathsf{L}^2_{-\ln}(\mathbb{R}^2)$. Hence $\varphi_n\to\varphi$ in $\mathsf{H}^1_{-\ln}(\mathbb{R}^2)$. Thus $\nabla\varphi=0$, i.e., φ is constant, and $\langle\varphi,1\rangle_{\mathsf{L}^2_{-\ln}(\mathbb{R}^2)}=0$, i.e., $\varphi=0$. Finally (11) shows

$$1=|\varphi_n|_{\mathsf{L}^2_{-\ln}(\mathbb{R}^2)}\le c|\nabla\varphi_n|_{\mathsf{L}^2(\mathbb{R}^2)}+c|\varphi_n|_{\mathsf{L}^2(\operatorname{supp}\nabla\xi)}\to 0,$$

a contradiction. □

From now on, in principle, all arguments from Section 2.3 to Section 2.6 carry over from the case $N\ge3$ just by replacing Lemma 2.5 by Lemma 2.14 and exchanging the respective Sobolev spaces. There is a slight change. Due to

$$N(\nabla)=\mathbb{R}\subset\mathsf{H}^1_{-\ln}(\mathbb{R}^2),$$

where $\nabla : \mathsf{H}^1_{-\ln}(\mathbb{R}^2) \to \mathsf{L}^2(\mathbb{R}^2)$, the non-trivial eigenspace requires Fredholm's alternative for a proper solution theory. In this case we have

$$\mathrm{A} = -\operatorname{div}\Theta\nabla : \mathsf{H}^1_{-\ln}(\mathbb{R}^2) \to \mathsf{H}^1_{-\ln}(\mathbb{R}^2)', \quad v \mapsto \mathrm{A}\, v := B(v, \cdot\,) := \langle \Theta\nabla v, \nabla\,\cdot\,\rangle_{\mathsf{L}^2(\mathbb{R}^N)}$$

with kernel $N(\mathrm{A}) = \mathbb{R}$ and closed range $R(\mathrm{A})$ since $R(\nabla)$ is closed by Lemma 2.14, cf. Remark 2.9(ii). More precisely, the range is given as annihilator, this is

$$R(\mathrm{A}) = N(\mathrm{A}')^\circ = N(\mathrm{A}\,\mathcal{I}^{-1}_{\mathsf{H}^1_{-\ln}(\mathbb{R}^2)})^\circ = (\mathcal{I}_{\mathsf{H}^1_{-\ln}(\mathbb{R}^2)}\,\mathbb{R})^\circ, \tag{12}$$

cf. Remark 2.10(i). Hence for $G \in \mathsf{H}^1_{-\ln}(\mathbb{R}^2)'$

$$G \in R(\mathrm{A}) \qquad \Leftrightarrow \qquad 0 = \mathcal{I}_{\mathsf{H}^1_{-1}(\mathbb{R}^N)}(1)G = G(1).$$

Let $f \in \mathsf{L}^2_{\ln}(\mathbb{R}^2)$. By Theorem 2.11 we obtain for our special $F \in \mathsf{H}^1_{-\ln}(\mathbb{R}^2)'$ the constraint

$$\begin{aligned} F \in R(\mathrm{A}) \quad \Leftrightarrow \quad 0 = F(1) &= \langle f, 1\rangle_{\mathsf{L}^2(\mathbb{R}^2)} - \langle \widetilde{\Theta}\nabla\widetilde{g}, \nabla 1\rangle_{\mathsf{L}^2(\widetilde{\Omega})} + \langle \widetilde{h}, \nabla 1\rangle_{\mathsf{L}^2(\widetilde{\Omega})} + \langle \operatorname{div}\widetilde{h}, 1\rangle_{\mathsf{L}^2(\widetilde{\Omega})} \\ &= \langle f, 1\rangle_{\mathsf{L}^2(\mathbb{R}^2)} + \langle \operatorname{div}\widetilde{h}, 1\rangle_{\mathsf{L}^2(\widetilde{\Omega})}, \end{aligned}$$

which reads in the smooth case as

$$\langle\!\langle f, 1\rangle\!\rangle_{\mathbb{R}^2} + \langle\!\langle h_\Gamma, 1\rangle\!\rangle_\Gamma = 0,$$

and classically, if $h_\Gamma \in \mathsf{L}^2(\Gamma)$,

$$\int_{\mathbb{R}^2} f + \int_\Gamma h_\Gamma = 0.$$

Lemma 2.15 (Solution Operator)**.** *For any $G \in \mathsf{H}^1_{-\ln}(\mathbb{R}^2)'$ such that $G(1) = 0$ there exists a unique $u_0 \in \mathsf{H}^1_{-\ln,\perp}(\mathbb{R}^2) := \mathsf{H}^1_{-\ln}(\mathbb{R}^2) \cap \mathbb{R}^{\perp_{\mathsf{L}^2_{-\ln}(\mathbb{R}^2)}}$ solving $\mathrm{A}\,u_0 = G$. The solution operator*

$$\mathcal{A}^{-1} : R(\mathrm{A}) \to \mathsf{H}^1_{-\ln,\perp}(\mathbb{R}^2),$$

cf. (12)*, is an isometric isomorphism.*

Proof. Lemma 2.14 shows that B is positive/coercive on $\mathsf{H}^1_{-\ln,\perp}(\mathbb{R}^2)$ since Θ is elliptic. Hence B defines an inner product for $\mathsf{H}^1_{-\ln,\perp}(\mathbb{R}^2)$. Riesz' representation theorem yields a unique solution $u_0 \in \mathsf{H}^1_{-\ln,\perp}(\mathbb{R}^2)$ of (8) with $|u_0|_{\mathsf{H}^1_{-\ln}(\mathbb{R}^2)} = |G|_{\mathsf{H}^1_{-\ln}(\mathbb{R}^2)'}$ for any $G \in R(\mathrm{A})$. □

Setting $u := u_0 + g$ shows:

Theorem 2.16 (V-W-S-Solution)**.** *Let $f \in \mathsf{L}^2_{\ln}(\mathbb{R}^2)$ and let g, h and F be defined as in Section 2.3. Moreover, let $\langle f, 1\rangle_{\mathsf{L}^2(\mathbb{R}^2)} + \langle \operatorname{div}\widetilde{h}, 1\rangle_{\mathsf{L}^2(\widetilde{\Omega})} = 0$. Then there exists a unique variational/weak solution u of* (TP) *with $u - g \in \mathsf{H}^1_{-\ln,\perp}(\mathbb{R}^2)$, and these two coincide. More precisely, the solution operator $\mathcal{A}^{-1}$ mapping $F \mapsto u_0 = u - g$ is an isometric isomorphism, and it holds $\langle u_0, 1\rangle_{\mathsf{L}^2_{-\ln}(\mathbb{R}^2)} = 0$.*

If Γ is Lipschitz, then there exists a unique strong solution of (TP)*, and this one coincides with the weak and the variational one.*

All operator theoretical results and remarks from Section 2.5 and Section 2.6 remain true with obvious modifications.

3. A Posteriori Error Equalities and Estimates

Our a posteriori error estimates rely on the pioneering work of Sergey Repin during the last decades, see, e.g., his books [5, 7].

We introduce the ellipticity constant $c_\Theta > 0$ and the trace constant $c_\Gamma > 0$ for the exterior domain $\widehat{\Omega}$ such that

$$\begin{aligned} &\forall\, \tau \in \mathsf{L}^2(\mathbb{R}^N) &\qquad& |\tau|_{\mathsf{L}^2(\mathbb{R}^N)} \leq c_\Theta |\tau|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}, \\ &\forall\, \varphi \in \mathsf{H}^1_{-1}(\widehat{\Omega}) && |\widehat{\operatorname{tr}}_{\mathsf{s}}\varphi|_{\mathsf{H}^{1/2}(\Gamma)} \leq c_\Gamma |\nabla\varphi|_{\mathsf{L}^2(\widehat{\Omega})}. \end{aligned}$$

Note that $|\nabla\,\cdot\,|_{\mathsf{L}^2(\widehat{\Omega})}$ is a norm for $\mathsf{H}^1_{-1}(\widehat{\Omega})$ due to Lemma 2.5. Moroever, we set

$$c_{N,\Theta} := c_N c_\Theta, \qquad c_{\Gamma,\Theta} := c_\Gamma c_\Theta.$$

Remark 3.1. *Again, there is some freedom in the choice of the trace operator and the correspond trace constant, cf. Remark 2.2. We may also pick*

$$\forall\,\varphi\in\mathsf{H}^1_{-1}(\widetilde{\Omega})\qquad |\widetilde{\mathrm{tr}}_{\mathsf{s}}\varphi|_{\mathsf{H}^{1/2}(\Gamma)}\leq c_\Gamma|\varphi|_{\mathsf{H}^1(\widetilde{\Omega})}\leq\widetilde{c}_\Gamma\Big(|\nabla\varphi|_{\mathsf{L}^2(\widetilde{\Omega})}+\big|\langle\varphi,1\rangle_{\mathsf{L}^2(\widetilde{\Omega})}\big|\Big)$$

with different constants c_Γ and $\widetilde{c}_\Gamma$.

Let us consider generally *non-conforming approximations*

$$v\in\mathsf{L}^2_{-1}(\mathbb{R}^N),\qquad \sigma\in\mathsf{L}^2_\Theta(\mathbb{R}^N)$$

of the unique exact solution u and its gradient, respectively. We shall call

$$e:=u-v=u_0+g-v,\qquad \eta:=\nabla u-\sigma=\nabla u_0+\nabla g-\sigma$$

the *scalar error* and the *gradient error*, respectively.

According to Lemma 2.13 we decompose the error term orthogonally into

$$\mathsf{L}^2_\Theta(\mathbb{R}^N)\ni\eta=\eta_\nabla\oplus\eta_0,\qquad \eta_\nabla\in\nabla\mathsf{H}^1_{-1}(\mathbb{R}^N),\quad \eta_0\in N(\mathrm{div}_\Theta),$$

more precisely, $\eta_\nabla=\nabla e_0$ with $e_0\in\mathsf{H}^1_{-1}(\mathbb{R}^N)$ and $\eta_0\in\mathsf{H}_0(\mathrm{div}_\Theta,\mathbb{R}^N)$ with $\mathrm{div}\,\Theta\eta_0=0$.

3.1. **Upper Bound.** Let $\varphi\in\mathsf{H}^1_{-1}(\mathbb{R}^N)$ and $\Theta\tau\in\mathsf{H}_0(\mathrm{div},\mathbb{R}^N_\Gamma)$. We have by orthogonality

$$\begin{aligned}\langle\nabla e_0,\nabla\varphi\rangle_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}&=\langle\eta,\nabla\varphi\rangle_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}\\&=\underbrace{\langle\Theta\nabla u_0,\nabla\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}}_{B(u_0,\varphi)=F(\varphi)}+\langle\widetilde{\Theta}\nabla\widetilde{g},\nabla\varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})}-\langle\sigma,\nabla\varphi\rangle_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}\\&=\langle f,\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}+\langle\widetilde{h},\nabla\varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})}+\langle\mathrm{div}\,\widetilde{h},\varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})}-\langle\sigma,\nabla\varphi\rangle_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}\\&=\langle f+\mathrm{div}\,\Theta\tau,\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}+\big\langle\pi_\nabla(\tau-\sigma),\nabla\varphi\big\rangle_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}\\&\qquad+\langle h-\Theta\tau,\nabla\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}+\langle\mathrm{div}\,h-\mathrm{div}\,\Theta\tau,\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}.\end{aligned}\tag{13}$$

If Γ is smooth enough, e.g., Lipschitz, we see (note that $\widehat{\mathrm{tr}}_{\mathsf{s}}\varphi=\widetilde{\mathrm{tr}}_{\mathsf{s}}\varphi$)

$$\begin{aligned}&\langle h-\Theta\tau,\nabla\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}+\langle\mathrm{div}\,h-\mathrm{div}\,\Theta\tau,\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}\\&=\langle\widetilde{h}-\widetilde{\Theta}\widetilde{\tau},\nabla\varphi\rangle_{\mathsf{L}^2(\widetilde{\Omega})}+\big\langle\mathrm{div}(\widetilde{h}-\widetilde{\Theta}\widetilde{\tau}),\varphi\big\rangle_{\mathsf{L}^2(\widetilde{\Omega})}-\langle\widehat{\Theta}\widehat{\tau},\nabla\varphi\rangle_{\mathsf{L}^2(\widehat{\Omega})}-\langle\mathrm{div}\,\widehat{\Theta}\widehat{\tau},\varphi\rangle_{\mathsf{L}^2(\widehat{\Omega})}\\&=\langle\!\langle\widetilde{\mathrm{tr}}_{\mathsf{n}}\widetilde{h}-\widetilde{\mathrm{tr}}_{\mathsf{n}}\widetilde{\Theta}\widetilde{\tau},\widetilde{\mathrm{tr}}_{\mathsf{s}}\varphi\rangle\!\rangle_\Gamma+\langle\!\langle\widehat{\mathrm{tr}}_{\mathsf{n}}\widehat{\Theta}\widehat{\tau},\widehat{\mathrm{tr}}_{\mathsf{s}}\varphi\rangle\!\rangle_\Gamma\\&=\big\langle\!\big\langle h_\Gamma-(\widetilde{\mathrm{tr}}_{\mathsf{n}}\widetilde{\Theta}\widetilde{\tau}-\widehat{\mathrm{tr}}_{\mathsf{n}}\widehat{\Theta}\widehat{\tau}),\widetilde{\mathrm{tr}}_{\mathsf{s}}\varphi\big\rangle\!\big\rangle_\Gamma,\end{aligned}\tag{14}$$

and hence by Lemma 2.5

$$\begin{aligned}&\langle\nabla e_0,\nabla\varphi\rangle_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}\\&=\langle f+\mathrm{div}\,\Theta\tau,\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}+\big\langle\pi_\nabla(\tau-\sigma),\nabla\varphi\big\rangle_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}+\big\langle\!\big\langle h_\Gamma-(\widetilde{\mathrm{tr}}_{\mathsf{n}}\widetilde{\Theta}\widetilde{\tau}-\widehat{\mathrm{tr}}_{\mathsf{n}}\widehat{\Theta}\widehat{\tau}),\widetilde{\mathrm{tr}}_{\mathsf{s}}\varphi\big\rangle\!\big\rangle_\Gamma\\&\leq|f+\mathrm{div}\,\Theta\tau|_{\mathsf{L}^2_1(\mathbb{R}^N)}\underbrace{|\varphi|_{\mathsf{L}^2_{-1}(\mathbb{R}^N)}}_{\leq c_N|\nabla\varphi|_{\mathsf{L}^2(\mathbb{R}^N)}}+\big|\pi_\nabla(\tau-\sigma)\big|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}|\nabla\varphi|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}\\&\qquad+\big|h_\Gamma-(\widetilde{\mathrm{tr}}_{\mathsf{n}}\widetilde{\Theta}\widetilde{\tau}-\widehat{\mathrm{tr}}_{\mathsf{n}}\widehat{\Theta}\widehat{\tau})\big|_{\mathsf{H}^{-1/2}(\Gamma)}\underbrace{|\widehat{\mathrm{tr}}_{\mathsf{s}}\varphi|_{\mathsf{H}^{1/2}(\Gamma)}}_{\leq c_\Gamma|\nabla\varphi|_{\mathsf{L}^2(\widehat{\Omega})}\leq c_\Gamma|\nabla\varphi|_{\mathsf{L}^2(\mathbb{R}^N)}}\\&\leq\mathcal{M}_{+,\nabla}(\sigma,\tau)\,|\nabla\varphi|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)},\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}_{+,\nabla}(\sigma,\tau):=c_{N,\Theta}|f+\mathrm{div}\,\Theta\tau|_{\mathsf{L}^2_1(\mathbb{R}^N)}+\big|\pi_\nabla(\tau-\sigma)\big|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}\\+c_{\Gamma,\Theta}\big|h_\Gamma-(\widetilde{\mathrm{tr}}_{\mathsf{n}}\widetilde{\Theta}\widetilde{\tau}-\widehat{\mathrm{tr}}_{\mathsf{n}}\widehat{\Theta}\widehat{\tau})\big|_{\mathsf{H}^{-1/2}(\Gamma)}.\end{aligned}\tag{15}$$

Again by orthogonality we see

$$\eta_0=\pi_0\eta=\pi_0(\nabla u_0+\nabla g-\sigma)=\pi_0(\nabla g-\sigma).$$

Setting $\varphi = e_0$ we get:

Theorem 3.2 (Upper Bound for Non-Conforming Approximations)**.** *Let* Γ *be Lipschitz and let* $\sigma \in \mathsf{L}^2_\Theta(\mathbb{R}^N)$ *be a non-conforming approximation of the broken gradient* ∇u *of the exact solution* $u \in \mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma)$*. Then*

$$\eta = \nabla u - \sigma = \nabla u_0 + \nabla g - \sigma = \eta_\nabla \oplus \eta_0$$

with $\eta_\nabla = \pi_\nabla \eta = \nabla e_0$*,* $e_0 \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$ *and* $\eta_0 = \pi_0 \eta = \pi_0(\nabla g - \sigma) \in N(\operatorname{div}_\Theta)$*. Moreover,*

$$|\eta|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} = |\eta_\nabla|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} + |\eta_0|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}, \qquad |\eta_\nabla|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} = \min_{\Theta\tau \in \mathsf{H}_0(\operatorname{div},\mathbb{R}^N_\Gamma)} \mathcal{M}_{+,\nabla}(\sigma,\tau),$$

where the upper bound $\mathcal{M}_{+,\nabla}$ *is given by* (15)*. The minimum is attained at* $\tau = \nabla u$*.*

Note that for $\psi \in N(\operatorname{div}_\Theta)$ and $\varphi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$ we have $\pi_\nabla \psi = 0$ and $\pi_0 \nabla \varphi = 0$. Hence

$$\begin{aligned} \big|\pi_\nabla(\tau - \sigma)\big|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} &\le \big|\psi + \tau - \sigma\big|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}, \\ \big|\pi_0(\nabla g - \sigma)\big|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} &\le |\nabla \varphi + \nabla g - \sigma|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} =: \mathcal{M}_{+,0}(\sigma,\varphi). \end{aligned} \tag{16}$$

By $\pi_\nabla + \pi_0 = \mathrm{id}$ we obtain:

Theorem 3.3 (Upper Bound for Non-Conforming Approximations)**.** *Let* Γ *be Lipschitz. Then*

$$\begin{aligned} \big|\pi_\nabla(\tau - \sigma)\big|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} &= \min_{\psi \in N(\operatorname{div}_\Theta)} \big|\psi + \tau - \sigma\big|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}, \\ |\eta_0|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} = \big|\pi_0(\nabla g - \sigma)\big|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} &= \min_{\varphi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)} \mathcal{M}_{+,0}(\sigma,\varphi), \end{aligned}$$

where the upper bound $\mathcal{M}_{+,0}$ *is given by* (16)*. The minima are attained at* $\psi = \pi_0(\sigma - \tau)$ *and* $\nabla \varphi = \pi_\nabla(\sigma - \nabla g)$*, respectively.*

3.2. **Lower Bound.** Our lower bound is given by the simple fact, that in any Hilbert space H it holds

$$|x|^2_{\mathsf{H}} = \max_{y \in \mathsf{H}} \big(2\langle x, y\rangle_{\mathsf{H}} - |y|^2_{\mathsf{H}}\big).$$

For $\mathsf{H} = \nabla \mathsf{H}^1_{-1}(\mathbb{R}^N)$ resp. $\mathsf{H} = N(\operatorname{div}_\Theta)$ we get

$$\begin{aligned} |\eta_\nabla|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} &= \max_{\varphi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)} \Big(2\langle \nabla e_0, \nabla \varphi\rangle_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} - |\nabla \varphi|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}\Big), \\ |\eta_0|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} &= \max_{\psi \in N(\operatorname{div}_\Theta)} \Big(2\langle \eta_0, \psi\rangle_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} - |\psi|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}\Big). \end{aligned}$$

Recall the situation in Theorem 3.2. For $\varphi, \phi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$ and $\Theta\tau \in \mathsf{H}_0(\operatorname{div}, \mathbb{R}^N_\Gamma)$ we have by (13) and orthogonality

$$\begin{aligned} \langle \eta_\nabla, \nabla \varphi\rangle_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} &= \langle f + \operatorname{div}\Theta\tau, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} + \big\langle \pi_\nabla(\tau - \sigma), \nabla \varphi\big\rangle_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} \\ &\qquad + \langle h - \Theta\tau, \nabla \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} + \langle \operatorname{div} h - \operatorname{div}\Theta\tau, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}, \\ \langle \eta_0, \psi\rangle_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} &= \langle \eta, \psi\rangle_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} = \langle \nabla \phi + \nabla g - \sigma, \psi\rangle_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}. \end{aligned}$$

We conclude:

Theorem 3.4 (Lower Bound for Non-Conforming Approximations)**.** *Let* $\sigma \in \mathsf{L}^2_\Theta(\mathbb{R}^N)$ *be a non-conforming approximation of the broken gradient* ∇u *of the exact solution* $u \in \mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma)$*. Then*

$$\eta = \nabla u - \sigma = \nabla u_0 + \nabla g - \sigma = \eta_\nabla \oplus \eta_0$$

with $\eta_\nabla = \pi_\nabla \eta = \nabla e_0 \in \nabla \mathsf{H}^1_{-1}(\mathbb{R}^N)$ *and* $\eta_0 = \pi_0 \eta = \pi_0(\nabla g - \sigma) \in N(\operatorname{div}_\Theta)$*. Moreover,*

$$|\eta|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} = |\eta_\nabla|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} + |\eta_0|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}$$

with

$$|\eta_\nabla|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} = \max_{\varphi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)} \mathcal{M}_{-,\nabla}(\sigma,\varphi,\tau), \qquad |\eta_0|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} = \max_{\psi \in N(\operatorname{div}_\Theta)} \mathcal{M}_{-,0}(\sigma,\psi,\phi)$$

for all $\Theta\tau \in \mathsf{H}_0(\operatorname{div}, \mathbb{R}^N_\Gamma)$ *and all* $\phi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$*, where*

$$\begin{aligned}\mathcal{M}_{-,\nabla}(\sigma,\varphi,\tau) &:= 2\Big(\langle f + \operatorname{div}\Theta\tau, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} + \langle \tau - \sigma - \frac{1}{2}\nabla\varphi, \nabla\varphi\rangle_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} \\ &\qquad + \langle h - \Theta\tau, \nabla\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} + \langle \operatorname{div} h - \operatorname{div}\Theta\tau, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)}\Big), \\ \mathcal{M}_{-,0}(\sigma,\psi,\phi) &:= 2\langle \nabla\phi + \nabla g - \sigma - \frac{1}{2}\psi, \psi\rangle_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}.\end{aligned}$$

The minima are attained at $\varphi = e_0$ *and* $\psi = \eta_0$*.*

Remark 3.5 (Lower Bound for Non-Conforming Approximations). *If* Γ *is Lipschitz,* (14) *yields*

$$\langle h - \Theta\tau, \nabla\varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} + \langle \operatorname{div} h - \operatorname{div}\Theta\tau, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} = \big\langle\!\big\langle h_\Gamma - (\widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\widetilde{\tau} - \widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{\Theta}\widehat{\tau}), \widetilde{\operatorname{tr}}_{\mathsf{s}}\varphi\big\rangle\!\big\rangle_\Gamma.$$

3.3. **Two-Sided Bounds.** A combination of Theorem 3.2, Theorem 3.3, and Theorem 3.4 reads a follows:

Corollary 3.6 (Two-Sided Bounds for Non-Conforming Approximations). *Let* Γ *be Lipschitz and let* $\sigma \in \mathsf{L}^2_\Theta(\mathbb{R}^N)$ *be a non-conforming approximation of the broken gradient* ∇u *of the exact solution* $u \in \mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma)$*. Then*

$$\eta = \nabla u - \sigma = \nabla u_0 + \nabla g - \sigma = \eta_\nabla \oplus \eta_0$$

with $\eta_\nabla = \pi_\nabla\eta = \nabla e_0 \in \nabla\mathsf{H}^1_{-1}(\mathbb{R}^N)$ *and* $\eta_0 = \pi_0\eta = \pi_0(\nabla g - \sigma) \in N(\operatorname{div}_\Theta)$*. Moreover,*

$$|\eta|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} = |\eta_\nabla|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} + |\eta_0|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}$$

with

$$\begin{aligned}\max_{\varphi\in\mathsf{H}^1_{-1}(\mathbb{R}^N)} \mathcal{M}_{-,\nabla}(\sigma,\varphi,\vartheta) &= |\eta_\nabla|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} = \min_{\Theta\tau\in\mathsf{H}_0(\operatorname{div},\mathbb{R}^N_\Gamma)} \mathcal{M}^2_{+,\nabla}(\sigma,\tau), \\ \max_{\psi\in N(\operatorname{div}_\Theta)} \mathcal{M}_{-,0}(\sigma,\psi,\phi) &= |\eta_0|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} = \min_{\xi\in\mathsf{H}^1_{-1}(\mathbb{R}^N)} \mathcal{M}^2_{+,0}(\sigma,\xi)\end{aligned}$$

for all $\Theta\vartheta \in \mathsf{H}_0(\operatorname{div}, \mathbb{R}^N_\Gamma)$ *and all* $\phi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$*, where* $\mathcal{M}_{+,\nabla}$*,* $\mathcal{M}_{+,0}$ *are given by* (15), (16) *and* $\mathcal{M}_{-,\nabla}$*,* $\mathcal{M}_{-,0}$ *by Theorem 3.4. The minima are attained at* $\tau = \nabla u$ *and* $\nabla\xi = \pi_\nabla(\sigma - \nabla g)$*, and the maxima at* $\varphi = e_0$ *and* $\psi = \eta_0$*. Furthermore, the term* $\big|\pi_\nabla(\tau - \sigma)\big|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}$ *of* $\mathcal{M}_{+,\nabla}(\sigma,\tau)$ *can be handled by Theorem 3.3.*

We note that the solenoidal part $\eta_0 = \pi_0(\nabla g - \sigma)$ of the gradient error η measures the error of the approximation of the Dirichlet transmission boundary data g_Γ only. On the other hand, the gradient part $\eta_\nabla = \nabla e_0$ of the gradient error η measures the error of the approximation of the volume data f and the Neumann transmission boundary data h_Γ. To illustrate this we observe (at least formally, if σ is not regular enough)

$$\begin{aligned}-\operatorname{div}\Theta\nabla e_0 &= f + \operatorname{div}\Theta\sigma, & -\operatorname{div}\Theta\eta_0 &= 0 & &\text{in } \mathbb{R}^N_\Gamma, \\ \widetilde{\operatorname{tr}}_{\mathsf{s}}\widetilde{e}_0 - \widehat{\operatorname{tr}}_{\mathsf{s}}\widehat{e}_0 &= 0, & & & &\text{on } \Gamma, \\ \widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\nabla\widetilde{e}_0 - \widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{\Theta}\nabla\widehat{e}_0 &= h_\Gamma - (\widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\widetilde{\sigma} - \widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{\Theta}\widehat{\sigma}), & \widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\widetilde{\eta}_0 - \widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{\Theta}\widehat{\eta}_0 &= 0 & &\text{on } \Gamma.\end{aligned}$$

More precisely, we have:

Lemma 3.7. *Let* Γ *be Lipschitz. For the error components* $\eta = \eta_\nabla + \eta_0$ *it holds:*

(i) *If* $\Theta\sigma \in \mathsf{H}_0(\operatorname{div}, \mathbb{R}^N_\Gamma)$*, then*

$$\begin{aligned}|\eta_\nabla|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} &\le c_{N,\Theta}|f + \operatorname{div}\Theta\sigma|_{\mathsf{L}^2_1(\mathbb{R}^N)} \\ &\qquad + c_{\Gamma,\Theta}\big|h_\Gamma - (\widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\widetilde{\sigma} - \widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{\Theta}\widehat{\sigma})\big|_{\mathsf{H}^{-1/2}(\Gamma)}, \\ |\eta_\nabla|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} &\ge 2\langle f + \operatorname{div}\Theta\sigma, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} - |\nabla\varphi|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} \\ &\qquad + 2\big\langle\!\big\langle h_\Gamma - (\widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\widetilde{\sigma} - \widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{\Theta}\widehat{\sigma}), \widetilde{\operatorname{tr}}_{\mathsf{s}}\varphi\big\rangle\!\big\rangle_\Gamma\end{aligned}$$

for all $\varphi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$*. Thus* η_∇ *is controlled by* $f + \operatorname{div}\Theta\sigma$ *and* $h_\Gamma - (\widetilde{\operatorname{tr}}_{\mathsf{n}}\widetilde{\Theta}\widetilde{\sigma} - \widehat{\operatorname{tr}}_{\mathsf{n}}\widehat{\Theta}\widehat{\sigma})$*.*

(ii) *If $\sigma = \nabla v$ with $v \in \mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma)$, then*

$$|\eta_0|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} \leq \check{c}_{\Gamma,\Theta}\big|g_\Gamma - (\widetilde{\mathrm{tr}}_{\mathsf{s}}\widetilde{v} - \widehat{\mathrm{tr}}_{\mathsf{s}}\widehat{v})\big|_{\mathsf{H}^{1/2}(\Gamma)},$$
$$|\eta_0|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} \geq \big\langle\!\big\langle g_\Gamma - (\widetilde{\mathrm{tr}}_{\mathsf{s}}\widetilde{v} - \widehat{\mathrm{tr}}_{\mathsf{s}}\widehat{v}), \widetilde{\mathrm{tr}}_{\mathsf{n}}\widetilde{\Theta}\widetilde{\psi}\big\rangle\!\big\rangle_\Gamma - |\psi|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}$$

for all $\psi \in N(\mathrm{div}_\Theta)$. Thus η_0 is controlled by $g_\Gamma - (\widetilde{\mathrm{tr}}_{\mathsf{s}}\widetilde{v} - \widehat{\mathrm{tr}}_{\mathsf{s}}\widehat{v})$.

Here $\check{c}_{\Gamma,\Theta}$ denotes the continuity constant of a right inverse

$$\check{\mathrm{tr}}_{\mathsf{s}} : \mathsf{H}^{1/2}(\Gamma) \to \big\{\zeta \in \mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma) \,\big|\, \widetilde{\zeta} \in \mathsf{H}^1(\widetilde{\Omega}) \wedge \widehat{\zeta} = 0\big\}$$

corresponding to the scalar trace $\mathrm{tr}_{\mathsf{s}} : \mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma) \to \mathsf{H}^{1/2}(\Gamma)$ with $\mathrm{tr}_{\mathsf{s}}\,\zeta = \widetilde{\mathrm{tr}}_{\mathsf{s}}\widetilde{\zeta}$. More precisely,

$$\forall\, \lambda \in \mathsf{H}^{1/2}(\Gamma) \qquad |\check{\mathrm{tr}}_{\mathsf{s}}\lambda|^2_{\mathsf{L}^2(\widetilde{\Omega})} + |\nabla\,\check{\mathrm{tr}}_{\mathsf{s}}\lambda|^2_{\mathsf{L}^2_\Theta(\widetilde{\Omega})} \leq \check{c}^2_{\Gamma,\Theta}|\lambda|^2_{\mathsf{H}^{1/2}(\Gamma)}.$$

Proof. Let $\Theta\sigma \in \mathsf{H}_0(\mathrm{div}, \mathbb{R}^N_\Gamma)$. In Corollary 3.6 we choose $\tau = \vartheta = \sigma$ and get for the terms $\mathcal{M}_{\pm,\nabla}$ estimatting $|\eta_\nabla|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}$

$$\mathcal{M}_{+,\nabla}(\sigma,\sigma) = c_{N,\Theta}|f + \mathrm{div}\,\Theta\sigma|_{\mathsf{L}^2_1(\mathbb{R}^N)} + c_{\Gamma,\Theta}\big|h_\Gamma - (\widetilde{\mathrm{tr}}_{\mathsf{n}}\widetilde{\Theta}\widetilde{\sigma} - \widehat{\mathrm{tr}}_{\mathsf{n}}\widehat{\Theta}\widehat{\sigma})\big|_{\mathsf{H}^{-1/2}(\Gamma)},$$
$$\mathcal{M}_{-,\nabla}(\sigma,\varphi,\sigma) = 2\langle f + \mathrm{div}\,\Theta\sigma, \varphi\rangle_{\mathsf{L}^2(\mathbb{R}^N)} - |\nabla\varphi|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} + 2\big\langle\!\big\langle h_\Gamma - (\widetilde{\mathrm{tr}}_{\mathsf{n}}\widetilde{\Theta}\widetilde{\sigma} - \widehat{\mathrm{tr}}_{\mathsf{n}}\widehat{\Theta}\widehat{\sigma}), \widetilde{\mathrm{tr}}_{\mathsf{s}}\varphi\big\rangle\!\big\rangle_\Gamma.$$

Let $\sigma = \nabla v$ with some $v \in \mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma)$. We set

$$\xi := v - g + \check{\mathrm{tr}}_{\mathsf{s}}\big(g_\Gamma - (\widetilde{\mathrm{tr}}_{\mathsf{s}}\widetilde{v} - \widehat{\mathrm{tr}}_{\mathsf{s}}\widehat{v})\big) \in \mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma).$$

Then

$$\widetilde{\mathrm{tr}}_{\mathsf{s}}\widetilde{\xi} - \widehat{\mathrm{tr}}_{\mathsf{s}}\widehat{\xi} = \widetilde{\mathrm{tr}}_{\mathsf{s}}\widetilde{v} - \widehat{\mathrm{tr}}_{\mathsf{s}}\widehat{v} - \widetilde{\mathrm{tr}}_{\mathsf{s}}\widetilde{g} + g_\Gamma - (\widetilde{\mathrm{tr}}_{\mathsf{s}}\widetilde{v} - \widehat{\mathrm{tr}}_{\mathsf{s}}\widehat{v}) = 0$$

and hence $\xi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$ by Remark 2.1(i). We conclude for the terms $\mathcal{M}_{\pm,0}$ estimatting $|\eta_0|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}$

$$\begin{aligned}\mathcal{M}_{+,0}(\nabla v,\xi) &= \underbrace{|\nabla(\xi + g - v)|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}}_{= \left|\nabla\,\check{\mathrm{tr}}_{\mathsf{s}}\left(g_\Gamma - (\widetilde{\mathrm{tr}}_{\mathsf{s}}\widetilde{v} - \widehat{\mathrm{tr}}_{\mathsf{s}}\widehat{v})\right)\right|_{\mathsf{L}^2_\Theta(\widetilde{\Omega})}} \leq \check{c}_{\Gamma,\Theta}\big|g_\Gamma - (\widetilde{\mathrm{tr}}_{\mathsf{s}}\widetilde{v} - \widehat{\mathrm{tr}}_{\mathsf{s}}\widehat{v})\big|_{\mathsf{H}^{1/2}(\Gamma)},\\ \mathcal{M}_{-,0}(\nabla v,\psi,\phi) &= 2\underbrace{\langle\nabla(\phi + g - v),\psi\rangle_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}}_{= \langle\!\langle g_\Gamma - (\widetilde{\mathrm{tr}}_{\mathsf{s}}\widetilde{v} - \widehat{\mathrm{tr}}_{\mathsf{s}}\widehat{v}), \widetilde{\mathrm{tr}}_{\mathsf{n}}\widetilde{\Theta}\widetilde{\psi}\rangle\!\rangle_\Gamma} - |\psi|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}.\end{aligned}$$

Note that $\phi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$ and $\psi \in N(\mathrm{div}_\Theta) \subset \mathsf{H}_0(\mathrm{div}\,\Theta, \mathbb{R}^N)$ which implies $\widetilde{\mathrm{tr}}_{\mathsf{n}}\widetilde{\Theta}\widetilde{\psi} = \widehat{\mathrm{tr}}_{\mathsf{n}}\widehat{\Theta}\widehat{\psi}$ by Remark 2.1(ii). □

For (unrealistic) conforming approximations we have:

Corollary 3.8 (Two-Sided Bounds for Conforming Approximations)**.** *Let Γ be Lipschitz and let v be a conforming approximation of the exact solution $u \in \mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma)$, i.e., $v \in \mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma)$ with $\Theta\nabla v \in \mathsf{H}_0(\mathrm{div}, \mathbb{R}^N_\Gamma)$ and $v - g \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$. Then $e = u - v \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$ with $\nabla e = \pi_\nabla\eta = \eta$ and*

$$\max_{\varphi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)} \mathcal{M}_{-,\nabla}(\nabla v,\varphi,\vartheta) = |\eta|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} = \min_{\Theta\tau \in \mathsf{H}_0(\mathrm{div},\mathbb{R}^N_\Gamma)} \mathcal{M}^2_{+,\nabla}(\nabla v,\tau)$$

for all $\Theta\vartheta \in \mathsf{H}_0(\mathrm{div}, \mathbb{R}^N_\Gamma)$, where $\mathcal{M}_{+,\nabla}$ is given by (15) *and $\mathcal{M}_{-,\nabla}$ by Theorem 3.4. The minimum is attained at $\tau = \nabla u$ and the maximum at $\varphi = e_0$. Moreover, the term $\big|\pi_\nabla(\tau - \nabla v)\big|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}$ of $\mathcal{M}_{+,\nabla}(\nabla v,\tau)$ can be handled by Theorem 3.3.*

Proof. Set $\sigma := \nabla v$. We see $e = u - v = u_0 + g - v \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$ and note $\eta = \nabla(u - v) = \nabla e$. Hence $\pi_\nabla\eta = \eta$ and $\pi_0\eta = 0$. Theorem 3.2 and Theorem 3.3 show the assertions. □

3.4. **Two-Sided Bounds for Hybrid Approximations.** A reasonable numerical approximation, in particular, for BEM-FEM couplings, is a hybrid of conforming and non-conforming approximations. A realistic scenario features

$$\widehat{\Theta} = 1, \qquad \widehat{f} = 0 \tag{17}$$

with an approximation v satisfying

$$v \in \mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma), \qquad \nabla \widehat{v} \in \mathsf{H}_0(\mathrm{div}, \widehat{\Omega}). \tag{18}$$

Hence we may set $\sigma := \nabla v \in \mathsf{L}^2_\Theta(\mathbb{R}^N)$. Then by Corollary 3.6:

Corollary 3.9 (Two-Sided Bounds for Hybrid Approximations). *Let Γ be Lipschitz and assume* (17)*. Moreover, let v be a hybrid approximation of the exact solution u, i.e.,* (18) *holds. Let $\sigma := \nabla v$. Then*

$$\eta = \nabla e = \nabla(u - v) = \nabla u_0 + \nabla(g - v) = \eta_\nabla \oplus \eta_0$$

with $\eta_\nabla = \pi_\nabla \eta = \nabla e_0 \in \nabla \mathsf{H}^1_{-1}(\mathbb{R}^N)$ and $\eta_0 = \pi_0 \eta = \pi_0 \nabla(g - v) \in N(\mathrm{div}_\Theta)$. Moreover,

$$|\eta|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} = |\eta_\nabla|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} + |\eta_0|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}$$

with

$$\max_{\varphi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)} \mathcal{M}_{-,\nabla}(\nabla v, \varphi, \vartheta) = |\eta_\nabla|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} = \min_{\Theta\tau \in \mathsf{H}_0(\mathrm{div}, \mathbb{R}^N_\Gamma)} \mathcal{M}^2_{+,\nabla}(\nabla v, \tau),$$

$$\max_{\psi \in N(\mathrm{div}_\Theta)} \mathcal{M}_{-,0}(\nabla v, \psi, \phi) = |\eta_0|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} = \min_{\xi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)} \mathcal{M}^2_{+,0}(\nabla v, \xi) = \min_{\substack{\zeta \in \mathsf{H}^1_{-1}(\mathbb{R}^N_\Gamma),\\ \widetilde{\mathrm{tr}}_{\mathsf{s}}\widetilde{\zeta} - \widehat{\mathrm{tr}}_{\mathsf{s}}\widehat{\zeta} = g_\Gamma - (\widetilde{\mathrm{tr}}_{\mathsf{s}}\widetilde{v} - \widehat{\mathrm{tr}}_{\mathsf{s}}\widehat{v})}} |\nabla \zeta|^2_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}$$

for all $\Theta\vartheta \in \mathsf{H}_0(\mathrm{div}, \mathbb{R}^N_\Gamma)$ and all $\phi \in \mathsf{H}^1_{-1}(\mathbb{R}^N)$, where $\mathcal{M}_{+,\nabla}$, $\mathcal{M}_{+,0}$ are given by (15)*,* (16) *and $\mathcal{M}_{-,\nabla}$, $\mathcal{M}_{-,0}$ by Theorem 3.4. The minima are attained at $\tau = \nabla u$ and $\nabla \xi = \pi_\nabla \nabla(v - g)$, $\zeta = e - e_0$, and the maxima at $\varphi = e_0$ and $\psi = \eta_0$. Furthermore, the term $\big|\pi_\nabla(\tau - \nabla v)\big|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}$ of $\mathcal{M}_{+,\nabla}(\nabla v, \tau)$ can be handled by Theorem 3.3.*

- **(i)** *If $\mathrm{div}\,\widehat{\tau} = 0$, then $f + \mathrm{div}\,\Theta\tau = 0$ in $\widehat{\Omega}$.*
- **(ii)** *If $\widehat{\tau} = \nabla \widehat{v}$, then $\tau - \nabla v = 0$ in $\widehat{\Omega}$.*
- **(iii)** *If $\widehat{\tau} = \nabla \widehat{v}$ and $\Delta \widehat{v} = 0$, then $\mathrm{div}\,\widehat{\tau} = \Delta \widehat{v} = 0$ and the assertions of* (i) *and* (ii) *hold.*

In particular, if (i) *and* (ii) *hold, then*

$$\mathcal{M}_{+,\nabla}(\nabla v, \tau) = c_{N,\Theta}\big|\rho(\widetilde{f} + \mathrm{div}\,\widetilde{\Theta}\widetilde{\tau})\big|_{\mathsf{L}^2(\widetilde{\Omega})} + \big|\pi_\nabla(\tau - \nabla v)\big|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}$$
$$+ c_{\Gamma,\Theta}\big|h_\Gamma - (\widetilde{\mathrm{tr}}_{\mathsf{n}}\widetilde{\Theta}\widetilde{\tau} - \widehat{\mathrm{tr}}_{\mathsf{n}}\widehat{\Theta}\widehat{\tau})\big|_{\mathsf{H}^{-1/2}(\Gamma)},$$

$$\big|\pi_\nabla(\tau - \nabla v)\big|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)} \leq \inf_{\widetilde{\psi} \in N(\mathring{\mathrm{div}}_{\widetilde{\Theta}})} |\widetilde{\psi} + \widetilde{\tau} - \nabla \widetilde{v}|_{\mathsf{L}^2_{\widetilde{\Theta}}(\widetilde{\Omega})},$$

since for ψ with $\widehat{\psi} = 0$ we have $\psi \in N(\mathrm{div}_\Theta) \Leftrightarrow \widetilde{\psi} \in N(\mathring{\mathrm{div}}_{\widetilde{\Theta}})$, and $\mathcal{M}_{-,\nabla}$ and $\mathcal{M}_{\pm,0}$ can be modified as well.

Remark 3.10 (Two-Sided Bounds for Hybrid Approximations). *While the error bounds $\mathcal{M}_{\pm,\nabla}$ and $\mathcal{M}_{\pm,0}$ reflect the original error bounds of Repin, cf.* [7, Chapter 3.2]*, the upper bound*

$$\min_\zeta |\nabla \zeta|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}$$

for $|\eta_0|_{\mathsf{L}^2_\Theta(\mathbb{R}^N)}$ corresponds to the new upper bound based on Dirichlet's principle for harmonic approximations presented in [3, (6), Theorem 4]*, which is particularly suited for BEM. In view of Lemma 3.7 this term estimates the error of the approximation of the Dirichlet transmission boundary data g_Γ.*

Acknowledgement

We cordially thank Sergey Igorevich Repin (St. Petersburg Department of the Steklov Mathematical Institute of the Russian Academy of Sciences) for bringing our attention to the so-called functional a posteriori error estimates, on which this work is based. We are deeply grateful for his friendship and enduring support for so many years.

References

[1] A. Freiszlinger, D. Pauly, and D. Praetorius. Convergence of adaptive boundary element methods driven by functional a posteriori error estimates. *arXiv,* `https://arxiv.org/abs/2506.10499`, 2025.

[2] R. Hiptmair, D. Pauly, and E. Schulz. Traces for Hilbert complexes. *J. Funct. Anal.*, 284(10):50pp, 2023.

[3] S. Kurz, D. Pauly, D. Praetorius, S. Repin, and D. Sebastian. Functional a posteriori error estimates for boundary element methods. *Numer. Math.*, 147(4):937–966, 2021.

[4] R. Leis. *Initial Boundary Value Problems in Mathematical Physics.* Teubner, Stuttgart, 1986.

[5] P. Neittaanmäki and S. Repin. *Reliable methods for computer simulation, error control and a posteriori estimates.* Elsevier, New York, 2004.

[6] D. Pauly and S. Repin. Functional a posteriori error estimates for elliptic problems in exterior domains. *J. Math. Sci. (N.Y.)*, 162(3):393–406, 2009.

[7] S. Repin. *A posteriori estimates for partial differential equations.* Walter de Gruyter (Radon Series Comp. Appl. Math.), Berlin, 2008.

[8] J. Saranen and K.-J. Witsch. Exterior boundary value problems for elliptic equations. *Ann. Acad. Sci. Fenn. Math.*, 8(1):3–42, 1983.

[9] S.A. Sauter and C. Schwab. *Boundary element methods*, volume 39 of *Springer Ser. Comput. Math.* Berlin: Springer, 2011.

Institut für Analysis, Technische Universität Dresden, Germany
Email address, Dirk Pauly: `dirk.pauly@tu-dresden.de`