FRIEDRICHS/GAFFNEY/POINCARÉ ESTIMATES FOR MAXWELL'S EQUATIONS IN PARTIALLY BOUNDED DOMAINS

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Dedicated to my Teacher and Dear Friend Rainer Picard on the occasion of his 80th birthday

ABSTRACT. We prove Friedrichs/Gaffney/Maxwell/Poincaré estimates and related results in partially bounded domains Ω for ∇ , rot, and div, cf. Theorem 3.17. Our main finding is that

 $\mathrm{rot}:\mathsf{H}(\mathrm{rot},\Omega)\subset\mathsf{L}^2(\Omega)\to\mathsf{L}^2(\Omega),\qquad \mathring{\mathrm{rot}}:\mathring{\mathsf{H}}(\mathrm{rot},\Omega)\subset\mathsf{L}^2(\Omega)\to\mathsf{L}^2(\Omega)$

have closed range for any domain $\Omega \subset \mathbb{R}^3$ being Lipschitz diffeomorphic to the unbounded cylinder (bounded in two directions)

$$\Omega_2 = \mathbb{R} \times (0, d)^2.$$

Remarkably (bigger domain and apparently weaker boundary condition), even for Ω being Lipschitz diffeomorphic to (bounded in just one direction)

$$\Omega_1 = \mathbb{R}^2 \times (0, d)$$

we obtain a result for mixed boundary conditions, namely that rot has closed range if the tangential boundary condition is prescribed on just one part corresponding to, e.g., $\mathbb{R}^2 \times \{0\}$.

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1. INTRODUCTION

Throughout this paper let $\Omega \subset \mathbb{R}^3$ be a domain (connected and open set).

With standard notation we introduce the densely defined and closed (unbounded) linear operators

$$\begin{split} \mathring{\mathrm{rot}} &: D(\mathring{\mathrm{rot}}) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \qquad \quad D(\mathring{\mathrm{rot}}) := \mathring{\mathsf{H}}(\mathrm{rot}, \Omega) := \overline{\mathring{\mathsf{C}}^{\infty}(\Omega)}^{\mathsf{H}(\mathrm{rot}, \Omega)}, \\ \mathrm{rot} &= \mathring{\mathrm{rot}}^* : D(\mathrm{rot}) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \qquad \quad D(\mathrm{rot}) := \mathsf{H}(\mathrm{rot}, \Omega). \end{split}$$

We denote the domain of definition, kernel, and range of a linear operator A by D(A), N(A), and R(A), respectively. Moreover, let

$$\mathsf{H} := \mathsf{L}^2(\Omega) \times \mathsf{L}^2(\Omega), \quad D(\mathsf{M}) := D(\mathring{\mathrm{rot}}) \times D(\mathrm{rot}).$$

It is well known that the Maxwell operator

$$\mathbf{M} := i \begin{bmatrix} 0 & -\operatorname{rot} \\ \operatorname{rot} & 0 \end{bmatrix} : D(\mathbf{M}) \subset \mathsf{H} \to \mathsf{H}$$

is selfadjoint with $0 \in \sigma(M) \subset \mathbb{R}$. Moreover, we have

$$\sigma(\mathbf{M}) = \begin{cases} \mathbb{R} & \text{if } \Omega = \mathbb{R}^3, \\ \{0, \pm \lambda_1^{1/2}, \pm \lambda_2^{1/2}, \dots\} & \text{if } \Omega \text{ bounded, (weakly) Lipschitz.} \end{cases}$$

Here $\sigma(\mathbf{M}) = \sigma_{\mathsf{p}}(\mathbf{M}) \dot{\cup} \sigma_{\mathsf{c}}(\mathbf{M})$ with $\sigma_{\mathsf{p}}(\mathbf{M}) = \{0\}$ and $\sigma_{\mathsf{c}}(\mathbf{M}) = \mathbb{R} \setminus \{0\}$ if $\Omega = \mathbb{R}^3$ which follows by considering, e.g., the vector field $E(x) = \exp(\lambda x_1)[0\ 1\ 0]^\top$, and a cutting technique together with Rellich's estimate and the principle of unique continuation. In this case, the ranges $R(\mathring{rot})$, $R(\operatorname{rot})$, and $R(\mathbf{M})$ are not closed.

If Ω is a bounded (weakly) Lipschitz domain then the spectrum is a pure point spectrum given by the strictly monotone increasing sequence

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_n \to \infty$$

being the pure point spectrum of the nonnegative operator rot rot resp. rot rot, cf. [8]. The pure point spectrum $\sigma(M) = \sigma_p(M)$ for Ω being bounded and Lipschitz is due to the spectral theorem for compact operators which is applicable by the compact embeddings

(1)
$$\check{\mathsf{H}}(\operatorname{rot},\Omega) \cap \mathsf{H}(\operatorname{div},\Omega) \hookrightarrow \mathsf{L}^{2}(\Omega), \quad \mathsf{H}(\operatorname{rot},\Omega) \cap \check{\mathsf{H}}(\operatorname{div},\Omega) \hookrightarrow \mathsf{L}^{2}(\Omega),$$

cf. [8] and [33, 32, 28, 6, 34, 7, 29] as well as [4, 5, 22]. In this case, the ranges R(rot), R(rot), and R(M) are closed. Note that the orthogonal Helmholtz type decomposition

$$\mathsf{H} = R(\mathsf{M}) \oplus N(\mathsf{M})$$

reduces M and that

$$R(M) \subset N(\operatorname{div}) \times N(\operatorname{div}),$$

which shows that $D(M) \cap R(M) \hookrightarrow H$ is compact by (1). See (10) for the underlying structure of the de Rham complex suited for Maxwell's equations.

It is well known that the gradient $\check{\nabla}$ with homogeneous Dirichlet boundary conditions has still closed range if the domain is bounded in at least one direction. To the best of the authors knowledge such results for rot were unknown. Recently, the authors noticed a result from [1, Example 10], namely, that the Maxwell operator M has closed range R(M) and that a punctured neighbourhood of the origin belongs to the resolvent set of M even for some unbounded domains (cylinders). Note that a cylinder could be seen as *in between* the latter two examples of a bounded domain and the whole space. While for a bounded Lipschitz domain the compact embedding (1) holds, it fails for the unbounded cylinder and the whole space. On the other hand, in case of the whole space, the continuous spectrum of M consists of the whole reals axis, while for a bounded Lipschitz domain and the unbounded cylinder a punctured neighbourhood of the origin belongs to the resolvent set of M.

Impressed by this, we present a different and more general result and proof of [1, Example 10] avoiding explicit calculations using Fourier transformations and eigenfunction expansions but using just simple estimates and some basic functional analysis instead.

Let us fix a general notation. Let d > 0 and let $\Omega_{\ell} \subset \mathbb{R}^3$, $\ell = 0, \ldots 3$, be defined by

$$\Omega_0 := \mathbb{R}^3, \quad \Omega_1 := \mathbb{R}^2 \times (0, d), \quad \Omega_2 := \mathbb{R} \times (0, d)^2, \quad \Omega_3 := (0, d)^3.$$

 Ω_{ℓ} shall be called (generalised) cylinders.

2. Preliminaries

2.1. Tiny FA-ToolBox. We recall parts of the FA-ToolBox from, e.g., [15, 17], cf. [14, 16, 22, 25, 26].

2.1.1. FA-ToolBox I. Let us consider a densely defined and closed linear operator

$$\mathbf{A}: D(\mathbf{A}) \subset \mathsf{H}_0 \to \mathsf{H}_1$$

between two Hilbert spaces H_0 and H_1 together with its (densely defined and closed) Hilbert space adjoint

$$A^*: D(A^*) \subset H_1 \to H_0.$$

Since $A^{**} = A$ we call (A, A^*) a dual pair. Note that R(A) is closed if and only if $R(A^*)$ is closed by the closed range theorem. Moreover, the projection theorem yields the orthogonal decompositions (Helmholtz type decompositions)

(2)
$$\mathsf{H}_0 = \overline{R(\mathbf{A}^*)} \oplus_{\mathsf{H}_0} N(\mathbf{A}), \quad \mathsf{H}_1 = \overline{R(\mathbf{A})} \oplus_{\mathsf{H}_1} N(\mathbf{A}^*),$$

which suggest to investigate the injective restrictions $\widehat{A} = A|_{N(A)^{\perp}}$ and $\widehat{A}^* = A^*|_{N(A^*)^{\perp}}$, more precisely, the injective reduced operators (restricted to / projected onto the respective orthogonal complements)

$$\widehat{\mathbf{A}} := \iota_{N(\mathbf{A}^*)^{\perp}}^* \mathbf{A} \,\iota_{N(\mathbf{A})^{\perp}} : D(\widehat{\mathbf{A}}) \subset N(\mathbf{A})^{\perp} \to \overline{R(\mathbf{A})} = N(\mathbf{A}^*)^{\perp}, \quad D(\widehat{\mathbf{A}}) := D(\mathbf{A}) \cap N(\mathbf{A})^{\perp},$$
$$\widehat{\mathbf{A}}^* := \iota_{N(\mathbf{A})^{\perp}}^* \mathbf{A}^* \,\iota_{N(\mathbf{A}^*)^{\perp}} : D(\widehat{\mathbf{A}}^*) \subset N(\mathbf{A}^*)^{\perp} \to \overline{R(\mathbf{A}^*)} = N(\mathbf{A})^{\perp}, \quad D(\widehat{\mathbf{A}}^*) := D(\mathbf{A}^*) \cap N(\mathbf{A}^*)^{\perp}.$$

 $(\widehat{A}, \widehat{A}^*)$ are also densely defined and closed forming another dual pair with dense ranges. Moreover, by (2) we have

~ *

(3)
$$D(\mathbf{A}) = D(\mathbf{A}) \oplus_{\mathsf{H}_0} N(\mathbf{A}), \quad D(\mathbf{A}^*) = D(\mathbf{A}^*) \oplus_{\mathsf{H}_1} N(\mathbf{A}^*),$$
$$R(\mathbf{A}) = R(\widehat{\mathbf{A}}), \qquad \qquad R(\mathbf{A}^*) = R(\widehat{\mathbf{A}}^*).$$

Here we have used the symbols $\overline{\cdot}, \oplus, \bot$, and ι for the closure, the orthogonal sum, the orthogonal complement, and the bounded embedding, respectively.

From [15, Lemma 4.1, Remark 4.2], see also [17, Lemma 2.1, Lemma 2.2] or [23, Lemma 2.1, Lemma 2.4], we cite the following elementary result.

Lemma 2.1 (fundamental FA-ToolBox lemma). The following assertions are equivalent:

- (i) $\exists c_{\mathbf{A}} > 0 \quad \forall x \in D(\widehat{\mathbf{A}})$ $|x|_{\mathsf{H}_0} \le c_{\mathsf{A}} |\operatorname{A} x|_{\mathsf{H}_1}$
- (i*) $\exists c_{A^*} > 0 \quad \forall y \in D(\widehat{A}^*) \qquad |y|_{H_1} \le c_{A^*} |A^* y|_{H_0}$
- (ii) R(A) is closed.
- (ii*) $R(\mathbf{A}^*)$ is closed. (iii) $\widehat{\mathbf{A}}^{-1}: R(\mathbf{A}) \to D(\widehat{\mathbf{A}})$ is bounded.
- (iii*) $(\widehat{A}^*)^{-1} : R(A^*) \to D(\widehat{A}^*)$ is bounded.

Moreover, if (i) holds with c_A then (ii) holds with $c_{A^*} \leq c_A$ and vice versa. For the best constants *it holds* $|\widehat{A}^{-1}|_{R(A)\to R(A^*)} = c_A = c_{A^*} = |(\widehat{A}^*)^{-1}|_{R(A^*)\to R(A)}$.

2.1.2. FA-ToolBox II. Let H_2 be another Hilbert space and let

(4)
$$\cdots \xrightarrow{\dots} H_0 \xrightarrow{A_0} H_1 \xrightarrow{A_1} H_2 \xrightarrow{\dots} \cdots$$

be a primal and dual Hilbert complex, i.e.,

$$\begin{aligned} \mathbf{A}_0 : D(\mathbf{A}_0) \subset \mathbf{H}_0 \to \mathbf{H}_1, & \mathbf{A}_1 : D(\mathbf{A}_1) \subset \mathbf{H}_1 \to \mathbf{H}_2, \\ \mathbf{A}_0^* : D(\mathbf{A}_0^*) \subset \mathbf{H}_1 \to \mathbf{H}_0, & \mathbf{A}_1^* : D(\mathbf{A}_1^*) \subset \mathbf{H}_2 \to \mathbf{H}_1 \end{aligned}$$

are densely defined and closed linear operators satisfying the complex property

$$A_1 A_0 \subset 0.$$

Note that (5) is equivalent to $R(A_0) \subset N(A_1)$ which is equivalent to $R(A_1^*) \subset N(A_0^*)$ (dual complex property) as $R(A_1^*) \subset \overline{R(A_1^*)} = N(A_1)^{\perp_{H_1}} \subset R(A_0)^{\perp_{H_1}} = N(A_0^*)$ and vice versa.

Defining the cohomology group

$$N_{0,1} := N(A_1) \cap N(A_0^*)$$

we get the following orthogonal Helmholtz-type decompositions, cf. (2).

Lemma 2.2 (Helmholtz decomposition). The orthogonal Helmholtz-type decompositions

$$\begin{array}{l} \mathsf{H}_{1} = \overline{R(A_{0})} \oplus_{\mathsf{H}_{1}} N(\mathbf{A}_{0}^{*}), & \mathsf{H}_{1} = N(\mathbf{A}_{1}) \oplus_{\mathsf{H}_{1}} \overline{R(\mathbf{A}_{1}^{*})}, \\ (6) & N(\mathbf{A}_{1}) = \overline{R(A_{0})} \oplus_{\mathsf{H}_{1}} N_{0,1}, & N(\mathbf{A}_{0}^{*}) = N_{0,1} \oplus_{\mathsf{H}_{1}} \overline{R(\mathbf{A}_{1}^{*})}, \\ D(\mathbf{A}_{1}) = \overline{R(A_{0})} \oplus_{\mathsf{H}_{1}} \left(D(\mathbf{A}_{1}) \cap N(\mathbf{A}_{0}^{*}) \right), & D(\mathbf{A}_{0}^{*}) = \left(N(\mathbf{A}_{1}) \cap D(\mathbf{A}_{0}^{*}) \right) \oplus_{\mathsf{H}_{1}} \overline{R(\mathbf{A}_{1}^{*})}, \\ D(\mathbf{A}_{0}^{*}) = D(\widehat{\mathbf{A}}_{0}^{*}) \oplus_{\mathsf{H}_{1}} N(\mathbf{A}_{0}^{*}), & D(\mathbf{A}_{1}) = N(\mathbf{A}_{1}) \oplus_{\mathsf{H}_{1}} D(\widehat{\mathbf{A}}_{1}), \end{array}$$

as well as $R(\widehat{A}_0^*) = R(A_0^*)$ and $R(\widehat{A}_1) = R(A_1)$ hold. Moreover,

(7)

$$\begin{aligned}
\mathsf{H}_1 &= R(\mathsf{A}_0) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} R(\mathsf{A}_1^*), \\
D(\mathsf{A}_0^*) &= D(\widehat{\mathsf{A}}_0^*) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} \overline{R(\mathsf{A}_1^*)}, \\
D(\mathsf{A}_1) &= \overline{R(\mathsf{A}_0)} \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} D(\widehat{\mathsf{A}}_1), \\
D(\mathsf{A}_1) &\cap D(\mathsf{A}_0^*) &= D(\widehat{\mathsf{A}}_0^*) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} D(\widehat{\mathsf{A}}_1).
\end{aligned}$$

Summarising the latter results we get the following theorem.

Theorem 2.3 (mini FA-ToolBox). Let $R(A_0)$ and $R(A_1)$ be closed. Then:

- (i) $R(A_0^*)$ and $R(A_1^*)$ are closed.
- (ii) The inverse operators $(\widehat{A}_0)^{-1}$, $(\widehat{A}_0^*)^{-1}$ and $(\widehat{A}_1)^{-1}$, $(\widehat{A}_1^*)^{-1}$ are bounded.
- (iii) The orthogonal Helmholtz-type decompositions (6) and (7) hold, in particular,

$$\mathsf{H}_1 = R(\mathsf{A}_0) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} R(\mathsf{A}_1^*).$$

(iv) There exist $c_{A_0}, c_{A_1} > 0$ such that

$$\begin{aligned} \forall x \in D(\widehat{A}_0) &= D(A_0) \cap N(A_0)^{\perp_{H_0}} = D(A_0) \cap R(A_0^*) & |x|_{H_0} \leq c_{A_0} |A_0 x|_{H_1} \\ \forall y \in D(\widehat{A}_0^*) &= D(A_0^*) \cap N(A_0^*)^{\perp_{H_1}} = D(A_0^*) \cap R(A_0) & |y|_{H_1} \leq c_{A_0} |A_0^* y|_{H_0} \\ \forall y \in D(\widehat{A}_1) &= D(A_1) \cap N(A_1)^{\perp_{H_1}} = D(A_1) \cap R(A_1^*) & |y|_{H_1} \leq c_{A_1} |A_1 y|_{H_2} \\ \forall z \in D(\widehat{A}_1^*) &= D(A_1^*) \cap N(A_1^*)^{\perp_{H_2}} = D(A_1^*) \cap R(A_1) & |z|_{H_2} \leq c_{A_1} |A_1^* z|_{H_1} \end{aligned}$$

(v) With c_{A_0} and c_{A_1} from (v) it holds

$$\forall y \in D(\mathcal{A}_1) \cap D(\mathcal{A}_0^*) \cap N_{0,1}^{\perp_{\mathcal{H}_1}} \qquad |y|_{\mathcal{H}_1}^2 \leq c_{\mathcal{A}_0}^2 |\mathcal{A}_0^* y|_{\mathcal{H}_0}^2 + c_{\mathcal{A}_1}^2 |\mathcal{A}_1 y|_{\mathcal{H}_2}^2.$$

2.1.3. FA-ToolBox III. In the following let H be an Hilbert space and let $\lambda \in \mathbb{C}$. Moreover, let $T: D(T) \subset H \to H$

be selfadjoint (or skew-selfadjoint) with selfadjoint (or skew-selfadjoint) reduced operator

$$\widehat{\mathbf{T}}: D(\widehat{\mathbf{T}}) \subset N(\mathbf{T})^{\perp} \to \overline{R(\mathbf{T})} = N(\mathbf{T})^{\perp}.$$

Remark 2.4 ((skew-)selfadjoint operators). Typical examples in our mind are the following: Let A be densely defined and closed. Then $T := A^* A$ and $T := \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$ are selfadjoint and $S := \begin{bmatrix} 0 & -A^* \\ A & 0 \end{bmatrix}$ is skew-selfadjoint. Note that i S is also selfadjoint.

If R(T) is closed then by Lemma 2.1

$$\widehat{\mathbf{T}}^{-1}: R(\mathbf{T}) \to D(\widehat{\mathbf{T}})$$

is bounded. Thus $0 \in \rho(\widehat{T})$ (resolvent set). Hence there is an open neighbourhood U of 0 such that

$$(\widehat{\mathbf{T}} - \lambda)^{-1} : R(\mathbf{T}) \to D(\widehat{\mathbf{T}})$$

is bounded for all $\lambda \in U$. We may choose $U := B(0, 1/c_T)$. More precisely, we have the following result for solving the equation

$$(\mathbf{T} - \lambda)x = f \in R(\mathbf{T}).$$

Lemma 2.5 (spectrum around the origin for the reduced operator). Let R(T) be closed and let $|\lambda| < 1/c_{\mathrm{T}}$ with $c_{\mathrm{T}} := |\widehat{\mathrm{T}}^{-1}|_{B(\mathrm{T}) \to B(\mathrm{T})}$. Then

$$\forall x \in D(\widehat{\mathbf{T}}) \qquad |x|_{\mathsf{H}} \leq \widehat{c}_{\mathrm{T},\lambda} \left| (\mathrm{T} - \lambda)x \right|_{\mathsf{H}}, \qquad \widehat{c}_{\mathrm{T},\lambda} := \frac{c_{\mathrm{T}}}{1 - c_{\mathrm{T}}|\lambda|},$$

and

$$N(\widehat{\mathbf{T}} - \lambda) = \{0\}, \quad R(\widehat{\mathbf{T}} - \lambda) = R(\mathbf{T}),$$

in particular, $R(\widehat{T} - \lambda)$ is closed. Moreover, the inverse $(\widehat{T} - \lambda)^{-1} : R(T) \to D(\widehat{T})$ is bounded with $|(\widehat{T} - \lambda)^{-1}|_{R(T) \to R(T)} \leq \widehat{c}_{T,\lambda}$. In other words $B(0, 1/c_T) \subset \rho(\widehat{T})$.

Proof. For convenience we point out the short argument. As mentioned above by Lemma 2.1 we have that $\widehat{\mathbf{T}}^{-1}: R(\mathbf{T}) \to D(\widehat{\mathbf{T}})$ is bounded and that for $x \in D(\widehat{\mathbf{T}})$ it holds

$$|\mathbf{x}|_{\mathsf{H}} \leq c_{\mathrm{T}} |\mathrm{T} x|_{\mathsf{H}} \leq c_{\mathrm{T}} |(\mathrm{T} - \lambda)x|_{\mathsf{H}} + c_{\mathrm{T}} |\lambda| |x|_{\mathsf{H}}$$

showing the estimate for all $|\lambda| < 1/c_{\rm T}$. Hence $N(\widehat{\rm T} - \lambda) = \{0\}$ and $R(\widehat{\rm T} - \lambda)$ is closed with $R(\widehat{\mathbf{T}} - \lambda) = N(\widehat{\mathbf{T}}^* - \overline{\lambda})^{\perp_{R(\mathbf{T})}} = N(\widehat{\mathbf{T}} - \overline{\lambda})^{\perp_{R(\mathbf{T})}} = R(\mathbf{T}).$

Thus $(\widehat{\widehat{T} - \lambda}) = \widehat{T} - \lambda$ by Lemma 2.1 applied to $A := \widehat{T} - \lambda$. Therefore,

$$(T - \lambda) = (T - \lambda)^{-1} : R(T - \lambda) = R(T - \lambda) = R(T) \to D(T - \lambda) = D(T - \lambda) = D(T)$$

is bounded by the latter estimate and Lemma 2.1.

Next we want to solve the equation

$$(\mathbf{T} - \lambda)x = f \in \mathbf{H}.$$

Lemma 2.6 (spectrum around the origin). Let R(T) be closed and let $0 < |\lambda| < 1/c_T$. Then $N(T-\lambda) = 0$ and $R(T-\lambda) = H$. Moreover,

$$(\mathbf{T} - \lambda)^{-1} : \mathbf{H} \to D(\mathbf{T})$$

is bounded with $|(\mathbf{T}-\lambda)^{-1}|_{\mathsf{H}\to\mathsf{H}} \leq c_{\mathbf{T},\lambda}$, where $c_{\mathbf{T},\lambda} := \sqrt{\widehat{c}_{\mathbf{T},\lambda}^2 + |\lambda|^{-2}}$. In particular, $\forall x \in D(\mathbf{T}) \qquad |x|_{\mathsf{H}} \leq c_{\mathbf{T},\lambda} |(\mathbf{T}-\lambda)x|_{\mathbf{T},\lambda}$

$$\in D(T)$$
 $|x|_{\mathsf{H}} \leq c_{\mathsf{T},\lambda} |(\mathsf{T}-\lambda)x|_{\mathsf{H}}.$

In other words $B(0, 1/c_{\rm T}) \setminus \{0\} \subset \rho({\rm T})$.

Proof. Let $x \in D(T)$ with $(T - \lambda)x = f \in H$. According to (2) and (3) we decompose

$$\begin{split} D(\mathbf{T}) &\ni x = x_R + x_N \in D(\widehat{\mathbf{T}}) \oplus_\mathsf{H} N(\mathbf{T}), \\ \mathsf{H} &\ni f = f_R + f_N \in R(\mathbf{T}) \oplus_\mathsf{H} N(\mathbf{T}), \end{split} \qquad \qquad D(\widehat{\mathbf{T}}) = D(\mathbf{T}) \cap R(\mathbf{T}), \end{split}$$

and obtain the equation $(T - \lambda)x_R - \lambda x_N = f_R + f_N$, which separates into the two equations

$$(\widehat{\mathbf{T}} - \lambda)x_R = f_R \in R(\mathbf{T}), \quad -\lambda x_N = f_N \in N(\mathbf{T})$$

by orthogonality. Lemma 2.5 yields

$$x_R = (\widehat{\mathbf{T}} - \lambda)^{-1} f_R, \quad x_N = -\frac{1}{\lambda} f_N$$

and thus $|x|_{\mathsf{H}}^2 = |x_R|_{\mathsf{H}}^2 + |x_N|_{\mathsf{H}}^2 \leq \hat{c}_{\mathrm{T},\lambda}^2 |f_R|_{\mathsf{H}}^2 + |\lambda|^{-2} |f_N|_{\mathsf{H}}^2 \leq c_{\mathrm{T},\lambda}^2 |f|_{\mathsf{H}}^2$. We conclude¹ $N(\mathrm{T}-\lambda) = 0$ and $\widehat{\mathrm{T}-\lambda} = \mathrm{T}-\lambda$. Lemma 2.1 shows $R(\mathrm{T}-\lambda) = \mathsf{H}$ and that $(\mathrm{T}-\lambda)^{-1} : \mathsf{H} \to D(\mathrm{T})$ is bounded with $|(\mathrm{T}-\lambda)^{-1}|_{\mathsf{H}\to\mathsf{H}} \leq c_{\mathrm{T},\lambda}$. We emphasise that indeed $x := x_R + x_N = (\widehat{\mathrm{T}}-\lambda)^{-1}f_R - \frac{1}{\lambda}f_N$ for $f \in \mathsf{H}$ solves $(\mathrm{T}-\lambda)x = f_R + f_N = f$.

Remark 2.7. The latter proof shows

$$(\mathbf{T}-\lambda)^{-1} = (\widehat{\mathbf{T}}-\lambda)^{-1}\pi_{R(\mathbf{T})} - \frac{1}{\lambda}\pi_{N(\mathbf{T})}$$

with orthogonal projectors $\pi_{R(T)}$ and $\pi_{N(T)}$ onto the range and kernel of T, respectively.

Lemma 2.8 (low frequency asymptotics). Let R(T) be closed and let $0 < |\lambda| < 1/c_T$. Then

$$(\mathbf{T}-\lambda)^{-1} = (\widehat{\mathbf{T}}-\lambda)^{-1}\pi_{R(\mathbf{T})} - \frac{1}{\lambda}\pi_{N(\mathbf{T})} = \sum_{n=0}^{\infty} \lambda^{n}\widehat{\mathbf{T}}^{-n-1}\pi_{R(\mathbf{T})} - \frac{1}{\lambda}\pi_{N(\mathbf{T})}$$
$$= \sum_{n=0}^{k-1} \lambda^{n}\widehat{\mathbf{T}}^{-n-1}\pi_{R(\mathbf{T})} + \lambda^{k}\widehat{\mathbf{T}}^{-k-1}\sum_{n=0}^{\infty} \lambda^{n}\widehat{\mathbf{T}}^{-n}\pi_{R(\mathbf{T})} - \frac{1}{\lambda}\pi_{N(\mathbf{T})}$$

and

$$\left| (\mathbf{T} - \lambda)^{-1} - \sum_{n=0}^{k-1} \lambda^n \widehat{\mathbf{T}}^{-n-1} \pi_{R(\mathbf{T})} + \frac{1}{\lambda} \pi_{N(\mathbf{T})} \right|_{\mathsf{H} \to \mathsf{H}} \leq \widehat{c}_{\mathbf{T},\lambda} c_{\mathbf{T}}^k |\lambda|^k = \mathcal{O}(\lambda^k) \quad (\text{for } \lambda \to 0).$$

Proof. We observe $(\widehat{\mathbf{T}} - \lambda) = \widehat{\mathbf{T}}(1 - \lambda \widehat{\mathbf{T}}^{-1})$ and $|\lambda \widehat{\mathbf{T}}^{-1}|_{R(\mathbf{T}) \to R(\mathbf{T})} = |\lambda|c_{\mathbf{T}} < 1$. Thus by Neumann's series

$$(\widehat{\mathbf{T}} - \lambda)^{-1} = (1 - \lambda \widehat{\mathbf{T}}^{-1})^{-1} \widehat{\mathbf{T}}^{-1} = \sum_{n=0}^{\infty} \lambda^n \widehat{\mathbf{T}}^{-n-1} = \sum_{n=0}^{k-1} \lambda^n \widehat{\mathbf{T}}^{-n-1} + \lambda^k \widehat{\mathbf{T}}^{-k-1} \sum_{n=0}^{\infty} \lambda^n \widehat{\mathbf{T}}^{-n},$$

which shows together with Remark 2.7 the equations. Moreover,

$$\left|\widehat{\mathbf{T}}^{-k-1}\sum_{n=0}^{\infty}\lambda^{n}\widehat{\mathbf{T}}^{-n}\pi_{R(\mathbf{T})}\right|_{\mathsf{H}\to\mathsf{H}} \le c_{\mathbf{T}}^{k+1}\left|\sum_{n=0}^{\infty}\lambda^{n}\widehat{\mathbf{T}}^{-n}\right|_{R(\mathbf{T})\to R(\mathbf{T})} \le \frac{c_{\mathbf{T}}^{k+1}}{1-\lambda|c_{\mathbf{T}}} = \widehat{c}_{\mathbf{T},\lambda}c_{\mathbf{T}}^{k},$$

concluding the proof.

2.2. Friedrichs and Gaffney Type Estimates.

Lemma 2.9 (Friedrichs estimate). Let $u \in H^1(\Omega_1)$ be such that $u|_{\mathbb{R}^2 \times \{0\}} = 0$. Then

$$|u|_{\mathsf{L}^2(\Omega_1)} \le c_d |\nabla u|_{\mathsf{L}^2(\Omega_1)}, \quad c_d := \frac{d}{\sqrt{2}}.$$

¹Note that for $\lambda \neq 0$ we always have $N(T - \lambda) = N(\widehat{T} - \lambda)$.

Proof. There exists a sequence $(\phi_n) \subset \mathsf{C}_0^{\infty}(\mathbb{R}^2 \times (0,\infty))$ such that $\phi_n \to u$ in $\mathsf{H}^1(\Omega_1)$. Then we have for all $x_1, x_2 \in \mathbb{R}$ and for all $x_3 \in (0,d)$

$$\phi_n(x_1, x_2, x_3) = \int_0^{x_3} \partial_3 \phi_n(x_1, x_2, \cdot)$$

and thus $|\phi_n(x_1, x_2, x_3)|^2 \le x_3 \int_0^d |\partial_3 \phi_n(x_1, x_2, \cdot)|^2$, which implies

(8)
$$\int_{0}^{d} |\phi_{n}(x_{1}, x_{2}, \cdot)|^{2} \leq \frac{d^{2}}{2} \int_{0}^{d} |\partial_{3} \phi_{n}(x_{1}, x_{2}, \cdot)|^{2}.$$

Integration over \mathbb{R}^2 shows $|\phi_n|^2_{L^2(\Omega_1)} \leq c_d^2 |\partial_3 \phi_n|^2_{L^2(\Omega_1)}$ which yields the assertion for $n \to \infty$. \Box

Remark 2.10 (best Friedrichs constant). The best Friedrichs constant c_F in Lemma 2.9 is given by

$$\frac{1}{c_F} = \lambda := \min_{\substack{0 \neq u \in \mathsf{H}^1(\Omega_1), \\ u|_{\mathbb{R}^2 \times \{0\}} = 0}} \frac{|\nabla u|_{\mathsf{L}^2(\Omega_1)}}{|u|_{\mathsf{L}^2(\Omega_1)}},$$

which is the square root of the first positive eigenvalue of the mixed Dirichlet/Neumann negative Laplacian

$$-\Delta u = \lambda u \text{ in } \Omega_1, \qquad u|_{\mathbb{R}^2 \times \{0\}} = 0, \quad n \times \nabla u|_{\mathbb{R}^2 \times \{d\}} = 0 \text{ at } \partial \Omega_1.$$

To get a better constant than c_d we note that the best Friedrichs constant c_f in (8) is given by

$$\frac{1}{c_f} = \lambda := \min_{\substack{0 \neq u \in \mathsf{H}^1(I), \\ u(0)=0}} \frac{|u'|_{\mathsf{L}^2(I)}}{|u|_{\mathsf{L}^2(I)}}, \qquad I := (0, d),$$

which is the square root of the first positive eigenvalue of the mixed Dirichlet/Neumann negative 1D-Laplacian

$$-u'' = \lambda u \text{ in } I, \qquad u(0) = 0, \quad u'(d) = 0.$$

Hence c_d can be replaced by the (slightly) better constant² c_f with

$$\frac{c_f}{d} = \frac{2}{\pi} < 0.637 < 0.707 < \frac{1}{\sqrt{2}} = \frac{c_d}{d}.$$

Note that for full Dirichlet boundary conditions, i.e., $u|_{\partial \Omega_1} = 0$ resp. u(0) = u(d) = 0, or full Neumann boundary conditions, i.e., $n \times \nabla u|_{\partial \Omega_1} = 0$ resp. u'(0) = u'(d) = 0, we get the well known Friedrichs/Poincaré constant³ $c_f = d/\pi$, cf. [27].

Remark 2.11 (Friedrichs estimate). The same proof works also for Ω_2 and $u|_{\mathbb{R}\times(0,d)\times\{0\}} = 0$ and for Ω_3 and $u|_{(0,d)^2\times\{0\}} = 0$. Hence Lemma 2.9 holds also in those cases.

Remark 2.12 (Friedrichs estimate in $\Omega_0 = \mathbb{R}^3$). In case of the whole space we have polynomially weighted versions of the Friedrichs estimates. For example, for all $u \in H^1_{-1}(\Omega_0)$, i.e., $u \in H^1_{loc}(\Omega_0)$ such that $(1 + |\cdot|^2)^{-1/2} u \in L^2(\Omega_0)$ and $\nabla u \in L^2(\Omega_0)$, it holds

$$|u|_{\mathsf{L}^{2}_{-1}(\Omega_{0})} = |(1+|\cdot|^{2})^{-1/2}u|_{\mathsf{L}^{2}(\Omega_{0})} \le 2|\nabla u|_{\mathsf{L}^{2}(\Omega_{0})},$$

cf. [8, p. 57] or [31], [20, 21]. Moreover,

$$\overline{\nabla \,\mathsf{H}^1(\Omega_0)} = \nabla \,\mathsf{H}^1_{-1}(\Omega_0).$$

Similar estimates hold also for rot and div and extend to exterior domains, i.e., domains with compact complement, as well as to mixed boundary conditions, cf. [9, 10, 12, 11, 13, 18, 19].

From [30] and [2, Theorem 2.17], see also [16, Lemma 3.2 and Appendix A] for a proof, we cite the following result.

²The eigenfunction is $\sin(\lambda \cdot)$.

³The eigenfunctions are $\sin(\lambda \cdot)$ resp. $\cos(\lambda \cdot)$.

Lemma 2.13 (Gaffney estimate for bounded convex domains). Let $\Omega \subset \mathbb{R}^3$ be a bounded and convex domain and let either $E \in \check{H}(rot,\Omega) \cap H(\operatorname{div},\Omega)$ or $E \in H(rot,\Omega) \cap \check{H}(\operatorname{div},\Omega)$. Then $E \in \mathsf{H}^1(\Omega)$ and

$$|\nabla E|^2_{\mathsf{L}^2(\Omega)} \le |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathsf{L}^2(\Omega)}$$

We can skip the boundedness of the domain.

Lemma 2.14 (Gaffney estimate for convex domains). Let Ω be a convex domain and let either $E \in \mathring{H}(\mathrm{rot}, \Omega) \cap \mathsf{H}(\mathrm{div}, \Omega) \text{ or } E \in \mathsf{H}(\mathrm{rot}, \Omega) \cap \mathring{H}(\mathrm{div}, \Omega).$ Then $E \in \mathsf{H}^1(\Omega)$ and

$$|\nabla E|^2_{\mathsf{L}^2(\Omega)} \leq |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathsf{L}^2(\Omega)}$$

 $|\nabla E|_{\mathsf{L}^{2}(\Omega)} \leq |\operatorname{rot} E|_{\mathsf{L}^{2}(\Omega)} + |\operatorname{div} E|_{\mathsf{L}^{2}(\Omega)}.$ If $\Omega = \Omega_{n}$ for $n \in \{0, 1, 2, 3\}$, then even $|\nabla E|_{\mathsf{L}^{2}(\Omega)}^{2} = |\operatorname{rot} E|_{\mathsf{L}^{2}(\Omega)}^{2} + |\operatorname{div} E|_{\mathsf{L}^{2}(\Omega)}^{2}.$

Proof. Assume that Ω is unbounded. Let $\varphi \in \mathsf{C}^{\infty}(\mathbb{R},[0,1])$ be a cut-off-function such that $\varphi|_{(-\infty,1)} = 1$ and $\varphi|_{(2,\infty)} = 0$ and define $\varphi_r \in \mathsf{C}_0^\infty(\mathbb{R}^3)$ for r > 0 by $\varphi_r(x) := \varphi(|x|/r)$. Then $\varphi_r|_{B(0,r)} = 1$ and $\varphi_r|_{\mathbb{R}^3 \setminus B(0,2r)} = 0$. Note that $\operatorname{supp} \nabla \varphi_r \subset \overline{B(0,2r)} \setminus B(0,r)$ and $|\nabla \varphi_r| \leq c/r$. Let $\Omega_r := \Omega \cap B(0, r)$. As

$$\partial_j(\varphi_r E) = \varphi_r \,\partial_j E + (\partial_j \,\varphi_r)E, \qquad j = 1, 2, 3,$$

$$\operatorname{rot}(\varphi_r E) = \varphi_r \operatorname{rot} E + (\nabla \,\varphi_r) \times E,$$

$$\operatorname{div}(\varphi_r E) = \varphi_r \operatorname{div} E + (\nabla \,\varphi_r) \cdot E$$

we have $\varphi_r E \in \mathring{H}(rot, \Omega_{2r}) \cap H(\operatorname{div}, \Omega_{2r})$ or $\varphi_r E \in H(rot, \Omega_{2r}) \cap \mathring{H}(\operatorname{div}, \Omega_{2r})$. Thus Lemma 2.13 shows $\varphi_r E \in \mathsf{H}^1(\Omega_{2r})$ with

(9)
$$\left| \nabla(\varphi_r E) \right|_{\mathsf{L}^2(\Omega)}^2 \le \left| \operatorname{rot}(\varphi_r E) \right|_{\mathsf{L}^2(\Omega)}^2 + \left| \operatorname{div}(\varphi_r E) \right|_{\mathsf{L}^2(\Omega)}^2$$

By Lebesgue's dominated convergence theorem we have

 $\varphi_r E \xrightarrow{r \to \infty} E, \quad \varphi_r \operatorname{rot} E \xrightarrow{r \to \infty} \operatorname{rot} E, \quad \varphi_r \operatorname{div} E \xrightarrow{r \to \infty} \operatorname{div} E, \quad (\partial_i \varphi_r) E \xrightarrow{r \to \infty} 0$ with convergence in $L^2(\Omega)$. Moreover, (9) yields

$$\begin{aligned} |\varphi_r \nabla E|_{\mathsf{L}^2(\Omega)} &\leq c \Big(\big| \nabla(\varphi_r E) \big|_{\mathsf{L}^2(\Omega)} + \sum_{j=1}^3 \big| (\partial_j \varphi_r) E \big|_{\mathsf{L}^2(\Omega)} \Big) \\ &\leq c \Big(\big| \operatorname{rot}(\varphi_r E) \big|_{\mathsf{L}^2(\Omega)} + \big| \operatorname{div}(\varphi_r E) \big|_{\mathsf{L}^2(\Omega)} + \sum_{j=1}^3 \big| (\partial_j \varphi_r) E \big|_{\mathsf{L}^2(\Omega)} \Big) \\ &\leq c \Big(\big| \operatorname{rot} E) \big|_{\mathsf{L}^2(\Omega)} + \big| \operatorname{div} E) \big|_{\mathsf{L}^2(\Omega)} + \sum_{j=1}^3 \big| (\partial_j \varphi_r) E \big|_{\mathsf{L}^2(\Omega)} \Big). \end{aligned}$$

Hence using the monotone convergence theorem we conclude $E \in H^1(\Omega)$ for $r \to \infty$. Therefore,

$$\nabla(\varphi_r E) \xrightarrow{r \to \infty} \nabla E, \quad \operatorname{rot}(\varphi_r E) \xrightarrow{r \to \infty} \operatorname{rot} E, \quad \operatorname{div}(\varphi_r E) \xrightarrow{r \to \infty} \operatorname{div} E$$

by Lebesgue's dominated convergence theorem and (9) gives the asserted estimate for $r \to \infty$.

Now let $\Omega = \Omega_n$ for $n \in \{0, 1, 2\}$. [3, Lemma 13], cf. [2], shows that (9) holds even with equality since we can interpret the integration over $\Omega_{2r} := \Omega \cap (-2r, 2r)^3$ instead over Ω_{2r} and since the normal vector at $\partial \widetilde{\Omega}_{2r}$ is constant almost everywhere (flat boundary parts). Lebesgue's dominated convergence theorem proves the assertion. For $\Omega = \Omega_3$ the proof even simplifies as we need no cutting.

The latter proof, in particular, [3, Lemma 13], shows the following result.

Lemma 2.15 (Gaffney estimate for convex flat domains). Let Ω be convex and piecewise flat, and let either $E \in \mathring{H}(\operatorname{rot}, \Omega) \cap \mathsf{H}(\operatorname{div}, \Omega)$ or $E \in \mathsf{H}(\operatorname{rot}, \Omega) \cap \mathring{\mathsf{H}}(\operatorname{div}, \Omega)$. Then $E \in \mathsf{H}^1(\Omega)$ and

$$|\nabla E|^2_{\mathsf{L}^2(\Omega)} = |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathsf{L}^2(\Omega)}.$$

2.3. The De Rham Complex. We intend to apply the latter results to ∇ , rot, and div, i.e., to the classical de Rham complex

(10)
$$L^{2}(\Omega) \xrightarrow{\overset{\circ}{\nabla}} L^{2}(\Omega) \xrightarrow{\overset{\circ}{\operatorname{rot}}} L^{2}(\Omega) \xrightarrow{\overset{\circ}{\operatorname{rot}}} L^{2}(\Omega) \xrightarrow{\overset{\circ}{\operatorname{trot}}} L^{2}(\Omega).$$

We consider the following dual pairs

 $(\mathring{\mathrm{rot}},\mathrm{rot}),\quad (\mathring{\nabla},-\operatorname{div}),\quad (\mathring{\operatorname{div}},-\nabla)$

together with their reduced versions, more precisely: We introduce the densely defined and closed dual pairs

$$\begin{split} \bullet & \mathring{\nabla}: D(\mathring{\nabla}) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \qquad D(\mathring{\nabla}) := \mathring{\mathsf{H}}^1(\Omega) := \overline{\mathring{\mathsf{C}}^{\infty}(\Omega)}^{\mathsf{H}^1(\Omega)}, \\ -\operatorname{div} = \mathring{\nabla}^*: D(\operatorname{div}) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \qquad D(\operatorname{div}) = \mathsf{H}(\operatorname{div}, \Omega), \\ \bullet & \operatorname{rot}: D(\operatorname{rot}) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \qquad D(\operatorname{rot}) := \mathring{\mathsf{H}}(\operatorname{rot}, \Omega) := \overline{\mathring{\mathsf{C}}^{\infty}(\Omega)}^{\mathsf{H}(\operatorname{rot}, \Omega)}, \\ \operatorname{rot} = \operatorname{rot}^*: D(\operatorname{rot}) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \qquad D(\operatorname{rot}) = \mathsf{H}(\operatorname{rot}, \Omega), \\ \bullet & \operatorname{div}: D(\operatorname{div}) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \qquad D(\operatorname{div}) := \mathring{\mathsf{H}}(\operatorname{div}, \Omega) := \overline{\mathring{\mathsf{C}}^{\infty}(\Omega)}^{\mathsf{H}(\operatorname{div}, \Omega)}, \\ -\nabla = \operatorname{div}^*: D(\nabla) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \qquad D(\nabla) = \mathsf{H}^1(\Omega). \end{split}$$

The densely defined and closed reduced dual pairs are then given by

$$\begin{split} \bullet & \hat{\nabla} = \iota_{N(\operatorname{div})^{\perp}}^* \stackrel{\circ}{\nabla} \iota_{N(\hat{\nabla})^{\perp}} : D(\hat{\nabla}) \subset N(\hat{\nabla})^{\perp} \to \overline{R(\hat{\nabla})} = N(\operatorname{div})^{\perp}, \\ -\widehat{\operatorname{div}} = -\iota_{N(\hat{\nabla})^{\perp}}^* \operatorname{div} \iota_{N(\operatorname{div})^{\perp}} : D(\widehat{\operatorname{div}}) \subset N(\operatorname{div})^{\perp} \to \overline{R(\operatorname{div})} = N(\hat{\nabla})^{\perp}, \\ \bullet & \widehat{\operatorname{rot}} = \iota_{N(\operatorname{rot})^{\perp}}^* \stackrel{\circ}{\operatorname{rot}} \iota_{N(\operatorname{rot})^{\perp}} : D(\widehat{\operatorname{rot}}) \subset N(\operatorname{rot})^{\perp} \to \overline{R(\operatorname{rot})} = N(\operatorname{rot})^{\perp}, \\ \widehat{\operatorname{rot}} = \iota_{N(\operatorname{rot})^{\perp}}^* \operatorname{rot} \iota_{N(\operatorname{rot})^{\perp}} : D(\widehat{\operatorname{rot}}) \subset N(\operatorname{rot})^{\perp} \to \overline{R(\operatorname{rot})} = N(\operatorname{rot})^{\perp}, \\ \widehat{\operatorname{div}} = \iota_{N(\nabla)^{\perp}}^* \operatorname{div} \iota_{N(\operatorname{div})^{\perp}} : D(\widehat{\operatorname{rot}}) \subset N(\operatorname{rot})^{\perp} \to \overline{R(\operatorname{rot})} = N(\operatorname{rot})^{\perp}, \\ -\widehat{\nabla} = -\iota_{N(\operatorname{div})^{\perp}}^* \nabla \iota_{N(\nabla)^{\perp}} : D(\widehat{\nabla}) \subset N(\nabla)^{\perp} \to \overline{R(\nabla)} = N(\operatorname{div})^{\perp}. \end{split}$$

with

$$\begin{split} D(\mathring{\nabla}) &= D(\mathring{\nabla}) \cap N(\mathring{\nabla})^{\perp}, & D(\widehat{\operatorname{div}}) = D(\operatorname{div}) \cap N(\operatorname{div})^{\perp}, \\ D(\widehat{\operatorname{rot}}) &= D(\operatorname{rot}) \cap N(\operatorname{rot})^{\perp}, & D(\widehat{\operatorname{rot}}) = D(\operatorname{rot}) \cap N(\operatorname{rot})^{\perp}, \\ D(\widehat{\operatorname{div}}) &= D(\operatorname{div}) \cap N(\operatorname{div})^{\perp}, & D(\widehat{\nabla}) = D(\nabla) \cap N(\nabla)^{\perp}. \end{split}$$

Note that always $N(\mathring{\nabla}) = \{0\}$ and $\overline{R(\operatorname{div})} = N(\mathring{\nabla})^{\perp} = \mathsf{L}^2(\Omega)$. Moreover, for Ω with finite volume we have $N(\nabla) = \mathbb{R}$ and $\overline{R(\operatorname{div})} = N(\nabla)^{\perp} = \mathsf{L}^2(\Omega) \cap \mathbb{R}^{\perp}$, but for Ω with infinite volume $N(\nabla) = \{0\}$ and $\overline{R(\operatorname{div})} = N(\nabla)^{\perp} = \mathsf{L}^2(\Omega)$.

Let us emphasise that

 $-\operatorname{div} \overset{\circ}{\nabla}, -\overset{\circ}{\nabla} \operatorname{div}, \operatorname{rot} \operatorname{rot}, \operatorname{rot} \operatorname{rot}, -\operatorname{div} \nabla, -\nabla \operatorname{div}, \operatorname{rot} \operatorname{rot} -\overset{\circ}{\nabla} \operatorname{div}, \operatorname{rot} \operatorname{rot} - \nabla \operatorname{div}$ are selfadjoint and non-negative and that

$$\begin{bmatrix} 0 & \operatorname{div} \\ \mathring{\nabla} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\operatorname{rot} \\ \operatorname{rot} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \operatorname{div} \\ \nabla & 0 \end{bmatrix}$$

are skew-selfadjoint.

2.4. The Transformation Theorem. Let $\Xi \subset \mathbb{R}^3$ be another domain and let $\Phi \in C^{0,1}(\mathbb{R}^3, \mathbb{R}^3)$ be such that its restriction to Ξ , still denoted by

$$\Phi:\Xi\to\Phi(\Xi)=\Omega,$$

is bi-Lipschitz, bounded, and regular, i.e., $\Phi \in \mathsf{C}^{0,1}_{\mathsf{bd}}(\overline{\Xi},\overline{\Omega})$ and $\Phi^{-1} \in \mathsf{C}^{0,1}_{\mathsf{bd}}(\overline{\Omega},\overline{\Xi})$ with

$$J_{\Phi} = \Phi' = (\nabla \Phi)^{\top}, \quad \det J_{\Phi} > 0.$$

Such regular bi-Lipschitz transformations Φ will be called *admissible*. For admissible Φ the inverse and adjunct matrix of J_{Φ} shall be denoted by

$$J_{\Phi}^{-1}, \qquad \text{adj} J_{\Phi} := (\det J_{\Phi}) J_{\Phi}^{-1},$$

respectively. We denote the composition with Φ by tilde, i.e., for any tensor field ψ we define

$$\widetilde{\psi} := \psi \circ \Phi.$$

We introduce a new notation

$$\dot{H} = \dot{H}$$
 or $\dot{H} = H$

to handle spaces with and without boundary conditions simultaneously. In the following, let Φ be admissible.

Theorem 2.16 (transformation theorem). Let $u \in \dot{H}^1(\Omega)$, $E \in \dot{H}(rot, \Omega)$, and $H \in \dot{H}(div, \Omega)$. Then

$$\begin{split} \tau^0_\Phi u &:= \widetilde{u} \in \dot{\mathsf{H}}^1(\Xi) \qquad and \qquad \nabla \, \tau^0_\Phi u = \tau^1_\Phi \, \nabla \, u, \\ \tau^1_\Phi E &:= J_\Phi^\top \widetilde{E} \in \dot{\mathsf{H}}(\mathrm{rot}, \Xi) \qquad and \qquad \mathrm{rot} \, \tau^1_\Phi E = \tau^2_\Phi \, \mathrm{rot} \, E, \\ \tau^2_\Phi H &:= (\mathrm{adj} \, J_\Phi) \widetilde{H} \in \dot{\mathsf{H}}(\mathrm{div}, \Xi) \qquad and \qquad \mathrm{div} \, \tau^2_\Phi H = \tau^3_\Phi \, \mathrm{div} \, H \end{split}$$

with $\tau_{\Phi}^3 f := (\det J_{\Phi}) \widetilde{f} = (\det J_{\Phi}) \tau_{\Phi}^0 f \in \mathsf{L}^2(\Xi)$ for $f \in \mathsf{L}^2(\Omega)$. Moreover,

$$\begin{split} \tau_{\Phi}^{0} &: \dot{\mathsf{H}}^{1}(\Omega) \to \dot{\mathsf{H}}^{1}(\Xi), & \tau_{\Phi}^{1} : \dot{\mathsf{H}}(\mathrm{rot}, \Omega) \to \dot{\mathsf{H}}(\mathrm{rot}, \Xi), \\ \tau_{\Phi}^{3} : \mathsf{L}^{2}(\Omega) \to \mathsf{L}^{2}(\Xi), & \tau_{\Phi}^{2} : \dot{\mathsf{H}}(\mathrm{div}, \Omega) \to \dot{\mathsf{H}}(\mathrm{div}, \Xi) \end{split}$$

are topological isomorphisms with norms depending on Ξ and J_{Φ} . The inverse operators and the L^2 -adjoints, i.e., the Hilbert space adjoints of $\tau_{\Phi}^q : L^2(\Omega) \to L^2(\Xi)$, $q \in \{0, 1, 2, 3\}$, are given by

$$(\tau_{\Phi}^{q})^{-1} = \tau_{\Phi^{-1}}^{q}, \qquad (\tau_{\Phi}^{q})^{*} = \tau_{\Phi^{-1}}^{N-q}.$$

A proof for differential forms can be found in the appendix of [5].

Proof. Recall Rademacher's theorem on Lipschitz functions.

• For $u \in \dot{\mathsf{C}}^{0,1}(\Omega)$ we have by Rademacher's theorem $\widetilde{u} \in \dot{\mathsf{C}}^{0,1}(\Xi)$ and the standard chain rule $(\widetilde{u})' = \widetilde{u'}\Phi'$ holds, i.e.,

(11)
$$\nabla \widetilde{u} = \nabla \Phi \widetilde{\nabla u} = J_{\Phi}^{\top} \widetilde{\nabla u}.$$

For $u \in \dot{H}^1(\Omega)$ we pick a sequence $(u^\ell) \subset \dot{C}^{0,1}(\Omega)$ such that $u^\ell \to E$ in $\dot{H}^1(\Omega)$. Then $\tilde{u^\ell} \to \tilde{E}$ and $\tilde{\nabla u^\ell} \to \tilde{\nabla u}$ in $L^2(\Xi)$ by the standard transformation theorem. We have $\tilde{u^\ell} \in \dot{C}^{0,1}(\Xi) \subset \dot{H}^1(\Xi)$ by (11) with

$$\widetilde{u^{\ell}} \to \widetilde{u}, \quad \nabla \widetilde{u^{\ell}} = J_{\Phi}^{\top} \widetilde{\nabla u^{\ell}} \to J_{\Phi}^{\top} \widetilde{\nabla u} \qquad \text{in } \mathsf{L}^{2}(\Xi).$$

Since $\dot{\nabla}: \dot{H}^1(\Xi) \subset L^2(\Xi) \to L^2(\Xi)$ is closed, we conclude $\widetilde{u} \in \dot{H}^1(\Xi)$ and

$$\nabla \widetilde{u} = J_{\Phi}^{\top} \nabla u.$$

• Let $E \in \dot{\mathsf{C}}^{0,1}(\Omega)$. Then $\widetilde{E} \in \dot{\mathsf{C}}^{0,1}(\Xi)$ and

$$J_{\Phi}^{\top}\widetilde{E} = \nabla \Phi \widetilde{E} = [\nabla \Phi_1 \ \nabla \Phi_2 \ \nabla \Phi_3]\widetilde{E} = \sum_n \widetilde{E}_n \nabla \Phi_n$$

As $\nabla \Phi_n \in R(\nabla) \subset N(\operatorname{rot}) \subset \mathsf{H}(\operatorname{rot}, \Xi)$ we conclude $J_{\Phi}^{\top} \widetilde{E} \in \mathsf{H}(\operatorname{rot}, \Xi)$ and also $J_{\Phi}^{\top} \widetilde{E} \in \dot{\mathsf{H}}(\operatorname{rot}, \Xi)$ by mollification as well as

$$\operatorname{rot}(J_{\Phi}^{\top}\widetilde{E}) = \sum_{n} \nabla \widetilde{E}_{n} \times \nabla \Phi_{n} = \sum_{n} (J_{\Phi}^{\top} \widetilde{\nabla E_{n}}) \times \nabla \Phi_{n}$$

$$(12) \qquad = \sum_{n} ([\nabla \Phi_{1} \ \nabla \Phi_{2} \ \nabla \Phi_{3}] \widetilde{\nabla E_{n}}) \times \nabla \Phi_{n}$$

$$= \sum_{n,m} \widetilde{\partial_{m} E_{n}} \nabla \Phi_{m} \times \nabla \Phi_{n} = \sum_{n < m} (\widetilde{\partial_{m} E_{n}} - \widetilde{\partial_{n} E_{m}}) \nabla \Phi_{m} \times \nabla \Phi_{n}$$

$$= [\nabla \Phi_{2} \times \nabla \Phi_{3} \quad \nabla \Phi_{3} \times \nabla \Phi_{1} \quad \nabla \Phi_{1} \times \nabla \Phi_{2}] \operatorname{rot} \widetilde{E} = (\operatorname{adj} J_{\Phi}) \operatorname{rot} \widetilde{E}.$$

For $E \in \dot{\mathsf{H}}(\mathrm{rot},\Omega)$ we pick a sequence $(E^{\ell}) \subset \dot{\mathsf{C}}^{0,1}(\Omega)$ such that $E^{\ell} \to E$ in $\mathsf{H}(\mathrm{rot},\Omega)$. Then $\widetilde{E^{\ell}} \to \widetilde{E}$ and $\widetilde{\operatorname{rot} E^{\ell}} \to \widetilde{\operatorname{rot} E}$ in $\mathsf{L}^2(\Xi)$. Hence by (12) $J_{\Phi}^{\top} \widetilde{E^{\ell}} \in \dot{\mathsf{H}}(\operatorname{rot}, \Xi)$ with

$$J_{\Phi}^{\top} \widetilde{E^{\ell}} \to J_{\Phi}^{\top} \widetilde{E}, \quad \operatorname{rot}(J_{\Phi}^{\top} \widetilde{E^{\ell}}) = (\operatorname{adj} J_{\Phi}) \operatorname{\widetilde{rot}} \widetilde{E^{\ell}} \to (\operatorname{adj} J_{\Phi}) \operatorname{\widetilde{rot}} \widetilde{E} \qquad \text{in } \mathsf{L}^{2}(\Xi).$$

Since $\operatorname{rot} : \dot{\mathsf{H}}(\operatorname{rot}, \Xi) \subset \mathsf{L}^2(\Xi) \to \mathsf{L}^2(\Xi)$ is closed, we conclude $J_{\Phi}^{\top} \widetilde{E} \in \dot{\mathsf{H}}(\operatorname{rot}, \Xi)$ and

$$\operatorname{rot}(J_{\Phi}^{+}E) = (\operatorname{adj} J_{\Phi}) \operatorname{rot} E$$

$$\operatorname{rot}(J_{\Phi}^{\top} \widetilde{E}) = (\operatorname{adj} J_{\Phi}) \operatorname{\widetilde{rot}} \widetilde{E}.$$

• Let $H \in \dot{\mathsf{C}}^{0,1}(\Omega)$. Then $\widetilde{H} \in \dot{\mathsf{C}}^{0,1}(\Xi)$ and
 $(\operatorname{adj} J_{\Phi})\widetilde{H} = [\nabla \Phi_2 \times \nabla \Phi_3 \quad \nabla \Phi_3 \times \nabla \Phi_1 \quad \nabla \Phi_1 \times \nabla \Phi_2]\widetilde{H} = \sum_{(n,m,l)} \widetilde{H}_n \nabla \Phi_m \times \nabla \Phi_l,$

cf. (12), where the summation is over the three even permutations (n, m, l) of (1, 2, 3). As we have $\nabla \Phi_m \times \nabla \Phi_l = \operatorname{rot}(\Phi_m \nabla \Phi_l) \in R(\operatorname{rot}) \subset N(\operatorname{div}) \subset \mathsf{H}(\operatorname{div}, \Xi) \text{ we conclude } (\operatorname{adj} J_{\Phi})H \in \mathsf{H}(\operatorname{div}, \Xi)$ and thus also $(\operatorname{adj} J_{\Phi})\widetilde{H} \in \dot{\mathsf{H}}(\operatorname{div}, \Xi)$ by mollification as well as

$$\operatorname{div}\left((\operatorname{adj} J_{\Phi})\widetilde{H}\right) = \sum_{(n,m,l)} \nabla \widetilde{H}_{n} \cdot (\nabla \Phi_{m} \times \nabla \Phi_{l}) = \sum_{(n,m,l)} (J_{\Phi}^{\top} \widetilde{\nabla} \widetilde{H}_{n}) \cdot (\nabla \Phi_{m} \times \nabla \Phi_{l})$$

$$(13) \qquad \qquad = \sum_{(n,m,l)} \left([\nabla \Phi_{1} \ \nabla \Phi_{2} \ \nabla \Phi_{3}] \widetilde{\nabla H}_{n} \right) \cdot (\nabla \Phi_{m} \times \nabla \Phi_{l})$$

$$= \sum_{(n,m,l),k} \widetilde{\partial_{k} H_{n}} \nabla \Phi_{k} \cdot (\nabla \Phi_{m} \times \nabla \Phi_{l})$$

$$\overset{k=n}{=} (\operatorname{det} \nabla \Phi) \widetilde{\operatorname{div}} \widetilde{H} = (\operatorname{det} J_{\Phi}) \widetilde{\operatorname{div}} \widetilde{H}.$$

For $H \in \dot{\mathsf{H}}(\operatorname{div},\Omega)$ we pick a sequence $(H^{\ell}) \subset \dot{\mathsf{C}}^{0,1}(\Omega)$ such that $H^{\ell} \to H$ in $\mathsf{H}(\operatorname{div},\Omega)$. Then $\widetilde{H^{\ell}} \to \widetilde{H}$ and $\widetilde{\operatorname{div} H^{\ell}} \to \widetilde{\operatorname{div} H}$ in $L^2(\Xi)$. By (13) we get $(\operatorname{adj} J_{\Phi})\widetilde{H^{\ell}} \in \dot{H}(\operatorname{div}, \Xi)$ and also $(\operatorname{adj} J_{\Phi})\widetilde{H^{\ell}} \to (\operatorname{adj} J_{\Phi})\widetilde{H}$ and $\operatorname{div} ((\operatorname{adj} J_{\Phi})\widetilde{H^{\ell}}) = (\operatorname{det} J_{\Phi})\widetilde{\operatorname{div} H^{\ell}} \to (\operatorname{det} J_{\Phi})\widetilde{\operatorname{div} H}$ in $L^{2}(\Xi)$. Since $\operatorname{div}: \dot{\mathsf{H}}(\operatorname{div},\Xi) \subset \mathsf{L}^2(\Xi) \to \mathsf{L}^2(\Xi)$ is closed, we conclude that it holds $(\operatorname{adj} J_{\Phi})\widetilde{H} \in \dot{\mathsf{H}}(\operatorname{div},\Xi)$ and

$$\operatorname{div}\left((\operatorname{adj} J_{\Phi})\widetilde{H}\right) = (\det J_{\Phi})\operatorname{div} H.$$

• Concerning the inverse operators and L^2 -adjoints we consider, e.g., q = 1. Then using $J_{\Phi^{-1}} = J_{\Phi}^{-1} \circ \Phi^{-1}$ we compute

$$\tau_{\Phi^{-1}}^1 \tau_{\Phi}^1 E = \tau_{\Phi^{-1}}^1 J_{\Phi}^\top \widetilde{E} = J_{\Phi^{-1}}^\top \left((J_{\Phi}^\top \widetilde{E}) \circ \Phi^{-1} \right) = \left(J_{\Phi}^{-\top} J_{\Phi}^\top \widetilde{E} \right) \circ \Phi^{-1} = E_{\Phi^{-1}}^\top J_{\Phi}^\top \widetilde{E} = J_{\Phi^{-1}}^\top J_{\Phi}^\top J_{\Phi}^\top \widetilde{E} = J_{\Phi^{-1}}^\top J_{\Phi}^\top J_{\Phi}^\top \widetilde{E} = J_{\Phi^{-1}}^\top$$

i.e.,
$$(\tau_{\Phi}^1)^{-1} = \tau_{\Phi^{-1}}^1$$
, and

$$\begin{aligned} \langle \tau_{\Phi}^{1} E, \Psi \rangle_{\mathsf{L}^{2}(\Xi)} &= \langle J_{\Phi}^{\top} \widetilde{E}, \Psi \rangle_{\mathsf{L}^{2}(\Xi)} = \left\langle E, (\det J_{\Phi^{-1}}) (J_{\Phi} \Psi) \circ \Phi^{-1} \right\rangle_{\mathsf{L}^{2}(\Omega)} \\ &= \left\langle E, (\det J_{\Phi^{-1}}) J_{\Phi^{-1}}^{-1} (\Psi \circ \Phi^{-1}) \right\rangle_{\mathsf{L}^{2}(\Omega)} \\ &= \left\langle E, (\operatorname{adj} J_{\Phi^{-1}}) (\Psi \circ \Phi^{-1}) \right\rangle_{\mathsf{L}^{2}(\Omega)} = \langle E, \tau_{\Phi^{-1}}^{2} \Psi \rangle_{\mathsf{L}^{2}(\Omega)}, \end{aligned}$$

i.e., $(\tau_{\Phi}^1)^* = \tau_{\Phi^{-1}}^2$.

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Remark 2.17 (transformation theorem). For the divergence there is also a duality argument leading to the result of Theorem 2.16. For this, let $H \in H(\operatorname{div}, \Omega)$ and pick some $\psi \in \mathring{C}^{0,1}(\Xi)$. Then $\phi := \psi \circ \Phi^{-1} \in \mathring{C}^{0,1}(\Omega)$ and $\tilde{\phi} = \psi$. By (11) we compute

$$\langle H, \nabla \phi \rangle_{\mathsf{L}^{2}(\Omega)} = -\langle \operatorname{div} H, \phi \rangle_{\mathsf{L}^{2}(\Omega)} = -\langle (\det J_{\Phi}) \operatorname{div} H, \psi \rangle_{\mathsf{L}^{2}(\Xi)}$$
$$= \langle (\det J_{\Phi}) \widetilde{H}, \widetilde{\nabla \phi} \rangle_{\mathsf{L}^{2}(\Xi)} = \langle (\det J_{\Phi}) \widetilde{H}, J_{\Phi}^{-\top} \nabla \widetilde{\phi} \rangle_{\mathsf{L}^{2}(\Xi)} = \langle (\operatorname{adj} J_{\Phi}) \widetilde{H}, \nabla \psi \rangle_{\mathsf{L}^{2}(\Xi)}.$$

Hence, $(\operatorname{adj} J_{\Phi})\widetilde{H} \in \mathsf{H}(\operatorname{div}, \Xi)$ and $\operatorname{div}((\operatorname{adj} J_{\Phi})\widetilde{H}) = (\operatorname{det} J_{\Phi})\widetilde{\operatorname{div} H}$.

2.4.1. Transformation Theorem for Maxwell Operators. Often material properties/constitutive laws (dielectricity and permeability) enter Maxwell's equations. Hence we introduce tensor fields $\varepsilon, \mu \in \mathsf{L}^{\infty}(\Omega, \mathbb{R}^{3\times 3})$ which are symmetric and positive with respect to the $\mathsf{L}^{2}(\Omega)$ -inner product. Such matrix fields ε, μ shall be called *admissible*. A positive function $\nu \in \mathsf{L}^{\infty}(\Omega, \mathbb{R}^{3})$ is also called admissible.

In Section 2.3 we introduced the primal and dual vector de Rham Hilbert complex (10). This complex can easily be modified by inserting material properties, i.e., admissible tensor-valued fields $\varepsilon, \mu, \nu, \theta$. Then we arrive at the primal and dual vector de Rham Hilbert complex

(14)
$$\mathsf{L}^{2}_{\nu}(\Omega) \xrightarrow[-\nu^{-1}\operatorname{div}\varepsilon]{\varepsilon} \mathsf{L}^{2}_{\varepsilon}(\Omega) \xrightarrow[\varepsilon^{-1}\operatorname{rot}]{\mu^{-1}\operatorname{rot}} \mathsf{L}^{2}_{\mu}(\Omega) \xrightarrow[\varepsilon^{-1}\operatorname{div}\mu]{\theta^{-1}\operatorname{div}\mu} \mathsf{L}^{2}_{\theta}(\Omega),$$

consisting again of densely defined and closed linear operators and their adjoints.

Corollary 2.18 (transformation theorem for Maxwell operators). Let ε be admissible and let $E \in \mathring{H}(\operatorname{rot}, \Omega) \cap \varepsilon^{-1} H(\operatorname{div}, \Omega)$ resp. $E \in H(\operatorname{rot}, \Omega) \cap \varepsilon^{-1} \mathring{H}(\operatorname{div}, \Omega)$. Then

$$\tau_{\Phi}^{1}E \in \mathring{\mathsf{H}}(\mathrm{rot},\Xi) \cap \varepsilon_{\Phi}^{-1}\mathsf{H}(\mathrm{div},\Xi) \quad \textit{resp.} \quad \tau_{\Phi}^{1}E \in \mathsf{H}(\mathrm{rot},\Xi) \cap \varepsilon_{\Phi}^{-1}\mathring{\mathsf{H}}(\mathrm{div},\Xi)$$

and it holds

$$\operatorname{rot} \tau_{\Phi}^{1} E = \tau_{\Phi}^{2} \operatorname{rot} E, \qquad \operatorname{div} \varepsilon_{\Phi} \tau_{\Phi}^{1} E = \tau_{\Phi}^{3} \operatorname{div} \varepsilon E, \qquad \varepsilon_{\Phi} \tau_{\Phi}^{1} = \tau_{\Phi}^{2} \varepsilon$$

with $\varepsilon_{\Phi} := \tau_{\Phi}^2 \varepsilon \tau_{\Phi^{-1}}^1 = (\det J_{\Phi}) J_{\Phi}^{-1} \widetilde{\varepsilon} J_{\Phi}^{-\top} = (\operatorname{adj} J_{\Phi}) \widetilde{\varepsilon} J_{\Phi}^{-\top}$. Moreover,

$$\begin{split} \tau_{\Phi}^{1}: \mathsf{H}(\mathrm{rot},\Omega) \cap \varepsilon^{-1}\mathsf{H}(\mathrm{div},\Omega) \to \mathsf{H}(\mathrm{rot},\Xi) \cap \varepsilon_{\Phi}^{-1}\mathsf{H}(\mathrm{div},\Xi) \\ resp. \quad \tau_{\Phi}^{1}: \mathsf{H}(\mathrm{rot},\Omega) \cap \varepsilon^{-1}\mathring{\mathsf{H}}(\mathrm{div},\Omega) \to \mathsf{H}(\mathrm{rot},\Xi) \cap \varepsilon_{\Phi}^{-1}\mathring{\mathsf{H}}(\mathrm{div},\Xi) \end{split}$$

is a topological isomorphism with norm depending on Ξ , ε , and J_{Φ} . Its inverse is given by $\tau_{\Phi^{-1}}^1$.

Proof. Using Theorem 2.16 we compute for $\varepsilon E \in \dot{\mathsf{H}}(\operatorname{div}, \Omega)$

$$\tau_{\Phi}^{3}\operatorname{div}\varepsilon E = \operatorname{div}\tau_{\Phi}^{2}\varepsilon E = \operatorname{div}\tau_{\Phi}^{2}\varepsilon\tau_{\Phi^{-1}}^{1}\tau_{\Phi}^{1}E = \operatorname{div}\varepsilon_{\Phi}\tau_{\Phi}^{1}E$$

with $\varepsilon_{\Phi} = \tau_{\Phi}^2 \varepsilon \tau_{\Phi^{-1}}^1 = (\operatorname{adj} J_{\Phi}) \varepsilon \tau_{\Phi^{-1}}^1 = (\operatorname{adj} J_{\Phi}) \widetilde{\varepsilon} J_{\Phi}^{-\top} = (\det J_{\Phi}) J_{\Phi}^{-1} \widetilde{\varepsilon} J_{\Phi}^{-\top}.$

Remark 2.19 (transformation theorem). More explicitly, in Theorem 2.16 and Corollary 2.18 it holds

$$\begin{aligned} \forall u \in \dot{\mathsf{H}}^{1}(\Omega) & \nabla \widetilde{u} = J_{\Phi}^{\top} \widetilde{\nabla u}, \\ \forall E \in \dot{\mathsf{H}}(\operatorname{rot}, \Omega) & \operatorname{rot}(J_{\Phi}^{\top} \widetilde{E}) = (\operatorname{adj} J_{\Phi}) \operatorname{rot} \widetilde{E}, \\ \forall H \in \dot{\mathsf{H}}(\operatorname{div}, \Omega) & \operatorname{div}\left((\operatorname{adj} J_{\Phi}) \widetilde{H}\right) = (\operatorname{det} J_{\Phi}) \operatorname{div} \widetilde{H}, \\ \forall E \in \varepsilon^{-1} \dot{\mathsf{H}}(\operatorname{div}, \Omega) & \operatorname{div}(\varepsilon_{\Phi} J_{\Phi}^{\top} \widetilde{E}) = (\operatorname{det} J_{\Phi}) \operatorname{div} \widetilde{\varepsilon_{E}}. \end{aligned}$$

2.4.2. Transformation Theorem for Laplacians.

Corollary 2.20 (transformation theorem for Laplacians). Let ε and ν be admissible. Moreover, let $u \in D(\operatorname{div}_{\Omega} \varepsilon \overset{\circ}{\nabla}_{\Omega})$ resp. $u \in D(\operatorname{div}_{\Omega} \varepsilon \nabla_{\Omega})$, i.e.,

$$u \in \mathring{H}^{1}(\Omega), \quad \varepsilon \nabla u \in \mathsf{H}(\operatorname{div}, \Omega) \quad resp. \quad u \in \mathsf{H}^{1}(\Omega), \quad \varepsilon \nabla u \in \mathring{\mathsf{H}}(\operatorname{div}, \Omega)$$

Then $\tau_{\Phi}^{0} u \in D(\operatorname{div}_{\Xi} \varepsilon_{\Phi} \overset{\circ}{\nabla}_{\Xi})$ resp. $\tau_{\Phi}^{0} u \in D(\operatorname{div}_{\Xi} \varepsilon_{\Phi} \nabla_{\Xi})$, i.e.,

$$\begin{split} \tau_{\Phi}^{0} u \in \mathring{H}^{1}(\Xi), \quad \varepsilon_{\Phi} \, \nabla \, \tau_{\Phi}^{0} u \in \mathsf{H}(\mathrm{div}, \Xi) \quad resp. \quad \tau_{\Phi}^{0} u \in \mathsf{H}^{1}(\Xi), \quad \varepsilon_{\Phi} \, \nabla \, \tau_{\Phi}^{0} u \in \mathring{\mathsf{H}}(\mathrm{div}, \Xi), \\ and \ \nu_{\Phi}^{-1} \operatorname{div} \varepsilon_{\Phi} \, \nabla \, \tau_{\Phi}^{0} u = \tau_{\Phi}^{0} \nu^{-1} \operatorname{div} \varepsilon \, \nabla \, u \ with \ \nu_{\Phi} := \tau_{\Phi}^{3} \nu \tau_{\Phi^{-1}}^{0} = (\det J_{\Phi}) \widetilde{\nu}. \ Moreover, \end{split}$$

$$\begin{aligned} \tau_{\Phi}^{0} : D(\operatorname{div}_{\Omega} \varepsilon \overset{\circ}{\nabla}_{\Omega}) \to D(\operatorname{div}_{\Xi} \varepsilon_{\Phi} \overset{\circ}{\nabla}_{\Xi}) \\ resp. \quad \tau_{\Phi}^{0} : D(\operatorname{div}_{\Omega} \varepsilon \nabla_{\Omega}) \to D(\operatorname{div}_{\Xi} \varepsilon_{\Phi} \nabla_{\Xi}) \end{aligned}$$

is a topological isomorphism with norm depending on Ξ , ε , and J_{Φ} . Its inverse is given by $\tau_{\Phi^{-1}}^0$.

Proof. Let $E := \nabla u$. Then $E \in \mathring{H}(\operatorname{rot}, \Omega) \cap \varepsilon^{-1} \mathsf{H}(\operatorname{div}, \Omega)$ resp. $E \in \mathsf{H}(\operatorname{rot}, \Omega) \cap \varepsilon^{-1} \mathring{H}(\operatorname{div}, \Omega)$. Theorem 2.16 and Corollary 2.18 show $\tau_{\Phi}^{0} u \in \mathring{H}^{1}(\Xi)$ resp. $\tau_{\Phi}^{0} u \in \mathsf{H}^{1}(\Xi)$ with $\tau_{\Phi}^{1} \nabla u = \nabla \tau_{\Phi}^{0} u$ and

$$\tau_{\Phi}^{1}E \in \mathring{\mathsf{H}}(\mathrm{rot},\Xi) \cap \varepsilon_{\Phi}^{-1}\mathsf{H}(\mathrm{div},\Xi) \quad \mathrm{resp.} \quad \tau_{\Phi}^{1}E \in \mathsf{H}(\mathrm{rot},\Xi) \cap \varepsilon_{\Phi}^{-1}\mathring{\mathsf{H}}(\mathrm{div},\Xi)$$

with
$$\tau_{\Phi}^{0}\nu^{-1} \operatorname{div} \varepsilon \nabla u = \nu_{\Phi}^{-1} \tau_{\Phi}^{3} \operatorname{div} \varepsilon \nabla u = \nu_{\Phi}^{-1} \operatorname{div} \varepsilon_{\Phi} \tau_{\Phi}^{1} \nabla u = \nu_{\Phi}^{-1} \operatorname{div} \varepsilon_{\Phi} \nabla \tau_{\Phi}^{0} u.$$

2.4.3. Transformation Theorems for Vector Laplacians.

Corollary 2.21 (transformation theorem for vector Laplacians). Let ε and μ be admissible. Moreover, let $E \in D(\operatorname{rot}_{\Omega} \varepsilon^{-1} \operatorname{rot}_{\Omega})$ resp. $E \in D(\operatorname{rot}_{\Omega} \varepsilon^{-1} \operatorname{rot}_{\Omega})$, i.e.,

$$E \in \dot{\mathsf{H}}(\mathrm{rot},\Omega), \quad \varepsilon^{-1} \operatorname{rot} E \in \mathsf{H}(\mathrm{rot},\Omega) \quad resp. \quad E \in \mathsf{H}(\mathrm{rot},\Omega), \quad \varepsilon^{-1} \operatorname{rot} E \in \dot{\mathsf{H}}(\mathrm{rot},\Omega).$$

Then $\tau_{\Phi}^{1} E \in D(\operatorname{rot}_{\Xi} \varepsilon_{\Phi}^{-1} \operatorname{rot}_{\Xi}) \quad resp. \quad \tau_{\Phi}^{1} E \in D(\operatorname{rot}_{\Xi} \varepsilon_{\Phi}^{-1} \operatorname{rot}_{\Xi}), \ i.e.,$

 $\tau_{\Phi}^{1}E \in \mathring{\mathsf{H}}(\operatorname{rot}, \Xi), \quad \varepsilon_{\Phi}^{-1}\operatorname{rot}\tau_{\Phi}^{1}E \in \mathsf{H}(\operatorname{rot}, \Xi) \quad resp. \quad \tau_{\Phi}^{1}E \in \mathsf{H}(\operatorname{rot}, \Xi), \quad \varepsilon_{\Phi}^{-1}\operatorname{rot}\tau_{\Phi}^{1}E \in \mathring{\mathsf{H}}(\operatorname{rot}, \Xi),$ and $\mu_{\Phi}^{-1}\operatorname{rot}\varepsilon_{\Phi}^{-1}\operatorname{rot}\tau_{\Phi}^{1}E = \tau_{\Phi}^{1}\mu^{-1}\operatorname{rot}\varepsilon^{-1}\operatorname{rot} E \quad Moreover,$

$$\begin{aligned} \tau_{\Phi}^{1} &: D(\operatorname{rot}_{\Omega} \varepsilon^{-1} \operatorname{rot}_{\Omega}) \to D(\operatorname{rot}_{\Xi} \varepsilon_{\Phi}^{-1} \operatorname{rot}_{\Xi}) \\ sp. \quad \tau_{\Phi}^{1} &: D(\operatorname{rot}_{\Omega} \varepsilon^{-1} \operatorname{rot}_{\Omega}) \to D(\operatorname{rot}_{\Xi} \varepsilon_{\Phi}^{-1} \operatorname{rot}_{\Xi}) \end{aligned}$$

is a topological isomorphism with norm depending on Ξ , ε , and J_{Φ} . Its inverse is given by $\tau_{\Phi^{-1}}^1$.

Proof. Let $H := \varepsilon^{-1} \operatorname{rot} E$. Then $H \in \varepsilon^{-1} \mathring{\mathsf{H}}(\operatorname{div}, \Omega) \cap \mathsf{H}(\operatorname{rot}, \Omega)$ resp. $H \in \varepsilon^{-1} \mathsf{H}(\operatorname{div}, \Omega) \cap \mathring{\mathsf{H}}(\operatorname{rot}, \Omega)$. Theorem 2.16 shows $\tau_{\Phi}^1 E \in \mathring{\mathsf{H}}(\operatorname{rot}, \Xi)$ resp. $\tau_{\Phi}^1 E \in \mathsf{H}(\operatorname{rot}, \Xi)$ with $\tau_{\Phi}^2 \operatorname{rot} E = \operatorname{rot} \tau_{\Phi}^1 E$ and Corollary 2.18 yields

$$\tau_{\Phi}^{1}H \in \varepsilon_{\Phi}^{-1}\mathsf{H}(\operatorname{div},\Xi) \cap \mathsf{H}(\operatorname{rot},\Xi) \quad \operatorname{resp.} \quad \tau_{\Phi}^{1}H \in \varepsilon_{\Phi}^{-1}\mathsf{H}(\operatorname{div},\Xi) \cap \mathsf{H}(\operatorname{rot},\Xi)$$
$$\iota^{-1}\operatorname{rot}\varepsilon^{-1}\operatorname{rot} E = \mu_{\Phi}^{-1}\tau_{\Phi}^{2}\operatorname{rot}\varepsilon^{-1}\operatorname{rot} E = \mu_{\Phi}^{-1}\operatorname{rot}\tau_{\Phi}^{1}\varepsilon^{-1}\tau_{\Phi}^{2}\operatorname{rot} E = \mu_{\Phi}^{-1}\operatorname{rot}\tau_{\Phi}^{1}E$$

with $\tau_{\Phi}^{1}\mu^{-1}$ rot ε^{-1} rot $E = \mu_{\Phi}^{-1}\tau_{\Phi}^{2}$ rot ε^{-1} rot $E = \mu_{\Phi}^{-1}$ rot $\tau_{\Phi}^{1}\varepsilon^{-1}\tau_{\Phi}^{2}$ rot $E = \mu_{\Phi}^{-1}$ rot ε_{Φ}^{-1} rot $\tau_{\Phi}^{1}E$ as $\varepsilon_{\Phi}^{-1} = \tau_{\Phi}^{1}\varepsilon^{-1}\tau_{\Phi^{-1}}^{2}$. Note that $\tau_{\Phi}^{1}H = \tau_{\Phi}^{1}\varepsilon^{-1}$ rot $E = \varepsilon_{\Phi}^{-1}\tau_{\Phi}^{2}$ rot $E = \varepsilon_{\Phi}^{-1}$ rot $\tau_{\Phi}^{1}E$.

Corollary 2.22 (transformation theorem for vector Laplacians). Let ε and ν^{-1} be admissible. Moreover, let $E \in D(\nabla_{\Omega} \nu^{-1} \operatorname{div}_{\Omega} \varepsilon)$ resp. $E \in D(\mathring{\nabla}_{\Omega} \nu^{-1} \operatorname{div}_{\Omega} \varepsilon)$, i.e.,

$$\varepsilon E \in \mathring{H}(\operatorname{div}, \Omega), \quad \nu^{-1} \operatorname{div} \varepsilon E \in H^{1}(\Omega) \quad resp. \quad \varepsilon E \in H(\operatorname{div}, \Omega), \quad \nu^{-1} \operatorname{div} \varepsilon E \in \mathring{H}^{1}(\Omega).$$

Then $\tau_{\Phi}^1 E \in D(\nabla_{\Xi} \nu_{\Phi}^{-1} \operatorname{div}_{\Xi} \varepsilon_{\Phi})$ resp. $\tau_{\Phi}^1 E \in D(\mathring{\nabla}_{\Xi} \nu_{\Phi}^{-1} \operatorname{div}_{\Xi} \varepsilon_{\Phi})$, *i.e.*,

$$\varepsilon_{\Phi}\tau_{\Phi}^{\perp}E \in \mathsf{H}(\operatorname{div},\Xi), \qquad \qquad \nu_{\Phi}^{-1}\operatorname{div}\varepsilon_{\Phi}\tau_{\Phi}^{\perp}E \in \mathsf{H}^{1}(\Xi)$$
$$p. \quad \varepsilon_{\Phi}\tau_{\Phi}^{\perp}E \in \mathsf{H}(\operatorname{div},\Xi), \qquad \qquad \nu_{\Phi}^{-1}\operatorname{div}\varepsilon_{\Phi}\tau_{\Phi}^{\perp}E \in \mathring{\mathsf{H}}^{1}(\Xi),$$

and $\nabla \nu_{\Phi}^{-1} \operatorname{div} \varepsilon_{\Phi} \tau_{\Phi}^{1} E = \tau_{\Phi}^{1} \nabla \nu^{-1} \operatorname{div} \varepsilon E$. Moreover,

res

$$\tau_{\Phi}^{1}: D(\nabla_{\Omega} \nu^{-1} \operatorname{div}_{\Omega} \varepsilon) \to D(\nabla_{\Xi} \nu_{\Phi}^{-1} \operatorname{div}_{\Xi} \varepsilon_{\Phi})$$

resp.
$$\tau_{\Phi}^{1}: D(\mathring{\nabla}_{\Omega} \nu^{-1} \operatorname{div}_{\Omega} \varepsilon) \to D(\mathring{\nabla}_{\Xi} \nu_{\Phi}^{-1} \operatorname{div}_{\Xi} \varepsilon_{\Phi})$$

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is a topological isomorphism with norm depending on Ξ , ε , ν^{-1} , and J_{Φ} . Its inverse is given by $\tau_{\Phi^{-1}}^1$.

Proof. Let $u := \nu^{-1} \operatorname{div} \varepsilon E$. Then $u \in \mathsf{H}^1(\Omega)$ resp. $u \in \mathring{\mathsf{H}}^1(\Omega)$. Theorem 2.16 and Corollary 2.18 show $\tau_{\Phi}^0 u \in \mathring{\mathsf{H}}^1(\Xi)$ resp. $\tau_{\Phi}^0 u \in \mathsf{H}^1(\Xi)$ with $\tau_{\Phi}^1 \nabla u = \nabla \tau_{\Phi}^0 u$ and

$$\varepsilon_{\Phi}\tau_{\Phi}^{1}E \in \mathring{\mathsf{H}}(\operatorname{div},\Xi)$$
 resp. $\varepsilon_{\Phi}\tau_{\Phi}^{1}E \in \mathsf{H}(\operatorname{div},\Xi)$

with $\tau_{\Phi}^1 \nabla \nu^{-1} \operatorname{div} \varepsilon E = \nabla \tau_{\Phi}^0 \nu^{-1} \operatorname{div} \varepsilon E = \nabla \nu_{\Phi}^{-1} \tau_{\Phi}^3 \operatorname{div} \varepsilon E = \nabla \nu_{\Phi}^{-1} \operatorname{div} \varepsilon_{\Phi} \tau_{\Phi}^1 E.$

Corollary 2.23 (transformation theorem for vector Laplacians). Let ε , μ , and ν^{-1} be admissible. Moreover, let

$$E \in D(\varepsilon^{-1}\operatorname{rot}_{\Omega}\mu^{-1}\operatorname{rot}_{\Omega}-\nabla_{\Omega}\nu^{-1}\operatorname{div}_{\Omega}\varepsilon)$$

resp. $H \in D(\mu^{-1}\operatorname{rot}_{\Omega}\varepsilon^{-1}\operatorname{rot}_{\Omega}-\nabla_{\Omega}\nu^{-1}\operatorname{div}_{\Omega}\mu)$

i.e., $E \in \mathring{H}(\operatorname{rot}, \Omega) \cap \varepsilon^{-1} \mathbb{H}(\operatorname{div}, \Omega)$ and $\mu^{-1} \operatorname{rot} E \in \mathbb{H}(\operatorname{rot}, \Omega)$, $\nu^{-1} \operatorname{div} \varepsilon E \in \mathring{H}^{1}(\Omega)$ resp. $H \in \mathbb{H}(\operatorname{rot}, \Omega) \cap \mu^{-1} \mathring{H}(\operatorname{div}, \Omega)$ and $\varepsilon^{-1} \operatorname{rot} H \in \mathring{H}(\operatorname{rot}, \Omega)$, $\nu^{-1} \operatorname{div} \mu H \in \mathbb{H}^{1}(\Omega)$. Then

$$\tau_{\Phi}^{1}E \in D(\varepsilon_{\Phi}^{-1}\operatorname{rot}_{\Xi}\mu_{\Phi}^{-1}\operatorname{rot}_{\Xi}-\check{\nabla}_{\Xi}\nu_{\Phi}^{-1}\operatorname{div}_{\Xi}\varepsilon_{\Phi})$$

esp.
$$\tau_{\Phi}^{1}H \in D(\mu_{\Phi}^{-1}\operatorname{rot}_{\Xi}\varepsilon_{\Phi}^{-1}\operatorname{rot}_{\Xi}-\nabla_{\Xi}\nu_{\Phi}^{-1}\operatorname{div}_{\Xi}\mu_{\Phi}),$$

i.e., $\tau_{\Phi}^{1}E \in \mathring{H}(\operatorname{rot},\Xi) \cap \varepsilon_{\Phi}^{-1}\mathsf{H}(\operatorname{div},\Xi)$ and $\mu_{\Phi}^{-1}\operatorname{rot}\tau_{\Phi}^{1}E \in \mathsf{H}(\operatorname{rot},\Xi)$, $\nu_{\Phi}^{-1}\operatorname{div}\varepsilon_{\Phi}\tau_{\Phi}^{1}E \in \mathring{H}^{1}(\Xi)$ resp. $\tau_{\Phi}^{1}H \in \mathsf{H}(\operatorname{rot},\Xi) \cap \mu_{\Phi}^{-1}\mathring{H}(\operatorname{div},\Xi)$ and $\varepsilon_{\Phi}^{-1}\operatorname{rot}\tau_{\Phi}^{1}H \in \mathring{H}(\operatorname{rot},\Xi)$, $\nu_{\Phi}^{-1}\operatorname{div}\varepsilon_{\Phi}\tau_{\Phi}^{1}H \in \mathsf{H}^{1}(\Xi)$ as well as

$$(\varepsilon_{\Phi}^{-1} \operatorname{rot} \mu_{\Phi}^{-1} \operatorname{rot} - \nabla \nu_{\Phi}^{-1} \operatorname{div} \varepsilon_{\Phi}) \tau_{\Phi}^{1} = \tau_{\Phi}^{1} (\varepsilon^{-1} \operatorname{rot} \mu^{-1} \operatorname{rot} - \nabla \nu^{-1} \operatorname{div} \varepsilon).$$

Moreover,

$$\tau_{\Phi}^{1}: D(\varepsilon^{-1}\operatorname{rot}_{\Omega}\mu^{-1}\operatorname{rot}_{\Omega}-\overset{\circ}{\nabla}_{\Omega}\nu^{-1}\operatorname{div}_{\Omega}\varepsilon) \to D(\varepsilon_{\Phi}^{-1}\operatorname{rot}_{\Xi}\mu_{\Phi}^{-1}\operatorname{rot}_{\Xi}-\overset{\circ}{\nabla}_{\Xi}\nu_{\Phi}^{-1}\operatorname{div}_{\Xi}\varepsilon_{\Phi})$$

resp.
$$\tau_{\Phi}^{1}: D(\mu^{-1}\operatorname{rot}_{\Omega}\varepsilon^{-1}\operatorname{rot}_{\Omega}-\nabla_{\Omega}\nu^{-1}\operatorname{div}_{\Omega}\mu) \to D(\mu_{\Phi}^{-1}\operatorname{rot}_{\Xi}\varepsilon_{\Phi}^{-1}\operatorname{rot}_{\Xi}-\nabla_{\Xi}\nu_{\Phi}^{-1}\operatorname{div}_{\Xi}\mu_{\Phi})$$

 \square

is a topological isomorphism with norm depending on Ξ , ε , μ , ν^{-1} , and J_{Φ} with inverse $\tau^{1}_{\Phi^{-1}}$.

Proof. Corollary 2.21 and Corollary 2.22 yield the results.

2.5. The De Rham Complex under Lipschitz Transformations.

2.5.1. Closed Ranges under Lipschitz Transformations. Recall Section 2.4 and suppose that we have an admissible bi-Lipschitz transformation $\Phi : \Xi \to \Phi(\Xi) = \Omega$. We consider, e.g.,

$$\operatorname{rot}_{\Omega}: D(\operatorname{rot}_{\Omega}) \subset \mathsf{L}^{2}(\Omega) \to \mathsf{L}^{2}(\Omega), \quad \operatorname{rot}_{\Xi}: D(\operatorname{rot}_{\Xi}) \subset \mathsf{L}^{2}(\Xi) \to \mathsf{L}^{2}(\Xi).$$

Lemma 2.24 (range invariance). $R(\operatorname{rot}_{\Omega})$ is closed if and only if $R(\operatorname{rot}_{\Xi})$ is closed. The same holds for $R(\nabla_{\Omega})$, $R(\operatorname{div}_{\Omega})$, and $R(\overset{\circ}{\nabla}_{\Omega})$, $R(\operatorname{rot}_{\Omega})$, $R(\operatorname{div}_{\Omega})$.

Proof. Assume that $R(\operatorname{rot}_{\Xi})$ is closed and let $E_n \in D(\operatorname{rot}_{\Omega}) = \mathsf{H}(\operatorname{rot},\Omega)$ be a sequence such that $\operatorname{rot} E_n \to F$ in $\mathsf{L}^2(\Omega)$. By Theorem 2.16 $\tau_{\Phi}^1 E_n \in \mathsf{H}(\operatorname{rot},\Xi)$ and $\operatorname{rot} \tau_{\Phi}^1 E_n = \tau_{\Phi}^2 \operatorname{rot} E_n \to \tau_{\Phi}^2 F$ in $\mathsf{L}^2(\Xi)$. As $R(\operatorname{rot}_{\Xi})$ is closed we get $\tau_{\Phi}^2 F = \operatorname{rot} H \in R(\operatorname{rot}_{\Xi})$ with $H \in D(\operatorname{rot}_{\Xi}) = \mathsf{H}(\operatorname{rot},\Xi)$. Then $\tau_{\Phi^{-1}}^1 H \in \mathsf{H}(\operatorname{rot},\Omega) = D(\operatorname{rot}_{\Omega})$ and $\operatorname{rot} \tau_{\Phi^{-1}}^1 H = \tau_{\Phi^{-1}}^2$ rot H = F by Theorem 2.16 and thus $F \in R(\operatorname{rot}_{\Omega})$.

Similarly, we see the corresponding results for $R(\nabla_{\Omega})$ and $R(\operatorname{div}_{\Omega})$. Duality (Lemma 2.1) yields the assertions for $R(\mathring{\nabla}_{\Omega})$, $R(\operatorname{rot}_{\Omega})$, $R(\operatorname{div}_{\Omega})$.

2.5.2. Dirichlet/Neumann fields under Lipschitz Transformations. In Section 2.3 and Section 2.4.1 we introduced the primal and dual vector de Rham Hilbert complexes (10) and (14). According to this we define the Dirichlet and Neumann fields (cohomology groups) by

$$\mathcal{H}_{\mathsf{D},\varepsilon}(\Omega) = N(\mathring{\mathrm{rot}}) \cap N(\operatorname{div} \varepsilon), \qquad \qquad \mathcal{H}_{\mathsf{N},\mu}(\Omega) = N(\operatorname{rot}) \cap N(\operatorname{div} \mu).$$

Lemma 2.25 (dimension invariance of Dirichlet/Neumann fields). The dimension of the Dirichlet/Neumann fields is independent

• of the material properties

• and under bi-Lipschitz transformations of the domain.

More precisely: Let $\Phi: \Xi \to \Phi(\Xi) = \Omega$ be an admissible bi-Lipschitz transformation and let ε and μ be admissible. Then

$$\dim \mathcal{H}_{\mathsf{D}/\mathsf{N},\varepsilon}(\Omega) = \dim \mathcal{H}_{\mathsf{D}/\mathsf{N},\mathrm{id}}(\Omega) = \dim \mathcal{H}_{\mathsf{D}/\mathsf{N},\mathrm{id}}(\Xi) = \dim \mathcal{H}_{\mathsf{D}/\mathsf{N},\mu}(\Xi).$$

Proof. Independence of ε is well known, cf. [24, Lemma 5.1, Remark 6.11] for a general proof. For convenience and as an example we give here a particular proof of

$$\dim \mathcal{H}_{\mathsf{D},\mu}(\Omega) = \dim \mathcal{H}_{\mathsf{D},\varepsilon}(\Omega).$$

For $A := \check{\nabla}$ and $A^* = -$ div the Helmholtz type decomposition (2) reads

$$\mathsf{L}^{2}_{\varepsilon}(\Omega) = R(\mathring{\nabla}) \oplus_{\mathsf{L}^{2}_{\varepsilon}(\Omega)} N(\operatorname{div} \varepsilon).$$

By the complex property $R(\check{\nabla}) \subset N(\mathring{rot})$ we have the refinement

(15)
$$N(\operatorname{rot}) = R(\check{\nabla}) \oplus_{\mathsf{L}^2_{\varepsilon}(\Omega)} \mathcal{H}_{\mathsf{D},\varepsilon}(\Omega).$$

cf. (6). Let us denote the orthogonal projector onto $N(\operatorname{div} \varepsilon)$ resp. $\mathcal{H}_{\mathsf{D},\varepsilon}(\Omega)$ by π . Then

$$\pi_{\mathcal{H}}: \mathcal{H}_{\mathsf{D},\mu}(\Omega) \to \mathcal{H}_{\mathsf{D},\varepsilon}(\Omega); \quad H \to \pi H$$

is well defined and injective as $\langle \mu H, H \rangle_{\mathsf{L}^{2}(\Omega)} = 0$ for $H \in \mathcal{H}_{\mathsf{D},\mu}(\Omega) \subset N(\operatorname{div} \mu)$ with $\pi H = 0$, i.e., $H \in \overline{R(\overset{\circ}{\nabla})}$. Hence $\operatorname{dim} \mathcal{H}_{\mathsf{D},\mu}(\Omega) \leq \operatorname{dim} \mathcal{H}_{\mathsf{D},\varepsilon}(\Omega)$ and by symmetry $\operatorname{dim} \mathcal{H}_{\mathsf{D},\mu}(\Omega) = \operatorname{dim} \mathcal{H}_{\mathsf{D},\varepsilon}(\Omega)$. Now we show the independence under Lipschitz transformations. Let $E \in \mathcal{H}_{\mathsf{D},\varepsilon}(\Omega)$. Corollary 2.18 yields

$$\tau_{\Phi}^{1}E \in N(\operatorname{rot}) \cap N(\operatorname{div} \varepsilon_{\Phi}) = \mathcal{H}_{\mathsf{D},\varepsilon_{\Phi}}(\Xi)$$

Thus

$$\tau_{\Phi}^{1}: \mathcal{H}_{\mathsf{D},\varepsilon}(\Omega) \to \mathcal{H}_{\mathsf{D},\varepsilon_{\Phi}}(\Xi)$$

is well defined and injective. Hence $\dim \mathcal{H}_{\mathsf{D},\varepsilon}(\Omega) \leq \mathcal{H}_{\mathsf{D},\varepsilon\Phi}(\Xi) = \mathcal{H}_{\mathsf{D},\mu}(\Xi)$ by the previous independence on μ . Symmetry yields $\dim \mathcal{H}_{\mathsf{D},\varepsilon}(\Omega) = \mathcal{H}_{\mathsf{D},\mu}(\Xi)$.

The same (similar) proofs work for the Neumann fields as well.

Lemma 2.26 (trivial Dirichlet/Neumann fields). Let Ω be convex and either bounded or of infinite volume. Then $\mathcal{H}_{D,id}(\Omega) = \mathcal{H}_{N,id}(\Omega) = \{0\}$

Note that the result is well known for bounded Ω .

Proof. Let $E \in \mathcal{H}_{\mathsf{D},\mathrm{id}}(\Omega) \cup \mathcal{H}_{\mathsf{N},\mathrm{id}}(\Omega)$. Lemma 2.14 yields that $E \in \mathsf{H}^1(\Omega)$ is constant. Hence E = 0 due to the boundary conditions in case Ω is bounded and E = 0 due to integrability in case Ω is unbounded.

3. Main Results

3.1. Friedrichs/Gaffney Type Estimates and Closed Ranges for Cylinders.

3.1.1. Friedrichs/Gaffney Type Estimates.

Lemma 3.1 (tangential Friedrichs/Gaffney estimate for rot and div). Let Ω_2 be as above and let $E \in \mathring{H}(rot, \Omega_2) \cap H(\operatorname{div}, \Omega_2)$. Then $E \in H^1(\Omega_2)$ and

$$|E|^{2}_{\mathsf{L}^{2}(\Omega_{2})} \leq c_{d}^{2} |\nabla E|^{2}_{\mathsf{L}^{2}(\Omega_{2})} = c_{d}^{2} (|\operatorname{rot} E|^{2}_{\mathsf{L}^{2}(\Omega_{2})} + |\operatorname{div} E|^{2}_{\mathsf{L}^{2}(\Omega_{2})}).$$

 Ω_2 can be replaced by Ω_3 .

Proof. As Ω_2 is convex Lemma 2.14 yields $E \in \mathsf{H}^1(\Omega_2)$ and

$$|\nabla E|^2_{\mathsf{L}^2(\Omega_2)} = |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega_2)} + |\operatorname{div} E|^2_{\mathsf{L}^2(\Omega_2)}$$

The tangential boundary condition implies $E_1 = E_3 = 0$ at $\mathbb{R} \times \{0, d\} \times (0, d)$ and $E_1 = E_2 = 0$ at $\mathbb{R} \times (0, d) \times \{0, d\}$, in particular, $E_1 = E_3 = 0$ at $\mathbb{R} \times \{0\} \times (0, d)$ and $E_2 = 0$ at $\mathbb{R} \times (0, d) \times \{0\}$. Hence Friedrichs' estimate (Lemma 2.9 and Remark 2.11) shows the assertion. **Lemma 3.2** (normal Friedrichs/Gaffney estimate for rot and div). Let Ω_3 be as above and let $E \in \mathsf{H}(\mathrm{rot}, \Omega_3) \cap \mathring{\mathsf{H}}(\mathrm{div}, \Omega_3)$. Then $E \in \mathsf{H}^1(\Omega_3)$ and

$$|E|_{\mathsf{L}^{2}(\Omega_{3})}^{2} \leq c_{d}^{2} |\nabla E|_{\mathsf{L}^{2}(\Omega_{3})}^{2} = c_{d}^{2} (|\operatorname{rot} E|_{\mathsf{L}^{2}(\Omega_{3})}^{2} + |\operatorname{div} E|_{\mathsf{L}^{2}(\Omega_{3})}^{2}).$$

Proof. We copy the proof of Lemma 3.1 with the only difference that now the normal boundary condition implies $E_1 = 0$ at $\{0, d\} \times (0, d)^2$ and $E_2 = 0$ at $(0, d) \times \{0, d\} \times (0, d)$ and $E_3 = 0$ at $(0, d)^2 \times \{0, d\}$.

Lemma 2.9, Lemma 3.1, and Lemma 3.2 show that the number of bounded directions of the domain for which the L²-Friedrichs/Gaffney type estimates for ∇ , rot, and div remain valid depends essentially on the boundary conditions. Let us summarise.

Theorem 3.3 (Friedrichs/Gaffney estimates for ∇ , rot, and div). Let $u \in H^1(\Omega_1)$ with boundary condition $u|_{\mathbb{R}^2 \times \{0\}} = 0$ and let $E \in \mathring{H}(\operatorname{rot}, \Omega_2) \cap H(\operatorname{div}, \Omega_2)$ and $H \in H(\operatorname{rot}, \Omega_3) \cap \mathring{H}(\operatorname{div}, \Omega_3)$. Then $E \in H^1(\Omega_2)$ and $H \in H^1(\Omega_3)$ and it holds:

• $|u|_{\mathsf{L}^2(\Omega_1)} \le c_d |\nabla u|_{\mathsf{L}^2(\Omega_1)}$ ($\Omega_1 \ bd \ in \ one \ dir$)

•
$$|E|_{\mathsf{L}^{2}(\Omega_{2})} \le c_{d} |\nabla E|_{\mathsf{L}^{2}(\Omega_{2})} = c_{d} (|\operatorname{rot} E|^{2}_{\mathsf{L}^{2}(\Omega_{2})} + |\operatorname{div} E|^{2}_{\mathsf{L}^{2}(\Omega_{2})})^{1/2}$$
 ($\Omega_{2} \ bd \ in \ two \ dir)$

• $|H|_{\mathsf{L}^{2}(\Omega_{3})} \leq c_{d} |\nabla H|_{\mathsf{L}^{2}(\Omega_{3})} = c_{d} (|\operatorname{rot} H|^{2}_{\mathsf{L}^{2}(\Omega_{3})} + |\operatorname{div} H|^{2}_{\mathsf{L}^{2}(\Omega_{3})})^{1/2}$ (Ω_{3} bd in all dir)

 Ω_1 can be replaced by Ω_2 or Ω_3 , and Ω_2 can be replaced by Ω_3 .

3.1.2. Rotation in Ω_2 .

Theorem 3.4 (Friedrichs estimate for rot). Let the dual pair (rot, rot) be considered in Ω_2 and let $E \in D(\widehat{rot}) \cup D(\widehat{rot})$. Then $E \in H^1(\Omega_2)$ and

$$|E|_{\mathsf{L}^{2}(\Omega_{2})} \leq c_{d} | \operatorname{rot} E|_{\mathsf{L}^{2}(\Omega_{2})}, \quad | \operatorname{rot} E|_{\mathsf{L}^{2}(\Omega_{2})} = |\nabla E|_{\mathsf{L}^{2}(\Omega_{2})}.$$

 Ω_2 can be replaced by Ω_3 .

Proof. Recall $D(\operatorname{rot}) = D(\operatorname{rot}) \cap \overline{R(\operatorname{rot})} \subset D(\operatorname{rot}) \cap N(\operatorname{div}) \subset \operatorname{\mathsf{H}}(\operatorname{rot}, \Omega_2) \cap \operatorname{\mathsf{H}}(\operatorname{div}, \Omega_2)$ and apply Lemma 3.1 to $E \in D(\operatorname{rot})$. Moreover, Lemma 2.1 shows the result⁴ for $E \in D(\operatorname{rot})$ with the same constant.

Theorem 3.5 (closed range of rot). Let the dual pair (rot, rot) be considered in Ω_2 . Then R(rot) and R(rot) are closed. Moreover,

$$\widehat{\operatorname{rot}}^{-1}: R(\operatorname{rot}) \to D(\widehat{\operatorname{rot}}), \quad \widehat{\operatorname{rot}}^{-1}: R(\operatorname{rot}) \to D(\widehat{\operatorname{rot}})$$

are bounded. Ω_2 can be replaced by Ω_3 .

Proof. Theorem 3.4 and Lemma 2.1 yield the assertions.

3.1.3. Gradient and Divergence in Ω_1 .

Theorem 3.6 (Friedrichs estimate for ∇ and div). Let the dual pair $(\mathring{\nabla}, -\operatorname{div})$ be considered in Ω_1 and let $u \in D(\widehat{\nabla})$ and $H \in D(\widehat{\operatorname{div}})$. Then $H \in \mathsf{H}^1(\Omega_1)$ and

 $|u|_{\mathsf{L}^{2}(\Omega_{1})} \leq c_{d} |\nabla u|_{\mathsf{L}^{2}(\Omega_{1})}, \quad |H|_{\mathsf{L}^{2}(\Omega_{1})} \leq c_{d} |\operatorname{div} H|_{\mathsf{L}^{2}(\Omega_{1})}.$

 Ω_1 can be replaced by Ω_2 or Ω_3 .

Proof. Recall $D(\hat{\nabla}) = D(\hat{\nabla}) \cap \overline{R(\operatorname{div})} = \mathring{H}^1(\Omega_1)$ and apply Lemma 2.9 to $u \in D(\hat{\nabla})$. Lemma 2.1 shows the result⁵ for $H \in D(\widehat{\operatorname{div}})$ with the same constant.

⁴For completeness we note that $D(\widehat{\operatorname{rot}}) = D(\operatorname{rot}) \cap \overline{R(\operatorname{rot})} \subset D(\operatorname{rot}) \cap N(\operatorname{div}) \subset \mathsf{H}(\operatorname{rot},\Omega_2) \cap \mathring{\mathsf{H}}(\operatorname{div},\Omega_2).$

⁵For completeness we note that $D(\widehat{\operatorname{div}}) = D(\operatorname{div}) \cap R(\overset{\circ}{\nabla}) \subset D(\operatorname{div}) \cap N(\operatorname{rot}) \subset H(\operatorname{div},\Omega_1) \cap \overset{\circ}{\mathsf{H}}(\operatorname{rot},\Omega_1).$

Theorem 3.7 (closed range of ∇ and div). Let the dual pair $(\mathring{\nabla}, -\operatorname{div})$ be considered in Ω_1 . Then $R(\mathring{\nabla})$ and $R(\operatorname{div})$ are closed. Moreover,

$$\widehat{\nabla}^{-1}: R(\widehat{\nabla}) \to D(\widehat{\nabla}), \quad \widehat{\operatorname{div}}^{-1}: R(\operatorname{div}) \to D(\widehat{\operatorname{div}})$$

are bounded. Ω_1 can be replaced by Ω_2 or Ω_3 .

Proof. Apply Theorem 3.6 and Lemma 2.1.

3.1.4. Gradient and Divergence in Ω_3 .

Theorem 3.8 (Friedrichs estimate for ∇ and div). Let the dual pair $(\operatorname{div}, -\nabla)$ be considered in Ω_3 and let $H \in D(\widehat{\operatorname{div}})$ and $u \in D(\widehat{\nabla})$. Then $H \in \mathsf{H}^1(\Omega_3)$ and

$$|H|_{\mathsf{L}^{2}(\Omega_{3})} \leq c_{d} |\operatorname{div} H|_{\mathsf{L}^{2}(\Omega_{3})}, \quad |u|_{\mathsf{L}^{2}(\Omega_{3})} \leq c_{d} |\nabla u|_{\mathsf{L}^{2}(\Omega_{3})}.$$

Proof. Recall $D(\widehat{\operatorname{div}}) = D(\operatorname{div}) \cap \overline{R(\nabla)} \subset D(\operatorname{div}) \cap N(\operatorname{rot}) \subset \mathring{H}(\operatorname{div}, \Omega_3) \cap H(\operatorname{rot}, \Omega_3)$ and apply Lemma 3.2 to $H \in D(\widehat{\operatorname{div}})$. Lemma 2.1 shows the result⁶ for $u \in D(\widehat{\nabla})$ with the same constant. \Box

Remark 3.9 (Friedrichs/Poincaré estimate for ∇). Theorem 3.8 gives for u the well-known Poincaré estimate for bounded convex domains, cf. [27].

Theorem 3.10 (closed range of ∇ and div). Let the dual pair (div, $-\nabla$) be considered in Ω_3 . Then R(div) and $R(\nabla)$ are closed. Moreover,

$$\widehat{\operatorname{div}}^{-1}: R(\operatorname{div}) \to D(\widehat{\operatorname{div}}), \quad \widehat{\nabla}^{-1}: R(\nabla) \to D(\widehat{\nabla})$$

are bounded.

Proof. Use Theorem 3.8 and Lemma 2.1.

3.1.5. Summing Up the Friedrichs/Gaffney Estimates.

Corollary 3.11 (Friedrichs/Gaffney estimates for ∇ , rot, and div). Let

$u \in H^{1}(\Omega_{1}),$	$B \in H(\mathrm{rot}, \Omega_1) \cap H(\mathrm{div}, \Omega_1),$
$E \in \mathring{H}(\mathrm{rot}, \Omega_2) \cap H(\mathrm{div}, \Omega_2),$	$D \in H(\mathrm{rot}, \Omega_2) \cap \mathring{H}(\mathrm{div}, \Omega_2),$
$H \in H(\mathrm{rot}, \Omega_3) \cap \mathring{H}(\mathrm{div}, \Omega_3),$	$v \in H^1(\Omega_3) \cap \mathbb{R}^\perp$

with div E = div D = 0 and rot H = rot B = 0. Then $B \in H^1(\Omega_1)$, $E, D \in H^1(\Omega_2)$, and $H \in H^1(\Omega_3)$ and it holds:

•	$ u _{L^2(\Omega_1)} \le c_d \nabla u _{L^2(\Omega_1)}$	$(\Omega_1 \ bd \ in \ one \ dir)$
•	$ B _{L^{2}(\Omega_{1})} \leq c_{d} \nabla B _{L^{2}(\Omega_{1})} = c_{d} \operatorname{div} B _{L^{2}(\Omega_{1})}$	$(\Omega_1 \ bd \ in \ one \ dir)$
•	$ E _{L^2(\Omega_2)} \le c_d \nabla E _{L^2(\Omega_2)} = c_d \operatorname{rot} E _{L^2(\Omega_2)}$	$(\Omega_2 \ bd \ in \ two \ dir)$
•	$ D _{L^{2}(\Omega_{2})} \leq c_{d} \nabla D _{L^{2}(\Omega_{2})} = c_{d} \operatorname{rot} D _{L^{2}(\Omega_{2})}$	$(\Omega_2 \ bd \ in \ two \ dir)$
•	$ H _{L^{2}(\Omega_{3})} \leq c_{d} \nabla H _{L^{2}(\Omega_{3})} = c_{d} \operatorname{div} H _{L^{2}(\Omega_{3})}$	$(\Omega_3 \ bd \ in \ all \ dir)$
•	$ v _{L^2(\Omega_3)} \le c_d \nabla v _{L^2(\Omega_3)}$	$(\Omega_3 \ bd \ in \ all \ dir)$

All respective ranges are closed. Ω_1 can be replaced by Ω_2 or Ω_3 , and Ω_2 can be replaced by Ω_3 .

Proof. The assertions for u, v, E, and H follow by Theorem 3.3 and Theorem 3.8.

For B we have by Lemma 2.14 $B \in H^1(\Omega_1)$ and $|\nabla B|_{L^2(\Omega_1)} = |\operatorname{div} B|_{L^2(\Omega_1)}$. Moreover, by (15), Theorem 3.7, and Lemma 2.26

$$B \in D(\operatorname{div}_{\Omega_1}) \cap N(\operatorname{rot}_{\Omega_1}) = D(\operatorname{div}_{\Omega_1}) \cap \left(R(\check{\nabla}_{\Omega_1}) \oplus_{\mathsf{L}^2(\Omega)} \mathcal{H}_{\mathsf{D},\operatorname{id}}(\Omega_1)\right)$$
$$= D(\operatorname{div}_{\Omega_1}) \cap R(\check{\nabla}_{\Omega_1}) = D(\widehat{\operatorname{div}}_{\Omega_1}),$$

⁶For completeness we note that $D(\widehat{\nabla}) = D(\nabla) \cap \overline{R(\operatorname{div})} = \mathsf{H}^1(\Omega_3) \cap \mathbb{R}^{\perp}$.

and thus by Theorem 3.6 $|B|_{\mathsf{L}^2(\Omega_1)} \leq c_d |\operatorname{div} B|_{\mathsf{L}^2(\Omega_1)}$.

For D we have by Lemma 2.14 $D \in \mathsf{H}^1(\Omega_2)$ and $|\nabla D|_{\mathsf{L}^2(\Omega_2)} = |\operatorname{rot} D|_{\mathsf{L}^2(\Omega_2)}$. Moreover, similar to (15), and by Theorem 3.5 and Lemma 2.26

$$D \in D(\operatorname{rot}_{\Omega_2}) \cap N(\operatorname{div}_{\Omega_2}) = D(\operatorname{rot}_{\Omega_2}) \cap \left(R(\operatorname{rot}_{\Omega_2}) \oplus_{\mathsf{L}^2(\Omega)} \mathcal{H}_{\mathsf{N},\operatorname{id}}(\Omega_2)\right)$$
$$= D(\operatorname{rot}_{\Omega_2}) \cap R(\operatorname{rot}_{\Omega_2}) = D(\widehat{\operatorname{rot}}_{\Omega_2}),$$

and thus by Theorem 3.4 $|D|_{\mathsf{L}^2(\Omega_2)} \leq c_d |\operatorname{rot} D|_{\mathsf{L}^2(\Omega_2)}$.

3.1.6. Spectrum Around the Origin. Recall Section 2.3 and Remark 2.4.

• In Ω_1 , Ω_2 or Ω_3 we consider the selfadjoint and non-negative operators

$$\mathbf{T}_{1,D} := -\operatorname{div} \dot{\nabla}, \quad \mathbf{T}_{1,N} := -\dot{\nabla} \operatorname{div},$$

and the skew-selfadjoint operator $S_1 := \begin{bmatrix} 0 & div \\ \nabla & 0 \end{bmatrix}$.

• In Ω_2 or Ω_3 we consider the selfadjoint and non-negative operators

$$T_{2,D} := \operatorname{rot} \operatorname{rot}, \quad T_{2,N} := \operatorname{rot} \operatorname{rot}, \quad T_2 := \operatorname{rot} \operatorname{rot} - \mathring{\nabla} \operatorname{div},$$

and the skew-selfadjoint operator $S_2 := \begin{bmatrix} 0 & -\operatorname{rot} \\ \operatorname{rot} & 0 \end{bmatrix}$.

• In Ω_3 we consider the selfadjoint and non-negative operators

$$\mathbf{T}_{3,D} := -\operatorname{div} \nabla, \quad \mathbf{T}_{3,N} := -\nabla \operatorname{div}, \quad \mathbf{T}_3 := \operatorname{rot} \operatorname{rot} - \nabla \operatorname{div},$$

and the skew-selfadjoint operator $S_3 := \begin{bmatrix} 0 & d i v \\ \nabla & 0 \end{bmatrix}$.

Let

$$\mathbf{T} \in \{\mathbf{T}_{1,D}, \mathbf{T}_{1,N}, \mathbf{T}_{2,D}, \mathbf{T}_{2,N}, \mathbf{T}_{2}, \mathbf{T}_{3,D}, \mathbf{T}_{3,N}, \mathbf{T}_{3}, \mathbf{S}_{1}, \mathbf{S}_{2}, \mathbf{S}_{3}\}.$$

By the latter results, cf. Corollary 3.11, the range R(T) is closed. Hence we have the following results about the spectrum of T.

Corollary 3.12. Lemma 2.5, Lemma 2.6, and Lemma 2.8 hold with $c_{\rm T} = c_d$

3.2. The Maxwell Operator in Lipschitz Cylinders. From now on let ε , μ , and ν , θ be admissible. We recall the de Rham complex (14). Picking $\Xi \in \{\Omega_1, \Omega_2, \Omega_3\}$ we obtain the following result for Lipschitz cylinders.

3.2.1. Closed Ranges.

Theorem 3.13 (closed range of ∇ and div). Let Ω allow for an admissible bi-Lipschitz transformation $\Phi : \Omega_1 \to \Phi(\Omega_1) = \Omega$ and consider the dual pair $(\mathring{\nabla}_{\Omega}, -\operatorname{div}_{\Omega})$ in Ω . Then $R(\mathring{\nabla}_{\Omega})$ and $R(\operatorname{div}_{\Omega})$ are closed. Moreover,

$$\widehat{\check{\nabla}}_{\Omega}^{-1}: R(\check{\nabla}_{\Omega}) \to D(\widehat{\check{\nabla}}_{\Omega}), \quad \widehat{\operatorname{div}}_{\Omega}^{-1}: R(\operatorname{div}_{\Omega}) \to D(\widehat{\operatorname{div}}_{\Omega})$$

are bounded. Ω_1 can be replaced by Ω_2 or Ω_3 .

Proof. Combine Lemma 2.24, Theorem 3.7, and Lemma 2.1.

Theorem 3.14 (closed range of rot). Let Ω allow for an admissible bi-Lipschitz transformation $\Phi: \Omega_2 \to \Phi(\Omega_2) = \Omega$ and consider the dual pair $(\operatorname{rot}_{\Omega}, \operatorname{rot}_{\Omega})$ in Ω . Then $R(\operatorname{rot}_{\Omega})$ and $R(\operatorname{rot}_{\Omega})$ are closed. Moreover,

$$\widehat{\mathring{\operatorname{rot}}}_{\Omega}^{-1}: R(\mathring{\operatorname{rot}}_{\Omega}) \to D(\widehat{\mathring{\operatorname{rot}}}_{\Omega}), \quad \widehat{\operatorname{rot}}_{\Omega}^{-1}: R(\operatorname{rot}_{\Omega}) \to D(\widehat{\operatorname{rot}}_{\Omega})$$

are bounded. Ω_2 can be replaced by Ω_3 .

Proof. Combine Lemma 2.24, Theorem 3.5, and Lemma 2.1.

Theorem 3.15 (closed range of ∇ and div). Let Ω allow for an admissible bi-Lipschitz transformation $\Phi: \Omega_3 \to \Phi(\Omega_3) = \Omega$ and consider the dual pair $(\operatorname{div}_{\Omega}, -\nabla_{\Omega})$ in Ω . Then $R(\operatorname{div}_{\Omega})$ and $R(\nabla_{\Omega})$ are closed. Moreover,

$$\widehat{\operatorname{div}}_{\Omega}^{-1} : R(\operatorname{div}_{\Omega}) \to D(\widehat{\operatorname{div}}_{\Omega}), \quad \widehat{\nabla}_{\Omega}^{-1} : R(\nabla_{\Omega}) \to D(\widehat{\nabla}_{\Omega})$$

are bounded.

Proof. Combine Lemma 2.24, Theorem 3.10, and Lemma 2.1.

3.2.2. The Static Maxwell Operator.

Lemma 3.16 (trivial Dirichlet/Neumann fields). Let $n \in \{0, 1, 2, 3\}$ and let Ξ be convex and either bounded or of infinite volume. Moreover, let Ω allow for an admissible bi-Lipschitz transformation

- $\Phi: \Omega_n \to \Phi(\Omega_n) = \Omega$
- or $\Phi: \Xi \to \Phi(\Xi) = \Omega$.

Then $\mathcal{H}_{\mathsf{D},\varepsilon}(\Omega) = \mathcal{H}_{\mathsf{N},\mu}(\Omega) = \{0\}.$

Proof. Apply Lemm 2.25 and Lemm 2.26.

The latter results, in particular, Lemma 3.16, together with Theorem 2.3 and Lemma 2.2 show:

Theorem 3.17 (mini FA-ToolBox). Let $n \in \{2,3\}$. Moreover, let Ω allow for an admissible bi-Lipschitz transformation $\Phi: \Omega_n \to \Phi(\Omega_n) = \Omega$ and consider the dual pairs $(\check{\nabla}_{\Omega}, -\nu^{-1} \operatorname{div}_{\Omega} \varepsilon)$, $(\mu^{-1} \operatorname{rot}_{\Omega}, \varepsilon^{-1} \operatorname{rot}_{\Omega}), \text{ and } (\theta^{-1} \operatorname{div}_{\Omega} \mu, -\nabla_{\Omega}) \text{ in } \Omega.$ Then:

- (i) The ranges $R(\mathring{\nabla}_{\Omega})$, $R(\nu^{-1}\operatorname{div}_{\Omega}\varepsilon) = \mathsf{L}^{2}_{\nu}(\Omega)$, and $R(\mu^{-1}\operatorname{rot}_{\Omega})$, $R(\varepsilon^{-1}\operatorname{rot}_{\Omega})$ are closed. (ii) The inverse operators $\widehat{\nabla}_{\Omega}^{-1}$, $\nu^{-1}\operatorname{div}_{\Omega}\varepsilon^{-1}$, and $\mu^{-1}\operatorname{rot}_{\Omega}^{-1}$, $\varepsilon^{-1}\operatorname{rot}_{\Omega}^{-1}$ are bounded.
- (iii) The orthogonal Helmholtz decompositions

$$L^{2}_{\varepsilon}(\Omega) = N(\operatorname{rot}_{\Omega}) \oplus_{L^{2}_{\varepsilon}(\Omega)} N(\operatorname{div}_{\Omega} \varepsilon), L^{2}_{\mu}(\Omega) = N(\operatorname{rot}_{\Omega}) \oplus_{L^{2}_{\omega}(\Omega)} N(\operatorname{div}_{\Omega} \mu)$$

hold, where

$N(\operatorname{rot}_{\Omega}) = R(\check{\nabla}_{\Omega}),$	$N(\operatorname{div}_{\Omega}\varepsilon) = \varepsilon^{-1}R(\operatorname{rot}_{\Omega}),$
$N(\operatorname{rot}_{\Omega}) = \overline{R(\nabla_{\Omega})},$	$N(\operatorname{div}_{\Omega}\mu) = \mu^{-1}R(\operatorname{rot}_{\Omega}).$

Moreover, $R(\theta^{-1}\operatorname{div}_{\Omega}\mu) = \mathsf{L}^{2}_{\theta}(\Omega) \cap \mathbb{R}^{\perp_{\theta}}$, *i.e.*, $R(\operatorname{div}_{\Omega}\mu) = \mathsf{L}^{2}(\Omega) \cap \mathbb{R}^{\perp}$.

- (iv) There are $c_1 := c_{\nabla}$ and $c_2 := c_{rot} > 0$ such that for all
 - $u \in D(\mathring{\nabla}_{\Omega}) = \mathring{H}^{1}(\Omega),$ $B \in D(\nu^{-1}\operatorname{div}_{\Omega} \varepsilon) = D(\operatorname{div}_{\Omega} \varepsilon) \cap N(\operatorname{rot}_{\Omega}),$ $E \in D(\mu^{-1} \operatorname{rot}_{\Omega}) = D(\operatorname{rot}_{\Omega}) \cap N(\operatorname{div}_{\Omega} \varepsilon),$ $D \in D(\widehat{\varepsilon^{-1}\operatorname{rot}}_{\Omega}) = D(\operatorname{rot}_{\Omega}) \cap N(\operatorname{div}_{\Omega} \mu)$

it holds

$$|u|_{\mathsf{L}^{2}_{\nu}(\Omega)} \leq c_{1} | \nabla u|_{\mathsf{L}^{2}_{\varepsilon}(\Omega)}, \qquad |B|_{\mathsf{L}^{2}_{\varepsilon}(\Omega)} \leq c_{1} | \nu^{-1} \operatorname{div} \varepsilon B|_{\mathsf{L}^{2}_{\nu}(\Omega)}, |E|_{\mathsf{L}^{2}_{\varepsilon}(\Omega)} \leq c_{2} | \mu^{-1} \operatorname{rot} E|_{\mathsf{L}^{2}_{\mu}(\Omega)}, \qquad |D|_{\mathsf{L}^{2}_{\mu}(\Omega)} \leq c_{2} | \varepsilon^{-1} \operatorname{rot} D|_{\mathsf{L}^{2}_{\varepsilon}(\Omega)}.$$

(v) For all
$$E \in D(\mu^{-1} \operatorname{rot}_{\Omega}) \cap D(\nu^{-1} \operatorname{div}_{\Omega} \varepsilon) = \operatorname{H}(\operatorname{rot}, \Omega) \cap \varepsilon^{-1} \operatorname{H}(\operatorname{div}, \Omega)$$
 it holds
$$|E|^{2}_{\mathsf{L}^{2}_{\varepsilon}(\Omega)} \leq c_{2}^{2} |\mu^{-1} \operatorname{rot} E|^{2}_{\mathsf{L}^{2}_{\mu}(\Omega)} + c_{1}^{2} |\nu^{-1} \operatorname{div} \varepsilon E|^{2}_{\mathsf{L}^{2}_{\nu}(\Omega)}.$$

In case n = 3 we have additionally:

- (i') The ranges $R(\operatorname{div}_{\Omega} \mu) = L^2(\Omega) \cap \mathbb{R}^{\perp}$ and $R(\nabla_{\Omega})$ are closed. (ii') The inverse operators $\theta^{-1} \operatorname{div}_{\Omega} \mu^{-1}$ and $\widehat{\nabla}_{\Omega}^{-1}$ are bounded.

 \square

(iv') There is $c_3 := c_{\text{div}} > 0$ such that for all $H \in D(\theta^{-1} \operatorname{div}_{\Omega} \mu) = D(\operatorname{div}_{\Omega} \mu) \cap N(\operatorname{rot}_{\Omega}), \qquad v \in D(\widehat{\nabla}_{\Omega}) = \mathsf{H}^{1}(\Omega) \cap \mathbb{R}^{\perp}$ it holds $|H|_{\mathsf{L}^2_{\mu}(\Omega)} \leq c_3 |\theta^{-1} \operatorname{div} \mu H|_{\mathsf{L}^2_{\mu}(\Omega)},$ $|v|_{\mathsf{L}^2_{\mathsf{a}}(\Omega)} \leq c_3 |\nabla v|_{\mathsf{L}^2_{\mathsf{a}}(\Omega)}.$ (v') For all $H \in D(\varepsilon^{-1} \operatorname{rot}_{\Omega}) \cap D(\theta^{-1} \operatorname{div}_{\Omega} \mu) = \mathsf{H}(\operatorname{rot}, \Omega) \cap \mu^{-1} \mathring{\mathsf{H}}(\operatorname{div}, \Omega)$

$$|H|^2_{\mathsf{L}^2_{\varepsilon}(\Omega)} \leq c_2^2 |\varepsilon^{-1} \operatorname{rot} H|^2_{\mathsf{L}^2_{\varepsilon}(\Omega)} + c_3^2 |\theta^{-1} \operatorname{div} \mu H|^2_{\mathsf{L}^2_{\varepsilon}(\Omega)}$$

3.2.3. Spectrum Around the Origin. We recall the selfadjoint and skew-selfadjoint operators T.

and $S_{...}$ from Section 3.1.6 acting on $\Omega_1, \Omega_2, \Omega_3$ and redefined those by including inhomogeneities. Next we show that the results about the spectrum, cf. Corollary 3.12, hold also in the following general situations.

Corollary 3.18.

• Let us pick $n \in \{1,2,3\}$ and let Ω allow for an admissible bi-Lipschitz transformation $\Phi: \Omega_n \to \Phi(\Omega_n) = \Omega$. In Ω we consider the selfadjoint and non-negative operators

 $T_{1,D} := -\nu^{-1} \operatorname{div} \varepsilon \overset{\circ}{\nabla}, \quad T_{1,N} := - \overset{\circ}{\nabla} \nu^{-1} \operatorname{div} \varepsilon,$

and the skew-selfadjoint operator $S_1 := \begin{bmatrix} 0 & \nu^{-1} \operatorname{div} \varepsilon \\ \mathring{\nabla} & 0 \end{bmatrix}$. Let $T \in \{T_{1,D}, T_{1,N}, S_1\}$. Then Lemma 2.5, Lemma 2.6, and Lemma 2.8 hold with with $c_{\mathrm{T}} = c_1$.

• Let us choose $n \in \{2,3\}$ and let Ω allow for an admissible bi-Lipschitz transformation $\Phi: \Omega_n \to \Phi(\Omega_n) = \Omega$. In Ω we consider the selfadjoint and non-negative operators

$$\mathbf{T}_{2,D} := \varepsilon^{-1} \operatorname{rot} \mu^{-1} \mathring{\operatorname{rot}}, \quad \mathbf{T}_{2,N} := \mu^{-1} \mathring{\operatorname{rot}} \varepsilon^{-1} \operatorname{rot}, \quad \mathbf{T}_2 := \varepsilon^{-1} \operatorname{rot} \mu^{-1} \mathring{\operatorname{rot}} - \mathring{\nabla} \nu^{-1} \operatorname{div} \varepsilon,$$

and the skew-selfadjoint operator $S_2 := \begin{bmatrix} 0 & -\varepsilon^{-1} \operatorname{rot} \\ \mu^{-1} \operatorname{rot} & 0 \end{bmatrix}$.

Let $T \in \{T_{2,D}, T_{2,N}, T_2, S_2\}$. Then Lemma 2.5, Lemma 2.6, and Lemma 2.8 hold with with $c_{\rm T} = c_2 \ resp. \ c_{\rm T} = \max\{c_1, c_2\} \ for \ {\rm T} = {\rm T}_2.$

• Let Ω allow for an admissible bi-Lipschitz transformation $\Phi: \Omega_3 \to \Phi(\Omega_3) = \Omega$. In Ω we consider the selfadjoint and non-negative operators

$$\mathbf{T}_{3,D} := -\theta^{-1} \operatorname{div} \mu \, \nabla, \quad \mathbf{T}_{3,N} := -\, \nabla \, \theta^{-1} \operatorname{div} \mu, \quad \mathbf{T}_3 := \mu^{-1} \operatorname{rot} \varepsilon^{-1} \operatorname{rot} - \nabla \, \theta^{-1} \operatorname{div} \mu,$$

and the skew-selfadjoint operator $S_3 := \begin{bmatrix} 0 & \theta^{-1} \operatorname{div} \mu \\ \nabla & 0 \end{bmatrix}$.

Let $T \in {T_{3,D}, T_{3,N}, T_3, S_3}$. Then Lemma 2.5, Lemma 2.6, and Lemma 2.8 hold with with $c_{\rm T} = c_3$ resp. $c_{\rm T} = \max\{c_2, c_3\}$ for ${\rm T} = {\rm T}_3$.

Proof. The latter results follow by Theorem 3.13 and Theorem 3.17 as R(T) is closed.

4. Additional Results for Mixed Boundary Conditions

We recall Lemma 3.1 and extend a modified version even to

$$\Omega_1 = \mathbb{R}^2 \times (0, d)$$

with, e.g.⁷,

$$\Gamma := \mathbb{R}^2 \times \{0\}, \quad \widetilde{\Gamma} := \mathbb{R}^2 \times \{d\}, \qquad \Gamma \dot{\cup} \widetilde{\Gamma} = \partial \,\Omega_1 = \mathbb{R}^2 \times \{0, d\}.$$

For full (or equivalently empty) boundary conditions the proof of Lemma 3.1 fails as even for $E \in \mathring{H}(rot, \Omega_1) \cap H^1(\Omega_1)$ the tangential boundary condition just implies $E_1 = E_2 = 0$ at $\partial \Omega_1$, i.e., a boundary condition for E_3 is missing.

 $^{^{7}\}Gamma$ and $\tilde{\Gamma}$ may be interchanged.

For trying mixed boundary conditions we introduce the Sobolev spaces⁸

$$\mathring{\mathsf{H}}^{1}_{\Gamma}(\Omega_{1}), \quad \mathring{\mathsf{H}}_{\Gamma}(\mathrm{rot},\Omega_{1}), \quad \mathring{\mathsf{H}}_{\Gamma}(\mathrm{div},\Omega_{1})$$

as closures in $H^1(\Omega_1)$, $H(rot, \Omega_1)$, resp. $H(\operatorname{div}, \Omega_1)$ of (restrictions to Ω of) test spaces $\check{C}^{\infty}(\mathbb{R}^3)$ vanishing in a neighbourhood of Γ .

Recall from Section 2.3 and Section 2.4.1 the primal and dual vector de Rham Hilbert complexes (10) and (14). Here we have the primal and dual vector de Rham Hilbert complex

(16)
$$\mathsf{L}^{2}(\Omega_{1}) \xleftarrow{\nabla_{\Gamma}}{\leftarrow \operatorname{div}_{\widetilde{\Gamma}}} \mathsf{L}^{2}(\Omega_{1}) \xleftarrow{\operatorname{rot}_{\Gamma}}{\leftarrow \operatorname{rot}_{\widetilde{\Gamma}}} \mathsf{L}^{2}(\Omega_{1}) \xleftarrow{\operatorname{div}_{\Gamma}}{\leftarrow -\nabla_{\widetilde{\Gamma}}} \mathsf{L}^{2}(\Omega_{1}),$$

where we consider the densely defined and closed linear realisations of ∇ , rot, and div, cf. [4, 22],

$$\begin{split} \mathring{\nabla}_{\Gamma} &: \mathring{H}^{1}_{\Gamma}(\Omega_{1}) \subset \mathsf{L}^{2}(\Omega_{1}) \to \mathsf{L}^{2}(\Omega_{1}), \qquad -\operatorname{div}_{\widetilde{\Gamma}} = \mathring{\nabla}^{*}_{\Gamma} : \mathring{H}_{\widetilde{\Gamma}}(\operatorname{div},\Omega_{1}) \subset \mathsf{L}^{2}(\Omega_{1}) \to \mathsf{L}^{2}(\Omega_{1}), \\ \operatorname{rot}_{\Gamma} &: \mathring{H}_{\Gamma}(\operatorname{rot},\Omega_{1}) \subset \mathsf{L}^{2}(\Omega_{1}) \to \mathsf{L}^{2}(\Omega_{1}), \qquad \operatorname{rot}_{\widetilde{\Gamma}} = \operatorname{rot}_{\Gamma}^{*} : \mathring{H}_{\widetilde{\Gamma}}(\operatorname{rot},\Omega_{1}) \subset \mathsf{L}^{2}(\Omega_{1}) \to \mathsf{L}^{2}(\Omega_{1}), \\ \operatorname{div}_{\Gamma} : \mathring{H}_{\Gamma}(\operatorname{div},\Omega_{1}) \subset \mathsf{L}^{2}(\Omega_{1}) \to \mathsf{L}^{2}(\Omega_{1}), \qquad -\mathring{\nabla}_{\widetilde{\Gamma}} = \mathring{\nabla}^{*}_{\Gamma} : \mathring{H}_{\widetilde{\Gamma}}^{1}(\Omega_{1}) \subset \mathsf{L}^{2}(\Omega_{1}) \to \mathsf{L}^{2}(\Omega_{1}). \end{split}$$

The extension including ε , μ , ν , and θ is straight forward and may be omitted here.

Lemma 4.1 (tangential/normal Friedrichs/Gaffney estimate for rot and div). We consider Ω_1 . Let $E \in \mathring{H}_{\Gamma}(\operatorname{rot}, \Omega_1) \cap \mathring{H}_{\widetilde{\Gamma}}(\operatorname{div}, \Omega_1)$. Then $E \in H^1(\Omega_1)$ and

$$|E|_{\mathsf{L}^{2}(\Omega_{1})} \leq c_{d} |\nabla E|_{\mathsf{L}^{2}(\Omega_{1})} = c_{d} \sqrt{|\operatorname{rot} E|^{2}_{\mathsf{L}^{2}(\Omega_{1})} + |\operatorname{div} E|^{2}_{\mathsf{L}^{2}(\Omega_{1})}}$$

Proof. We follow the proof of Lemma 3.1 and recall the proof of Lemma 2.14, in particular, the cut-off function φ_r and $\widetilde{\Omega}_r := \Omega \cap (-r, r)^3$.

Let $\phi \in \mathsf{C}^{\infty}(\mathbb{R}^3, [0, 1])$ with $\phi = 1$ near Γ and $\phi = 0$ near $\widetilde{\Gamma}$. Then $E = \phi E + (1 - \phi)E$ as well as (by mollification) $\phi E \in \mathring{\mathsf{H}}(\operatorname{rot}, \Omega_1) \cap \mathsf{H}(\operatorname{div}, \Omega_1)$ and $(1 - \phi)E \in \mathsf{H}(\operatorname{rot}, \Omega_1) \cap \mathring{\mathsf{H}}(\operatorname{div}, \Omega_1)$. As Ω_1 is convex Lemma 2.14 yields ϕE , $(1 - \phi)E \in \mathsf{H}^1(\Omega_1)$, i.e., $E \in \mathsf{H}^1(\Omega_1)$.

[3, Lemma 13], cf. [2], shows that (9), i.e.,

$$\left| \nabla(\varphi_r E) \right|_{\mathsf{L}^2(\Omega_1)}^2 = \left| \operatorname{rot}(\varphi_r E) \right|_{\mathsf{L}^2(\Omega_1)}^2 + \left| \operatorname{div}(\varphi_r E) \right|_{\mathsf{L}^2(\Omega_1)}^2$$

(integration just over $\tilde{\Omega}_{2r}$, flat boundaries, and mixed boundary conditions), holds. Lebesgue's dominated convergence theorem yields for $r \to \infty$

$$|\nabla E|^2_{\mathsf{L}^2(\Omega_1)} = |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega_1)} + |\operatorname{div} E|^2_{\mathsf{L}^2(\Omega_1)}.$$

The tangential boundary condition implies $E_1 = E_2 = 0$ at Γ and the normal boundary condition shows $E_3 = 0$ at $\widetilde{\Gamma}$. Hence Friedrichs' estimate (Lemma 2.9) shows the assertion.

Theorem 4.2 (Friedrichs estimate for rot). Let the dual pair $(\mathring{rot}_{\Gamma}, \mathring{rot}_{\widetilde{\Gamma}})$ be considered in Ω_1 .

(i) $E \in D(\operatorname{rot}_{\Gamma})$ implies $E \in H^1(\Omega_1)$ and

$$|E|_{\mathsf{L}^2(\Omega_1)} \le c_d |\operatorname{rot} E|_{\mathsf{L}^2(\Omega_1)}, \quad |\operatorname{rot} E|_{\mathsf{L}^2(\Omega_1)} = |\nabla E|_{\mathsf{L}^2(\Omega_1)}.$$

(ii) $R(\operatorname{rot}_{\Gamma})$ is closed and $\widehat{\operatorname{rot}}_{\Gamma}^{-1} : R(\operatorname{rot}_{\Gamma}) \to D(\widehat{\operatorname{rot}}_{\Gamma})$ is bounded.

Proof. Note $D(\widehat{\operatorname{rot}}_{\Gamma}) = D(\operatorname{rot}_{\Gamma}) \cap \overline{R(\operatorname{rot}_{\widetilde{\Gamma}})} \subset D(\operatorname{rot}_{\Gamma}) \cap N(\operatorname{div}_{\widetilde{\Gamma}}) \subset \operatorname{H}_{\Gamma}(\operatorname{rot},\Omega_{1}) \cap \operatorname{H}_{\widetilde{\Gamma}}(\operatorname{div},\Omega_{1})$ and apply Lemma 4.1 to $E \in D(\operatorname{rot}_{\Gamma})$. Lemma 2.1 concludes the proof.

Theorem 4.3 (Friedrichs estimate for ∇ and div). Let the dual pair $(\mathring{\nabla}_{\Gamma}, -\mathring{\operatorname{div}}_{\widetilde{\Gamma}})$ be considered in Ω_1 .

(i) Let
$$u \in D(\hat{\nabla}_{\Gamma})$$
 and $H \in D(\hat{\operatorname{div}}_{\Gamma})$. Then $H \in H^{1}(\Omega_{1})$ and
 $|u|_{\mathsf{L}^{2}(\Omega_{1})} \leq c_{d} | \nabla u|_{\mathsf{L}^{2}(\Omega_{1})}, \quad |H|_{\mathsf{L}^{2}(\Omega_{1})} \leq c_{d} | \operatorname{div} H|_{\mathsf{L}^{2}(\Omega_{1})}, \quad |\operatorname{div} H|_{\mathsf{L}^{2}(\Omega_{1})} = | \nabla H|_{\mathsf{L}^{2}(\Omega_{1})}.$

⁸Same for $\widetilde{\Gamma}$.

(ii) $R(\mathring{\nabla}_{\Gamma})$ is closed and $\widehat{\check{\nabla}}_{\Gamma}^{-1} : R(\mathring{\nabla}_{\Gamma}) \to D(\widehat{\check{\nabla}}_{\Gamma})$ is bounded.

(iii) $R(\operatorname{div}_{\widetilde{\Gamma}})$ is closed and $\widehat{\operatorname{div}_{\widetilde{\Gamma}}}^{-1} : R(\operatorname{div}_{\widetilde{\Gamma}}) \to D(\widehat{\operatorname{div}_{\widetilde{\Gamma}}})$ is bounded.

Proof. Apply Lemma 2.9 and Lemma 2.1.

As before we define the Dirichlet and Neumann fields by

$$\mathcal{H}_{\mathsf{D},\Gamma,\mathrm{id}}(\Omega_1) := N(\mathring{\mathrm{rot}}_{\Gamma}) \cap N(\mathring{\mathrm{div}}_{\widetilde{\Gamma}}), \qquad \mathcal{H}_{\mathsf{N},\widetilde{\Gamma},\mathrm{id}}(\Omega_1) := N(\mathring{\mathrm{rot}}_{\widetilde{\Gamma}}) \cap N(\mathring{\mathrm{div}}_{\Gamma}) = \mathcal{H}_{\mathsf{D},\widetilde{\Gamma},\mathrm{id}}(\Omega_1).$$

Lemma 4.4 (trivial Dirichlet/Neumann fields). $\mathcal{H}_{D,\Gamma,id}(\Omega_1) = \{0\}$

Proof. Let
$$E \in \mathcal{H}_{D,\Gamma,id}(\Omega_1)$$
. Lemma 4.1 yields that $E \in H^1(\Omega)$ is constant. Hence $E = 0$.

Recall Section 3.1.6. We introduce T as one of the following operators in Ω_1 :

$$-\operatorname{div}_{\widetilde{\Gamma}} \overset{\circ}{\nabla}_{\Gamma}, \quad -\overset{\circ}{\nabla}_{\Gamma} \operatorname{div}_{\widetilde{\Gamma}}, \quad \operatorname{rot}_{\widetilde{\Gamma}} \operatorname{rot}_{\Gamma}, \quad \operatorname{rot}_{\widetilde{\Gamma}} \operatorname{rot}_{\Gamma} - \overset{\circ}{\nabla}_{\Gamma} \operatorname{div}_{\widetilde{\Gamma}}, \quad \begin{bmatrix} 0 & \operatorname{div}_{\widetilde{\Gamma}} \\ \operatorname{div}_{\Gamma} & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -\operatorname{rot}_{\widetilde{\Gamma}} \\ \operatorname{rot}_{\Gamma} & 0 \end{bmatrix}$$

Corollary 4.5. Lemma 2.5, Lemma 2.6, and Lemma 2.8 hold with $c_{\rm T} = c_d$

Proof. By the latter results R(T) is closed.

It is now clear, that all results of Section 3.2 carry over to the case that

• Ω allows for an admissible bi-Lipschitz transformation

$$\Phi:\Omega_1\to\Phi(\Omega_1)=\Omega$$

with
$$\gamma := \Phi(\Gamma)$$
 and $\widetilde{\gamma} := \Phi(\Gamma)$,

• $\mathring{H}^{1}_{\gamma}(\Omega)$, $\mathring{H}_{\gamma}(\operatorname{rot}, \Omega)$, and $\mathring{H}_{\widetilde{\gamma}}(\operatorname{div}, \Omega)$, • $\mathsf{L}^{2}_{\nu}(\Omega) \xrightarrow{\mathring{\nabla}_{\gamma}} \underset{(-\nu^{-1}\operatorname{div}_{\widetilde{\gamma}}\varepsilon)}{\overset{}\leftarrow} \mathsf{L}^{2}_{\varepsilon}(\Omega) \xrightarrow{\mu^{-1}\operatorname{rot}_{\gamma}} \underset{\varepsilon^{-1}\operatorname{rot}_{\widetilde{\gamma}}}{\overset{}\leftarrow} \mathsf{L}^{2}_{\mu}(\Omega) \xrightarrow{\theta^{-1}\operatorname{div}_{\gamma}\mu} \mathsf{L}^{2}_{\theta}(\Omega).$

In particular, respective versions of Lemma 3.16, Theorem 3.17, and Corollary 3.18 hold.

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