BIHARMONIC EQUATIONS

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ABSTRACT. In this note we devise and analyse well-posed variational formulations and operator theoretical methods for boundary value problems associated to the biharmonic operator Δ^2. Of particular interest are Neumann type and over-/underdetermined (maximal/minimal) boundary value problems.

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1. Introduction

Recently, in the new edition of the book of Valli [23], the second author discussed a Neumann boundary value problem for the biharmonic equation and presented proofs for the well-posedness of related variational formulations under very strong regularity assumptions (C^4-boundary), cf. [23, Section 5.6].

In this context some interesting questions arose which we will answer in this contribution using basic methods from functional analysis, cf. Section 2.1. In particular, we can reduce the regularity to Lipschitz boundaries and even beyond. More precisely, we shall show that even under very weak regularity and boundedness assumptions on the underlying domain (admissible domains, cf. Defintion 2.6 and Remark 2.7) we can answer the latter questions and extend the results to a whole zoo of biharmonic operators, including, among others, the Neumann case.

For simplicity and readability we restrict our analysis to homogeneous boundary conditions. Note that as soon as proper trace and extension operators are available, the inhomogeneous boundary value problems can be easily formulated as homogeneous ones. A very general trace theory, meeting our needs, has been developed recently by the first author and his collaborators in [7].

Throughout this paper – unless otherwise stated – let $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, be a domain (connected and open set) with boundary $\Gamma := \partial\,\Omega$.

1.1. **Historical Remarks.**

Let us consider a Neumann boundary value problems for the biharmonic operator, that is

$$\Delta^2 u = f \text{ in } \Omega \qquad \text{and} \qquad \Delta u = 0, \quad n \cdot \nabla\,\Delta u = 0 \text{ on } \Gamma,$$

more closely. Later, the corresponding proper Neumann biharmonic operator will be denoted by

$$\mathcal{B}_{\mathtt{N}} = \mathcal{B}_{\circ,\cdot} = \mathring{\mathcal{L}}\,\mathcal{L},$$

see Section 2.2.7. From the physical point of view this problem is not the most interesting, as the model which describes the equilibrium position of an elastic thin plate, unconstrained on the boundary, involves other second order and third order boundary operators, in which the Poisson ratio also has a role (see Courant and Hilbert [5, p. 250] and more recently, e.g., Verchota [24], Provenzano [21]; the original physical model even dates back to Kirchhoff and Kelvin).

Instead of

$$\Delta^2 = \Delta\Delta,$$

the bi-Laplacian can also be factorised by means of

$$\Delta^2 = \operatorname{divDiv} \nabla\nabla,$$

cf. [18, 16], with different Neumann boundary conditions and hence different physical interpretations. We shall come back later to this alternative, cf. Section 1.3 and Section 2.3.

However, despite these remarks, we think that the Neumann problem for the biharmonic operator has a nice and simple mathematical structure, similar to that of other classical problems, and we find it interesting from the mathematical point of view. Moreover, it is the limiting case, for the Poisson ratio going to 1 of suitable physical models.

Let us first compare a bit the Neumann problems for $-\Delta$ and Δ^2. As it is well-known, the Neumann boundary value problems for the negative Laplacian $-\Delta$ and what we call the bi-Laplacian/biharmonic operator Δ^2, i.e.,

$$\begin{aligned} -\Delta v &= g, & \Delta^2 u &= f & &\text{in } \Omega,\\ n \cdot \nabla\, v &= 0, & \Delta u &= 0 & &\text{on } \Gamma,\\ & & n \cdot \nabla\,\Delta u &= 0, & &\text{on } \Gamma, \end{aligned}$$

have similarities and differences. In particular, for both of them the solution is not unique: adding to v a constant $c \in \mathbb{R}$ resp. to u a harmonic function $h \in \mathbb{H}$, where

$$\mathbb{H} = \big\{\varphi \in \mathsf{L}^2(\Omega) : \Delta\varphi = 0\big\},$$

gives another solution. Moreover, for both of them the data have to satisfy a compatibility condition (Fredholm alternative): we have

$$g \perp \mathbb{R} \qquad \text{for } -\Delta$$

and

$$f \perp \mathbb{H} \qquad \text{for} \quad \Delta^2,$$

where $\perp$ denotes orthogonality in $\mathsf{L}^2(\Omega)$. Note that in the smooth case we have

$$\langle g, 1\rangle_{\mathsf{L}^2(\Omega)} = -\int_\Omega \operatorname{div} \nabla v = -\int_\Gamma n \cdot \nabla v = 0,$$
$$\langle f, h\rangle_{\mathsf{L}^2(\Omega)} = \int_\Omega (\Delta^2 u) h = \int_\Gamma (n \cdot \nabla \Delta u) h - \int_\Gamma \Delta u \, (n \cdot \nabla h) = 0.$$

A first difference concerns the type of the boundary conditions: the Neumann boundary condition for $-\Delta$ satisfies the so-called complementing condition of Agmon, Douglis, and Nirenberg [1], that in the present situation simply says that the polynomial t is not divisible by $t - i$. This is not true for the Neumann boundary condition for Δ^2: the complementing condition would require that the polynomials $1 + t^2$ and $t + t^3$ were linearly independent modulo $(t - i)^2$, and it is well-known that this is not the case. For the ease of the reader, let us readily show this last statement: we have

$$\frac{1 + t^2}{(t - i)^2} = 1 + \frac{2(1 + it)}{(t - i)^2}, \qquad \frac{t + t^3}{(t - i)^2} = t + 2i + \frac{2(i - t)}{(t - i)^2},$$

and the second remainder $2(i - t)$ is proportional to the first remainder $2(1 + it)$ by a factor i. Note that the difference between the two Neumann problems still shows up when considering the so-called Lopatinskiĭ–Šapiro condition (see Wloka [25, Examples 11.2 and 11.8]); in fact, it is known that the complementing condition and the Lopatinskiĭ–Šapiro condition are equivalent (see, e.g., Negrón-Marrero and Montes-Pizarro [9, Appendix A]).

Another difference seems to be related to well-posedness: in fact, the Neumann boundary value problem associated to $-\Delta$ is well-posed in a suitable space where uniqueness is recovered and for data which satisfy the necessary compatibility condition (Fredholm alternative). This well-known result is an easy consequence of the Poincaré inequality, cf. Definition 2.6, and the Riesz representation theorem, or of more elaborate arguments using linear functional analysis, cf. the discussion of the Neumann Laplacian $\mathcal{L}_{\mathtt{N}}$ from Section 2.2.4. On the contrary, well-posedness for the Neumann boundary value problem associated to the Δ^2 operator seemed to be questionable so far (see, e.g., what is explicitly reported in Verchota [24, p. 217 and Sc. 21], and in a more indirect way in Renardy and Rogers [22, Sc. 9.4.2 and Example 9.30], Gazzola, Grunau and Sweers [6, Sc. 2.3], Provenzano [21, p. 1006]). In addition to this, it can be noted that Begehr [4] presents a long list of *twelve* boundary value problems for the biharmonic operator that are either well-posed or solvable under suitable compatibility conditions, and in that list the Neumann problem is not included.

Going a little bit more in depth, in Renardy and Rogers [22], Gazzola, Grunau and Sweers [6], and Provenzano [21] the comments about the fact that the Neumann problem for the biharmonic operator Δ^2 is possibly not well-posed are related to the fact that the complementing condition is not satisfied (in particular, this condition is assumed in the existence and uniqueness Theorems 2.16 and 2.20 in [6]; there see also Remark 2.17). This is apparently to be meaningful, as in Agmon, Douglis and Nirenberg [1, Sc. 10] it is explicitly proved that the complementing condition is necessary for obtaining higher order a-priori estimates in Hölder and L^p spaces (in this respect, see also Lions and Magenes [8, Chap. 2, Sc. 8.3 and Remark 9.8]).

However, rather surprisingly, it turns out that this condition is not necessary for well-posedness in suitable Hilbert spaces, as the following example shows: Consider the operator

$$\Delta^2 + 1.$$

Since the complementing condition only depends on the principal parts of the spatial and boundary operators, we are in the same situation of the Neumann problem for the biharmonic operator;

therefore the complementing condition is also not satisfied. However, the weak formulation of the problem (with homogeneous boundary data) reads: For $f \in \mathsf{L}^2(\Omega)$ find

$$u \in \mathsf{V} := \mathsf{H}(\Delta, \Omega) := \big\{\varphi \in \mathsf{L}^2(\Omega) : \Delta\varphi \in \mathsf{L}^2(\Omega)\big\} \tag{1}$$

such that

$$\forall\, \varphi \in \mathsf{V} \qquad \langle \Delta u, \Delta\varphi\rangle_{\mathsf{L}^2(\Omega)} + \langle u, \varphi\rangle_{\mathsf{L}^2(\Omega)} = \langle f, \varphi\rangle_{\mathsf{L}^2(\Omega)},$$

which is uniquely solvable by Riesz' representation theorem with stability estimate $|u|_{\mathsf{V}} \leq |f|_{\mathsf{L}^2(\Omega)}$.

Being now evident that the complementing condition is not necessary for well-posedness in a suitable Hilbert space, in this paper we shall show that indeed, among others, the Neumann boundary value problem for the biharmonic operator is well-posed.

1.2. **Approach by Bi-Laplacians.** The approach in [23, Section 5.6] is based on the representation of the biharmonic operator as bi-Laplacian Δ^2 with Dirichlet/Neumann type and over-/underdetermined boundary conditions.

The story begins with the (full, not mixed) Dirichlet and Neumann boundary conditions for the negative Laplacian $-\Delta$. These are the two reasonable boundary conditions for

$$-\Delta = -\operatorname{div}\nabla,$$

resulting in the apparently four "reasonable" (full) boundary conditions for the biharmonic operator interpreted as bi-Laplacian. More precisely: For a possible solution u of

$$\Delta^2 u = f \qquad \text{in } \Omega$$

we may impose a couple of the boundary conditions

$$u = 0 \qquad \text{on } \Gamma, \qquad \big(\text{Dirichlet on } u, \text{ i.e., } u \in \mathring{\mathsf{H}}^1(\Omega)\big) \tag{2}$$

$$n \cdot \nabla u = 0 \qquad \text{on } \Gamma, \qquad \big(\text{Neumann on } u, \text{ i.e., } \nabla u \in \mathring{\mathsf{H}}(\operatorname{div}, \Omega)\big) \tag{3}$$

$$\Delta u = 0 \qquad \text{on } \Gamma, \qquad \big(\text{Dirichlet on } \Delta u, \text{ i.e., } \Delta u \in \mathring{\mathsf{H}}^1(\Omega)\big) \tag{4}$$

$$n \cdot \nabla \Delta u = 0 \qquad \text{on } \Gamma, \qquad \big(\text{Neumann on } \Delta u, \text{ i.e., } \nabla \Delta u \in \mathring{\mathsf{H}}(\operatorname{div}, \Omega)\big) \tag{5}$$

as presented in the equations [23, (5.14)-(5.27)]. Some of these six pairs have prominent names:

Dirichlet:	(2) $\wedge$ (3)	Navier:	(2) $\wedge$ (4)
Neumann:	(4) $\wedge$ (5)	Riquier:	(3) $\wedge$ (5)

As we will see later, cf. Section 2.2.7, the latter four pairs as well as (2) $\wedge$ (5), then denoted by

$$\mathcal{L}\,\mathring{\mathcal{L}}, \quad \mathring{\mathcal{L}}\,\mathcal{L}, \quad \mathcal{L}_{\mathtt{D}}\,\mathcal{L}_{\mathtt{D}}, \quad \mathcal{L}_{\mathtt{N}}\,\mathcal{L}_{\mathtt{N}}, \quad \mathcal{L}_{\mathtt{N}}\,\mathcal{L}_{\mathtt{D}},$$

make Δ^2 well-posed, while (3) $\wedge$ (4), then denoted by $\mathcal{L}_{\mathtt{D}}\,\mathcal{L}_{\mathtt{N}}$, does not, at least without further restrictions on the domain of definition. The second author proposes variational formulations for the Dirichlet, Neumann, Navier, and Riquier biharmonic problems, but notices that for (2) $\wedge$ (5) and (3) $\wedge$ (4) no well-posed variational formulations are at hand.

[23, Section 5.6.1] is then specifically devoted to the Neumann problem for the biharmonic equation (4) $\wedge$ (5) ($\mathring{\mathcal{L}}\,\mathcal{L}$). The natural Sobolev spaces for the variational method are then given by V from (1) as well as $\mathsf{V}_{\#} = \mathsf{H}(\Delta, \Omega) \cap \mathbb{H}^{\perp}$ and $\mathsf{L}^2_{\#}(\Omega) = \mathsf{L}^2(\Omega) \cap \mathbb{H}^{\perp}$. For $f \in \mathsf{L}^2_{\#}(\Omega)$ the variational formulation is then to find $u \in \mathsf{V}_{\#}$ such that

$$\forall\, \varphi \in \mathsf{V} \text{ (or } \mathsf{V}_{\#}) \qquad \langle \Delta u, \Delta\varphi\rangle_{\mathsf{L}^2(\Omega)} = \langle f, \varphi\rangle_{\mathsf{L}^2(\Omega)},$$

cf. (25) in Section 2.2.6. It has been emphasised that the problem is well-posed if the Poincaré type estimate

$$\exists\, c > 0 \quad \forall\, \varphi \in \mathsf{V}_{\#} \qquad |\varphi|_{\mathsf{L}^2(\Omega)} \leq c|\Delta\varphi|_{\mathsf{L}^2(\Omega)} \tag{6}$$

holds, which follows if the embedding

$$\mathsf{V}_{\#} \hookrightarrow \mathsf{L}^2(\Omega) \tag{7}$$

is compact. The second author could show by regularity and Rellich's selection theorem that this is indeed true if Ω is bounded and of class C^4, since then the embedding $\mathsf{V}_\# \hookrightarrow \mathsf{H}^2(\Omega)$ is continuous, more precisely,

$$\mathsf{V}_\# = \Delta\big(\mathsf{H}^4(\Omega) \cap \mathring{\mathsf{H}}^2(\Omega)\big) = \mathsf{H}^2(\Omega) \cap \mathbb{H}^\perp = \mathsf{H}^2_\#(\Omega) \tag{8}$$

with equivalent norms. It has also been noted that $\mathsf{V} \hookrightarrow \mathsf{L}^2(\Omega)$ is in general not compact even for the unit ball and $N = 2$. The space $\Delta\big(\mathsf{H}^4(\Omega) \cap \mathring{\mathsf{H}}^2(\Omega)\big)$ (of high regularity) has already been proposed by Provenzano [21] for studying an eigenvalue problem for the Neumann biharmonic operator.

A theory relying on C^4-boundaries is unsatisfying. In this contribution we shall present a solution theory for quite a zoo of biharmonic problems – including the Neumann type – which works fine with bounded Lipschitz domains Ω or even beyond, cf. the notion of admissible domains in Defintion 2.6 and Remark 2.7, which need neither to be bounded nor smooth (Lipschitz).

1.2.1. *Sketch of Basic Ideas and Concepts.* We promote a more operator theoretical method using basic functional analysis. For simplicity, in this introduction, let us assume that Ω is a bounded Lipschitz domain.

We consider the two gradients (minimal and maximal)

$$\begin{aligned} \nabla_\Gamma &: \mathring{\mathsf{H}}^1(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega); & \phi &\mapsto \nabla\phi, \\ \nabla_\emptyset &: \mathsf{H}^1(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \end{aligned}$$

cf. Section 2.2.4 and [14] for more details and results on the de Rham complex. Note that we have the Friedrichs' and Poincaré's estimates

$$\begin{aligned} &\exists\, c_{\mathsf{f}} > 0 && \forall\, \varphi \in \mathring{\mathsf{H}}^1(\Omega) && |\varphi|_{\mathsf{L}^2(\Omega)} \le c_{\mathsf{f}} |\nabla\varphi|_{\mathsf{L}^2(\Omega)}, \\ & && \forall\, \varphi \in \mathring{\mathsf{H}}^2(\Omega) && |\varphi|_{\mathsf{L}^2(\Omega)} \le c_{\mathsf{f}}^2 |\Delta\varphi|_{\mathsf{L}^2(\Omega)}, \\ &\exists\, c_{\mathsf{p}} > 0 && \forall\, \varphi \in \widehat{\mathsf{H}}^1(\Omega) = \mathsf{H}^1(\Omega) \cap \mathbb{R}^\perp && |\varphi|_{\mathsf{L}^2(\Omega)} \le c_{\mathsf{p}} |\nabla\varphi|_{\mathsf{L}^2(\Omega)}. \end{aligned} \tag{9}$$

With (9) ∇_Γ and $\nabla_\emptyset$ are well-defined, i.e., densely defined and closed linear operators, with closed ranges, and so are their Hilbert space adjoints (maximal and minimal divergence)

$$\begin{aligned} -\operatorname{div}_\emptyset = (\nabla_\Gamma)^* &: \mathsf{H}(\operatorname{div}, \Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega); & \Phi &\mapsto \operatorname{div}\Phi, \\ -\operatorname{div}_\Gamma = (\nabla_\emptyset)^* &: \mathring{\mathsf{H}}(\operatorname{div}, \Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega). \end{aligned}$$

Then the Dirichlet and Neumann negative Laplacians

$$\begin{aligned} \mathrm{L}_{\mathtt{D}} = \nabla_\Gamma^* \nabla_\Gamma = -\operatorname{div}_\emptyset \nabla_\Gamma &: D(\mathrm{L}_{\mathtt{D}}) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega); & \phi &\mapsto -\operatorname{div}\nabla\phi, \\ \mathrm{L}_{\mathtt{N}} = \nabla_\emptyset^* \nabla_\emptyset = -\operatorname{div}_\Gamma \nabla_\emptyset &: D(\mathrm{L}_{\mathtt{N}}) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \end{aligned}$$

are selfadjoint and non-negative with kernels $N(\mathrm{L}_{\mathtt{D}}) = \{0\}$ and $N(\mathrm{L}_{\mathtt{N}}) = \mathbb{R}$. With the closed ranges we have the Fredholm alternatives

$$R(\mathrm{L}_{\mathtt{D}}) = N(\mathrm{L}_{\mathtt{D}})^\perp = \mathsf{L}^2(\Omega), \qquad R(\mathrm{L}_{\mathtt{N}}) = N(\mathrm{L}_{\mathtt{N}})^\perp = \mathsf{L}^2(\Omega) \cap \mathbb{R}^\perp = \widehat{\mathsf{L}}^2(\Omega).$$

Therefore, $\mathrm{L}_{\mathtt{D}}$ and $\mathcal{L}_{\mathtt{N}} = \mathrm{L}_{\mathtt{N}}|_{\mathbb{R}^\perp}$ are bijective and the inverse operators

$$\mathrm{L}_{\mathtt{D}}^{-1} : \mathsf{L}^2(\Omega) \to D(\mathrm{L}_{\mathtt{D}}), \qquad \mathcal{L}_{\mathtt{N}}^{-1} : \widehat{\mathsf{L}}^2(\Omega) \to D(\mathrm{L}_{\mathtt{N}}) \cap \mathbb{R}^\perp = D(\mathcal{L}_{\mathtt{N}})$$

are bounded by (9).

We may also consider over- and underdetermined negative Laplacians $\mathring{\mathrm{L}}$ and L, respectively, cf. Section 2.2.5. Now (9), cf. Lemma 2.9 and Lemma 2.1, yields that

$$\begin{aligned} \mathring{\mathrm{L}} &: \mathring{\mathsf{H}}^2(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega); & \varphi &\mapsto -\Delta\varphi, \\ \mathrm{L} := \mathring{\mathrm{L}}^* &: \mathsf{H}(\Delta, \Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega) \end{aligned}$$

are densely defined and closed with kernels $N(\mathring{\mathrm{L}}) = \{0\}$ and $N(\mathrm{L}) = \mathbb{H}$ and closed ranges

$$R(\mathring{\mathrm{L}}) = N(\mathrm{L})^\perp = \mathsf{L}^2(\Omega) \cap \mathbb{H}^\perp = \widetilde{\mathsf{L}}^2(\Omega), \qquad R(\mathrm{L}) = N(\mathring{\mathrm{L}})^\perp = \mathsf{L}^2(\Omega).$$

Thus $\mathring{\mathrm{L}}$ and $\mathcal{L} = \mathrm{L}\,|_{\mathbb{H}^\perp}$ are bijective and the inverse operators

$$\mathring{\mathrm{L}}^{-1} : \widetilde{\mathsf{L}}^2(\Omega) \to D(\mathring{\mathrm{L}}), \qquad \mathcal{L}^{-1} : \mathsf{L}^2(\Omega) \to D(\mathrm{L}) \cap \mathbb{H}^\perp = D(\mathcal{L})$$

are bounded by (9).

One could say that the dual (adjoint) pair $\mathring{\mathrm{L}}$ and L defines the minimal and the maximal $\mathsf{L}^2(\Omega)$-realisations of the negative Laplacian, in the sense

$$\mathring{\mathrm{L}} \subset \mathrm{L}_{\mathtt{D}}, \mathrm{L}_{\mathtt{N}} \subset \mathrm{L}$$

with the two selfadjoint realisations $\mathrm{L}_{\mathtt{D}}$ and $\mathrm{L}_{\mathtt{N}}$ in-between.

Let $f \in \mathsf{L}^2(\Omega)$, $\widehat{f} \in \widehat{\mathsf{L}}^2(\Omega)$, and $\widetilde{f} \in \widetilde{\mathsf{L}}^2(\Omega)$. Then:

- $u_{\mathtt{D}} = \mathrm{L}_{\mathtt{D}}^{-1} f \in D(\mathrm{L}_{\mathtt{D}})$ is the unique solution of the Dirichlet negative Laplace boundary value problem $\mathrm{L}_{\mathtt{D}}\, u_{\mathtt{D}} = f$.
- $u = \mathcal{L}^{-1} f \in D(\mathcal{L})$ is the unique solution of the underdetermined negative Laplace boundary value problem $\mathcal{L}\, u = f$.
- $u_{\mathtt{N}} = \mathcal{L}_{\mathtt{N}}^{-1} \widehat{f} \in D(\mathcal{L}_{\mathtt{N}})$ is the unique solution of the Neumann negative Laplace boundary value problem $\mathcal{L}_{\mathtt{N}}\, u_{\mathtt{N}} = \widehat{f}$.
- $\mathring{u} = \mathring{\mathcal{L}}^{-1} \widetilde{f} \in D(\mathring{\mathrm{L}})$ is the unique solution of the overdetermined negative Laplace boundary value problem $\mathring{\mathcal{L}} \mathring{u} = \widetilde{f}$.

In classical terms we have

$$\begin{array}{rrrrl} -\Delta \mathring{u} = \widetilde{f}, & -\Delta u_{\mathtt{D}} = f, & -\Delta u_{\mathtt{N}} = \widehat{f}, & -\Delta u = f & \text{in } \Omega, \\ \mathring{u} = 0, & u_{\mathtt{D}} = 0 & & & \text{on } \Gamma, \\ n \cdot \nabla \mathring{u} = 0, & & n \cdot \nabla u_{\mathtt{N}} = 0 & & \text{on } \Gamma, \\ & & u_{\mathtt{N}} \perp \mathbb{R}, & u \perp \mathbb{H}. & \end{array}$$

To find $u_{\mathtt{D}} \in D(\mathrm{L}_{\mathtt{D}}) \subset \mathring{\mathsf{H}}^1(\Omega)$ and $u_{\mathtt{N}} \in D(\mathcal{L}_{\mathtt{N}}) \subset \widehat{\mathsf{H}}^1(\Omega)$ by variational methods one may consider the well-known text book formulations

$$\begin{aligned} &\forall\, \varphi \in \mathring{\mathsf{H}}^1(\Omega) && \langle \nabla u_{\mathtt{D}}, \nabla \varphi \rangle_{\mathsf{L}^2(\Omega)} = \langle f, \varphi \rangle_{\mathsf{L}^2(\Omega)}, \\ &\forall\, \psi \in \mathsf{H}^1(\Omega) \ \big(\text{or } \widehat{\mathsf{H}}^1(\Omega)\big) && \langle \nabla u_{\mathtt{N}}, \nabla \psi \rangle_{\mathsf{L}^2(\Omega)} = \langle \widehat{f}, \psi \rangle_{\mathsf{L}^2(\Omega)}. \end{aligned}$$

In Remark 2.14 we suggest variational methods to find the two solutions $\mathring{u} \in D(\mathring{\mathrm{L}}) = \mathring{\mathsf{H}}^2(\Omega)$ and $u \in D(\mathcal{L}) = \mathsf{H}(\Delta, \Omega) \cap \mathbb{H}^\perp$.

Let us go back to the different biharmonic operators from the beginning of Section 1.2, cf. Section 2.2.6 and Section 2.2.7. From our operator theoretical point of view, these six different boundary value problems for the biharmonic equation, are given by the following table:

Dirichlet:	$(2) \wedge (3)$	given by $\mathrm{B}_{\mathtt{D}} = \mathrm{L}\,\mathring{\mathrm{L}}$	with bd right inverse $\mathcal{B}_{\mathtt{D}}^{-1} = \mathring{\mathcal{L}}^{-1}\,\mathcal{L}^{-1}$
Neumann:	$(4) \wedge (5)$	given by $\mathrm{B}_{\mathtt{N}} = \mathring{\mathrm{L}}\,\mathrm{L}$	with bd right inverse $\mathcal{B}_{\mathtt{N}}^{-1} = \mathcal{L}^{-1}\,\mathring{\mathcal{L}}^{-1}$
Navier:	$(2) \wedge (4)$	given by $\mathrm{B}_{\mathtt{D},\mathtt{D}} = \mathrm{L}_{\mathtt{D}}\,\mathrm{L}_{\mathtt{D}}$	with bd right inverse $\mathcal{B}_{\mathtt{D},\mathtt{D}}^{-1} = \mathcal{L}_{\mathtt{D}}^{-1}\,\mathcal{L}_{\mathtt{D}}^{-1}$
Riquier:	$(3) \wedge (5)$	given by $\mathrm{B}_{\mathtt{N},\mathtt{N}} = \mathrm{L}_{\mathtt{N}}\,\mathrm{L}_{\mathtt{N}}$	with bd right inverse $\mathcal{B}_{\mathtt{N},\mathtt{N}}^{-1} = \mathcal{L}_{\mathtt{N}}^{-1}\,\mathcal{L}_{\mathtt{N}}^{-1}$
N-D:	$(2) \wedge (5)$	given by $\mathrm{B}_{\mathtt{N},\mathtt{D}} = \mathrm{L}_{\mathtt{N}}\,\mathrm{L}_{\mathtt{D}}$	with bd right inverse $\mathcal{B}_{\mathtt{N},\mathtt{D}}^{-1} = \mathcal{L}_{\mathtt{D}}^{-1}\,\mathcal{L}_{\mathtt{N}}^{-1}$
D-N:	$(3) \wedge (4)$	given by $\mathrm{B}_{\mathtt{D},\mathtt{N}} = \mathrm{L}_{\mathtt{D}}\,\mathrm{L}_{\mathtt{N}}$	

By our theory the inverses of the first five biharmonic operators are well-defined, while the last inverse of the biharmonic operator $\mathrm{B}_{\mathtt{D},\mathtt{N}}$ is not in terms of $\mathcal{L}_{\mathtt{N}}^{-1}\,\mathcal{L}_{\mathtt{D}}^{-1}$. Since

$$\mathring{\mathrm{L}}^* = \mathrm{L}, \qquad \mathrm{L}^* = \mathring{\mathrm{L}}, \qquad \mathrm{L}_{\mathtt{D}}^* = \mathrm{L}_{\mathtt{D}}, \qquad \mathrm{L}_{\mathtt{N}}^* = \mathrm{L}_{\mathtt{N}},$$

we see that the operators corresponding to the Dirichlet, Neumann, Navier, and Riquier biharmonic problems are selfadjoint (and non-negative), while the others are not.

Let $f \in \mathsf{L}^2(\Omega)$, $\widehat{f} \in \widehat{\mathsf{L}}^2(\Omega)$, and $\widetilde{f} \in \widetilde{\mathsf{L}}^2(\Omega)$. Then:

- $u_{\mathtt{D}} = \mathrm{B}_{\mathtt{D}}^{-1} = \mathring{\mathcal{L}}^{-1}\,\mathcal{L}^{-1} f \in D(\mathrm{B}_{\mathtt{D}})$ is the unique solution of the Dirichlet biharmonic boundary value problem $\mathrm{B}_{\mathtt{D}}\, u_{\mathtt{D}} = f$.

- $u_{\mathtt{DD}} = \mathrm{B}_{\mathtt{D},\mathtt{D}}^{-1} = \mathrm{L}_{\mathtt{D}}^{-1}\,\mathrm{L}_{\mathtt{D}}^{-1}\,f \in D(\mathrm{B}_{\mathtt{D},\mathtt{D}})$ is the unique solution of the Navier biharmonic boundary value problem $\mathrm{B}_{\mathtt{D},\mathtt{D}}\,u_{\mathtt{DD}} = f$.
- $u_{\mathtt{N}} = \mathcal{B}_{\mathtt{N}}^{-1} = \mathcal{L}^{-1}\,\mathring{\mathcal{L}}^{-1}\,\widetilde{f} \in D(\mathcal{B}_{\mathtt{N}})$ is the unique solution of the Neumann biharmonic boundary value problem $\mathcal{B}_{\mathtt{N}}\,u_{\mathtt{N}} = \widetilde{f}$.
- $u_{\mathtt{NN}} = \mathcal{B}_{\mathtt{N},\mathtt{N}}^{-1} = \mathcal{L}_{\mathtt{N}}^{-1}\,\mathcal{L}_{\mathtt{N}}^{-1}\,\widehat{f} \in D(\mathcal{B}_{\mathtt{N},\mathtt{N}})$ is the unique solution of the Riquier biharmonic boundary value problem $\mathcal{B}_{\mathtt{N},\mathtt{N}}\,u_{\mathtt{NN}} = \widehat{f}$.
- $u_{\mathtt{ND}} = \mathcal{B}_{\mathtt{N},\mathtt{D}}^{-1} = \mathcal{L}_{\mathtt{D}}^{-1}\,\mathcal{L}_{\mathtt{N}}^{-1}\,\widehat{f} \in D(\mathcal{B}_{\mathtt{N},\mathtt{D}})$ is the unique solution of the N-D biharmonic boundary value problem $\mathcal{B}_{\mathtt{N},\mathtt{D}}\,u_{\mathtt{ND}} = \widehat{f}$.

In classical terms we have

$$\begin{array}{rrrrrl}
\Delta^2 u_{\mathtt{D}} = f, & \Delta^2 u_{\mathtt{DD}} = f, & \Delta^2 u_{\mathtt{N}} = \widetilde{f}, & \Delta^2 u_{\mathtt{NN}} = \widehat{f}, & \Delta^2 u_{\mathtt{ND}} = \widehat{f} & \text{in } \Omega,\\
u_{\mathtt{D}} = 0, & u_{\mathtt{DD}} = 0, & & & u_{\mathtt{ND}} = 0 & \text{on } \Gamma,\\
n\cdot\nabla\, u_{\mathtt{D}} = 0, & & & n\cdot\nabla\, u_{\mathtt{NN}} = 0 & & \text{on } \Gamma,\\
& \Delta u_{\mathtt{DD}} = 0, & \Delta u_{\mathtt{N}} = 0 & & & \text{on } \Gamma,\\
& & n\cdot\nabla\,\Delta u_{\mathtt{N}} = 0, & n\cdot\nabla\,\Delta u_{\mathtt{NN}} = 0, & n\cdot\nabla\,\Delta u_{\mathtt{ND}} = 0 & \text{on } \Gamma,\\
& & u_{\mathtt{N}} \perp \mathbb{H}, & u_{\mathtt{NN}} \perp \mathbb{R}, & & \\
& & & \Delta u_{\mathtt{NN}} \perp \mathbb{R}, & \Delta u_{\mathtt{ND}} \perp \mathbb{R}. &
\end{array}$$

In fact, in Section 2.2.7 and Section 2.2.8 we present *eighteen* different biharmonic operators, and it turns out that *thirteen* of those are well defined and lead to uniquely solvable boundary value problems for the bi-Laplacian. Our general theory extends also to mixed boundary conditions giving even more well defined bi-Laplacians, cf. Section 2.2.9.

Remark 1.1. *Concerning* (6)*,* (7)*, and* (8) *we can show remarkable results for all bounded domains* Ω*. Note that no(!) regularity assumptions on* Ω *are needed at all.*

(i) *In Theorem 2.13 we prove that indeed*

$$D(\mathcal{L}) = \mathsf{H}(\Delta,\Omega)\cap\mathbb{H}^{\perp} = \mathsf{V}_{\#} \hookrightarrow \mathsf{L}^2(\Omega)$$

is compact.

(ii) *Moreover, for all* $\varphi \in D(\mathring{\mathrm{L}}) = \mathring{\mathsf{H}}^2(\Omega)$ *and all* $\varphi \in D(\mathcal{L})$ *we have*

$$|\varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{f}}^2 |\Delta\varphi|_{\mathsf{L}^2(\Omega)},$$

cf. Remark 2.10.

(iii) *For the variational space we see*

$$\mathsf{V}_{\#} = \mathring{\mathcal{L}}\mathring{\mathcal{L}}^{-1}\mathsf{V}_{\#} = \Delta\big\{\varphi\in\mathring{\mathsf{H}}^2(\Omega) : \Delta\varphi\in D(\mathcal{L})\big\} = \Delta\big\{\varphi\in\mathring{\mathsf{H}}^2(\Omega) : \Delta^2\varphi\in\mathsf{L}^2(\Omega)\big\},$$

which – in case of a C^4*-boundary* Γ *– equals by standard regularity theory for the Dirichlet bi-Laplacian* $\Delta\big(\mathsf{H}^4(\Omega)\cap\mathring{\mathsf{H}}^2(\Omega)\big)$*.*

Remark 1.2. *Analogously we have:*

(i) *Analogously, Theorem 2.17 shows that*

$$D(\mathcal{B}) = \mathsf{H}(\Delta^2,\Omega)\cap\mathbb{BH}^{\perp} \hookrightarrow \mathsf{L}^2(\Omega)$$

is compact.

(ii) *For all* $\varphi\in D(\mathring{\mathrm{B}}) = \mathring{\mathsf{H}}^4(\Omega)$ *and all* $\varphi\in D(\mathcal{B})$ *we have*

$$|\varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{f}}^4|\Delta^2\varphi|_{\mathsf{L}^2(\Omega)}.$$

(iii) *For the variational space we have*

$$D(\mathcal{B}) = \mathring{\mathcal{B}}\mathring{\mathcal{B}}^{-1}D(\mathcal{B}) = \Delta^2\big\{\varphi\in\mathring{\mathsf{H}}^4(\Omega) : \Delta^2\varphi\in D(\mathcal{B})\big\} = \Delta^2\big\{\varphi\in\mathring{\mathsf{H}}^4(\Omega) : \Delta^4\varphi\in\mathsf{L}^2(\Omega)\big\},$$

which – in case of a C^8*-boundary* Γ *– equals by standard regularity theory for higher order elliptic Dirichlet problems* $\Delta^2\big(\mathsf{H}^8(\Omega)\cap\mathring{\mathsf{H}}^4(\Omega)\big)$*.*

1.2.2. *Shifting Boundary Conditions by Laplacians.* Using the latter four different Laplacians, cf. (26), we can solve the Neumann biharmonic problem by the Dirichlet biharmonic problem and vice versa. For example,

$$\mathcal{B}_{\mathtt{N}}^{-1} = \mathring{\mathcal{L}}\mathring{\mathcal{L}}^{-1}\,\mathcal{L}^{-1}\,\mathring{\mathcal{L}}^{-1} = \mathring{\mathcal{L}}\,\mathcal{B}_{\mathtt{D}}^{-1}\,\mathring{\mathcal{L}}^{-1},$$

i.e., the solution $u = \mathcal{B}_{\mathtt{N}}^{-1} f$ of the Neumann biharmonic problem can be found by solving for the overdetermined Laplacian $\mathring{\mathcal{L}}^{-1} f$, then for the Dirichlet biharmonic problem $v = \mathcal{B}_{\mathtt{D}}^{-1}\,\mathring{\mathcal{L}}^{-1} f$, and finally taking $u = \Delta v$. Analogously, we have

$$\mathcal{B}_{\mathtt{D}}^{-1} = \mathcal{L}\,\mathcal{L}^{-1}\,\mathring{\mathcal{L}}^{-1}\,\mathcal{L}^{-1} = \mathcal{L}\,\mathcal{B}_{\mathtt{N}}^{-1}\,\mathring{\mathcal{L}}^{-1}.$$

This "trick" can extended to a lot more "allowed" combinations, such as

$$\mathcal{B}_{\mathtt{N},\mathtt{D}}^{-1} = \mathcal{L}_{\mathtt{D}}\,\mathcal{L}_{\mathtt{D}}^{-1}\,\mathcal{L}_{\mathtt{D}}^{-1}\,\mathcal{L}_{\mathtt{N}}^{-1} = \mathcal{L}_{\mathtt{D}}\,\mathcal{B}_{\mathtt{D},\mathtt{D}}^{-1}\,\mathcal{L}_{\mathtt{N}}^{-1},$$

just to mention one of many options.

1.3. **Alternative Approach by the Hessian Complex.** For $\phi \in \mathsf{C}^\infty(\mathbb{R}^3)$ we have point-wise

$$\Delta^2\phi = \sum_{i,j} \partial_i^2\,\partial_j^2\,\phi = \sum_{i,j} \partial_i\,\partial_j\,\partial_i\,\partial_j\,\phi = \operatorname{divDiv}\nabla\nabla\,\phi, \tag{10}$$

which then extends also to distributions ϕ. Here, $\nabla\nabla\,\phi$ denotes the Hessian of ϕ and Div acts as row-wise incarnation of div. Note that divDiv is the formal adjoint of $\nabla\nabla$. In the following we use again the basic concepts of functional analysis, cf. Section 2.1.

By (10) another way to look at the biharmonic equation, which respects more the underlining geometry (Hessian Hilbert complex) of the problem and the corresponding operators, is to investigate the two Hessian operators

$$\begin{aligned} \nabla\nabla_\Gamma &: \mathring{\mathsf{H}}^2(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega); & \phi \mapsto \nabla\nabla\,\phi,\\ \nabla\nabla_\emptyset &: \mathsf{H}^2(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \end{aligned}$$

cf. Section 2.3 and [18, 2, 16] for more details and results on the Hessian complex. Note that Friedrichs' estimate (9) shows

$$\exists\, c_{\mathsf{f}} > 0 \quad \forall\, \varphi \in \mathring{\mathsf{H}}^2(\Omega) \qquad |\varphi|_{\mathsf{L}^2(\Omega)} \le c_{\mathsf{f}}|\,\nabla\,\varphi|_{\mathsf{L}^2(\Omega)} \le c_{\mathsf{f}}^2|\,\nabla\nabla\,\varphi|_{\mathsf{L}^2(\Omega)}. \tag{11}$$

Moreover, $\varphi \in \mathsf{L}^2(\Omega)$ with $\nabla\nabla\,\varphi \in \mathsf{L}^2(\Omega)$ implies $\varphi \in \mathsf{H}^2(\Omega)$ by the Nečas/Lions lemma, and we have Poincaré's estimate

$$\exists\, c_{\mathsf{p}} > 0 \quad \forall\, \varphi \in \widehat{\widehat{\mathsf{H}}}^2(\Omega) = \mathsf{H}^2(\Omega) \cap \mathsf{P}_1^\perp \qquad |\varphi|_{\mathsf{L}^2(\Omega)} \le c_{\mathsf{p}}|\,\nabla\,\varphi|_{\mathsf{L}^2(\Omega)} \le c_{\mathsf{p}}^2|\,\nabla\nabla\,\varphi|_{\mathsf{L}^2(\Omega)}, \tag{12}$$

where P_1 denotes the first order polynomials. With (11), Nečas' lemma, and (12) $\nabla\nabla_\Gamma$ and $\nabla\nabla_\emptyset$ are well-defined, i.e., densely defined and closed linear operators. Their Hilbert space adjoints are given by

$$\begin{aligned} \operatorname{divDiv}_\emptyset = (\nabla\nabla_\Gamma)^* &: \mathsf{H}(\operatorname{divDiv},\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega); & \Phi \mapsto \operatorname{divDiv}\Phi,\\ \operatorname{divDiv}_\Gamma = (\nabla\nabla_\emptyset)^* &: \mathring{\mathsf{H}}(\operatorname{divDiv},\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega). \end{aligned}$$

We can then investigate the biharmonic operators

$$\begin{aligned} \mathrm{B}_\Gamma = \nabla\nabla_\Gamma{}^*\,\nabla\nabla_\Gamma = \operatorname{divDiv}_\emptyset\,\nabla\nabla_\Gamma &: D(\mathrm{B}_\Gamma) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega); & \phi \mapsto \operatorname{divDiv}\nabla\nabla\,\phi,\\ \mathrm{B}_\emptyset = \nabla\nabla_\emptyset{}^*\,\nabla\nabla_\emptyset = \operatorname{divDiv}_\Gamma\,\nabla\nabla_\emptyset &: D(\mathrm{B}_\emptyset) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \end{aligned}$$

which may be called Dirichlet resp. Neumann biharmonic operator as well. Note that indeed by Remark 2.11 the "new" Dirichlet biharmonic operator equals the "old" one from above, i.e.,

$$\mathrm{B}_\Gamma = \mathrm{B}_{\mathtt{D}} = \mathrm{L}\,\mathring{\mathrm{L}},$$

but the "new" Neumann biharmonic operator $\mathrm{B}_\emptyset$ is not a biharmonic operator with any combination of the boundary conditions (2)-(5). In fact, while the boundary conditions of $\mathrm{B}_{\mathtt{N}} = \mathring{\mathrm{L}}\,\mathrm{L}$ are imposed on the scalar Δu, the boundary conditions of $\mathrm{B}_\emptyset$ are imposed on the symmetric tensor $S = \nabla\nabla\,u$.

B_Γ and $\mathrm{B}_\emptyset$ are selfadjoint and non-negative with kernels $N(\mathrm{B}_\Gamma)=\{0\}$ and $N(\mathrm{B}_\emptyset)=\mathsf{P}_1$. By (11) and (12) the ranges are closed and we have the Fredholm alternatives

$$R(\mathrm{B}_\Gamma)=N(\mathrm{B}_\Gamma)^\perp=\mathsf{L}^2(\Omega),\qquad R(\mathrm{B}_\emptyset)=N(\mathrm{B}_\emptyset)^\perp=\mathsf{L}^2(\Omega)\cap\mathsf{P}_1^\perp=\hat{\hat{\mathsf{L}}}^2(\Omega).$$

Therefore, B_Γ and $\mathcal{B}_\emptyset=\mathrm{B}_\emptyset|_{\mathsf{P}_1^\perp}$ are bijective and the inverse operators

$$\mathrm{B}_\Gamma^{-1}:\mathsf{L}^2(\Omega)\to D(\mathrm{B}_\Gamma),\qquad \mathcal{B}_\emptyset^{-1}:\hat{\hat{\mathsf{L}}}^2(\Omega)\to D(\mathrm{B}_\emptyset)\cap\mathsf{P}_1^\perp=D(\mathcal{B}_\emptyset)$$

are bounded, i.e., $|\,\mathrm{B}_\Gamma^{-1}\,|\leq c_{\mathsf{f}}^4$ and $|\,\mathcal{B}_\emptyset^{-1}\,|\leq c_{\mathsf{p}}^4$.

Let $f\in\mathsf{L}^2(\Omega)$ and $f_1\in\hat{\hat{\mathsf{L}}}^2(\Omega)$. Then $u_\Gamma=\mathrm{B}_\Gamma^{-1}f\in D(\mathrm{B}_\Gamma)$ is the unique solution of the Dirichlet biharmonic boundary value problem $\mathrm{B}_\Gamma\,u_\Gamma=f$, and $u_\emptyset=\mathcal{B}_\emptyset^{-1}f_1\in D(\mathcal{B}_\emptyset)$ is the unique solution of the Neumann biharmonic boundary value problem $\mathcal{B}_\emptyset\,u_\emptyset=f_1$. In classical terms we have

$$\begin{aligned}
\operatorname{divDiv}\nabla\nabla\,u_\Gamma&=f, & \operatorname{divDiv}\nabla\nabla\,u_\emptyset&=f_1 &&\text{in }\Omega,\\
u_\Gamma&=0, & (\nabla\nabla\,u_\emptyset)\,n&=0 &&\text{on }\Gamma,\\
\nabla\,u_\Gamma&=0, & \big((\nabla\nabla\,u_\emptyset)\,n\big)\cdot n&=0 &&\text{on }\Gamma,\\
&& u_\emptyset&\perp\mathsf{P}_1.
\end{aligned}$$

To find $u\in D(\mathrm{B}_\Gamma)\subset\mathring{\mathsf{H}}^2(\Omega)$ and $v\in D(\mathcal{B}_\emptyset)\subset\hat{\hat{\mathsf{H}}}^2(\Omega)$ by variational methods one may consider

$$\begin{aligned}
&\forall\,\varphi\in\mathring{\mathsf{H}}^2(\Omega) && \langle\nabla\nabla\,u,\nabla\nabla\,\varphi\rangle_{\mathsf{L}^2(\Omega)}=\langle f,\varphi\rangle_{\mathsf{L}^2(\Omega)},\\
&\forall\,\psi\in\mathsf{H}^2(\Omega)\ \big(\text{or }\hat{\hat{\mathsf{H}}}^2(\Omega)\big) && \langle\nabla\nabla\,v,\nabla\nabla\,\psi\rangle_{\mathsf{L}^2(\Omega)}=\langle g,\psi\rangle_{\mathsf{L}^2(\Omega)}.
\end{aligned}$$

It is worth noting that the results in [18, Section 4] show that the Dirichlet biharmonic problem

$$\mathrm{B}_\Gamma\,u=\operatorname{divDiv}_{\mathbb{S},\emptyset}\nabla\nabla_\Gamma\,u=f$$

splits up into a sequence of *three second order* (elliptic) boundary value problems indicated by the matrix representation

$$\begin{bmatrix}\mathrm{L}_{\mathsf{D}} & \operatorname{tr}\operatorname{symRot}_{\mathbb{T},\emptyset} & 3\\ 0 & \operatorname{Rot}_{\mathbb{S},\Gamma}\operatorname{symRot}_{\mathbb{T},\emptyset} & \operatorname{Rot}_{\mathbb{S},\Gamma}\operatorname{tr}^*\\ 0 & 0 & \mathrm{L}_{\mathsf{D}}\end{bmatrix}\begin{bmatrix}u\\E\\v\end{bmatrix}=\begin{bmatrix}0\\0\\f\end{bmatrix},$$

cf. Section 2.3 for definitions. The matrix has the structure

$$\begin{bmatrix}a^*a & c & 3\\ 0 & b^*b & c^*\\ 0 & 0 & a^*a\end{bmatrix}$$

with $a=\mathrm{A}_0=\mathring{\nabla}$ taken from the de Rham complex (21), and $b=\mathrm{A}_1^*=\operatorname{symRot}_{\mathbb{T},\emptyset}$ and $c=\operatorname{tr}b$ from the Hessian complex (33).

2. Operator Theory for Biharmonic Equations

Let us begin with some abstract basics.

2.1. **Tiny FA-ToolBox.**

We recall a few results from linear functional analysis. In particular, we use fundamental results from the so-called FA-ToolBox, see, e.g., [12, 13], cf. [10, 11, 14, 15, 16, 18, 19].

2.1.1. *Single Operators.* Let us consider a densely defined and closed linear operator A between two Hilbert spaces H_0 and H_1 together with its (densely defined and closed) Hilbert space adjoint A^*, that is

$$\mathrm{A} : D(\mathrm{A}) \subset \mathsf{H}_0 \to \mathsf{H}_1, \qquad \mathrm{A}^* : D(\mathrm{A}^*) \subset \mathsf{H}_1 \to \mathsf{H}_0.$$

In general, A and A^* are unbounded and characterised by

$$\forall\, x \in D(\mathrm{A}) \quad \forall\, y \in D(\mathrm{A}^*) \qquad \langle \mathrm{A}\, x, y\rangle_{\mathsf{H}_1} = \langle x, \mathrm{A}^*\, y\rangle_{\mathsf{H}_0}.$$

Since $\mathrm{A}^{**} = \mathrm{A}$ we call $(\mathrm{A}, \mathrm{A}^*)$ a dual pair. Note that $R(\mathrm{A})$ is closed if and only if $R(\mathrm{A}^*)$ is closed by the closed range theorem. Moreover, the projection theorem yields the orthogonal decompositions (Helmholtz type decompositions)

$$\mathsf{H}_0 = \overline{R(\mathrm{A}^*)} \oplus_{\mathsf{H}_0} N(\mathrm{A}), \qquad \mathsf{H}_1 = \overline{R(\mathrm{A})} \oplus_{\mathsf{H}_1} N(\mathrm{A}^*), \tag{13}$$

which suggest to investigate the injective restrictions, also called reduced operators and denoted by calligraphical letters,

$$\mathcal{A} := \mathrm{A}\,|_{N(\mathrm{A})^\perp}, \qquad \mathcal{A}^* := \mathrm{A}^*\,|_{N(\mathrm{A}^*)^\perp},$$

more precisely,

$$\begin{aligned} \mathcal{A} : D(\mathcal{A}) \subset N(\mathrm{A})^\perp \to \overline{R(\mathrm{A})} = N(\mathrm{A}^*)^\perp, &\qquad D(\mathcal{A}) := D(\mathrm{A}) \cap N(\mathrm{A})^\perp, \\ \mathcal{A}^* : D(\mathcal{A}^*) \subset N(\mathrm{A}^*)^\perp \to \overline{R(\mathrm{A}^*)} = N(\mathrm{A})^\perp, &\qquad D(\mathcal{A}^*) := D(\mathrm{A}^*) \cap N(\mathrm{A}^*)^\perp. \end{aligned}$$

$(\mathcal{A}, \mathcal{A}^*)$ are also densely defined and closed forming another dual pair with dense ranges. Moreover, by (13) we have

$$\begin{aligned} D(\mathrm{A}) &= D(\mathcal{A}) \oplus_{\mathsf{H}_0} N(\mathrm{A}), & D(\mathrm{A}^*) &= D(\mathcal{A}^*) \oplus_{\mathsf{H}_1} N(\mathrm{A}^*), \\ R(\mathrm{A}) &= R(\mathcal{A}), & R(\mathrm{A}^*) &= R(\mathcal{A}^*). \end{aligned} \tag{14}$$

Here we have used the symbols $\overline{\cdot}$, $\oplus$, and $\perp$ for the closure, the orthogonal sum, and the orthogonal complement, respectively.

From [11, Lemma 4.1, Remark 4.2], see also [13, Lemma 2.1, Lemma 2.2] or [17, Lemma 2.1, Lemma 2.4], we cite the following elementary result.

Lemma 2.1 (fundamental FA-ToolBox lemma 1)**.** *The following assertions are equivalent:*

- **(i)** $\exists\, c_{\mathrm{A}} > 0 \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathsf{H}_0} \le c_{\mathrm{A}} |\,\mathrm{A}\, x|_{\mathsf{H}_1}$
- **(i*)** $\exists\, c_{\mathrm{A}^*} > 0 \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathsf{H}_1} \le c_{\mathrm{A}^*} |\,\mathrm{A}^*\, y|_{\mathsf{H}_0}$
- **(ii)** $R(\mathrm{A})$ *is closed.*
- **(ii*)** $R(\mathrm{A}^*)$ *is closed.*
- **(iii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is bounded.*
- **(iii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is bounded.*

The latter assertions hold, if the embedding $D(\mathcal{A}) \hookrightarrow H_0$ is compact.

- **(iv)** $D(\mathcal{A}) \hookrightarrow H_0$ *is compact, if and only if* $D(\mathcal{A}^*) \hookrightarrow H_1$ *is compact.*

Remark 2.2 (fundamental FA-ToolBox lemma 1)**.** *If the estimate in (i) holds with c_{A} then (ii) holds with $c_{\mathrm{A}^*} = c_{\mathrm{A}}$, and vice versa. For the best constants we have*

$$|\,\mathcal{A}^{-1}\,|_{R(\mathrm{A}) \to \mathsf{H}_0} = c_{\mathrm{A}} = c_{\mathrm{A}^*} = \big|(\mathcal{A}^*)^{-1}\big|_{R(\mathrm{A}^*) \to \mathsf{H}_1}.$$

Lemma 2.1 shows that the key point to a proper solution theory in the sense of Hadamard is a close range (ii) or, equivalently, a Friedrichs/Poincaré type estimate (i).

Lemma 2.3 (automatic regularity)**.** $\mathrm{A}^*\,\mathrm{A}$*,* $\mathrm{A}\,\mathrm{A}^*$ *and* $\mathcal{A}^*\,\mathcal{A}$*,* $\mathcal{A}\,\mathcal{A}^*$ *are selfadjoint and nonnegative.* $\mathcal{A}^*\,\mathcal{A} = \mathrm{A}^*\,\mathcal{A}$ *and* $\mathcal{A}\,\mathcal{A}^* = \mathrm{A}\,\mathcal{A}^*$ *are positive (and hence injective) with dense ranges. Moreover, the automatic regularity* (14) *extends to*

$$\begin{aligned} \ldots = R(\mathrm{A}\,\mathrm{A}^*\,\mathrm{A}\ldots) = R(\mathrm{A}\,\mathrm{A}^*) = R(\mathrm{A}) = R(\mathcal{A}) = R(\mathcal{A}\,\mathcal{A}^*) = R(\mathcal{A}\,\mathcal{A}^*\,\mathcal{A}\ldots) = \ldots, \\ \ldots = R(\mathrm{A}^*\,\mathrm{A}\,\mathrm{A}^*\ldots) = R(\mathrm{A}^*\,\mathrm{A}) = R(\mathrm{A}^*) = R(\mathcal{A}^*) = R(\mathcal{A}^*\,\mathcal{A}) = R(\mathcal{A}^*\,\mathcal{A}\,\mathcal{A}^*\ldots) = \ldots \end{aligned}$$

and it holds

$$N(\mathrm{A}) = N(\mathrm{A}^*\,\mathrm{A}) = N(\mathrm{A}\,\mathrm{A}^*\,\mathrm{A}) = \ldots, \qquad N(\mathrm{A}^*) = N(\mathrm{A}\,\mathrm{A}^*) = N(\mathrm{A}^*\,\mathrm{A}\,\mathrm{A}^*) = \ldots.$$

Lemma 2.4 (fundamental FA-ToolBox lemma 2). *Let $R(\mathrm{A})$ be closed.*

(i) $\mathcal{A}^*\mathcal{A} : D(\mathcal{A}^*\mathcal{A}) \subset N(\mathrm{A})^\perp \to R(\mathrm{A}^*) = N(\mathrm{A})^\perp$ *is bijective with bounded inverse*

$$(\mathcal{A}^*\mathcal{A})^{-1} = \mathcal{A}^{-1}(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*\mathcal{A}), \qquad \big|(\mathcal{A}^*\mathcal{A})^{-1}\big|_{R(\mathrm{A}^*)\to\mathsf{H}_0} = c_{\mathrm{A}}^2.$$

For $x \in D(\mathcal{A}^*\mathcal{A})$ it holds

$$|x|_{\mathsf{H}_0} \leq c_{\mathrm{A}}|\,\mathrm{A}\,x|_{\mathsf{H}_1} \leq c_{\mathrm{A}}^2|\,\mathrm{A}^*\,\mathrm{A}\,x|_{\mathsf{H}_0}. \tag{15}$$

Moreover, $\mathcal{A}^*\mathcal{A} = \mathrm{A}^*\mathcal{A}$.

(i*) *Interchanging* A *and* A^* *we get a similar results for $\mathcal{A}^*\mathcal{A}$.*

Solving $\mathrm{A}^*\,\mathrm{A}\,x = f$*:*

(ii) *For $f \in R(\mathrm{A}^*)$ the unique solution $x := (\mathcal{A}^*\mathcal{A})^{-1}f \in D(\mathcal{A}^*\mathcal{A})$ of $\mathcal{A}^*\mathcal{A}\,x = f$ can be found by the variational formulation*

$$\forall\,\phi \in D(\mathcal{A}) \qquad \langle \mathrm{A}\,x, \mathrm{A}\,\phi\rangle_{\mathsf{H}_1} = \langle f, \phi\rangle_{\mathsf{H}_0},$$

which holds also for all $\phi \in D(\mathrm{A})$ as

$$\langle \mathrm{A}\,x, \mathrm{A}\,\phi\rangle_{\mathsf{H}_1} = \langle \mathrm{A}\,x, \mathrm{A}\,\pi_{\mathrm{A}^*}\phi\rangle_{\mathsf{H}_1} = \langle f, \pi_{\mathrm{A}^*}\phi\rangle_{\mathsf{H}_0}, = \langle \pi_{\mathrm{A}^*}f, \phi\rangle_{\mathsf{H}_0}, = \langle f, \phi\rangle_{\mathsf{H}_0}, \tag{16}$$

where π_{A^*} denotes the orthogonal projector onto $N(\mathrm{A})^\perp = R(\mathrm{A}^*)$, which implies also $\pi_{\mathrm{A}^*} : D(\mathrm{A}) \to D(\mathcal{A})$.

(ii*) *Interchanging* A *and* A^* *we get a similar variational formulation for $\mathcal{A}\,\mathcal{A}^*\,y = g$.*

Solving $\mathrm{A}\,x = g$*:*

(iii) *For $g \in R(\mathrm{A})$ the unique solution $x := \mathcal{A}^{-1}\,g \in D(\mathcal{A})$ of $\mathcal{A}\,x = g$ can be found by the variational formulation*

$$\forall\,\phi \in D(\mathcal{A}) \qquad \langle \mathrm{A}\,x, \mathrm{A}\,\phi\rangle_{\mathsf{H}_1} = \langle g, \mathrm{A}\,\phi\rangle_{\mathsf{H}_1},$$

which holds also for all $\phi \in D(\mathrm{A})$ since $R(\mathcal{A}) = R(\mathrm{A})$. Note that $\mathrm{A}\,x - g \in R(\mathrm{A})$ belongs to $N(\mathcal{A}^*) = \{0\}$.

Another way is to use a potential y with $\mathrm{A}^*\,y = x$, i.e., to compute the unique potential $y := (\mathcal{A}^*)^{-1}x = (\mathcal{A}^*)^{-1}\,\mathcal{A}^{-1}\,g \in D(\mathcal{A}\,\mathcal{A}^*)$ which satisfies $\mathcal{A}\,\mathcal{A}^*\,y = g$. By (ii) we can find $y \in D(\mathcal{A}^*)$ by the variational formation

$$\forall\,\psi \in D(\mathrm{A}^*) \qquad \langle \mathrm{A}^*\,y, \mathrm{A}^*\,\psi\rangle_{\mathsf{H}_0} = \langle g, \psi\rangle_{\mathsf{H}_1}.$$

(iii*) *Interchanging* A *and* A^* *we get similar variational formulations for $\mathcal{A}^*\,y = f$.*

(iv) *Solving the latter variational formulations leads to saddle point problems which are tricky to handle. A comprehensive theory can be found in* [13].

2.1.2. *Helmholtz Projections.* The projections in (13) and (14) can be computed as follows:

Let $R(\mathrm{A})$ be closed and let us consider, e.g., $\mathsf{H}_1 = R(\mathrm{A}) \oplus_{\mathsf{H}_1} N(\mathrm{A}^*)$.

For $g \in \mathsf{H}_1$ the variational formulation in Lemma 2.4(iii) computes the orthogonal projections $\pi_{\mathrm{A}}g$ onto $R(\mathrm{A})$ and $(1-\pi_{\mathrm{A}})g$ onto $N(\mathrm{A}^*) = R(\mathrm{A})^\perp$. More precisely, $x \in D(\mathrm{A})$ such that

$$\forall\,\phi \in D(\mathrm{A}) \qquad \langle \mathrm{A}\,x, \mathrm{A}\,\phi\rangle_{\mathsf{H}_1} = \langle g, \mathrm{A}\,\phi\rangle_{\mathsf{H}_1} \tag{17}$$

implies $\mathrm{A}\,x - g \in N(\mathrm{A}^*)$ and thus we get (13), i.e., with $\pi_{\mathrm{A}}g = \mathrm{A}\,x$

$$g = \mathrm{A}\,x - \mathrm{A}\,x + g = \pi_{\mathrm{A}}g + (1-\pi_{\mathrm{A}})g \in R(\mathrm{A}) \oplus_{\mathsf{H}_1} N(\mathrm{A}^*).$$

Note that $\mathrm{A}\,x - g \in N(\mathcal{A}^*) = \{0\}$ if and only if $g \in R(\mathrm{A})$.

Now let us consider the bounded linear operator $\mathrm{A} : D(\mathrm{A}) \to \mathsf{H}_1$ and its Banach space adjoint A'. Using the Riesz isometry $\mathcal{R}_{\mathsf{H}_1} : \mathsf{H}_1 \to \mathsf{H}_1'$ we introduce the modified adjoint

$$\mathrm{A}^\top := \mathrm{A}'\,\mathcal{R}_{\mathsf{H}_1} : \mathsf{H}_1 \to D(\mathrm{A})'; \qquad y \mapsto \mathrm{A}'\,\mathcal{R}_{\mathsf{H}_1}y(\,\cdot\,) = \mathcal{R}_{\mathsf{H}_1}y(\mathrm{A}\,\cdot\,) = \langle \mathrm{A}\,\cdot\,, y\rangle_{\mathsf{H}_1}.$$

Then A' and $\mathrm{A}^\top$ are bounded linear operators with $|\,\mathrm{A}^\top\,| = |\,\mathrm{A}'\,| = |\,\mathrm{A}\,|$ and $\mathrm{A}^\top$ is an extension of A^*. Moreover, $N(\mathrm{A}^\top) = R(\mathrm{A})^\perp = N(\mathrm{A}^*)$ and $R(\mathrm{A}^\top) = R(\mathrm{A}')$ is closed by the closed range theorem. Therefore, $\mathrm{A}^\top\,|_{R(\mathrm{A})}$ is boundedly invertible on $R(\mathrm{A}')$ by the bounded inverse theorem, i.e.,

$$\mathcal{A}^\top := \mathrm{A}^\top\,|_{R(\mathrm{A})} : R(\mathrm{A}) \to R(\mathrm{A}'); \qquad y \mapsto \mathrm{A}^\top\,y$$

is a topological isomorphism.

(17) translates equivalently to

$$\mathrm{A}^\top g = \mathrm{A}^\top \mathrm{A}\, x = \mathcal{A}^\top \mathrm{A}\, x.$$

Hence

$$\pi_{\mathrm{A}} g = \mathrm{A}\, x = (\mathcal{A}^\top)^{-1} \mathrm{A}^\top g = \mathrm{A}\, \mathcal{A}^{-1} (\mathcal{A}^\top)^{-1} \mathrm{A}^\top g = \mathrm{A} (\mathcal{A}^\top \mathcal{A})^{-1} \mathrm{A}^\top g.$$

Note that we have indeed $\pi_{\mathrm{A}}^2 = \mathrm{A}(\mathcal{A}^\top \mathcal{A})^{-1} \mathrm{A}^\top \mathrm{A}(\mathcal{A}^\top \mathcal{A})^{-1} \mathrm{A}^\top = \mathrm{A}(\mathcal{A}^\top \mathcal{A})^{-1} \mathrm{A}^\top = \pi_{\mathrm{A}}$ and that for $g \in R(\mathrm{A})$ it holds $\pi_{\mathrm{A}} g = (\mathcal{A}^\top)^{-1} \mathrm{A}^\top g = (\mathcal{A}^\top)^{-1} \mathcal{A}^\top g = g$. Finally,

$$\pi_{\mathrm{A}} = \mathrm{A}(\mathcal{A}^\top \mathcal{A})^{-1} \mathrm{A}^\top = (\mathcal{A}^\top)^{-1} \mathrm{A}^\top : \mathsf{H}_1 \to R(\mathrm{A}),$$

where $\mathrm{A}^\dagger := (\mathcal{A}^\top \mathcal{A})^{-1} \mathrm{A}^\top$ is often called Moore/Penrose inverse of A. Note that $\mathrm{A}^\dagger$ is a bounded right inverse of A, i.e.,

$$\mathrm{A}\,\mathrm{A}^\dagger = (\mathcal{A}^\top)^{-1} \mathrm{A}^\top, \qquad \mathrm{A}\,\mathrm{A}^\dagger\big|_{R(\mathrm{A})} = (\mathcal{A}^\top)^{-1} \mathcal{A}^\top = \mathrm{id}_{R(\mathrm{A})}.$$

2.1.3. *Operator Complexes.* Let H_2 be another Hilbert space and let

$$\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathsf{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathsf{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathsf{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots \tag{18}$$

be a primal and dual Hilbert complex, i.e.,

$$\begin{aligned} \mathrm{A}_0 &: D(\mathrm{A}_0) \subset \mathsf{H}_0 \to \mathsf{H}_1, & \mathrm{A}_1 &: D(\mathrm{A}_1) \subset \mathsf{H}_1 \to \mathsf{H}_2, \\ \mathrm{A}_0^* &: D(\mathrm{A}_0^*) \subset \mathsf{H}_1 \to \mathsf{H}_0, & \mathrm{A}_1^* &: D(\mathrm{A}_1^*) \subset \mathsf{H}_2 \to \mathsf{H}_1 \end{aligned}$$

are densely defined and closed linear operators satisfying the complex property

$$\mathrm{A}_1 \mathrm{A}_0 \subset 0. \tag{19}$$

Note that (19) is equivalent to $R(\mathrm{A}_0) \subset N(\mathrm{A}_1)$ which is equivalent to $R(\mathrm{A}_1^*) \subset N(\mathrm{A}_0^*)$ (dual complex property) as $R(\mathrm{A}_1^*) \subset \overline{R(\mathrm{A}_1^*)} = N(\mathrm{A}_1)^{\perp_{\mathsf{H}_1}} \subset R(\mathrm{A}_0)^{\perp_{\mathsf{H}_1}} = N(\mathrm{A}_0^*)$ and vice versa.

Defining the cohomology group

$$N_{0,1} := N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)$$

we get the following orthogonal Helmholtz-type decompositions, cf. (13).

Lemma 2.5 (Helmholtz decomposition)**.** *The orthogonal Helmholtz-type decompositions*

$$\begin{aligned} \mathsf{H}_1 &= \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N(\mathrm{A}_0^*), & \mathsf{H}_1 &= N(\mathrm{A}_1) \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)}, \\ N(\mathrm{A}_1) &= \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N_{0,1}, & N(\mathrm{A}_0^*) &= N_{0,1} \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)}, \\ D(\mathrm{A}_1) &= \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} \big(D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)\big), & D(\mathrm{A}_0^*) &= \big(N(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)\big) \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)}, \\ D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus_{\mathsf{H}_1} N(\mathrm{A}_0^*), & D(\mathrm{A}_1) &= N(\mathrm{A}_1) \oplus_{\mathsf{H}_1} D(\mathcal{A}_1), \end{aligned} \tag{20}$$

as well as $R(\mathcal{A}_0^*) = R(\mathrm{A}_0^*)$ *and* $R(\mathcal{A}_1) = R(\mathrm{A}_1)$ *hold. Moreover,*

$$\begin{aligned} \mathsf{H}_1 &= \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)}, \\ D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)}, \\ D(\mathrm{A}_1) &= \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} D(\mathcal{A}_1), \\ D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} D(\mathcal{A}_1). \end{aligned}$$

2.2. **Applications to the De Rham Complex.** For Lebesgue and Sobolev spaces we use standard notations $\mathsf{L}^2(\Omega)$, $\mathsf{H}^k(\Omega)$, $k \in \mathbb{N}$, and $\mathsf{H}(\mathrm{rot}, \Omega)$, $\mathsf{H}(\mathrm{div}, \Omega)$, repsectively, and introduce homogeneous boundary conditions by

$$\mathring{\mathsf{H}}^k(\Omega) := \overline{\mathring{\mathsf{C}}^\infty(\Omega)}^{\mathsf{H}^k(\Omega)}, \qquad \mathring{\mathsf{H}}(\mathrm{rot}, \Omega) := \overline{\mathring{\mathsf{C}}^\infty(\Omega)}^{\mathsf{H}(\mathrm{rot},\Omega)}, \qquad \mathring{\mathsf{H}}(\mathrm{div}, \Omega) := \overline{\mathring{\mathsf{C}}^\infty(\Omega)}^{\mathsf{H}(\mathrm{div},\Omega)}.$$

2.2.1. *Domains.*

Definition 2.6 (admissible domains). *Ω is called*

(i) *'Friedrichs admissible' if Friedrichs' estimate*

$$\exists\, c_{\mathsf{f}} > 0 \quad \forall\, \varphi \in \mathring{\mathsf{H}}^1(\Omega) \qquad |\varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{f}} |\nabla \varphi|_{\mathsf{L}^2(\Omega)}$$

(ii) *'Poincaré admissible' if Ω is bounded and Poincaré's estimate*

$$\exists\, c_{\mathsf{p}} > 0 \quad \forall\, \varphi \in \widehat{\mathsf{H}}^1(\Omega) \qquad |\varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{p}} |\nabla \varphi|_{\mathsf{L}^2(\Omega)}$$

holds. Here $\widehat{\mathsf{H}}^1(\Omega) := \mathsf{H}^1(\Omega) \cap \mathbb{R}^\perp$ and $\perp$ denotes orthogonality in $\mathsf{L}^2(\Omega)$.

From now on, let c_{f} and c_{p} denote the best possible constants in Defintion 2.6, cf. (9).

Remark 2.7 (admissible domains). *We note:*

(i) *Any Ω being bounded in at least one direction with diameter $d > 0$ is Friedrichs admissible. Moreover, we emphasise that no regularity of Γ is needed. It holds*

$$\frac{1}{c_{\mathsf{f}}} = \min_{0 \neq u \in \mathring{\mathsf{H}}^1(\Omega)} \frac{|\nabla u|_{\mathsf{L}^2(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)}},$$

which means that $1/c_{\mathsf{f}}$ is the square root of the first eigenvalue of the negative Dirichlet Laplacian.

(ii) *Any bounded (weak) Lipschitz domain Ω with diameter $d > 0$ is Poincaré admissible. We have*

$$\frac{1}{c_{\mathsf{p}}} = \min_{0 \neq u \in \widehat{\mathsf{H}}^1(\Omega)} \frac{|\nabla u|_{\mathsf{L}^2(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)}},$$

which means that $1/c_{\mathsf{p}}$ is the square root of the first positive eigenvalue of the negative Neumann Laplacian.

In both cases we have $c_{\mathsf{f}}, c_{\mathsf{p}} \leq d/\pi$, cf. [20].

2.2.2. *De Rham Complex.* In the following we shall apply the latter abstract results with different choices of A for various operators from the de Rham complex

(21)
$$\mathsf{L}^2(\Omega) \underset{\mathrm{A}_0^* = -\operatorname{div}}{\overset{\mathrm{A}_0 = \mathring{\nabla}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{\mathrm{A}_1^* = \operatorname{rot}}{\overset{\mathrm{A}_1 = \mathring{\operatorname{rot}}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{\mathrm{A}_2^* = -\nabla}{\overset{\mathrm{A}_2 = \mathring{\operatorname{div}}}{\rightleftarrows}} \mathsf{L}^2(\Omega),$$

cf. (18), with the densely defined and closed linear operators from vector calculus

$$\begin{aligned}
\mathring{\nabla} : \mathring{\mathsf{H}}^1(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \qquad & \nabla : \mathsf{H}^1(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega); \qquad & \phi \mapsto \nabla \phi, \\
\mathring{\operatorname{rot}} : \mathring{\mathsf{H}}(\operatorname{rot}, \Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \qquad & \operatorname{rot} : \mathsf{H}(\operatorname{rot}, \Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega); \qquad & \Phi \mapsto \operatorname{rot} \Phi, \\
\mathring{\operatorname{div}} : \mathring{\mathsf{H}}(\operatorname{div}, \Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \qquad & \operatorname{div} : \mathsf{H}(\operatorname{div}, \Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega); \qquad & \Phi \mapsto \operatorname{div} \Phi.
\end{aligned}$$

2.2.3. *Gradients and Divergences.* Let us start with ∇ and div.

- Let Ω be Friedrichs admissible. We consider

$$\mathrm{A} := \mathrm{A}_0 = \mathring{\nabla}, \qquad \mathrm{A}^* = -\operatorname{div}.$$

By Friedrichs' estimate and Lemma 2.1 $R(\mathrm{A})$ and $R(\mathrm{A}^*)$ are closed. Moreover,

$$N(\mathring{\nabla}) = N(\mathrm{A}) = \{0\}, \qquad R(\operatorname{div}) = R(\mathrm{A}^*) = N(\mathrm{A})^\perp = \mathsf{L}^2(\Omega)$$

and

$$\begin{aligned}
\mathcal{A} : \mathring{\mathsf{H}}^1(\Omega) \subset \mathsf{L}^2(\Omega) \to R(\mathrm{A}), \\
\mathcal{A}^* : \mathsf{H}(\operatorname{div}, \Omega) \cap R(\mathrm{A}) \subset R(\mathrm{A}) \to \mathsf{L}^2(\Omega).
\end{aligned}$$

Note that by (20)

$$\begin{aligned}
N(\mathrm{A}^*) = N(\operatorname{div}) = \big\{\Phi \in \mathsf{H}(\operatorname{div}, \Omega) : \operatorname{div} \Phi = 0\big\}, \\
R(\mathrm{A}) = R(\mathring{\nabla}) = N(\mathring{\operatorname{rot}}) \cap \mathcal{H}_{\mathsf{D}}(\Omega)^\perp = \big\{\Phi \in \mathring{\mathsf{H}}(\operatorname{rot}, \Omega) : \operatorname{rot} \Phi = 0 \,\wedge\, \Phi \perp \mathcal{H}_{\mathsf{D}}(\Omega)\big\},
\end{aligned}$$

where we denote the harmonic Dirichlet fields (cohomology group) by

$$\mathcal{H}_{\mathtt{D}}(\Omega) := N(\mathring{\mathrm{rot}}) \cap N(\mathrm{div}) = \big\{\Phi \in \mathring{\mathsf{H}}(\mathrm{rot},\Omega) \cap \mathsf{H}(\mathrm{div},\Omega) : \mathrm{rot}\,\Phi = 0 \,\wedge\, \mathrm{div}\,\Phi = 0\big\}.$$

$\mathcal{A}$ and $\mathcal{A}^*$ are bijective with bounded inverses

$$\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A}), \qquad |\,\mathcal{A}^{-1}\,|_{R(\mathrm{A})\to\mathsf{L}^2(\Omega)} = c_{\mathsf{f}},$$
$$(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*), \qquad \big|(\mathcal{A}^*)^{-1}\big|_{R(\mathrm{A}^*)\to\mathsf{L}^2(\Omega)} = c_{\mathsf{f}}.$$

Let $f \in \mathsf{L}^2(\Omega)$ and $G \in R(\mathring{\nabla})$. Then

$$u := \mathcal{A}^{-1}\,G \in D(\mathcal{A}) = \mathring{\mathsf{H}}^1(\Omega), \qquad E := (\mathcal{A}^*)^{-1}f \in D(\mathcal{A}^*) = \mathsf{H}(\mathrm{div},\Omega) \cap R(\mathring{\nabla})$$

are the unique solutions of the boundary value problems $\mathring{\nabla}\,u = G$ and $-\,\mathrm{div}\,|_{R(\mathring{\nabla})}E = f$, i.e.,

$$\begin{array}{lll} \nabla\,u = G, & -\,\mathrm{div}\,E = f & \text{in } \Omega, \\ & \mathrm{rot}\,E = 0 & \text{in } \Omega, \\ u = 0, & n \times E = 0 & \text{on } \Gamma, \\ & E \perp \mathcal{H}_{\mathtt{D}}(\Omega). & \end{array}$$

Extensions to right hand sides $f \in \mathsf{H}^{-1}(\Omega) := \mathring{\mathsf{H}}^1(\Omega)'$ and $G \in \mathsf{H}(\mathrm{div},\Omega)'$ are straight forward using Banach space adjoints.

- Let Ω be Poincaré admissible. We consider

$$\mathrm{A} := -\,\mathrm{A}_2^* = \nabla, \qquad \mathrm{A}^* = -\,\mathring{\mathrm{div}}\,.$$

Note that $N(\nabla) = N(\mathrm{A}) = \mathbb{R}$ and $\mathcal{A} = \mathrm{A}\,|_{\mathbb{R}^\perp}$. By Poincaré's estimate and Lemma 2.1 $R(\mathcal{A}) = R(\mathrm{A})$ and $R(\mathrm{A}^*)$ are closed. Moreover,

$$R(\mathring{\mathrm{div}}) = R(\mathrm{A}^*) = N(\mathrm{A})^\perp = \mathsf{L}^2(\Omega) \cap \mathbb{R}^\perp =: \widehat{\mathsf{L}}^2(\Omega)$$

and

$$\mathcal{A} : \widehat{\mathsf{H}}^1(\Omega) \subset \widehat{\mathsf{L}}^2(\Omega) \to R(\mathrm{A}), \qquad \widehat{\mathsf{H}}^1(\Omega) = \mathsf{H}^1(\Omega) \cap \mathbb{R}^\perp$$
$$\mathcal{A}^* : \mathring{\mathsf{H}}(\mathrm{div},\Omega) \cap R(\mathrm{A}) \subset R(\mathrm{A}) \to \widehat{\mathsf{L}}^2(\Omega).$$

By (20)

$$N(\mathrm{A}^*) = N(\mathring{\mathrm{div}}) = \big\{\Phi \in \mathring{\mathsf{H}}(\mathrm{div},\Omega) : \mathrm{div}\,\Phi = 0\big\},$$
$$R(\mathrm{A}) = R(\nabla) = N(\mathrm{rot}) \cap \mathcal{H}_{\mathtt{N}}(\Omega)^\perp = \big\{\Phi \in \mathsf{H}(\mathrm{rot},\Omega) : \mathrm{rot}\,\Phi = 0 \,\wedge\, \Phi \perp \mathcal{H}_{\mathtt{N}}(\Omega)\big\},$$

where we denote the harmonic Neumann fields (cohomology group) by

$$\mathcal{H}_{\mathtt{N}}(\Omega) := N(\mathrm{rot}) \cap N(\mathring{\mathrm{div}}) = \big\{\Phi \in \mathsf{H}(\mathrm{rot},\Omega) \cap \mathring{\mathsf{H}}(\mathrm{div},\Omega) : \mathrm{rot}\,\Phi = 0 \,\wedge\, \mathrm{div}\,\Phi = 0\big\}.$$

$\mathcal{A}$ and $\mathcal{A}^*$ are bijective with bounded inverses

$$\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A}), \qquad |\,\mathcal{A}^{-1}\,|_{R(\mathrm{A})\to\mathsf{L}^2(\Omega)} = c_{\mathsf{p}},$$
$$(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*), \qquad \big|(\mathcal{A}^*)^{-1}\big|_{R(\mathrm{A}^*)\to\mathsf{L}^2(\Omega)} = c_{\mathsf{p}}.$$

Let $f \in \widehat{\mathsf{L}}^2(\Omega)$ and $G \in R(\nabla)$. Then

$$u := \mathcal{A}^{-1}\,G \in D(\mathcal{A}) = \widehat{\mathsf{H}}^1(\Omega), \qquad E := (\mathcal{A}^*)^{-1}f \in D(\mathcal{A}^*) = \mathsf{H}(\mathring{\mathrm{div}},\Omega) \cap R(\nabla)$$

are the unique solutions of $\nabla\,|_{\widehat{\mathsf{L}}^2(\Omega)}u = G$ and $-\,\mathring{\mathrm{div}}\,|_{R(\nabla)}E = f$, i.e.,

$$\begin{array}{lll} \nabla\,u = G, & -\,\mathrm{div}\,E = f & \text{in } \Omega, \\ & \mathrm{rot}\,E = 0 & \text{in } \Omega, \\ & n \cdot E = 0 & \text{on } \Gamma, \\ u \perp \mathbb{R}, & E \perp \mathcal{H}_{\mathtt{N}}(\Omega). & \end{array}$$

Extensions to right hand sides $f \in \mathring{\mathsf{H}}^{-1}(\Omega) := \mathsf{H}^1(\Omega)'$ and $G \in \mathsf{H}(\mathring{\mathrm{div}},\Omega)'$ are straight forward using Banach space adjoints.

Remark 2.8 (duals)**.** *Note that in* [7] *it has been shown*

$$\begin{aligned} \mathsf{H}(\mathring{\operatorname{div}},\Omega)' &= \{\Phi\in\mathsf{H}^{-1}:\operatorname{rot}\Phi\in\mathsf{H}^{-1}\}, & \mathsf{H}^{-k}(\Omega) &:= \mathring{\mathsf{H}}^{k}(\Omega)',\\ \mathsf{H}(\mathring{\operatorname{rot}},\Omega)' &= \{\Phi\in\mathsf{H}^{-1}:\operatorname{div}\Phi\in\mathsf{H}^{-1}\}, & &\\ \mathsf{H}(\operatorname{div},\Omega)' &= \{\Phi\in\mathring{\mathsf{H}}^{-1}:\operatorname{rot}\Phi\in\mathring{\mathsf{H}}^{-1}\}, & \mathring{\mathsf{H}}^{-k}(\Omega) &:= \mathsf{H}^{k}(\Omega)',\\ \mathsf{H}(\operatorname{rot},\Omega)' &= \{\Phi\in\mathring{\mathsf{H}}^{-1}:\operatorname{div}\Phi\in\mathring{\mathsf{H}}^{-1}\}. & & \end{aligned}$$

2.2.4. *Dirichlet/Neumann Laplacians.* We use the latter results.

- Let Ω be Friedrichs admissible and let $\mathrm{A}:=\mathrm{A}_0=\mathring{\nabla}$. We introduce the negative Dirichlet Laplacian

$$\mathrm{L}_{\mathtt{D}}:=\mathrm{A}^*\mathrm{A}=-\operatorname{div}\mathring{\nabla}:D(\mathrm{L}_{\mathtt{D}})\subset\mathsf{L}^2(\Omega)\to\mathsf{L}^2(\Omega);\quad\varphi\mapsto-\Delta\varphi,$$

where

$$D(\mathrm{L}_{\mathtt{D}})=\{\varphi\in\mathring{\mathsf{H}}^1(\Omega):\nabla\varphi\in\mathsf{H}(\operatorname{div},\Omega)\}=\{\varphi\in\mathring{\mathsf{H}}^1(\Omega):\Delta\varphi\in\mathsf{L}^2(\Omega)\}.$$

Note that $\mathcal{L}_{\mathtt{D}}=\mathcal{A}^*\mathcal{A}=\mathrm{A}^*\mathcal{A}=\mathrm{A}^*\mathrm{A}=\mathrm{L}_{\mathtt{D}}$ with closed range $R(\mathrm{L}_{\mathtt{D}})=R(\mathrm{A}^*)=\mathsf{L}^2(\Omega)$. $\mathrm{L}_{\mathtt{D}}$ is selfadjoint, positive, and bijective with bounded inverse

$$\mathrm{L}_{\mathtt{D}}^{-1}=\mathcal{A}^{-1}(\mathcal{A}^*)^{-1}:\mathsf{L}^2(\Omega)\to D(\mathrm{L}_{\mathtt{D}}),\qquad|\,\mathrm{L}_{\mathtt{D}}^{-1}\,|_{\mathsf{L}^2(\Omega)\to\mathsf{L}^2(\Omega)}=c_{\mathsf{f}}^2.$$

(15) reads

$$\forall\,\varphi\in D(\mathrm{L}_{\mathtt{D}})\qquad|\varphi|_{\mathsf{L}^2(\Omega)}\le c_{\mathsf{f}}|\nabla\varphi|_{\mathsf{L}^2(\Omega)}\le c_{\mathsf{f}}^2|\Delta\varphi|_{\mathsf{L}^2(\Omega)}. \tag{22}$$

Let $f\in\mathsf{L}^2(\Omega)$. Then $u:=\mathrm{L}_{\mathtt{D}}^{-1}f\in D(\mathrm{L}_{\mathtt{D}})$ is the unique solution of the Dirichlet Laplace boundary value problem $\mathrm{L}_{\mathtt{D}}u=f$, i.e.,

$$\begin{aligned} -\Delta u&=f && \text{in }\Omega,\\ u&=0 && \text{on }\Gamma. \end{aligned}$$

Note that

$$u=\mathcal{A}^{-1}(\mathcal{A}^*)^{-1}f=-\mathring{\nabla}^{-1}\operatorname{div}|_{R(\mathring{\nabla})}^{-1}f,$$

cf. Section 2.2.3. To find $u\in\mathring{\mathsf{H}}^1(\Omega)$ by variational methods we may consider (16), i.e.,

$$\forall\,\varphi\in\mathring{\mathsf{H}}^1(\Omega)\qquad\langle\nabla u,\nabla\varphi\rangle_{\mathsf{L}^2(\Omega)}=\langle f,\varphi\rangle_{\mathsf{L}^2(\Omega)}.$$

Extensions to right hand sides $f\in\mathsf{H}^{-1}(\Omega)$ are straight forward using Banach space adjoints.

- Let Ω be Poincaré admissible and let $\mathrm{A}:=-\mathrm{A}_2^*=\nabla$. We introduce the negative Neumann Laplacian and its reduced version

$$\mathrm{L}_{\mathtt{N}}:=\mathrm{A}^*\mathrm{A}=-\mathring{\operatorname{div}}\,\nabla:D(\mathrm{L}_{\mathtt{N}})\subset\mathsf{L}^2(\Omega)\to\mathsf{L}^2(\Omega);\qquad\varphi\mapsto-\Delta\varphi,$$

$$\mathcal{L}_{\mathtt{N}}=\mathcal{A}^*\mathcal{A}=\mathrm{A}^*\mathcal{A}=-\mathring{\operatorname{div}}\,\nabla|_{\widehat{\mathsf{L}}^2(\Omega)}:D(\mathcal{L}_{\mathtt{N}})\subset\widehat{\mathsf{L}}^2(\Omega)\to\widehat{\mathsf{L}}^2(\Omega),$$

where

$$\begin{aligned} D(\mathrm{L}_{\mathtt{N}})&=\{\varphi\in\mathsf{H}^1(\Omega):\nabla\varphi\in\mathring{\mathsf{H}}(\operatorname{div},\Omega)\},\\ D(\mathcal{L}_{\mathtt{N}})&=\{\varphi\in\widehat{\mathsf{H}}^1(\Omega):\nabla\varphi\in\mathring{\mathsf{H}}(\operatorname{div},\Omega)\}. \end{aligned}$$

$\mathcal{L}_{\mathtt{N}}$ is selfadjoint, positive, and bijective with closed range $R(\mathrm{L}_{\mathtt{N}})=R(\mathrm{A}^*)=\widehat{\mathsf{L}}^2(\Omega)$ and bounded inverse

$$\mathcal{L}_{\mathtt{N}}^{-1}=\mathcal{A}^{-1}(\mathcal{A}^*)^{-1}:\widehat{\mathsf{L}}^2(\Omega)\to D(\mathcal{L}_{\mathtt{N}}),\qquad|\,\mathcal{L}_{\mathtt{N}}^{-1}\,|_{\widehat{\mathsf{L}}^2(\Omega)\to\mathsf{L}^2(\Omega)}=c_{\mathsf{p}}^2.$$

(15) reads

$$\forall\,\varphi\in D(\mathcal{L}_{\mathtt{N}})\qquad|\varphi|_{\mathsf{L}^2(\Omega)}\le c_{\mathsf{p}}|\nabla\varphi|_{\mathsf{L}^2(\Omega)}\le c_{\mathsf{p}}^2|\Delta\varphi|_{\mathsf{L}^2(\Omega)}.$$

Let $f\in\widehat{\mathsf{L}}^2(\Omega)$. Then $u:=\mathcal{L}_{\mathtt{N}}^{-1}f\in D(\mathrm{L}_{\mathtt{N}})$ is the unique solution of the Neumann Laplace boundary value problem $\mathcal{L}_{\mathtt{N}}u=f$, i.e.,

$$-\Delta u=f\qquad\text{in }\Omega,$$

$$n \cdot \nabla u = 0 \qquad \text{on } \Gamma,$$
$$u \perp \mathbb{R}.$$

Note that

$$u = \mathcal{A}^{-1}(\mathcal{A}^*)^{-1} f = -\nabla|^{-1}_{\widehat{\mathsf{L}}^2(\Omega)} \mathring{\operatorname{div}}|^{-1}_{R(\nabla)} f,$$

cf. Section 2.2.3. To find $u \in \widehat{\mathsf{H}}^1(\Omega)$ by variational methods we may consider (16), i.e.,

$$\forall\, \varphi \in \mathsf{H}^1(\Omega) \qquad \langle \nabla u, \nabla \varphi \rangle_{\mathsf{L}^2(\Omega)} = \langle f, \varphi \rangle_{\mathsf{L}^2(\Omega)}.$$

Extensions to right hand sides $f \in \mathring{\mathsf{H}}^{-1}(\Omega)$ are straight forward using Banach space adjoints.

2.2.5. *Over- and Underdetermined Laplacians.* Let Ω be Friedrichs admissible.

Lemma 2.9. *On $\mathring{\mathsf{H}}^2(\Omega)$ the norms $|\cdot|_{\mathsf{H}^2(\Omega)}$ and $|\Delta \cdot|_{\mathsf{L}^2(\Omega)}$ are equivalent. More precisely,*

$$\forall\, \varphi \in \mathring{\mathsf{H}}^2(\Omega) \qquad |\varphi|_{\mathsf{H}^2(\Omega)} \leq c_\Delta |\Delta\varphi|_{\mathsf{L}^2(\Omega)},$$

where $c_\Delta := \sqrt{1 + c_{\mathsf{f}}^2 + c_{\mathsf{f}}^4}$.

Proof. By (22)

$$\forall\, \varphi \in D(\mathrm{L}_{\mathsf{D}}) \qquad |\varphi|^2_{\mathsf{L}^2(\Omega)} + |\nabla \varphi|^2_{\mathsf{L}^2(\Omega)} + |\Delta\varphi|^2_{\mathsf{L}^2(\Omega)} \leq c_\Delta^2 |\Delta\varphi|^2_{\mathsf{L}^2(\Omega)}.$$

For $\varphi \in \mathring{\mathsf{C}}^\infty(\Omega)$ we observe

$$\sum_{i,j} \langle \partial_i \partial_j \varphi, \partial_i \partial_j \varphi \rangle_{\mathsf{L}^2(\Omega)} = \sum_{i,j} \langle \partial_i^2 \varphi, \partial_j^2 \varphi \rangle_{\mathsf{L}^2(\Omega)} = |\Delta\varphi|^2_{\mathsf{L}^2(\Omega)},$$

which extends to $\varphi \in \mathring{\mathsf{H}}^2(\Omega)$ by continuity and shows the stated estimate. □

Let

$$\mathsf{H}(\Delta, \Omega) := \{\varphi \in \mathsf{L}^2(\Omega) : \Delta\varphi \in \mathsf{L}^2(\Omega)\}, \qquad \mathbb{H} := \{\varphi \in \mathsf{L}^2(\Omega) : \Delta\varphi = 0\},$$

where the latter denotes the harmonic functions.

Lemma 2.9 and Lemma 2.1 yield that the over- and underdetermined Laplacians

$$\mathring{\mathrm{L}} : \mathring{\mathsf{H}}^2(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega); \qquad \varphi \mapsto -\Delta\varphi,$$
$$\mathrm{L} := \mathring{\mathrm{L}}^* : \mathsf{H}(\Delta, \Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega)$$

are densely defined and closed with closed ranges $R(\mathring{\mathrm{L}})$ and $R(\mathrm{L})$. Moreover,

$$\begin{aligned} N(\mathring{\mathrm{L}}) &= \{0\}, \quad R(\mathring{\mathrm{L}}) = N(\mathrm{L})^\perp = \mathsf{L}^2(\Omega) \cap \mathbb{H}^\perp =: \widetilde{\mathsf{L}}^2(\Omega), \\ N(\mathrm{L}) &= \mathbb{H}, \qquad R(\mathrm{L}) = N(\mathring{\mathrm{L}})^\perp = \mathsf{L}^2(\Omega). \end{aligned} \tag{23}$$

The reduced operators

$$\mathring{\mathcal{L}} : \mathring{\mathsf{H}}^2(\Omega) \subset \mathsf{L}^2(\Omega) \to \widetilde{\mathsf{L}}^2(\Omega),$$
$$\mathcal{L} : \widetilde{\mathsf{H}}(\Delta, \Omega) \subset \widetilde{\mathsf{L}}^2(\Omega) \to \mathsf{L}^2(\Omega), \qquad \widetilde{\mathsf{H}}(\Delta, \Omega) := \mathsf{H}(\Delta, \Omega) \cap \mathbb{H}^\perp$$

are bijective with bounded inverse operators

$$\mathring{\mathcal{L}}^{-1} : \widetilde{\mathsf{L}}^2(\Omega) \to D(\mathring{\mathrm{L}}) = \mathring{\mathsf{H}}^2(\Omega),$$
$$\mathcal{L}^{-1} : \mathsf{L}^2(\Omega) \to D(\mathcal{L}) = \widetilde{\mathsf{H}}(\Delta, \Omega), \qquad |\mathring{\mathcal{L}}^{-1}|_{\widetilde{\mathsf{L}}^2(\Omega) \to \mathsf{L}^2(\Omega)} = |\mathcal{L}^{-1}|_{\mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega)} \leq c_{\mathsf{f}}^2,$$

cf. (22).

Remark 2.10. *In particular, we have for all $\varphi \in D(\mathring{\mathcal{L}}) = \mathring{\mathsf{H}}^2(\Omega)$ and all $\varphi \in D(\mathcal{L}) = \widetilde{\mathsf{H}}(\Delta, \Omega)$*

$$|\varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{f}}^2 |\Delta\varphi|_{\mathsf{L}^2(\Omega)}.$$

Let $f \in \widetilde{\mathsf{L}}^2(\Omega)$. Then $u := \mathring{\mathcal{L}}^{-1} f \in D(\mathring{\mathrm{L}}) = \mathring{\mathsf{H}}^2(\Omega)$ is the unique solution of the overdetermined negative Laplace boundary value problem $\mathring{\mathcal{L}} u = f$, i.e.,

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \\ n \cdot \nabla u &= 0 && \text{on } \Gamma. \end{aligned}$$

Remark 2.11. *Note that* $u|_\Gamma = 0$ *implies* $n \times \nabla u|_\Gamma = 0$. *Hence, together with* $n \cdot \nabla u|_\Gamma = 0$ *we see* $\nabla u|_\Gamma = 0$. *In other words, for* $u \in \mathring{\mathsf{H}}^1(\Omega)$ *it holds*

$$u \in \mathring{\mathsf{H}}^2(\Omega) \quad \Leftrightarrow \quad \nabla u \in \mathring{\mathsf{H}}(\operatorname{div}, \Omega) \cap \mathsf{H}^1(\Omega).$$

Let $f \in \mathsf{L}^2(\Omega)$. Then $u := \mathcal{L}^{-1} f \in D(\mathcal{L}) = \widetilde{\mathsf{H}}(\Delta, \Omega)$ is the unique solution of the underdetermined negative Laplace boundary value problem $\mathcal{L}\, u = f$, i.e.,

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &\perp \mathbb{H}. \end{aligned}$$

Remark 2.12. *Note that for the different negative Laplacians we have* $\mathring{\mathrm{L}} \subset \mathrm{L}_{\mathtt{D}}, \mathrm{L}_{\mathtt{N}} \subset \mathrm{L}$, *that is*

$$\mathring{\mathrm{L}} \subset \mathrm{L}_{\mathtt{D}} = -\operatorname{div} \mathring{\nabla} \subset \mathrm{L}, \qquad \mathring{\mathrm{L}} \subset \mathrm{L}_{\mathtt{N}} = -\mathring{\operatorname{div}} \nabla \subset \mathrm{L}\,.$$

In this sense, $\mathring{\mathrm{L}}$ *and* L *are minimal and maximal* $\mathsf{L}^2(\Omega)$*-realisations of the negative Laplacian, respectively.*

Theorem 2.13. *Let* Ω *be bounded. Then* $D(\mathcal{L}) = \widetilde{\mathsf{H}}(\Delta, \Omega) \hookrightarrow \mathsf{L}^2(\Omega)$ *is compact.*

Proof. By Lemma 2.1 we have that the embedding $D(\mathring{\mathcal{L}}) = D(\mathring{\mathrm{L}}) = \mathring{\mathsf{H}}^2(\Omega) \hookrightarrow \mathsf{L}^2(\Omega)$ is compact if and only if the embedding $D(\mathcal{L}) = \widetilde{\mathsf{H}}(\Delta, \Omega) \hookrightarrow \mathsf{L}^2(\Omega)$ is compact. Hence the latter embedding is compact by Rellich's selection theorem for, e.g., $\mathring{\mathsf{H}}^1(\Omega)$. □

2.2.6. *Bi-Laplacians and Biharmonic Operators.* Let Ω be Friedrichs admissible and let us consider $\mathrm{A} = \mathring{\mathrm{L}}$ with $\mathrm{A}^* = \mathrm{L}$ from the latter section.

- We introduce the Dirichlet bi-Laplacian (Dirichlet biharmonic operator)

  $$\mathrm{B}_{\mathtt{D}} := \mathrm{A}^* \mathrm{A} = \mathrm{L}\, \mathring{\mathrm{L}} : D(\mathrm{B}_{\mathtt{D}}) \to \mathsf{L}^2(\Omega); \quad \varphi \mapsto \Delta^2 \varphi,$$

  where

  $$D(\mathrm{B}_{\mathtt{D}}) = \big\{ u \in \mathring{\mathsf{H}}^2(\Omega) : \Delta u \in \mathsf{H}(\Delta, \Omega) \big\} = \big\{ u \in \mathring{\mathsf{H}}^2(\Omega) : \Delta^2 u \in \mathsf{L}^2(\Omega) \big\}.$$

  Then $\mathcal{B}_{\mathtt{D}} = \mathcal{A}^* \mathcal{A} = \mathcal{L}\, \mathring{\mathcal{L}} = \mathrm{L}\, \mathring{\mathcal{L}} = \mathrm{L}\, \mathring{\mathrm{L}} = \mathrm{B}_{\mathtt{D}}$ with closed range $R(\mathrm{B}_{\mathtt{D}}) = R(\mathrm{L}) = \mathsf{L}^2(\Omega)$. $\mathrm{B}_{\mathtt{D}}$ is selfadjoint, positive, and bijective with bounded inverse

  $$\mathrm{B}_{\mathtt{D}}^{-1} := \mathcal{A}^{-1} (\mathcal{A}^*)^{-1} = \mathring{\mathcal{L}}^{-1} \mathcal{L}^{-1} : \mathsf{L}^2(\Omega) \to D(\mathrm{B}_{\mathtt{D}}), \qquad |\, \mathrm{B}_{\mathtt{D}}^{-1} \,|_{\mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega)} \leq c_{\mathsf{f}}^4.$$

  (15) reads

  $$\forall\, \varphi \in D(\mathrm{B}_{\mathtt{D}}) \qquad |\varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{f}}^2 |\Delta \varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{f}}^4 |\Delta^2 \varphi|_{\mathsf{L}^2(\Omega)}.$$

  Let $f \in \mathsf{L}^2(\Omega)$. Then $u := \mathrm{B}_{\mathtt{D}}^{-1} f \in D(\mathrm{B}_{\mathtt{D}})$ is the unique solution of the Dirichlet boundary value problem for the bi-Laplacian $\mathrm{B}_{\mathtt{D}}\, u = f$, i.e.,

  $$\begin{aligned} \Delta^2 u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \\ n \cdot \nabla u &= 0 && \text{on } \Gamma. \end{aligned}$$

  Note that $u = \mathring{\mathcal{L}}^{-1} \mathcal{L}^{-1} f$. To find $u \in \mathring{\mathsf{H}}^2(\Omega)$ by variational methods we may consider (16), i.e.,

  $$\forall\, \varphi \in \mathring{\mathsf{H}}^2(\Omega) \qquad \langle \Delta u, \Delta \varphi \rangle_{\mathsf{L}^2(\Omega)} = \langle f, \varphi \rangle_{\mathsf{L}^2(\Omega)}. \tag{24}$$

  Extensions to right hand sides $f \in \mathsf{H}^{-2}(\Omega)$ are straight forward using Banach space adjoints.

- Analogously, we may consider the Neumann bi-Laplacian (Neumann biharmonic operator) and its reduced version

$$\mathrm{B_N} := \mathrm{A}\,\mathrm{A}^* = \mathring{\mathrm{L}}\,\mathrm{L} : D(\mathrm{B_N}) \to \mathsf{L}^2(\Omega),$$
$$\mathcal{B}_{\mathtt{N}} := \mathcal{A}\,\mathcal{A}^* = \mathring{\mathcal{L}}\,\mathcal{L} : D(\mathcal{B}_{\mathtt{N}}) \to R(\mathring{\mathrm{L}}) = \widetilde{\mathsf{L}}^2(\Omega),$$

where we recall (23) and

$$D(\mathrm{B_N}) = \{u \in \mathsf{H}(\Delta,\Omega) : \Delta u \in \mathring{\mathsf{H}}^2(\Omega)\},$$
$$D(\mathcal{B}_{\mathtt{N}}) = \{u \in \widetilde{\mathsf{H}}(\Delta,\Omega) : \Delta u \in \mathring{\mathsf{H}}^2(\Omega)\}.$$

$\mathrm{B_N}$ and $\mathcal{B}_{\mathtt{N}}$ have closed range $R(\mathrm{B_N}) = R(\mathring{\mathrm{L}}) = \widetilde{\mathsf{L}}^2(\Omega)$. $\mathcal{B}_{\mathtt{N}}$ is selfadjoint, positive, and bijective with bounded inverse

$$\mathcal{B}_{\mathtt{N}}^{-1} := (\mathcal{A}^*)^{-1}\,\mathcal{A}^{-1} = \mathcal{L}^{-1}\,\mathring{\mathcal{L}}^{-1} : \widetilde{\mathsf{L}}^2(\Omega) \to D(\mathcal{B}_{\mathtt{N}}), \qquad |\,\mathcal{B}_{\mathtt{N}}^{-1}\,|_{\widetilde{\mathsf{L}}^2(\Omega)\to\mathsf{L}^2(\Omega)} = c_{\mathsf{f}}^4.$$

(15) reads

$$\forall\,\varphi \in D(\mathcal{B}_{\mathtt{N}}) \qquad |\varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{f}}^2|\Delta\varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{f}}^4|\Delta^2\varphi|_{\mathsf{L}^2(\Omega)}.$$

Let $f \in \widetilde{\mathsf{L}}^2(\Omega)$. Then $u := \mathcal{B}_{\mathtt{N}}^{-1} f \in D(\mathcal{B}_{\mathtt{N}})$ is the unique solution of the Neumann boundary value problem for the bi-Laplacian $\mathcal{B}_{\mathtt{N}}\,u = f$, i.e.,

$$\begin{aligned} \Delta^2 u &= f && \text{in } \Omega,\\ \Delta u &= 0 && \text{on } \Gamma,\\ n\cdot\nabla\,\Delta u &= 0 && \text{on } \Gamma,\\ u &\perp \mathbb{H}. \end{aligned}$$

As in Remark 2.11 we have $\nabla\,\Delta u|_\Gamma = 0$. Note that $u = \mathcal{L}^{-1}\,\mathring{\mathcal{L}}^{-1} f$. To find $u \in \widetilde{\mathsf{H}}(\Delta,\Omega)$ by variational methods we may consider (16), i.e.,

$$\forall\,\varphi \in \mathsf{H}(\Delta,\Omega) \qquad \langle\Delta u, \Delta\varphi\rangle_{\mathsf{L}^2(\Omega)} = \langle f,\varphi\rangle_{\mathsf{L}^2(\Omega)}. \tag{25}$$

Extensions to right hand sides $f \in \mathsf{H}(\Delta,\Omega)'$ are straight forward using Banach space adjoints.

Remark 2.14 (over- and underdetermined Laplacians)**.** *Recall Section 2.2.5.*

- *For* $u := \mathrm{B}_{\mathtt{D}}^{-1} f = \mathring{\mathcal{L}}^{-1}\,\mathcal{L}^{-1} f$ *we get that* $\widetilde{u} := \Delta u = \mathcal{L}^{-1} f$ *is the unique solution of the underdetermined Laplace problem. Hence we may solve the underdetermined Laplace problem using a variational formulation for the Dirichlet bi-Laplace problem* (24)*. In fact, we compute a potential* u *and set* $\widetilde{u} := \mathring{\mathcal{L}} u = \Delta u$*, cf. Lemma 2.4(iii).*
- *For* $u := \mathcal{B}_{\mathtt{N}}^{-1} f = \mathcal{L}^{-1}\,\mathring{\mathcal{L}}^{-1} f$ *we get that* $\widetilde{u} := \Delta v = \mathring{\mathcal{L}}^{-1} f$ *is the unique solution of the overdetermined Laplace problem. Thus we can solve the overdetermined Laplace problem using a variational formulation for the Neumann bi-Laplace problem* (25)*. Here, we compute a potential* u *and set* $\widetilde{u} := \mathcal{L}\,u = \Delta u$*, cf. Lemma 2.4(iii).*

2.2.7. *A Zoo of Biharmonic Operators.* Let Ω be Poincaré admissible. Note that $\mathbb{R} \subset \mathbb{H}$ and hence

$$\widetilde{\mathsf{L}}^2(\Omega) \subset \widehat{\mathsf{L}}^2(\Omega).$$

Let us recall the different reduced negative Laplacians

$$\mathring{\mathcal{L}}, \qquad \mathcal{L}_{\mathtt{D}} = \mathrm{L}_{\mathtt{D}} = -\operatorname{div}\mathring{\nabla}, \qquad \mathcal{L}_{\mathtt{N}} = -\operatorname{\mathring{div}}\nabla_{\widehat{\mathsf{L}}^2(\Omega)}, \qquad \mathcal{L}$$

with domains of definition

$$D(\mathring{\mathcal{L}}) = \mathring{\mathsf{H}}^2(\Omega), \qquad D(\mathcal{L}_{\mathtt{D}}) = \{\varphi \in \mathring{\mathsf{H}}^1(\Omega) : \Delta\varphi \in \mathsf{L}^2(\Omega)\},$$
$$D(\mathcal{L}) = \widetilde{\mathsf{H}}(\Delta,\Omega), \qquad D(\mathcal{L}_{\mathtt{N}}) = \{\varphi \in \widehat{\mathsf{H}}^1(\Omega) : \nabla\,\varphi \in \mathring{\mathsf{H}}(\operatorname{div},\Omega)\}$$

and their respective bounded inverse operators

$$\mathring{\mathcal{L}}^{-1} : \widetilde{\mathsf{L}}^2(\Omega) \to D(\mathring{\mathcal{L}}) \subset \mathsf{L}^2(\Omega), \quad \mathcal{L}_{\mathtt{D}}^{-1} : \mathsf{L}^2(\Omega) \to D(\mathcal{L}_{\mathtt{D}}) \subset \mathsf{L}^2(\Omega),$$
$$\mathcal{L}^{-1} : \mathsf{L}^2(\Omega) \to D(\mathcal{L}) \subset \widetilde{\mathsf{L}}^2(\Omega), \quad \mathcal{L}_{\mathtt{N}}^{-1} : \widehat{\mathsf{L}}^2(\Omega) \to D(\mathcal{L}_{\mathtt{N}}) \subset \widehat{\mathsf{L}}^2(\Omega). \tag{26}$$

In Section 2.2.6 we already discussed the Dirichlet and Neumann biharmonic operators

$$\mathcal{B}_{\cdot,\circ} := \mathcal{B}_{\mathtt{D}} = \mathcal{L}\,\mathring{\mathcal{L}} = \mathrm{L}\,\mathring{\mathrm{L}} = \mathrm{B}_{\mathtt{D}}, \qquad \mathcal{B}_{\circ,\cdot} := \mathcal{B}_{\mathtt{N}} = \mathring{\mathcal{L}}\,\mathcal{L},$$

being selfadjoint, positive, bijective and boundedly invertible with the well posed (well defined and uniquely solvable) inverse operators $\mathring{\mathcal{L}}^{-1}\,\mathcal{L}^{-1}$ and $\mathcal{L}^{-1}\,\mathring{\mathcal{L}}^{-1}$.

Combining the four different Laplacians we obtain a whole zoo of formally sixteen biharmonic operators. Due to the restrictions of $\widehat{\mathsf{L}}^2(\Omega)$ and $\widetilde{\mathsf{L}}^2(\Omega)$ some combinations are – even formally – not possible (without further restrictions), those are the five combinations

$$\mathring{\mathcal{L}}^{-1}\mathring{\mathcal{L}}^{-1}, \qquad \mathring{\mathcal{L}}^{-1}\,\mathcal{L}_{\mathtt{D}}^{-1}, \qquad \mathring{\mathcal{L}}^{-1}\,\mathcal{L}_{\mathtt{N}}^{-1}, \qquad \mathcal{L}_{\mathtt{N}}^{-1}\,\mathring{\mathcal{L}}^{-1}, \qquad \mathcal{L}_{\mathtt{N}}^{-1}\,\mathcal{L}_{\mathtt{D}}^{-1},$$

corresponding to $\mathring{\mathcal{L}}\mathring{\mathcal{L}}$, $\mathcal{L}_{\mathtt{D}}\,\mathring{\mathcal{L}}$, $\mathcal{L}_{\mathtt{N}}\,\mathring{\mathcal{L}}$, $\mathring{\mathcal{L}}\,\mathcal{L}_{\mathtt{N}}$, $\mathcal{L}_{\mathtt{D}}\,\mathcal{L}_{\mathtt{N}}$, respectively. One has to consider different operators to realise even stronger or weaker boundary conditions. We come back to this later.

We end up with only nine more well posed biharmonic operators, namely

$$\mathcal{B}_{\cdot,\cdot} := \mathcal{L}\,\mathcal{L}, \qquad \mathcal{B}_{\mathtt{D},\cdot} := \mathcal{L}_{\mathtt{D}}\,\mathcal{L}, \qquad \mathcal{B}_{\mathtt{N},\cdot} := \mathcal{L}_{\mathtt{N}}\,\mathcal{L}, \qquad \mathcal{B}_{\cdot,\mathtt{N}} := \mathcal{L}\,\mathcal{L}_{\mathtt{N}}, \qquad \mathcal{B}_{\mathtt{N},\mathtt{N}} := \mathcal{L}_{\mathtt{N}}\,\mathcal{L}_{\mathtt{N}},$$
$$\mathcal{B}_{\circ,\mathtt{D}} := \mathring{\mathcal{L}}\,\mathcal{L}_{\mathtt{D}}, \qquad \mathcal{B}_{\cdot,\mathtt{D}} := \mathcal{L}\,\mathcal{L}_{\mathtt{D}}, \qquad \mathcal{B}_{\mathtt{D},\mathtt{D}} := \mathcal{L}_{\mathtt{D}}\,\mathcal{L}_{\mathtt{D}}, \qquad \mathcal{B}_{\mathtt{N},\mathtt{D}} := \mathcal{L}_{\mathtt{N}}\,\mathcal{L}_{\mathtt{D}}\,.$$

All eleven biharmonic operators are bijective and boundedly invertible, more precisely

$$\mathcal{B}_{\cdot,\circ}^{-1} = \mathring{\mathcal{L}}^{-1}\,\mathcal{L}^{-1} : \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \qquad \mathcal{B}_{\circ,\cdot}^{-1} = \mathcal{L}^{-1}\,\mathring{\mathcal{L}}^{-1} : \widetilde{\mathsf{L}}^2(\Omega) \to \widetilde{\mathsf{L}}^2(\Omega),$$
$$\mathcal{B}_{\cdot,\cdot}^{-1} = \mathcal{L}^{-1}\,\mathcal{L}^{-1} : \mathsf{L}^2(\Omega) \to \widetilde{\mathsf{L}}^2(\Omega), \qquad \mathcal{B}_{\mathtt{D},\cdot}^{-1} = \mathcal{L}^{-1}\,\mathcal{L}_{\mathtt{D}}^{-1} : \mathsf{L}^2(\Omega) \to \widetilde{\mathsf{L}}^2(\Omega),$$
$$\mathcal{B}_{\mathtt{N},\cdot}^{-1} = \mathcal{L}^{-1}\,\mathcal{L}_{\mathtt{N}}^{-1} : \widehat{\mathsf{L}}^2(\Omega) \to \widetilde{\mathsf{L}}^2(\Omega), \qquad \mathcal{B}_{\cdot,\mathtt{N}}^{-1} = \mathcal{L}_{\mathtt{N}}^{-1}\,\mathcal{L}^{-1} : \mathsf{L}^2(\Omega) \to \widehat{\mathsf{L}}^2(\Omega),$$
$$\mathcal{B}_{\mathtt{N},\mathtt{N}}^{-1} = \mathcal{L}_{\mathtt{N}}^{-1}\,\mathcal{L}_{\mathtt{N}}^{-1} : \widehat{\mathsf{L}}^2(\Omega) \to \widehat{\mathsf{L}}^2(\Omega), \qquad \mathcal{B}_{\circ,\mathtt{D}}^{-1} = \mathcal{L}_{\mathtt{D}}^{-1}\,\mathring{\mathcal{L}}^{-1} : \widetilde{\mathsf{L}}^2(\Omega) \to \mathsf{L}^2(\Omega),$$
$$\mathcal{B}_{\cdot,\mathtt{D}}^{-1} = \mathcal{L}_{\mathtt{D}}^{-1}\,\mathcal{L}^{-1} : \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega), \qquad \mathcal{B}_{\mathtt{D},\mathtt{D}}^{-1} = \mathcal{L}_{\mathtt{D}}^{-1}\,\mathcal{L}_{\mathtt{D}}^{-1} : \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega),$$
$$\mathcal{B}_{\mathtt{N},\mathtt{D}}^{-1} = \mathcal{L}_{\mathtt{D}}^{-1}\,\mathcal{L}_{\mathtt{N}}^{-1} : \widehat{\mathsf{L}}^2(\Omega) \to \mathsf{L}^2(\Omega).$$

Let us write down the classical formulations of the latter eleven (some apparently over- and underdetermined) biharmonic operators as uniquely solvable boundary value problems for the bi-Laplacian:

(i) $\mathcal{B}_{\cdot,\circ}^{-1} = \mathring{\mathcal{L}}^{-1}\,\mathcal{L}^{-1} : \mathsf{L}^2(\Omega) \to \big\{u \in D(\mathring{\mathcal{L}}) = \mathring{\mathsf{H}}^2(\Omega) : \Delta u \in D(\mathcal{L}) = \widetilde{\mathsf{H}}(\Delta,\Omega)\big\}$
yields the unique solution u of (The last condition is redundant.)

$$\begin{aligned} \Delta^2 u = f \in \mathsf{L}^2(\Omega) &\qquad \text{in } \Omega,\\ u = 0 &\qquad \text{on } \Gamma,\\ n\cdot\nabla\, u = 0 &\qquad \text{on } \Gamma,\\ \Delta u \perp \mathbb{H}. & \end{aligned}$$

(ii) $\mathcal{B}_{\circ,\cdot}^{-1} = \mathcal{L}^{-1}\,\mathring{\mathcal{L}}^{-1} : \widetilde{\mathsf{L}}^2(\Omega) \to \big\{u \in D(\mathcal{L}) = \widetilde{\mathsf{H}}(\Delta,\Omega) : \Delta u \in D(\mathring{\mathcal{L}}) = \mathring{\mathsf{H}}^2(\Omega)\big\}$
yields the unique solution u of

$$\begin{aligned} \Delta^2 u = f \in \widetilde{\mathsf{L}}^2(\Omega) &\qquad \text{in } \Omega,\\ \Delta u = 0 &\qquad \text{on } \Gamma,\\ n\cdot\nabla\,\Delta u = 0 &\qquad \text{on } \Gamma,\\ u \perp \mathbb{H}. & \end{aligned}$$

(iii) $\mathcal{B}_{\cdot,\cdot}^{-1} = \mathcal{L}^{-1}\,\mathcal{L}^{-1} : \mathsf{L}^2(\Omega) \to \big\{u \in D(\mathcal{L}) = \widetilde{\mathsf{H}}(\Delta,\Omega) : \Delta u \in D(\mathcal{L}) = \widetilde{\mathsf{H}}(\Delta,\Omega)\big\}$
yields the unique solution u of

$$\begin{aligned} \Delta^2 u = f \in \mathsf{L}^2(\Omega) &\qquad \text{in } \Omega,\\ u \perp \mathbb{H}, & \end{aligned}$$

$$\Delta u \perp \mathbb{H}.$$

(iv) $\mathcal{B}_{\mathtt{D},\cdot}^{-1} = \mathcal{L}^{-1}\,\mathcal{L}_{\mathtt{D}}^{-1} : \mathsf{L}^2(\Omega) \to \big\{u \in D(\mathcal{L}) = \widetilde{\mathsf{H}}(\Delta,\Omega) : \Delta u \in D(\mathcal{L}_{\mathtt{D}})\big\}$
yields the unique solution u of

$$\begin{aligned} \Delta^2 u &= f \in \mathsf{L}^2(\Omega) && \text{in } \Omega,\\ \Delta u &= 0 && \text{on } \Gamma,\\ u &\perp \mathbb{H}. \end{aligned}$$

(v) $\mathcal{B}_{\mathtt{N},\cdot}^{-1} = \mathcal{L}^{-1}\,\mathcal{L}_{\mathtt{N}}^{-1} : \widehat{\mathsf{L}}^2(\Omega) \to \big\{u \in D(\mathcal{L}) = \widetilde{\mathsf{H}}(\Delta,\Omega) : \Delta u \in D(\mathcal{L}_{\mathtt{N}})\big\}$
yields the unique solution u of

$$\begin{aligned} \Delta^2 u &= f \in \widehat{\mathsf{L}}^2(\Omega) && \text{in } \Omega,\\ n \cdot \nabla\,\Delta u &= 0 && \text{on } \Gamma,\\ u &\perp \mathbb{H},\\ \Delta u &\perp \mathbb{R}. \end{aligned}$$

(vi) $\mathcal{B}_{\cdot,\mathtt{N}}^{-1} = \mathcal{L}_{\mathtt{N}}^{-1}\,\mathcal{L}^{-1} : \mathsf{L}^2(\Omega) \to \big\{u \in D(\mathcal{L}_{\mathtt{N}}) : \Delta u \in D(\mathcal{L}) = \widetilde{\mathsf{H}}(\Delta,\Omega)\big\}$
yields the unique solution u of

$$\begin{aligned} \Delta^2 u &= f \in \mathsf{L}^2(\Omega) && \text{in } \Omega,\\ n \cdot \nabla\,u &= 0 && \text{on } \Gamma,\\ u &\perp \mathbb{R},\\ \Delta u &\perp \mathbb{H}. \end{aligned}$$

(vii) $\mathcal{B}_{\mathtt{N},\mathtt{N}}^{-1} = \mathcal{L}_{\mathtt{N}}^{-1}\,\mathcal{L}_{\mathtt{N}}^{-1} : \widehat{\mathsf{L}}^2(\Omega) \to \big\{u \in D(\mathcal{L}_{\mathtt{N}}) : \Delta u \in D(\mathcal{L}_{\mathtt{N}})\big\}$
yields the unique solution u of

$$\begin{aligned} \Delta^2 u &= f \in \widehat{\mathsf{L}}^2(\Omega) && \text{in } \Omega,\\ n \cdot \nabla\,u &= 0 && \text{on } \Gamma,\\ n \cdot \nabla\,\Delta u &= 0 && \text{on } \Gamma,\\ u &\perp \mathbb{R},\\ \Delta u &\perp \mathbb{R}. \end{aligned}$$

(viii) $\mathcal{B}_{\circ,\mathtt{D}}^{-1} = \mathcal{L}_{\mathtt{D}}^{-1}\,\mathring{\mathcal{L}}^{-1} : \widetilde{\mathsf{L}}^2(\Omega) \to \big\{u \in D(\mathcal{L}_{\mathtt{D}}) : \Delta u \in D(\mathring{\mathcal{L}}) = \mathring{\mathsf{H}}^2(\Omega)\big\}$
yields the unique solution u of

$$\begin{aligned} \Delta^2 u &= f \in \widetilde{\mathsf{L}}^2(\Omega) && \text{in } \Omega,\\ u &= 0 && \text{on } \Gamma,\\ \Delta u &= 0 && \text{on } \Gamma,\\ n \cdot \nabla\,\Delta u &= 0 && \text{on } \Gamma. \end{aligned}$$

(ix) $\mathcal{B}_{\cdot,\mathtt{D}}^{-1} = \mathcal{L}_{\mathtt{D}}^{-1}\,\mathcal{L}^{-1} : \mathsf{L}^2(\Omega) \to \big\{u \in D(\mathcal{L}_{\mathtt{D}}) : \Delta u \in D(\mathcal{L}) = \widetilde{\mathsf{H}}(\Delta,\Omega)\big\}$
yields the unique solution u of

$$\begin{aligned} \Delta^2 u &= f \in \mathsf{L}^2(\Omega) && \text{in } \Omega,\\ u &= 0 && \text{on } \Gamma,\\ \Delta u &\perp \mathbb{H}. \end{aligned}$$

(x) $\mathcal{B}_{\mathtt{D},\mathtt{D}}^{-1} = \mathcal{L}_{\mathtt{D}}^{-1}\,\mathcal{L}_{\mathtt{D}}^{-1} : \mathsf{L}^2(\Omega) \to \big\{u \in D(\mathcal{L}_{\mathtt{D}}) : \Delta u \in D(\mathcal{L}_{\mathtt{D}})\big\}$
yields the unique solution u of

$$\begin{aligned} \Delta^2 u &= f \in \mathsf{L}^2(\Omega) && \text{in } \Omega,\\ u &= 0 && \text{on } \Gamma,\\ \Delta u &= 0 && \text{on } \Gamma. \end{aligned}$$

(xi) $\mathcal{B}_{\mathtt{N},\mathtt{D}}^{-1} = \mathcal{L}_{\mathtt{D}}^{-1}\,\mathcal{L}_{\mathtt{N}}^{-1} : \widehat{\mathsf{L}}^2(\Omega) \to \big\{u \in D(\mathcal{L}_{\mathtt{D}}) : \Delta u \in D(\mathcal{L}_{\mathtt{N}})\big\}$
yields the unique solution u of

$$\begin{aligned} \Delta^2 u = f \in \widehat{\mathsf{L}}^2(\Omega) & \qquad \text{in } \Omega, \\ u = 0 & \qquad \text{on } \Gamma, \\ n \cdot \nabla \Delta u = 0 & \qquad \text{on } \Gamma, \\ \Delta u \perp \mathbb{R}. & \end{aligned}$$

2.2.8. *Over- and Underdetermined Biharmonic Operators.* Let Ω be Friedrichs admissible. We shall follow the rationale from Section 2.2.5. Note that (22) shows

$$\forall\, \varphi \in \mathring{\mathsf{H}}^4(\Omega) \qquad |\varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{f}} |\nabla \varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{f}}^2 |\Delta \varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{f}}^3 |\Delta \nabla \varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{f}}^4 |\Delta^2 \varphi|_{\mathsf{L}^2(\Omega)}, \tag{27}$$

which holds also for all $\varphi \in \mathring{\mathsf{H}}^3(\Omega)$ with $\Delta^2 \varphi \in \mathsf{L}^2(\Omega)$.

Lemma 2.15. *On $\mathring{\mathsf{H}}^4(\Omega)$ the norms $|\cdot|_{\mathsf{H}^4(\Omega)}$ and $|\Delta^2 \cdot|_{\mathsf{L}^2(\Omega)}$ are equivalent. More precisely,*

$$\forall\, \varphi \in \mathring{\mathsf{H}}^4(\Omega) \qquad |\varphi|_{\mathsf{H}^4(\Omega)} \leq c_{\Delta^2} |\Delta^2 \varphi|_{\mathsf{L}^2(\Omega)},$$

where $c_{\Delta^2} := c_\Delta \sqrt{1 + c_{\mathsf{f}}^4} = \sqrt{1 + c_{\mathsf{f}}^2 + 2c_{\mathsf{f}}^4 + c_{\mathsf{f}}^6 + c_{\mathsf{f}}^8}$.

Proof. Note that $\varphi \in \mathring{\mathsf{H}}^4(\Omega)$ implies $\varphi,\ \partial_i \partial_j \varphi \in \mathring{\mathsf{H}}^2(\Omega)$. Lemma 2.9, its proof, and (22) show

$$|\varphi|_{\mathsf{H}^4(\Omega)}^2 = |\varphi|_{\mathsf{H}^2(\Omega)}^2 + \sum_{i,j} |\partial_i \partial_j \varphi|_{\mathsf{H}^2(\Omega)}^2 \leq c_\Delta^2 \Big(\underbrace{|\Delta \varphi|_{\mathsf{L}^2(\Omega)}^2}_{\leq c_{\mathsf{f}}^4 |\Delta^2 \varphi|_{\mathsf{L}^2(\Omega)}^2} + \underbrace{\sum_{i,j} |\partial_i \partial_j \Delta \varphi|_{\mathsf{L}^2(\Omega)}^2}_{= |\Delta^2 \varphi|_{\mathsf{L}^2(\Omega)}^2} \Big),$$

completing the proof. □

Let

$$\mathsf{H}(\Delta^2, \Omega) := \big\{\varphi \in \mathsf{L}^2(\Omega) : \Delta^2 \varphi \in \mathsf{L}^2(\Omega)\big\}, \qquad \mathbb{BH} := \big\{\varphi \in \mathsf{L}^2(\Omega) : \Delta^2 \varphi = 0\big\},$$

where the latter denotes the biharmonic functions.

Lemma 2.15 and Lemma 2.1 show that the over- and underdetermined biharmonic operators (bi-Laplacians)

$$\begin{aligned} \mathring{\mathrm{B}} : \mathring{\mathsf{H}}^4(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega); & \qquad \varphi \mapsto \Delta^2 \varphi, \\ \mathrm{B} := \mathring{\mathrm{B}}^* : \mathsf{H}(\Delta^2, \Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega) & \end{aligned}$$

are densely defined and closed with closed ranges $R(\mathring{\mathrm{B}})$ and $R(\mathrm{B})$. Moreover,

$$\begin{aligned} N(\mathring{\mathrm{B}}) &= \{0\}, \quad R(\mathring{\mathrm{B}}) = N(\mathrm{B})^\perp = \mathsf{L}^2(\Omega) \cap \mathbb{BH}^\perp =: \check{\mathsf{L}}^2(\Omega), \\ N(\mathrm{B}) &= \mathbb{BH}, \quad R(\mathrm{B}) = N(\mathring{\mathrm{B}})^\perp = \mathsf{L}^2(\Omega). \end{aligned} \tag{28}$$

The reduced operators

$$\begin{aligned} \mathring{\mathcal{B}} : \mathring{\mathsf{H}}^4(\Omega) \subset \mathsf{L}^2(\Omega) \to \check{\mathsf{L}}^2(\Omega), & \\ \mathcal{B} : \check{\mathsf{H}}(\Delta^2, \Omega) \subset \check{\mathsf{L}}^2(\Omega) \to \mathsf{L}^2(\Omega), & \qquad \check{\mathsf{H}}(\Delta^2, \Omega) := \mathsf{H}(\Delta^2, \Omega) \cap \mathbb{BH}^\perp \end{aligned}$$

are bijective with bounded inverse operators

$$\begin{aligned} \mathring{\mathcal{B}}^{-1} &: \check{\mathsf{L}}^2(\Omega) \to D(\mathring{\mathrm{B}}) = \mathring{\mathsf{H}}^4(\Omega), \\ \mathcal{B}^{-1} &: \mathsf{L}^2(\Omega) \to D(\mathcal{B}) = \check{\mathsf{H}}(\Delta^2, \Omega), \qquad |\mathring{\mathcal{B}}^{-1}|_{\check{\mathsf{L}}^2(\Omega) \to \mathsf{L}^2(\Omega)} = |\mathcal{B}^{-1}|_{\mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega)} \leq c_{\mathsf{f}}^4, \end{aligned}$$

cf. (27).

Let $f \in \check{\mathsf{L}}^2(\Omega)$. Then $u := \mathring{\mathcal{B}}^{-1} f \in D(\mathring{\mathrm{B}}) = \mathring{\mathsf{H}}^4(\Omega)$ is the unique solution of the overdetermined biharmonic boundary value problem $\mathring{\mathcal{B}} u = f$, i.e.,

$$\begin{aligned} \Delta^2 u = f & \qquad \text{in } \Omega, \\ \forall\, |\alpha| \leq 3 \qquad \partial^\alpha v = 0 & \qquad \text{on } \Gamma. \end{aligned}$$

Let $f \in \mathsf{L}^2(\Omega)$. Then $u := \mathcal{B}^{-1} f \in D(\mathcal{B}) = \check{\mathsf{H}}(\Delta, \Omega)$ is the unique solution of the underdetermined biharmonic boundary value problem $\mathcal{B} u = f$, i.e.,

$$\begin{aligned} \Delta^2 u &= f && \text{in } \Omega, \\ u &\perp \mathbb{BH}. \end{aligned}$$

Note that the latter two boundary value problems for Δ^2 are the numbers **(xii)** and **(xiii)** on the list from Section 2.2.7.

Remark 2.16. *Note that for the different biharmonic operators we have* $\mathring{\mathrm{B}} \subset \mathrm{B}_{\mathtt{D}}, \mathrm{B}_{\mathtt{N}} \subset \mathrm{B}$, *that is*

$$\mathring{\mathrm{B}} \subset \mathrm{B}_{\mathtt{D}} = \mathrm{L}\,\mathring{\mathrm{L}} \subset \mathrm{B}, \qquad \mathring{\mathrm{B}} \subset \mathrm{B}_{\mathtt{N}} = \mathring{\mathrm{L}}\,\mathrm{L} \subset \mathrm{B}\,.$$

In this sense, $\mathring{\mathrm{B}}$ *and* B *are minimal and maximal* $\mathsf{L}^2(\Omega)$*-realisations of the biharmonic operator, respectively.*

Recall Theorem 2.13.

Theorem 2.17. *Let* Ω *be bounded. Then* $D(\mathcal{B}) = \check{\mathsf{H}}(\Delta^2, \Omega) \hookrightarrow \mathsf{L}^2(\Omega)$ *is compact.*

Proof. By Lemma 2.1 we have that the embedding $D(\mathring{\mathcal{B}}) = D(\mathring{\mathrm{B}}) = \mathring{\mathsf{H}}^4(\Omega) \hookrightarrow \mathsf{L}^2(\Omega)$ is compact if and only if the embedding $D(\mathcal{B}) = \check{\mathsf{H}}(\Delta^2, \Omega) \hookrightarrow \mathsf{L}^2(\Omega)$ is compact. Hence the latter embedding is compact by Rellich's selection theorem for, e.g., $\mathring{\mathsf{H}}^1(\Omega)$. □

2.2.9. *Biharmonic Operators with Mixed Boundary Conditions.* In principle, everything works also with mixed boundary conditions. For example, let us consider a division of Γ into two relatively open parts $\Gamma_{\mathsf{t}} \neq \emptyset$ and $\Gamma_{\mathsf{n}} := \Gamma \setminus \overline{\Gamma_{\mathsf{t}}}$. We introduce $\mathsf{H}^1_{\Gamma_{\mathsf{t}}}(\Omega)$ as closure of $\mathring{\mathsf{C}}^\infty_{\Gamma_{\mathsf{t}}}(\mathbb{R}^3)$ (the compact support does not touch Γ_{t}) in $\mathsf{H}^1(\Omega)$. Analogously we define $\mathsf{H}_{\Gamma_{\mathsf{n}}}(\operatorname{div}, \Omega)$.

Let Ω be such that the embedding

$$\mathsf{H}^1(\Omega) \hookrightarrow \mathsf{L}^2(\Omega) \tag{29}$$

is compact, which holds, e.g., for bounded Lipschitz domains Ω.

By the compact embedding (29) we obtain the Friedrichs/Poincaré estimate

$$\exists\, c_{\mathsf{fp}} > 0 \quad \forall\, \varphi \in \mathsf{H}^1_{\Gamma_{\mathsf{t}}}(\Omega) \qquad |\varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{fp}} |\nabla \varphi|_{\mathsf{L}^2(\Omega)}, \tag{30}$$

cf. Defintion 2.6. As before, we assume c_{fp} to be the best possible constant.

Then

$$\mathrm{A} := \nabla_{\Gamma_{\mathsf{t}}} : \mathsf{H}^1_{\Gamma_{\mathsf{t}}}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega)$$

is densely defined and closed with adjoint

$$\mathrm{A}^* = \nabla^*_{\Gamma_{\mathsf{t}}} = -\operatorname{div}_{\Gamma_{\mathsf{n}}} : \mathsf{H}_{\Gamma_{\mathsf{n}}}(\operatorname{div}, \Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega),$$

cf. [3, 14]. Note that the cases $\Gamma_{\mathsf{t}} = \Gamma$ and $\Gamma_{\mathsf{t}} = \emptyset$ have already been discussed by $\mathring{\nabla} = \nabla_\Gamma$ and $\nabla = \nabla_\emptyset$, respectively. By (30) and Lemma 2.1 $R(\mathrm{A})$ and $R(\mathrm{A}^*)$ are closed and we have

$$N(\nabla_{\Gamma_{\mathsf{t}}}) = N(\mathrm{A}) = \{0\}, \qquad R(\operatorname{div}_{\Gamma_{\mathsf{n}}}) = R(\mathrm{A}^*) = N(\mathrm{A})^\perp = \mathsf{L}^2(\Omega).$$

The full de Rham complex reads, cf. (18) and (21),

$$\mathsf{L}^2(\Omega) \underset{\mathrm{A}_0^* = -\operatorname{div}_{\Gamma_{\mathsf{n}}}}{\overset{\mathrm{A}_0 = \nabla_{\Gamma_{\mathsf{t}}}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{\mathrm{A}_1^* = \operatorname{rot}_{\Gamma_{\mathsf{n}}}}{\overset{\mathrm{A}_1 = \operatorname{rot}_{\Gamma_{\mathsf{t}}}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{\mathrm{A}_2^* = -\nabla_{\Gamma_{\mathsf{n}}}}{\overset{\mathrm{A}_2 = \operatorname{div}_{\Gamma_{\mathsf{t}}}}{\rightleftarrows}} \mathsf{L}^2(\Omega). \tag{31}$$

The negative Dirichlet-Neumann-Laplacian

$$-\Delta_{\Gamma_{\mathsf{t}}} := \mathcal{A}^* \mathcal{A} = \mathrm{A}^* \mathrm{A} = -\operatorname{div}_{\Gamma_{\mathsf{n}}} \nabla_{\Gamma_{\mathsf{t}}} : D(\Delta_{\Gamma_{\mathsf{t}}}) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega),$$

where

$$D(\Delta_{\Gamma_{\mathsf{t}}}) = \big\{\varphi \in \mathsf{H}^1_{\Gamma_{\mathsf{t}}}(\Omega) : \nabla \varphi \in \mathsf{H}_{\Gamma_{\mathsf{n}}}(\operatorname{div}, \Omega)\big\},$$

is selfadjoint, positive, and bijective with bounded inverse

$$-\Delta^{-1}_{\Gamma_{\mathsf{t}}} = \mathcal{A}^{-1}(\mathcal{A}^*)^{-1} : \mathsf{L}^2(\Omega) \to D(\Delta_{\Gamma_{\mathsf{t}}}), \qquad \big|\Delta^{-1}_{\Gamma_{\mathsf{t}}}\big|_{\mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega)} = c^2_{\mathsf{fp}}.$$

(15) reads

$$\forall\, \varphi \in D(\Delta_{\Gamma_{\mathsf{t}}}) \qquad |\varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{fp}} |\nabla \varphi|_{\mathsf{L}^2(\Omega)} \leq c^2_{\mathsf{fp}} |\Delta \varphi|_{\mathsf{L}^2(\Omega)}. \tag{32}$$

Let $f \in \mathsf{L}^2(\Omega)$. Then $u := -\Delta_{\Gamma_t}^{-1} f \in D(\Delta_{\Gamma_t})$ is the unique solution of the Dirichlet-Neumann Laplace boundary value problem $-\Delta_{\Gamma_t} u = f$, i.e.,

$$\begin{aligned} \Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_t, \\ n \cdot \nabla u &= 0 && \text{on } \Gamma_n. \end{aligned}$$

To find $u \in \mathsf{H}^1_{\Gamma_t}(\Omega)$ by variational methods we may consider (16), i.e.,

$$\forall\, \varphi \in \mathsf{H}^1_{\Gamma_t}(\Omega) \qquad \langle \nabla u, \nabla \varphi \rangle_{\mathsf{L}^2(\Omega)} = \langle f, \varphi \rangle_{\mathsf{L}^2(\Omega)}.$$

Moreover, $v := \Delta_{\Gamma_t}^{-1} \Delta_{\gamma_t}^{-1} f \in D(\Delta_{\gamma_t} \Delta_{\Gamma_t})$ for some other boundary pair $\gamma_t \neq \emptyset$ and $\gamma_n := \Gamma \setminus \overline{\gamma_t}$ is the unique solution of the Dirichlet-Neumann biharmonic boundary value problem $\Delta_{\gamma_t} \Delta_{\Gamma_t} v = f$, i.e.,

$$\begin{aligned} \Delta v^2 &= f && \text{in } \Omega, \\ v &= 0 && \text{on } \Gamma_t, \\ n \cdot \nabla v &= 0 && \text{on } \Gamma_n, \\ \Delta v &= 0 && \text{on } \gamma_t, \\ n \cdot \nabla \Delta v &= 0 && \text{on } \gamma_n. \end{aligned}$$

2.3. **Applications to the Hessian Complex.** Let Ω be a bounded Lipschitz domain, and recall the boundary parts Γ_t and Γ_n and the definition of the Sobolev spaces als closures of test fields from the latter section. In the following we shall apply our theory to the Hessian complex (complex of the biharmonic equation and general relativity)

$$\mathsf{L}^2(\Omega) \underset{\mathrm{A}_0^*=\operatorname{divDiv}_{\mathbb{S},\Gamma_n}}{\overset{\mathrm{A}_0=\nabla\nabla_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}}(\Omega) \underset{\mathrm{A}_1^*=\operatorname{symRot}_{\mathbb{T},\Gamma_n}}{\overset{\mathrm{A}_1=\operatorname{Rot}_{\mathbb{S},\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}}(\Omega) \underset{\mathrm{A}_2^*=-\operatorname{dev}\nabla_{\Gamma_n}}{\overset{\mathrm{A}_2=\operatorname{Div}_{\mathbb{T},\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2(\Omega), \tag{33}$$

cf. (18) and (21), with the densely defined and closed linear operators (acting row-wise on tensors)

$$\begin{aligned} \nabla\nabla_{\Gamma_t} : \mathsf{H}^2_{\Gamma_t}(\Omega) \subset \mathsf{L}^2(\Omega) &\to \mathsf{L}^2_{\mathbb{S}}(\Omega); && \phi \mapsto \nabla\nabla\,\phi, \\ \operatorname{Rot}_{\mathbb{S},\Gamma_t} : \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot},\Omega) \subset \mathsf{L}^2_{\mathbb{S}}(\Omega) &\to \mathsf{L}^2_{\mathbb{T}}(\Omega); && \Phi \mapsto \operatorname{Rot}\Phi, \\ \operatorname{Div}_{\mathbb{T},\Gamma_t} : \mathsf{H}_{\mathbb{T},\Gamma_t}(\operatorname{Div},\Omega) \subset \mathsf{L}^2_{\mathbb{T}}(\Omega) &\to \mathsf{L}^2(\Omega); && \Psi \mapsto \operatorname{Div}\Psi, \\ \operatorname{divDiv}_{\mathbb{S},\Gamma_n} : \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{divDiv},\Omega) \subset \mathsf{L}^2_{\mathbb{S}}(\Omega) &\to \mathsf{L}^2(\Omega); && \Phi \mapsto \operatorname{divDiv}\Phi, \\ \operatorname{symRot}_{\mathbb{T},\Gamma_n} : \mathsf{H}_{\mathbb{T},\Gamma_n}(\operatorname{symRot},\Omega) \subset \mathsf{L}^2_{\mathbb{T}}(\Omega) &\to \mathsf{L}^2_{\mathbb{S}}(\Omega); && \Psi \mapsto \operatorname{Rot}\Psi, \\ \operatorname{dev}\nabla_{\Gamma_n} : \mathsf{H}^1_{\Gamma_n}(\Omega) \subset \mathsf{L}^2(\Omega) &\to \mathsf{L}^2_{\mathbb{T}}(\Omega); && \psi \mapsto \nabla\nabla\,\psi, \end{aligned}$$

cf. [16, 18] for well-posedness.

Recall the identity (10). Hence, another way to look at the biharmonic equation – underlining more the geometry (complex property) of the underlying operators – is to investigate the biharmonic operator

$$\mathrm{B}_{\Gamma_t} := \mathrm{A}_0^* \, \mathrm{A}_0 = \operatorname{divDiv}_{\mathbb{S},\Gamma_n} \nabla\nabla_{\Gamma_t} : D(\mathrm{A}_0^* \, \mathrm{A}_0) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega); \qquad \Phi \mapsto \operatorname{divDiv} \nabla\nabla\,\Phi,$$

where

$$D(\mathrm{B}_{\Gamma_t}) = \big\{ \varphi \in \mathsf{H}^2_{\Gamma_t}(\Omega) : \nabla\nabla\,\varphi \in \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{divDiv},\Omega) \big\}.$$

For $\Gamma_t = \Gamma$ we get back the Dirichlet biharmonic operator, i.e., $\mathrm{B}_{\Gamma} = \mathrm{B}_{\mathsf{D}}$. But if $\Gamma_t \neq \Gamma$ we obtain a different operator due to the boundary conditions being imposed on the scalars u and Δu for Δ^2, and on the other hand on the scalar u and the symmetric tensor $S := \nabla\nabla\, u$ for B_{Γ_t}.

For simplicity, let us assume $\emptyset \neq \Gamma_t \neq \Gamma$. The Friedrichs/Poincaré estimate (30) yields

$$\forall\, \varphi \in \mathsf{H}^2_{\Gamma_t}(\Omega) \qquad |\varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{fp}} |\nabla\,\varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{fp}}^2 |\nabla\nabla\,\varphi|_{\mathsf{L}^2(\Omega)}. \tag{34}$$

By (34) and Lemma 2.1 $R(\mathrm{A}_0)$ and $R(\mathrm{B}_{\Gamma_t}) = R(\mathrm{A}_0^*)$ are closed and

$$N(\mathrm{B}_{\Gamma_t}) = N(\mathrm{A}_0) = \{0\}, \qquad R(\mathrm{B}_{\Gamma_t}) = R(\mathrm{A}_0^*) = N(\mathrm{A}_0)^{\perp} = \mathsf{L}^2(\Omega),$$

which shows $\mathcal{B}_{\Gamma_t} = \mathrm{B}_{\Gamma_t}$. B_{Γ_t} is selfadjoint, positive, and bijective with bounded inverse

$$\mathrm{B}_{\Gamma_t}^{-1} = \mathcal{A}_0^{-1}(\mathcal{A}_0^*)^{-1} : \mathsf{L}^2(\Omega) \to D(\mathrm{B}_{\Gamma_t}), \qquad |\,\mathrm{B}_{\Gamma_t}^{-1}\,|_{\mathsf{L}^2(\Omega)\to\mathsf{L}^2(\Omega)} \leq c_{\mathsf{fp}}^4.$$

(15) reads

$$\forall\,\varphi \in D(\mathrm{B}_{\Gamma_t}) \qquad |\varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{fp}}^2|\,\nabla\nabla\,\varphi|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{fp}}^4|\,\operatorname{divDiv}\nabla\nabla\,\varphi|_{\mathsf{L}^2(\Omega)} = c_{\mathsf{fp}}^4|\Delta^2\varphi|_{\mathsf{L}^2(\Omega)}.$$

Let $f \in \mathsf{L}^2(\Omega)$. Then $u := \mathrm{B}_{\Gamma_t}^{-1} f \in D(\mathrm{B}_{\Gamma_t})$ is the unique solution of the Dirichlet/Neumann biharmonic boundary value problem $\mathrm{B}_{\Gamma_t} u = f$, i.e.,

$$\begin{aligned} \operatorname{divDiv}\nabla\nabla\,u = \Delta^2 u &= f && \text{in } \Omega,\\ u &= 0 && \text{on } \Gamma_{\mathsf{t}},\\ \nabla\,u &= 0 && \text{on } \Gamma_{\mathsf{t}},\\ (\nabla\nabla\,u)\,n &= 0 && \text{on } \Gamma_{\mathsf{n}},\\ \big((\nabla\nabla\,u)\,n\big)\cdot n &= 0 && \text{on } \Gamma_{\mathsf{n}}. \end{aligned}$$

To find $u \in \mathsf{H}^2_{\Gamma_t}(\Omega)$ by variational methods we may consider (16), i.e.,

$$\forall\,\varphi \in \mathsf{H}^2_{\Gamma_t}(\Omega) \qquad \langle\nabla\nabla\,u, \nabla\nabla\,\varphi\rangle_{\mathsf{L}^2(\Omega)} = \langle f, \varphi\rangle_{\mathsf{L}^2(\Omega)}.$$

Extensions to right hand sides $f \in \mathsf{H}^{-2}_{\Gamma_n}(\Omega)$ are straight forward using Banach space adjoints.

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