FA-ToolBox:
Solving PDEs with Hilbert Complexes

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FA-ToolBox: Solving PDEs with Hilbert Complexes

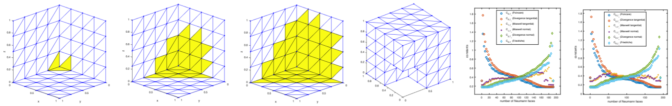
OVERVIEW

(I) general theory FA-ToolBox (Hilbert complexes, tailor-made functional analysis)

(II) applications to pdes

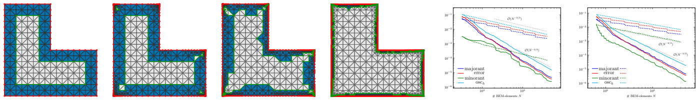
- $\cdots \, L^2 \underset{-\operatorname{div}}{\overset{\mathring{\nabla}}{\rightleftarrows}} L^2 \underset{\operatorname{rot}}{\overset{\mathring{\operatorname{rot}}}{\rightleftarrows}} L^2 \underset{-\nabla}{\overset{\mathring{\operatorname{div}}}{\rightleftarrows}} L^2 \, \cdots$ or $\cdots \, L^2 \underset{-\delta}{\overset{\mathring{d}}{\rightleftarrows}} L^2 \underset{-\delta}{\overset{\mathring{d}}{\rightleftarrows}} L^2 \, \cdots$ (de Rham complex)
- $\cdots \, L^2 \underset{-\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\operatorname{sym}\nabla}}{\rightleftarrows}} L^2_{\mathbb{S}} \underset{\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S}}}{\overset{\mathring{\operatorname{Rot}\operatorname{Rot}}{}^\top_{\mathbb{S}}}{\rightleftarrows}} L^2_{\mathbb{S}} \underset{-\operatorname{sym}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{S}}}{\rightleftarrows}} L^2 \, \cdots$ (elasticity complex)
- $\cdots \, L^2 \underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\nabla\nabla}}{\rightleftarrows}} L^2_{\mathbb{S}} \underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}} L^2_{\mathbb{T}} \underset{-\operatorname{dev}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}} L^2 \, \cdots$ (biharmonic/general relativity complex)
- $\cdots \, H_0 \underset{A_0^*}{\overset{A_0}{\rightleftarrows}} H_1 \underset{A_1^*}{\overset{A_1}{\rightleftarrows}} H_2 \, \cdots$ (. . . much more complexes)

(III) numerical applications to pdes

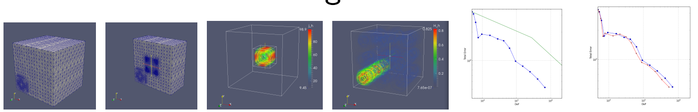


- Friedrichs/Poincaré/Maxwell constants (computations with FEM and with colleagues from Prag, Wien)

FEM



- functional a posteriori error estimates for BEM (computations with BEM and FEM and with colleagues from Darmstadt Wien, St. Petersburg, Bosch GmbH)



- functional a posteriori error estimates for electro-magneto static optimal control problems (computations with FEM and with colleagues from Essen)
- DEC: Discrete Exterior Calculus as discrete version of FA-ToolBox (Jyväskylä Group) $\rightsquigarrow$ `https://sites.google.com/jyu.fi/gfd/method/online-time-integrator`

Solving PDEs with Hilbert Complexes

Introduction and Motivation

Solving PDEs with Hilbert Complexes

FA-ToolBox

general observations

$$Ax = f$$

general observations

$$\mathrm{A}x = f$$

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ (lin, dd, cl) and $\mathrm{H}_0, \mathrm{H}_1$ Hilbert spaces

question: How to solve?

$$??? \qquad x = \mathrm{A}^{-1} f \qquad ???$$

general observations

$$\mathrm{A}x = f$$

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ (lin, dd, cl)

solution theory in the sense of Hadamard

- existence $\Leftrightarrow$ $f \in R(\mathrm{A})$
- uniqueness $\Leftrightarrow$ A inj $\Leftrightarrow$ $N(\mathrm{A}) = \{0\}$ $\Leftrightarrow$ A^{-1} exists
- cont dep on f $\Leftrightarrow$ A^{-1} cont

$\Rightarrow$ $x = \mathrm{A}^{-1} f \in D(\mathrm{A})$ and cont estimate (Friedrichs/Poincaré type estimate)

$$|x|_{\mathrm{H}_0} = |\mathrm{A}^{-1} f|_{\mathrm{H}_0} \leq c_{\mathrm{A}} |f|_{\mathrm{H}_1} = c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$$

$\Rightarrow$ best constant $c_{\mathrm{A}} = |\mathrm{A}^{-1}|_{R(\mathrm{A}),\mathrm{H}_0}$

general observations

$A : D(A) \subset H_0 \to H_1$ lin, dd, cl.

$A^* : D(A^*) \subset H_1 \to H_0$ Hilbert space adjoint

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$H_1 = \overline{R(A)} \oplus N(A^*), \qquad H_0 = N(A) \oplus \overline{R(A^*)}$$

$$Ax = f$$

solution theory in the sense of Hadamard

- existence $\Leftrightarrow$ $f \in R(A) = N(A^*)^\perp$ (Fredholm alt, if $R(A)$ cl)
- uniqueness $\Leftrightarrow$ A inj $\Leftrightarrow$ $N(A) = \{0\}$ $\Leftrightarrow$ A^{-1} exists
- cont dep on f $\Leftrightarrow$ A^{-1} cont $\Leftrightarrow$ $R(A)$ cl (cl graph theo)

fund range cond: $\boxed{R(A) = \overline{R(A)} \text{ closed}}$ (must hold $\rightsquigarrow$ right setting!)

kernel cond: $\boxed{N(A) = \{0\}}$ (fails in gen $\rightsquigarrow$ proj onto $N(A)^\perp = \overline{R(A^*)} = R(A^*)$)

general observations

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$\mathrm{H}_1 = \overline{R(\mathrm{A})} \oplus N(\mathrm{A}^*) \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

observations (from this perspective)

- **time-dependent problems** are simple

  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H} \to \mathrm{H}$, $\boxed{\mathrm{A} = \partial_t + \mathrm{T}}$ (gen T skw-sa, or at least Re $\mathrm{T} \geq 0$)

  $$N(\mathrm{A}) = \{0\} \qquad N(\mathrm{A}^*) = \{0\} \qquad R(\mathrm{A}) \text{ (cl) } = N(\mathrm{A}^*)^{\perp} = \mathrm{H}$$

- **time-harmonic problems** are more complicated

  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H} \to \mathrm{H}$, $\boxed{\mathrm{A} = -\omega + \mathrm{T}}$

  $$N(\mathrm{A}),\ N(\mathrm{A}^*) \text{ (fin dim)} \qquad R(\mathrm{A}) \text{ (cl, fin co-dim) } = N(\mathrm{A}^*)^{\perp}$$

  (Fredholm alternative)

- **static problems** are most complicated

  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$, $\boxed{\mathrm{A} = 0 + \mathrm{T}}$

  $\dim N(\mathrm{A}) = \dim N(\mathrm{A}^*) = \infty$ (possible/standard) $\quad R(\mathrm{A})$ (cl, infin co-dim) $= N(\mathrm{A}^*)^{\perp}$

FA-ToolBox for linear (first order) problems/systems

$$\boxed{Ax = f}$$

general theory

- solution theory
- closed ranges
- Friedrichs/Poincaré estimates and results about constants
- Helmholtz/Hodge/Weyl decompositions
- compact embeddings
- continuous and compact inverse operators
- regular potentials and regular decompositions (to show compact embeddings)
- variational formulations
- generalized div-curl-lemma
- index theorems
- dimensions and bases of cohomology groups
- functional a posteriori error estimates
- . . .

idea: solve problem with general and simple lin fa ($\Rightarrow$ FA-ToolBox) ...

literature: many parts probably very well known for ages, but hard to find ...

(Friedrichs, Weyl, Hörmander, Fredholm, von Neumann, Riesz, Banach, ... ?)

Why not rediscover and extend/modify for our purposes?

1st fundamental observations

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ lddc, $\mathrm{A}^* : D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0$ Hilbert space adjoint

$(\mathrm{A}, \mathrm{A}^*)$ dual pair as $(\mathrm{A}^*)^* = \overline{\mathrm{A}} = \mathrm{A}$

A, A^* may not be inj

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$\mathrm{H}_1 = N(\mathrm{A}^*) \oplus \overline{R(\mathrm{A})} \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

reduced operators restr to $N(\mathrm{A})^\perp$ and $N(\mathrm{A}^*)^\perp$

$$\boxed{\mathcal{A} := \mathrm{A}|_{N(\mathrm{A})^\perp} = \mathrm{A}|_{\overline{R(\mathrm{A}^*)}}} \qquad \boxed{\mathcal{A}^* := \mathrm{A}^*|_{N(\mathrm{A}^*)^\perp} = \mathrm{A}^*|_{\overline{R(\mathrm{A})}}}$$

$\mathcal{A}$, $\mathcal{A}^*$ inj $\quad \Rightarrow \quad \mathcal{A}^{-1}$, $(\mathcal{A}^*)^{-1}$ ex

1st fundamental observations

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1, \quad \mathrm{A}^* : D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0$ lddc $\qquad (\mathrm{A}, \mathrm{A}^*)$ dual pair

$$\mathrm{H}_1 = N(\mathrm{A}^*) \oplus \overline{R(\mathrm{A})} \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

more precisely

$$\mathcal{A} := \mathrm{A}|_{\overline{R(\mathrm{A}^*)}} : D(\mathcal{A}) \subset \overline{R(\mathrm{A}^*)} \to \overline{R(\mathrm{A})}, \qquad D(\mathcal{A}) := D(\mathrm{A}) \cap N(\mathrm{A})^{\perp} = D(\mathrm{A}) \cap \overline{R(\mathrm{A}^*)}$$
$$\mathcal{A}^* := \mathrm{A}^*|_{\overline{R(\mathrm{A})}} : D(\mathcal{A}^*) \subset \overline{R(\mathrm{A})} \to \overline{R(\mathrm{A}^*)}, \quad D(\mathcal{A}^*) := D(\mathrm{A}^*) \cap N(\mathrm{A}^*)^{\perp} = D(\mathrm{A}^*) \cap \overline{R(\mathrm{A})}$$

$(\mathcal{A}, \mathcal{A}^*)$ dual pair and $\mathcal{A}$, $\mathcal{A}^*$ inj $\Rightarrow$

inverse ops exist (and bij)

$$\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A}) \qquad (\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$$

refined decompositions

$$D(\mathrm{A}) = N(\mathrm{A}) \oplus D(\mathcal{A}) \qquad D(\mathrm{A}^*) = N(\mathrm{A}^*) \oplus D(\mathcal{A}^*)$$

$\Rightarrow$

$$R(\mathrm{A}) = R(\mathcal{A}) \qquad R(\mathrm{A}^*) = R(\mathcal{A}^*)$$

1st fundamental observations

closed range theorem & closed graph theorem $\Rightarrow$

Lemma (Friedrichs-Poincaré type est/cl range/cont inv)

The following assertions are equivalent:

- **(i)** $\exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathrm{H}_0} \leq c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$
- **(i*)** $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$
- **(ii)** $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_1.
- **(ii*)** $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_0.
- **(iii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is continuous and bijective.*
- **(iii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is continuous and bijective.*

Note: trivial equivalence of (i) or (i*) to inf-sup condition

1st fundamental observations

recall

$$\begin{aligned} &\textbf{(i)} \quad \exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \quad |x|_{\mathrm{H}_0} \le c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1} \\ &\textbf{(i}^*\textbf{)} \quad \exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \quad |y|_{\mathrm{H}_1} \le c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0} \end{aligned}$$

'best' consts in **(i)** and **(i*)** equal norms of the inv ops and Rayleigh quotients

$$c_{\mathrm{A}} = |\mathcal{A}^{-1}|_{R(\mathrm{A}),R(\mathrm{A}^*)} \qquad c_{\mathrm{A}^*} = |(\mathcal{A}^*)^{-1}|_{R(\mathrm{A}^*),R(\mathrm{A})}$$

$$\lambda_{\mathrm{A}} = \frac{1}{c_{\mathrm{A}}} = \inf_{0 \ne x \in D(\mathcal{A})} \frac{|\mathrm{A}x|_{\mathrm{H}_1}}{|x|_{\mathrm{H}_0}} \qquad \lambda_{\mathrm{A}^*} = \frac{1}{c_{\mathrm{A}^*}} = \inf_{0 \ne y \in D(\mathcal{A}^*)} \frac{|\mathrm{A}^* y|_{\mathrm{H}_0}}{|y|_{\mathrm{H}_1}}$$

Lemma (Friedrichs-Poincaré type const)

$c_{\mathrm{A}} = c_{\mathrm{A}^*}$

Remark (spectrum)

Even whole spectrum coincides, i.e.,
$\sigma(\mathrm{A}^*\mathrm{A})\setminus\{0\} = \sigma(\mathcal{A}^*\mathcal{A}) = \sigma(\mathcal{A}\mathcal{A}^*) = \sigma(\mathrm{A}\mathrm{A}^*)\setminus\{0\}$

1st fundamental observations

Lemma (Friedrichs-Poincaré type est/cl range/cont inv)

$\Downarrow$ $D(\mathcal{A}) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_0$ *compact*

- **(i)** $\exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathrm{H}_0} \le c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$
- **(i*)** $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathrm{H}_1} \le c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$
- **(ii)** $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_1.
- **(ii*)** $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_0.
- **(iii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is continuous and bijective.*
- **(iii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is continuous and bijective.*

(i)-(iii*) equi & the resp Helm deco hold & $|\mathcal{A}^{-1}| = c_{\mathrm{A}} = c_{\mathrm{A}^*} = |(\mathcal{A}^*)^{-1}|$

Lemma (cpt emb/cpt inv)

The following assertions are equivalent:

- **(i)** $D(\mathcal{A}) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_0$ *is compact.*
- **(i*)** $D(\mathcal{A}^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$ *is compact.*
- **(ii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to R(\mathrm{A}^*)$ *is compact.*
- **(ii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to R(\mathrm{A})$ *is compact.*

2nd fundamental observations

So far no complex...

$$A_0 : D(A_0) \subset H_0 \to H_1, \quad A_1 : D(A_1) \subset H_1 \to H_2 \text{ (lddc)}$$
$$A_0^* : D(A_0^*) \subset H_1 \to H_0, \quad A_1^* : D(A_1^*) \subset H_2 \to H_1 \text{ (lddc)}$$

general complex ($\boxed{A_1 A_0 = 0}$, i.e., $R(A_0) \subset N(A_1)$ and $R(A_1^*) \subset N(A_0^*)$)

$$\boxed{\cdots \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad H_0 \quad \underset{A_0^*}{\overset{A_0}{\rightleftarrows}} \quad H_1 \quad \underset{A_1^*}{\overset{A_1}{\rightleftarrows}} \quad H_2 \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad \cdots}$$

recall Helmholtz deco

$$\begin{aligned} H_1 &= \overline{R(A_0)} \oplus N(A_0^*) \\ &\quad\ \ \bigcap \qquad\quad \bigcup \\ &= N(A_1) \oplus \overline{R(A_1^*)} \end{aligned} \qquad \Rightarrow \text{ (e.g.)} \qquad N(A_1) = \overline{R(A_0)} \oplus \big(\underbrace{N(A_1) \cap N(A_0^*)}_{=:N_1} \big)$$

$\Rightarrow$ refined Helmholtz deco

$$\boxed{H_1 = \overline{R(A_0)} \oplus N_1 \oplus \overline{R(A_1^*)}}$$

2nd fundamental observations

$$N_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \qquad D(\mathrm{A}_1) = D(\mathcal{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \qquad D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \cap \overline{R(\mathrm{A}_0)}$$

Lemma (cpt emb II)

The following assertions are equivalent:

(i) $D(\mathcal{A}_0) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_0$, $D(\mathcal{A}_1) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1$, *and* $N_1 \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1$ *are compact.*

(ii) $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1$ *is compact.*

In this case $N_1 < \infty$.

Theorem (FA-ToolBox I)

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1 \text{ compact}}$

(i) *all emb cpt, i.e.,* $D(\mathcal{A}_0) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_0$, $D(\mathcal{A}_1) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1$, $D(\mathcal{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1$, $D(\mathcal{A}_1^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_2$ *cpt*

(ii) *cohomology group* N_1 *finite dim*

(iii) *all ranges closed, i.e.,* $R(\mathrm{A}_0)$, $R(\mathrm{A}_0^*)$, $R(\mathrm{A}_1)$, $R(\mathrm{A}_1^*)$ *cl*

(iv) *all Friedrichs-Poincaré type est hold*

(v) *all Hodge-Helmholtz-Weyl type deco I & II hold with closed ranges*

2nd fundamental observations

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

Theorem (FA-ToolBox I (Friedrichs-Poincaré type est))

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1 \text{ compact}}$ $\Rightarrow$ $\exists$ $|\mathcal{A}_i^{-1}| = c_{\mathrm{A}_i} = c_{\mathrm{A}_i^*} = |(\mathcal{A}_i^*)^{-1}| \in (0, \infty)$

(i) $\forall\, x \in D(\mathcal{A}_0)$ $\quad |x|_{\mathrm{H}_0} \leq c_{\mathrm{A}_0} |\mathrm{A}_0 x|_{\mathrm{H}_1}$

(i*) $\forall\, y \in D(\mathcal{A}_0^*)$ $\quad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}_0} |\mathrm{A}_0^* y|_{\mathrm{H}_0}$

(ii) $\forall\, y \in D(\mathcal{A}_1)$ $\quad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |\mathrm{A}_1 y|_{\mathrm{H}_2}$

(ii*) $\forall\, z \in D(\mathcal{A}_1^*)$ $\quad |z|_{\mathrm{H}_2} \leq c_{\mathrm{A}_1} |\mathrm{A}_1^* z|_{\mathrm{H}_1}$

(iii) $\forall\, y \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ $\quad |(1 - \pi_{N_1}) y|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |\mathrm{A}_1 y|_{\mathrm{H}_2} + c_{\mathrm{A}_0} |\mathrm{A}_0^* y|_{\mathrm{H}_0}$

note $\pi_{N_1} y \in N_1$ and $(1 - \pi_{N_1}) y \in N_1^{\perp}$

Remark

enough $R(\mathrm{A}_0)$ and $R(\mathrm{A}_1)$ cl

Solving PDEs with Hilbert Complexes

(Static) First Order Systems

(stat) first order system - solution theory

complex $\quad \cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

$$\boxed{\mathrm{A}_1 x = f} \qquad \dim N(\mathrm{A}_1) = \infty$$

find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ such that the fos

$$\begin{aligned} \mathrm{A}_1 x &= f \\ \mathrm{A}_0^* x &= g \\ \pi_{N_1} x &= k \end{aligned} \qquad \text{think of} \qquad \begin{aligned} (\mathring{\operatorname{rot}} E &= F) \\ (-\operatorname{div} E &= g) \\ (\pi_{\mathrm{D}} E &= K) \end{aligned}$$

kernel = cohomology group = $N_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)$

trivially necessary $\quad f \in R(\mathrm{A}_1) \quad g \in R(\mathrm{A}_0^*) \quad k \in N_1$

apply FA-ToolBox

(stat) first order system - solution theory

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ st fos $\boxed{\mathrm{A}_1 x = f \qquad \mathrm{A}_0^* x = g \qquad \pi_{N_1} x = k}$

Theorem (FA-ToolBox II (solution theory))

$\Downarrow \quad \boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1 \textit{ compact}}$

fos is uniq sol $\Leftrightarrow \quad f \in R(\mathrm{A}_1) \qquad g \in R(\mathrm{A}_0^*) \qquad k \in N_1$

$$x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus N_1 = D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$$

$$\boxed{x_f := \mathcal{A}_1^{-1} f} \in D(\mathcal{A}_1)$$

$$\boxed{x_g := (\mathcal{A}_0^*)^{-1} g} \in D(\mathcal{A}_0^*)$$

dep cont on data $|x|_{\mathrm{H}_1} \leq |x_f|_{\mathrm{H}_1} + |x_g|_{\mathrm{H}_1} + |k|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |f|_{\mathrm{H}_2} + c_{\mathrm{A}_0} |g|_{\mathrm{H}_0} + |k|_{\mathrm{H}_1}$

moreover

$$\pi_{R(\mathrm{A}_1^*)} x = x_f \qquad \pi_{R(\mathrm{A}_0)} x = x_g \qquad \pi_{N_1} x = k \qquad |x|_{\mathrm{H}_1}^2 = |x_f|_{\mathrm{H}_1}^2 + |x_g|_{\mathrm{H}_1}^2 + |k|_{\mathrm{H}_1}^2$$

Remark

enough $R(\mathrm{A}_0)$ *and* $R(\mathrm{A}_1)$ *cl*

(stat) first order system - a posteriori error estimates

problem: $\boxed{\text{find} \quad x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \quad \text{st} \quad \mathrm{A}_1 x = f \quad \mathrm{A}_0^* x = g \quad \pi_{N_1} x = k}$

'very' non-conforming 'approximation' of x: $\boxed{\tilde{x} \in \mathrm{H}_1}$

def., dcmp. err. $\boxed{e = x - \tilde{x}} = \pi_{R(\mathrm{A}_0)} e + \pi_{N_1} e + \pi_{R(\mathrm{A}_1^*)} e \in \mathrm{H}_1 = R(\mathrm{A}_0) \oplus N_1 \oplus R(\mathrm{A}_1^*)$

Theorem (sharp upper bounds)

Let $\tilde{x} \in \mathrm{H}_1$ *and* $e = x - \tilde{x}$. *Then*

$$|e|_{\mathrm{H}_1}^2 = |\pi_{R(\mathrm{A}_0)} e|_{\mathrm{H}_1}^2 + |\pi_{N_1} e|_{\mathrm{H}_1}^2 + |\pi_{R(\mathrm{A}_1^*)} e|_{\mathrm{H}_1}^2$$

$$|\pi_{R(\mathrm{A}_0)} e|_{\mathrm{H}_1} = \min_{\phi \in D(\mathrm{A}_0^*)} \big(c_{\mathrm{A}_0} |\mathrm{A}_0^* \phi - g|_{\mathrm{H}_0} + |\phi - \tilde{x}|_{\mathrm{H}_1} \big)$$

$\boxed{\textit{reg } (\mathrm{A}_0 \mathrm{A}_0^* + 1)\textit{-prbl in } D(\mathrm{A}_0^*)}$

$$|\pi_{R(\mathrm{A}_1^*)} e|_{\mathrm{H}_1} = \min_{\varphi \in D(\mathrm{A}_1)} \big(c_{\mathrm{A}_1} |\mathrm{A}_1 \varphi - f|_{\mathrm{H}_2} + |\varphi - \tilde{x}|_{\mathrm{H}_1} \big)$$

$\boxed{\textit{reg } (\mathrm{A}_1^* \mathrm{A}_1 + 1)\textit{-prbl in } D(\mathrm{A}_1)}$

$$|\pi_{N_1} e|_{\mathrm{H}_1} = |\pi_{N_1} \tilde{x} - k|_{\mathrm{H}_1} = \min_{\substack{\xi \in D(\mathrm{A}_0) \\ \zeta \in D(\mathrm{A}_1^*)}} |\mathrm{A}_0 \xi + \mathrm{A}_1^* \zeta + \tilde{x} - k|_{\mathrm{H}_1}$$

$\boxed{\textit{cpld } (\mathrm{A}_0^* \mathrm{A}_0)\text{-}(\mathrm{A}_1 \mathrm{A}_1^*)\textit{-sys in } D(\mathcal{A}_0)\text{-}D(\mathcal{A}_1^*)}$

Remark

Even $\pi_{N_1} e = k - \pi_{N_1} \tilde{x}$ *and the minima are attained at*

$$\hat{\phi} = \pi_{R(\mathrm{A}_0)} e + \tilde{x}, \qquad \hat{\varphi} = \pi_{R(\mathrm{A}_1^*)} e + \tilde{x}, \qquad \mathrm{A}_0 \hat{\xi} + \mathrm{A}_1^* \hat{\zeta} = (\pi_{N_1} - 1)\tilde{x}.$$

A_0^*-A_1-lemma (generalized global div-curl-lemma)

Lemma (A_0^*-A_1-lemma)

Let $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathrm{H}_1$ *be compact, and*

(i) (x_n) *bounded in* $D(\mathrm{A}_1)$,

(ii) (y_n) *bounded in* $D(\mathrm{A}_0^*)$.

$\Rightarrow \exists\ x \in D(\mathrm{A}_1),\ y \in D(\mathrm{A}_0^*)$ *and subsequences st*

$x_n \rightharpoonup x$ *in* $D(\mathrm{A}_1)$ *and* $y_n \rightharpoonup y$ *in* $D(\mathrm{A}_0^*)$ *as well as*

$$\langle x_n, y_n \rangle_{\mathrm{H}_1} \to \langle x, y \rangle_{\mathrm{H}_1}.$$

A_0^*-A_1-lemma (generalized global div-curl-lemma)

Lemma (generalized A_0^*-A_1-lemma)

Let $R(\mathrm{A}_0)$ and $R(\mathrm{A}_1$ be closed, and let N_1 be finite dimensional. Moreover, let $(x_n), (y_n) \subset \mathrm{H}_1$ be bounded such that

- **(i)** $(\widetilde{\mathrm{A}}_1 x_n)$ *is relatively compact in* $D(\mathrm{A}_1^*)'$,
- **(ii)** $(\widetilde{\mathrm{A}}_0^* y_n)$ *is relatively compact in* $D(\mathrm{A}_0)'$.

$\Rightarrow \exists\, x, y \in \mathrm{H}_1$ *and subsequences st* $x_n \rightharpoonup x$ *in* H_1 *and* $y_n \rightharpoonup y$ *in* H_1 *as well as*

$$\langle x_n, y_n\rangle_{\mathrm{H}_1} \to \langle x, y\rangle_{\mathrm{H}_1}.$$

proof uses key observation

Lemma

Let $R(\mathrm{A})$ be closed. For $(x_n) \subset \mathrm{H}_0$ the following statements are equivalent:

- **(i)** $(\widetilde{\mathrm{A}} x_n)$ *is relatively compact in* $D(\mathrm{A}^*)'$.
- **(ii)** $(\pi_{R(\mathrm{A}^*)} x_n)$ *is relatively compact in* $R(\mathrm{A}^*)$ *resp.* H_1.

If $x_n \rightharpoonup x$ in H_1, then either of cond. (i) or (ii) implies $\pi_{R(\mathrm{A}^*)} x_n \to \pi_{R(\mathrm{A}^*)} x$ in H_1.

nice results and joint work with Marcus Waurick

Solving PDEs with Hilbert Complexes

Applications: FOS & SOS (First and Second Order Systems)

classical de Rham complex in 3D (∇-rot-div-complex)

general complex property $A_1 A_0 = 0$, i.e., $\boxed{R(A_0) \subset N(A_1)}$

$$\cdots \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad H_0 \quad \underset{A_0^*}{\overset{A_0}{\rightleftarrows}} \quad H_1 \quad \underset{A_1^*}{\overset{A_1}{\rightleftarrows}} \quad H_2 \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad \cdots$$

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot\cup \Gamma_n}$

(electro-magneto dynamics, Maxwell's equations)

$$\{0\} \quad \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \quad L^2 \quad \underset{-\operatorname{div}}{\overset{\mathring{\nabla}}{\rightleftarrows}} \quad L^2 \quad \underset{\operatorname{rot}}{\overset{\mathring{\operatorname{rot}}}{\rightleftarrows}} \quad L^2 \quad \underset{-\nabla}{\overset{\mathring{\operatorname{div}}}{\rightleftarrows}} \quad L^2 \quad \underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}} \quad \mathbb{R}$$

mixed boundary conditions and inhomogeneous and anisotropic media

$$\{0\} \text{ or } \mathbb{R} \quad \underset{\pi}{\overset{\iota}{\rightleftarrows}} \quad L^2 \quad \underset{-\operatorname{div}_{\Gamma_n} \varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}} \quad L^2_{\varepsilon} \quad \underset{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}} \quad L^2 \quad \underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}} \quad L^2 \quad \underset{\iota}{\overset{\pi}{\rightleftarrows}} \quad \mathbb{R} \text{ or } \{0\}$$

classical de Rham complex in 3D (∇-rot-div-complex)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot\cup \Gamma_n}$

(electro-magneto dynamics, Maxwell's equations with mixed boundary conditions)

$$\{0\} \text{ or } \mathbb{R} \;\underset{\pi}{\overset{\iota}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\operatorname{div}_{\Gamma_n}\varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2_\varepsilon \;\underset{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota}{\overset{\pi}{\rightleftarrows}}\; \mathbb{R} \text{ or } \{0\}$$

related fos

$$\begin{array}{llllllllllll} \nabla_{\Gamma_t} u = A & \text{in } \Omega & | & \operatorname{rot}_{\Gamma_t} E = J & \text{in } \Omega & | & \operatorname{div}_{\Gamma_t} H = k & \text{in } \Omega & | & \pi v = b & \text{in } \Omega \\ \pi u = a & \text{in } \Omega & | & -\operatorname{div}_{\Gamma_n} \varepsilon E = j & \text{in } \Omega & | & \varepsilon^{-1}\operatorname{rot}_{\Gamma_n} H = K & \text{in } \Omega & | & -\nabla_{\Gamma_n} v = B & \text{in } \Omega \end{array}$$

related sos

$$\begin{array}{llllllll} -\operatorname{div}_{\Gamma_n} \varepsilon \nabla_{\Gamma_t} u = j & \text{in } \Omega & | & \varepsilon^{-1}\operatorname{rot}_{\Gamma_n}\operatorname{rot}_{\Gamma_t} E = K & \text{in } \Omega & | & -\nabla_{\Gamma_n}\operatorname{div}_{\Gamma_t} H = B & \text{in } \Omega \\ \pi u = a & \text{in } \Omega & | & -\operatorname{div}_{\Gamma_n} \varepsilon E = j & \text{in } \Omega & | & \varepsilon^{-1}\operatorname{rot}_{\Gamma_n} H = K & \text{in } \Omega \end{array}$$

corresponding compact embeddings:

$$D(\nabla_{\Gamma_t}) \cap D(\pi) = D(\nabla_{\Gamma_t}) = \mathsf{H}^1_{\Gamma_t} \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

$$D(\operatorname{rot}_{\Gamma_t}) \cap D(-\operatorname{div}_{\Gamma_n}\varepsilon) = \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n} \hookrightarrow \mathsf{L}^2_\varepsilon \qquad \text{(Weck's selection theorem, '74)}$$

$$D(\operatorname{div}_{\Gamma_t}) \cap D(\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}) = \mathsf{D}_{\Gamma_t} \cap \mathsf{R}_{\Gamma_n} \hookrightarrow \mathsf{L}^2 \qquad \text{(Weck's selection theorem, '74)}$$

$$D(\nabla_{\Gamma_n}) \cap D(\pi) = D(\nabla_{\Gamma_n}) = \mathsf{H}^1_{\Gamma_n} \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

Weck's selection theorem for weak Lip. dom. and mixed bc: Bauer/Py/Schomburg ('16)

Weck's selection theorem (Weck '74, (Habil. '72) stimulated by Rolf Leis)
(Weber '80, Picard '84, Costabel '90, Witsch '93, Jochmann '97, Kuhn '99, Picard/Weck/Witsch '01, Py '96, '03, '06, '07, '08)

classical de Rham complex in 3D (∇-rot-div-complex)

$$\begin{aligned} \operatorname{rot} E &= F && \text{in } \Omega \\ -\operatorname{div} \varepsilon E &= g && \text{in } \Omega \\ \nu \times E &= 0 && \text{at } \Gamma_t \\ \nu \cdot \varepsilon E &= 0 && \text{at } \Gamma_n \end{aligned}$$

non-trivial kernel $\mathcal{H}_{\mathtt{D},\varepsilon} = \{H \in \mathsf{L}^2 : \operatorname{rot} H = 0, \operatorname{div} \varepsilon H = 0, \nu \times H|_{\Gamma_t} = 0, \nu \cdot \varepsilon H|_{\Gamma_n} = 0\}$
additional condition on Dirichlet/Neumann fields for uniqueness

$$\pi_{\mathtt{D}} E = K \in \mathcal{H}_{\mathtt{D},\varepsilon}$$

$$\{0\} \text{ or } \mathbb{R} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{div}_{\Gamma_n} \varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_\varepsilon \underset{\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathbb{R} \text{ or } \{0\}$$

$$\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathsf{H}_{-1} \underset{\mathsf{A}^*_{-1}}{\overset{\mathsf{A}_{-1}}{\rightleftarrows}} \mathsf{H}_0 \underset{\mathsf{A}^*_0}{\overset{\mathsf{A}_0}{\rightleftarrows}} \mathsf{H}_1 \underset{\mathsf{A}^*_1}{\overset{\mathsf{A}_1}{\rightleftarrows}} \mathsf{H}_2 \underset{\mathsf{A}^*_2}{\overset{\mathsf{A}_2}{\rightleftarrows}} \mathsf{H}_3 \underset{\mathsf{A}^*_3}{\overset{\mathsf{A}_3}{\rightleftarrows}} \mathsf{H}_4 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$$

find $E \in \mathsf{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n}(\Omega)$ st (fos) find $x \in D(\mathsf{A}_1) \cap D(\mathsf{A}^*_0)$ st

$$\begin{aligned} \operatorname{rot}_{\Gamma_t} E &= F & \qquad & & \mathsf{A}_1 x &= f \\ -\operatorname{div}_{\Gamma_n} \varepsilon E &= g & \qquad & \text{translation} & \mathsf{A}^*_0 x &= g \\ \pi_{\mathsf{D/N}} E &= K & \qquad & & \pi_{N_1} x &= k \end{aligned}$$

classical de Rham complex in 3D (∇-rot-div-complex)

$c_{A_0} = c_{fp}$ (Friedrichs/Poincaré constant) and $c_{A_1} = c_m$ (Maxwell constant)

Lemma/Theorem $\Downarrow$ $\boxed{D(A_1) \cap D(A_0^*) \hookrightarrow\!\!\!\rightarrow L^2_\varepsilon(\Omega) \text{ compact}}$

(i) all Friedrichs-Poincaré type est hold

$$
\begin{aligned}
&\forall\, \varphi \in D(\mathcal{A}_0) && |\varphi|_{H_0} \le c_{A_0} |A_0 \varphi|_{H_1} && \Leftrightarrow && \forall\, \varphi \in H^1_{\Gamma_t} && |\varphi|_{L^2} \le c_{fp} |\nabla \varphi|_{L^2_\varepsilon} \\
&\forall\, \phi \in D(\mathcal{A}_0^*) && |\phi|_{H_1} \le c_{A_0} |A_0^* \phi|_{H_0} && \Leftrightarrow && \forall\, \Phi \in \varepsilon^{-1} D_{\Gamma_n} \cap \nabla H^1_{\Gamma_t} && |\Phi|_{L^2_\varepsilon} \le c_{fp} |\operatorname{div} \varepsilon \Phi|_{L^2} \\
&\forall\, \phi \in D(\mathcal{A}_1) && |\phi|_{H_1} \le c_{A_1} |A_1 \phi|_{H_2} && \Leftrightarrow && \forall\, \Phi \in R_{\Gamma_t} \cap \varepsilon^{-1} \operatorname{rot} R_{\Gamma_n} && |\Phi|_{L^2_\varepsilon} \le c_m |\operatorname{rot} \Phi|_{L^2} \\
&\forall\, \psi \in D(\mathcal{A}_1^*) && |\psi|_{H_2} \le c_{A_1} |A_1^* \psi|_{H_1} && \Leftrightarrow && \forall\, \Psi \in R_{\Gamma_n} \cap \operatorname{rot} R_{\Gamma_t} && |\Psi|_{L^2} \le c_m |\operatorname{rot} \Psi|_{L^2_\varepsilon}
\end{aligned}
$$

(ii) all ranges $R(A_0) = \nabla H^1_{\Gamma_t}$, $R(A_1) = \operatorname{rot} R_{\Gamma_t}$, $R(A_0^*) = \operatorname{div} D_{\Gamma_n}$ are cl in L^2

(iii) the inverse ops $(\widetilde{\nabla}_{\Gamma_t})^{-1}$, $(\widetilde{\operatorname{div}}_{\Gamma_n}\varepsilon)^{-1}$, $(\widetilde{\operatorname{rot}}_{\Gamma_t})^{-1}$, $(\widetilde{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}})^{-1}$ are cont, even cpt

(iv) all Helmholtz decomposition hold, e.g.,

$$H_1 = R(A_0) \oplus N_1 \oplus R(A_1^*) \quad \Leftrightarrow \quad L^2_\varepsilon = \nabla H^1_{\Gamma_t} \oplus_{L^2_\varepsilon} \mathcal{H}_{D,\varepsilon} \oplus_{L^2_\varepsilon} \varepsilon^{-1} \operatorname{rot} R_{\Gamma_n}$$

(v) solution theory

(vi) ...

classical de Rham complex in 3D (∇-rot-div-complex)

find $E \in \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n}$ s.t. / think of $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$

$$\begin{aligned} \operatorname{rot}_{\Gamma_t} E &= F \\ \operatorname{div}_{\Gamma_n} \varepsilon E &= g \\ \pi_{\mathcal{H}_{\mathsf{D},\varepsilon}} E &= K \end{aligned} \qquad / \qquad \text{think of} \qquad \begin{aligned} \mathrm{A}_1 x &= f \\ \mathrm{A}_0^* x &= g \\ \pi_{K_1} x &= k \end{aligned}$$

sol is simply $x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus K_1 = D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$

with $\boxed{x_f := \mathcal{A}_1^{-1} f} \in D(\mathcal{A}_1)$ and $\boxed{x_g := (\mathcal{A}_0^*)^{-1} g} \in D(\mathcal{A}_0^*)$

i.e., $E = E_F + E_g + K,$ where

$$\boxed{E_F := (\widetilde{\operatorname{rot}}_{\Gamma_t})^{-1} F} \in D(\widetilde{\operatorname{rot}}_{\Gamma_t}) = \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1} \operatorname{rot} \mathsf{R}_{\Gamma_n} = \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n,0} \cap \mathcal{H}_{\mathsf{D},\varepsilon}^{\perp},$$

$$\boxed{E_g := (\widetilde{\operatorname{div}}_{\Gamma_n}\varepsilon)^{-1} g} \in D(\widetilde{\operatorname{div}}_{\Gamma_n}\varepsilon) = \varepsilon^{-1}\mathsf{D}_{\Gamma_n} \cap \nabla \mathsf{H}^1_{\Gamma_t} = \varepsilon^{-1}\mathsf{D}_{\Gamma_n} \cap \mathsf{R}_{\Gamma_t,0} \cap \mathcal{H}_{\mathsf{D},\varepsilon}^{\perp}$$

classical de Rham complex in 3D (∇-rot-div-complex)

Theorem (sharp upper bounds)

Let $\boxed{\tilde{E} \in \mathrm{L}^2_\varepsilon}$ *(very non-conforming approximation of* E*!) and* $\boxed{e := E - \tilde{E}}$*. Then*

$$
\begin{aligned}
|e|^2_{\mathrm{L}^2_\varepsilon} &= |\pi_{R(\nabla_{\Gamma_t})} e|^2_{\mathrm{L}^2_\varepsilon} + |\pi_{R(\varepsilon^{-1}\operatorname{rot}_{\Gamma_n})} e|^2_{\mathrm{L}^2_\varepsilon} + |\pi_{\mathcal{H}_{\mathrm{D},\varepsilon}} e|^2_{\mathrm{L}^2_\varepsilon} \\
&= \min_{\Phi \in \varepsilon^{-1}\mathrm{D}_{\Gamma_n}} \big(c_{\mathrm{fp}} |\operatorname{div} \varepsilon\Phi + g|_{\mathrm{L}^2} + |\Phi - \tilde{E}|_{\mathrm{L}^2_\varepsilon} \big)^2 \\
&\qquad + \min_{\Phi \in \mathrm{R}_{\Gamma_t}} \big(c_{\mathrm{m}} |\operatorname{rot}\Phi - F|_{\mathrm{L}^2} + |\Phi - \tilde{E}|_{\mathrm{L}^2_\varepsilon} \big)^2 \\
&\qquad + \min_{\phi \in \mathrm{H}^1_{\Gamma_t}, \Psi \in \mathrm{R}_{\Gamma_n}} |\nabla\phi + \varepsilon^{-1}\operatorname{rot}\Psi + \tilde{E} - K|^2_{\mathrm{L}^2_\varepsilon}
\end{aligned}
$$

reg $(-\nabla_{\Gamma_t}\operatorname{div}_{\Gamma_n} + 1)$*-prbl in* D_{Γ_n}

reg $(\operatorname{rot}_{\Gamma_n}\operatorname{rot}_{\Gamma_t} + 1)$*-prbl in* R_{Γ_t}

cpld $(-\operatorname{div}_{\Gamma_n}\nabla_{\Gamma_t})$*-*$(\operatorname{rot}_{\Gamma_t}\operatorname{rot}_{\Gamma_n})$*-sys in* $\mathrm{H}^1_{\Gamma_t}$-R_{Γ_n}

Remark

- $(\operatorname{rot}_{\Gamma_t}\operatorname{rot}_{\Gamma_n})$*-prbl needs saddle point formulation*
- Ω *top trv* $\Rightarrow \pi_{\mathrm{D}} = 0$ *and* $\mathrm{R}_{\Gamma_t,0} = \nabla \mathrm{H}^1_{\Gamma_t}$ *and* $\mathrm{D}_{\Gamma_n,0} = \operatorname{rot}\mathrm{R}_{\Gamma_n}$
- $\boxed{\Omega \text{ convex and } \varepsilon = \mu = 1 \text{ and } \Gamma_t = \Gamma \text{ or } \Gamma_n = \Gamma \Rightarrow c_{\mathrm{f}} \le c_{\mathrm{m}} \le c_{\mathrm{p}} \le \frac{\operatorname{diam}_\Omega}{\pi}}$

div-curl-lemma

Lemma (div-curl-lemma (global version))

Assumptions:

- **(i)** (E_n) *bounded in* $\mathsf{L}^2(\Omega)$
- **(i')** (H_n) *bounded in* $\mathsf{L}^2(\Omega)$
- **(ii)** $(\operatorname{rot} E_n)$ *bounded in* $\mathsf{L}^2(\Omega)$
- **(ii')** $(\operatorname{div} \varepsilon H_n)$ *bounded in* $\mathsf{L}^2(\Omega)$
- **(iii)** $\nu \times E_n = 0$ *on* Γ_t, *i.e.,* $E_n \in \mathsf{R}_{\Gamma_t}(\Omega)$
- **(iii')** $\nu \cdot \varepsilon H_n = 0$ *on* Γ_n, *i.e.,* $H_n \in \varepsilon^{-1}\mathsf{D}_{\Gamma_n}(\Omega)$

$\Rightarrow \exists\ E, H$ *and subsequences st*

$E_n \rightharpoonup E$, $\operatorname{rot} E_n \rightharpoonup \operatorname{rot} E$ *and* $H_n \rightharpoonup H$, $\operatorname{div} H_n \rightharpoonup \operatorname{div} H$ *in* $\mathsf{L}^2(\Omega)$ *and*

$$\langle E_n, H_n \rangle_{\mathsf{L}^2_\varepsilon(\Omega)} \to \langle E, H \rangle_{\mathsf{L}^2_\varepsilon(\Omega)}$$

$\Rightarrow$ classical local version

Solving PDEs with Hilbert Complexes

APPENDIX I: Friedrichs/Poincaré/Maxwell constants (numerics)

joint work with Jan Valdman (Prag) and Carl-Martin Pfeiler (TU Wien)

Friedrichs/Poincaré/Maxwell constants

assumption: $\varepsilon = \mu = 1$ and $\boxed{\Gamma_t = \Gamma, \text{ i.e., } c_{\mathrm{fp}} = c_{\mathrm{f}}}$ or $\boxed{\Gamma_n = \Gamma, \text{ i.e., } c_{\mathrm{fp}} = c_{\mathrm{p}}}$

Lemma (Maxwell-Poincaré constants)

Ω *convex and bounded* $\Rightarrow$ $\boxed{c_{\mathrm{m}} \leq c_{\mathrm{p}} \leq \frac{\operatorname{diam}_\Omega}{\pi}}$

Mild Conjecture (Maxwell-Poincaré constants)

Ω convex and bounded $\Rightarrow$ $c_{\mathrm{f}} \leq c_{\mathrm{m}} \leq c_{\mathrm{p}} \leq \frac{\operatorname{diam}_\Omega}{\pi}$

Theorem (FA-ToolBox / Friedrichs-Poincaré type estimates and constants)

$$\forall\, \varphi \in D(\mathcal{A}_0) \quad |\varphi|_{\mathrm{H}_0} \leq c_{\mathrm{A}_0} |\mathrm{A}_0 \varphi|_{\mathrm{H}_1} \quad \Leftrightarrow \quad \forall\, \varphi \in \mathrm{H}^1_\Gamma \quad |\varphi|_{\mathrm{L}^2} \leq c_{\mathrm{f}} |\nabla \varphi|_{\mathrm{L}^2}$$

$$\forall\, \phi \in D(\mathcal{A}_0^*) \quad |\phi|_{\mathrm{H}_1} \leq c_{\mathrm{A}_0} |\mathrm{A}_0^* \phi|_{\mathrm{H}_0} \quad \Leftrightarrow \quad \forall\, \Phi \in \mathrm{D} \cap \nabla \mathrm{H}^1_\Gamma \quad |\Phi|_{\mathrm{L}^2} \leq c_{\mathrm{f}} |\operatorname{div} \Phi|_{\mathrm{L}^2}$$

$$\forall\, \phi \in D(\mathcal{A}_1) \quad |\phi|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |\mathrm{A}_1 \phi|_{\mathrm{H}_2} \quad \Leftrightarrow \quad \forall\, \Phi \in \mathrm{R}_\Gamma \cap \operatorname{rot} \mathrm{R} \quad |\Phi|_{\mathrm{L}^2} \leq c_{\mathrm{m}} |\operatorname{rot} \Phi|_{\mathrm{L}^2}$$

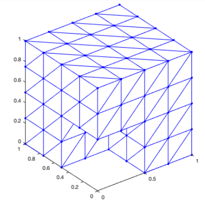
$$\forall\, \psi \in D(\mathcal{A}_1^*) \quad |\psi|_{\mathrm{H}_2} \leq c_{\mathrm{A}_1} |\mathrm{A}_1^* \psi|_{\mathrm{H}_1} \quad \Leftrightarrow \quad \forall\, \Psi \in \mathrm{R} \cap \operatorname{rot} \mathrm{R}_\Gamma \quad |\Psi|_{\mathrm{L}^2} \leq c_{\mathrm{m}} |\operatorname{rot} \Psi|_{\mathrm{L}^2}$$

$$\forall\, \psi \in D(\mathcal{A}_2) \quad |\psi|_{\mathrm{H}_2} \leq c_{\mathrm{A}_2} |\mathrm{A}_2 \psi|_{\mathrm{H}_3} \quad \Leftrightarrow \quad \forall\, \Psi \in \mathrm{D}_\Gamma \cap \nabla \mathrm{H}^1 \quad |\Psi|_{\mathrm{L}^2} \leq c_{\mathrm{p}} |\operatorname{div} \Psi|_{\mathrm{L}^2}$$

$$\forall\, \xi \in D(\mathcal{A}_2^*) \quad |\xi|_{\mathrm{H}_3} \leq c_{\mathrm{A}_2} |\mathrm{A}_2^* \xi|_{\mathrm{H}_2} \quad \Leftrightarrow \quad \forall\, \zeta \in \mathrm{H}^1 \cap \mathbb{R}^\perp \quad |\zeta|_{\mathrm{L}^2} \leq c_{\mathrm{p}} |\nabla \zeta|_{\mathrm{L}^2}$$

Friedrichs/Poincaré/Maxwell constants

surprise numerical tests show even for non-convex domains and mixed bc
e.g., Fichera corner domain

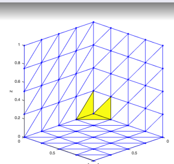
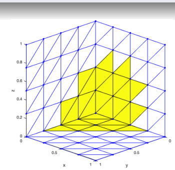
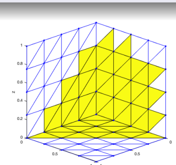
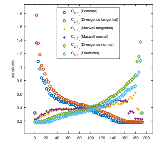
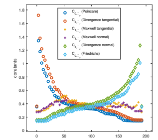


Conjecture (Maxwell-Poincaré constants)

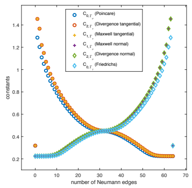
$$c_{\mathrm{f}} \leq \min\{c_{\mathrm{fp}}, c_{\mathrm{pf}}\} \leq c_{\mathrm{m}} \leq \max\{c_{\mathrm{fp}}, c_{\mathrm{pf}}\} \leq \sup_{\Gamma_t \neq \emptyset}\{c_{\mathrm{fp}}\} < \infty$$

Theorem (FA-ToolBox / Friedrichs-Poincaré type estimates and constants)

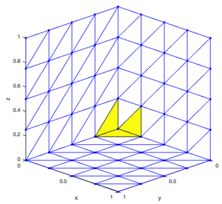
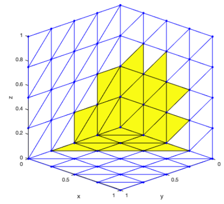
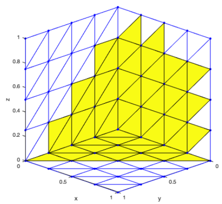
$$\begin{array}{llcll}
\forall\, \varphi \in D(\mathcal{A}_0) & |\varphi|_{\mathrm{H}_0} \leq c_{\mathrm{A}_0} |\mathrm{A}_0 \varphi|_{\mathrm{H}_1} & \Leftrightarrow & \forall\, \varphi \in \mathrm{H}^1_{\Gamma_t} & |\varphi|_{\mathrm{L}^2} \leq c_{\mathrm{fp}} |\nabla \varphi|_{\mathrm{L}^2} \\
\forall\, \phi \in D(\mathcal{A}_0^*) & |\phi|_{\mathrm{H}_1} \leq c_{\mathrm{A}_0} |\mathrm{A}_0^* \phi|_{\mathrm{H}_0} & \Leftrightarrow & \forall\, \Phi \in \mathrm{D}_{\Gamma_n} \cap \nabla \mathrm{H}^1_{\Gamma_t} & |\Phi|_{\mathrm{L}^2} \leq c_{\mathrm{fp}} |\operatorname{div} \Phi|_{\mathrm{L}^2} \\
\forall\, \phi \in D(\mathcal{A}_1) & |\phi|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |\mathrm{A}_1 \phi|_{\mathrm{H}_2} & \Leftrightarrow & \forall\, \Phi \in \mathrm{R}_{\Gamma_t} \cap \operatorname{rot} \mathrm{R}_{\Gamma_n} & |\Phi|_{\mathrm{L}^2} \leq c_{\mathrm{m}} |\operatorname{rot} \Phi|_{\mathrm{L}^2} \\
\forall\, \psi \in D(\mathcal{A}_1^*) & |\psi|_{\mathrm{H}_2} \leq c_{\mathrm{A}_1} |\mathrm{A}_1^* \psi|_{\mathrm{H}_1} & \Leftrightarrow & \forall\, \Psi \in \mathrm{R}_{\Gamma_n} \cap \operatorname{rot} \mathrm{R}_{\Gamma_t} & |\Psi|_{\mathrm{L}^2} \leq c_{\mathrm{m}} |\operatorname{rot} \Psi|_{\mathrm{L}^2} \\
\forall\, \psi \in D(\mathcal{A}_2) & |\psi|_{\mathrm{H}_2} \leq c_{\mathrm{A}_2} |\mathrm{A}_2 \psi|_{\mathrm{H}_3} & \Leftrightarrow & \forall\, \Psi \in \mathrm{D}_{\Gamma_t} \cap \nabla \mathrm{H}^1_{\Gamma_n} & |\Psi|_{\mathrm{L}^2} \leq c_{\mathrm{pf}} |\operatorname{div} \Psi|_{\mathrm{L}^2} \\
\forall\, \xi \in D(\mathcal{A}_2^*) & |\xi|_{\mathrm{H}_3} \leq c_{\mathrm{A}_2} |\mathrm{A}_2^* \xi|_{\mathrm{H}_2} & \Leftrightarrow & \forall\, \zeta \in \mathrm{H}^1_{\Gamma_n} & |\zeta|_{\mathrm{L}^2} \leq c_{\mathrm{pf}} |\nabla \zeta|_{\mathrm{L}^2}
\end{array}$$

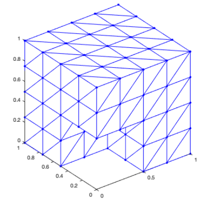
Friedrichs/Poincaré/Maxwell constants



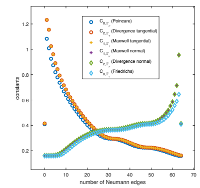
2D unit square

3D unit cube



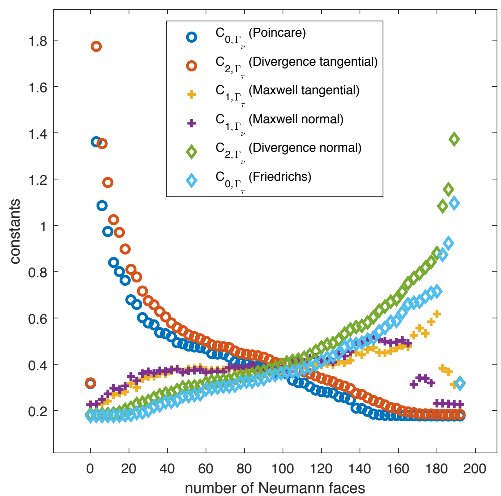
3D Fichera corner



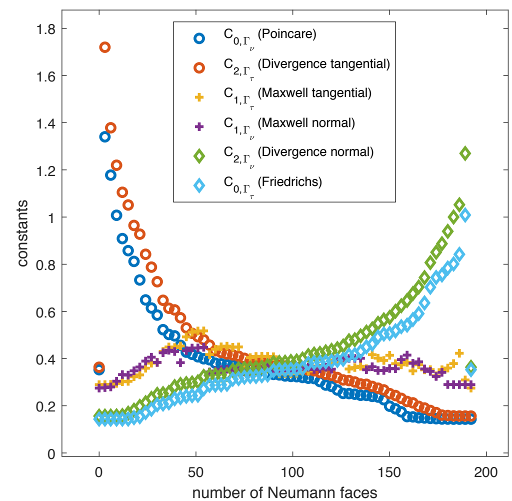
2D L-shape

Conjecture (Maxwell-Poincaré constants)

$$c_{\mathrm{f}} \leq \min\{c_{\mathrm{fp}}, c_{\mathrm{pf}}\} \leq c_{\mathrm{m}} \leq \max\{c_{\mathrm{fp}}, c_{\mathrm{pf}}\} \leq \sup_{\Gamma_t \neq \emptyset}\{c_{\mathrm{fp}}\} < \infty$$



3D unit cube



3D Fichera corner domain

Solving PDEs with Hilbert Complexes

APPENDIX II: More Complexes

elasticity complex in 3D ($\operatorname{sym}\nabla$-$\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}$-$\operatorname{Div}_{\mathbb{S}}$-complex)

general complex property $A_1 A_0 = 0$, i.e., $\boxed{R(A_0) \subset N(A_1)}$

$$\cdots \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad H_0 \quad \underset{A_0^*}{\overset{A_0}{\rightleftarrows}} \quad H_1 \quad \underset{A_1^*}{\overset{A_1}{\rightleftarrows}} \quad H_2 \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad \cdots$$

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \quad \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \quad L^2 \quad \underset{-\operatorname{Div}_{\mathbb{S}}}{\overset{\operatorname{sym}\mathring{\nabla}}{\rightleftarrows}} \quad L^2_{\mathbb{S}} \quad \underset{\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}}{\overset{\mathring{\operatorname{Rot}}\mathring{\operatorname{Rot}}_{\mathbb{S}}^{\top}}{\rightleftarrows}} \quad L^2_{\mathbb{S}} \quad \underset{-\operatorname{sym}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{S}}}{\rightleftarrows}} \quad L^2 \quad \underset{\iota_{RM}}{\overset{\pi_{RM}}{\rightleftarrows}} \quad RM$$

elasticity complex in 3D (sym ∇-Rot $\mathrm{Rot}_{\mathbb{S}}^{\top}$-$\mathrm{Div}_{\mathbb{S}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathrm{L}^2 \underset{-\mathrm{Div}_{\mathbb{S}}}{\overset{\mathring{\mathrm{sym}}\,\nabla}{\rightleftarrows}} \mathrm{L}^2_{\mathbb{S}} \underset{\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top}}{\overset{\mathrm{Rot}\,\mathring{\mathrm{Rot}}_{\mathbb{S}}^{\top}}{\rightleftarrows}} \mathrm{L}^2_{\mathbb{S}} \underset{-\mathrm{sym}\,\nabla}{\overset{\mathring{\mathrm{Div}}_{\mathbb{S}}}{\rightleftarrows}} \mathrm{L}^2 \underset{\iota_{\mathrm{RM}}}{\overset{\pi_{\mathrm{RM}}}{\rightleftarrows}} \mathrm{RM}$$

related fos ($\mathrm{Rot}\,\mathring{\mathrm{Rot}}_{\mathbb{S}}^{\top}$, $\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top}$ first order operators!)

$\mathring{\mathrm{sym}}\,\nabla v = M$ in Ω	$\mathrm{Rot}\,\mathring{\mathrm{Rot}}_{\mathbb{S}}^{\top} M = F$ in Ω	$\mathring{\mathrm{Div}}_{\mathbb{S}} N = g$ in Ω	$\pi v = r$ in Ω
$\pi v = 0$ in Ω	$-\mathrm{Div}_{\mathbb{S}} M = f$ in Ω	$\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top} N = G$ in Ω	$-\mathrm{sym}\,\nabla v = M$ in Ω

related sos ($\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top}\,\mathrm{Rot}\,\mathring{\mathrm{Rot}}_{\mathbb{S}}^{\top}$ second order operator!)

$-\mathrm{Div}_{\mathbb{S}}\,\mathring{\mathrm{sym}}\,\nabla v = f$ in Ω	$\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top}\,\mathrm{Rot}\,\mathring{\mathrm{Rot}}_{\mathbb{S}}^{\top} M = G$ in Ω	$-\mathrm{sym}\,\nabla\,\mathring{\mathrm{Div}}_{\mathbb{S}} N = M$ in Ω
$\pi v = 0$ in Ω	$-\mathrm{Div}_{\mathbb{S}} M = f$ in Ω	$\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top} N = G$ in Ω

corresponding compact embeddings:

$$D(\mathring{\mathrm{sym}}\,\nabla) \cap D(\pi) = D(\mathring{\nabla}) = \mathring{\mathrm{H}}^1 \hookrightarrow \mathrm{L}^2$$ (Rellich's selection theorem and Korn ineq.)

$$D(\mathrm{Rot}\,\mathring{\mathrm{Rot}}_{\mathbb{S}}^{\top}) \cap D(\mathrm{Div}_{\mathbb{S}}) \hookrightarrow \mathrm{L}^2_{\mathbb{S}}$$ (new selection theorem)

$$D(\mathring{\mathrm{Div}}_{\mathbb{S}}) \cap D(\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top}) \hookrightarrow \mathrm{L}^2_{\mathbb{S}}$$ (new selection theorem)

$$D(\pi) \cap D(\mathrm{sym}\,\nabla) = D(\nabla) = \mathrm{H}^1 \hookrightarrow \mathrm{L}^2$$ (Rellich's selection theorem and Korn ineq.)

two new selection theorems for strong Lip. dom.: Py/Schomburg/Zulehner ('18)

elasticity complex in 3D ($\operatorname{sym}\nabla$-$\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}$-$\operatorname{Div}_{\mathbb{S}}$-complex)

Lemma/Theorem $\Downarrow$ $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1, \quad D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_2 \quad \text{cpt}$

(i) all Friedrichs-Poincaré type est hold

est for $\mathcal{A}_0$ $\Leftrightarrow$ $\forall\, \varphi \in D(\operatorname{sym}\mathring{\nabla}) \cap R(\operatorname{Div}_{\mathbb{S}}) = \mathring{\mathrm{H}}^1$ $\quad |\varphi|_{\mathrm{L}^2} \leq c_0 |\operatorname{sym}\nabla\varphi|_{\mathrm{L}^2}$

est for $\mathcal{A}_0^*$ $\Leftrightarrow$ $\forall\, \Phi \in D(\operatorname{Div}_{\mathbb{S}}) \cap R(\operatorname{sym}\mathring{\nabla})$ $\quad |\Phi|_{\mathrm{L}^2} \leq c_0 |\operatorname{Div}\Phi|_{\mathrm{L}^2}$

est for $\mathcal{A}_1$ $\Leftrightarrow$ $\forall\, \Phi \in D(\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^{\top}) \cap R(\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top})$ $\quad |\Phi|_{\mathrm{L}^2} \leq c_1 |\operatorname{Rot}\operatorname{Rot}^{\top}\Phi|_{\mathrm{L}^2}$

est for $\mathcal{A}_1^*$ $\Leftrightarrow$ $\forall\, \Phi \in D(\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}) \cap R(\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^{\top})$ $\quad |\Phi|_{\mathrm{L}^2} \leq c_1 |\operatorname{Rot}\operatorname{Rot}^{\top}\Phi|_{\mathrm{L}^2}$

est for $\mathcal{A}_2$ $\Leftrightarrow$ $\forall\, \Phi \in D(\mathring{\operatorname{Div}}_{\mathbb{S}}) \cap R(\operatorname{sym}\nabla)$ $\quad |\Phi|_{\mathrm{L}^2} \leq c_2 |\operatorname{Div}\Phi|_{\mathrm{L}^2}$

est for $\mathcal{A}_2^*$ $\Leftrightarrow$ $\forall\, \varphi \in D(\operatorname{sym}\nabla) \cap R(\mathring{\operatorname{Div}}_{\mathbb{S}}) = \mathrm{H}^1 \cap \mathrm{RM}^{\perp}$ $\quad |\varphi|_{\mathrm{L}^2} \leq c_2 |\operatorname{sym}\nabla\varphi|_{\mathrm{L}^2}$

(ii) all ranges $R(\mathrm{A}_n) = R(\mathcal{A}_n)$, $R(\mathrm{A}_n^*) = R(\mathcal{A}_n^*)$ are cl in L^2
(iii) all inverse ops $\mathcal{A}_n^{-1}$, $(\mathcal{A}_n^*)^{-1}$ are cont, even cpt
(iv) all Helmholtz decomposition hold, e.g.,

$$\mathrm{H}_1 = R(\mathrm{A}_0) \oplus N_1 \oplus R(\mathrm{A}_1^*) \quad \Leftrightarrow \quad \mathrm{L}^2 = R(\operatorname{sym}\mathring{\nabla}) \oplus_{\mathrm{L}^2} \mathcal{H}_{\mathrm{D},\mathbb{S}} \oplus_{\mathrm{L}^2} R(\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top})$$

(v) solution theories
(vi) variational formulations
(vii) functional a posteriori error estimates
(viii) div-curl-lemmas
(ix) ...

biharmonic / general relativity complex in 3D ($\nabla\nabla$-Rot$_{\mathbb{S}}$-Div$_{\mathbb{T}}$-complex)

general complex property $A_1 A_0 = 0$, i.e., $\boxed{R(A_0) \subset N(A_1)}$

$$\cdots \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad H_0 \quad \underset{A_0^*}{\overset{A_0}{\rightleftarrows}} \quad H_1 \quad \underset{A_1^*}{\overset{A_1}{\rightleftarrows}} \quad H_2 \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad \cdots$$

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \quad \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \quad L^2 \quad \underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\nabla\nabla}}{\rightleftarrows}} \quad L^2_{\mathbb{S}} \quad \underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \quad L^2_{\mathbb{T}} \quad \underset{-\operatorname{dev}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}} \quad L^2 \quad \underset{\iota_{RT}}{\overset{\pi_{RT}}{\rightleftarrows}} \quad RT$$

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\mathrm{Rot}_{\mathbb{S}}$-$\mathrm{Div}_{\mathbb{T}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathrm{L}^2 \underset{\operatorname{div}\mathrm{Div}_{\mathbb{S}}}{\overset{\mathring{\nabla}\nabla}{\rightleftarrows}} \mathrm{L}^2_{\mathbb{S}} \underset{\operatorname{sym}\mathrm{Rot}_{\mathbb{T}}}{\overset{\mathring{\mathrm{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \mathrm{L}^2_{\mathbb{T}} \underset{-\operatorname{dev}\nabla}{\overset{\mathring{\mathrm{Div}}_{\mathbb{T}}}{\rightleftarrows}} \mathrm{L}^2 \underset{\iota_{\mathrm{RT}}}{\overset{\pi_{\mathrm{RT}}}{\rightleftarrows}} \mathrm{RT}$$

related fos ($\mathring{\nabla}\nabla$, $\operatorname{div}\mathrm{Div}_{\mathbb{S}}$ first order operators!)

$\mathring{\nabla}\nabla u = M$	in Ω	$\mathring{\mathrm{Rot}}_{\mathbb{S}} M = F$	in Ω	$\mathring{\mathrm{Div}}_{\mathbb{T}} N = g$	in Ω	$\pi v = r$	in Ω
$\pi u = 0$	in Ω	$\operatorname{div}\mathrm{Div}_{\mathbb{S}} M = f$	in Ω	$\operatorname{sym}\mathrm{Rot}_{\mathbb{T}} N = G$	in Ω	$-\operatorname{dev}\nabla v = T$	in Ω

related sos ($\operatorname{div}\mathrm{Div}_{\mathbb{S}}\,\mathring{\nabla}\nabla = \mathring{\Delta}^2$ second order operator!)

$\operatorname{div}\mathrm{Div}_{\mathbb{S}}\,\mathring{\nabla}\nabla u = \mathring{\Delta}^2 u = f$	in Ω	$\operatorname{sym}\mathrm{Rot}_{\mathbb{T}}\,\mathring{\mathrm{Rot}}_{\mathbb{S}} M = G$	in Ω	$-\operatorname{dev}\nabla\mathring{\mathrm{Div}}_{\mathbb{T}} N = T$	in Ω
$\pi u = 0$	in Ω	$\operatorname{div}\mathrm{Div}_{\mathbb{S}} M = f$	in Ω	$\operatorname{sym}\mathrm{Rot}_{\mathbb{T}} N = G$	in Ω

corresponding compact embeddings:

$$D(\mathring{\nabla}\nabla) \cap D(\pi) = D(\mathring{\nabla}\nabla) = \mathring{\mathrm{H}}^2 \hookrightarrow \mathrm{L}^2 \qquad \text{(Rellich's selection theorem)}$$

$$D(\mathring{\mathrm{Rot}}_{\mathbb{S}}) \cap D(\operatorname{div}\mathrm{Div}_{\mathbb{S}}) \hookrightarrow \mathrm{L}^2_{\mathbb{S}} \qquad \text{(new selection theorem)}$$

$$D(\mathring{\mathrm{Div}}_{\mathbb{T}}) \cap D(\operatorname{sym}\mathrm{Rot}_{\mathbb{T}}) \hookrightarrow \mathrm{L}^2_{\mathbb{T}} \qquad \text{(new selection theorem)}$$

$$D(\pi) \cap D(\operatorname{dev}\nabla) = D(\operatorname{dev}\nabla) = D(\nabla) = \mathrm{H}^1 \hookrightarrow \mathrm{L}^2 \qquad \text{(Rellich's selection theorem and Korn type ineq.)}$$

two new selection theorems for strong Lip. dom. and Korn Type ineq.: Py/Zulehner ('16)

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\operatorname{Rot}_{\mathbb{S}}$-$\operatorname{Div}_{\mathbb{T}}$-complex)

Lemma/Theorem $\Downarrow$ $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1, \quad D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_2 \quad \text{cpt}$

(i) all Friedrichs-Poincaré type est hold

est for $\mathcal{A}_0$	$\Leftrightarrow$	$\forall\, \varphi \in D(\mathring{\nabla}\nabla) \cap R(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) = \mathring{\mathrm{H}}^2$	$\lvert\varphi\rvert_{\mathrm{L}^2} \le c_0 \lvert\nabla\nabla\varphi\rvert_{\mathrm{L}^2}$
est for $\mathcal{A}_0^*$	$\Leftrightarrow$	$\forall\, \Phi \in D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \cap R(\mathring{\nabla}\nabla)$	$\lvert\Phi\rvert_{\mathrm{L}^2} \le c_0 \lvert\operatorname{div}\operatorname{Div}\Phi\rvert_{\mathrm{L}^2}$
est for $\mathcal{A}_1$	$\Leftrightarrow$	$\forall\, \Phi \in D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \cap R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}})$	$\lvert\Phi\rvert_{\mathrm{L}^2} \le c_1 \lvert\operatorname{Rot}\Phi\rvert_{\mathrm{L}^2}$
est for $\mathcal{A}_1^*$	$\Leftrightarrow$	$\forall\, \Phi \in D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \cap R(\mathring{\operatorname{Rot}}_{\mathbb{S}})$	$\lvert\Phi\rvert_{\mathrm{L}^2} \le c_1 \lvert\operatorname{sym}\operatorname{Rot}\Phi\rvert_{\mathrm{L}^2}$
est for $\mathcal{A}_2$	$\Leftrightarrow$	$\forall\, \Phi \in D(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap R(\operatorname{dev}\nabla)$	$\lvert\Phi\rvert_{\mathrm{L}^2} \le c_2 \lvert\operatorname{Div}\Phi\rvert_{\mathrm{L}^2}$
est for $\mathcal{A}_2^*$	$\Leftrightarrow$	$\forall\, \varphi \in D(\operatorname{dev}\nabla) \cap R(\mathring{\operatorname{Div}}_{\mathbb{T}}) = \mathrm{H}^1 \cap \mathrm{RT}^{\perp}$	$\lvert\varphi\rvert_{\mathrm{L}^2} \le c_2 \lvert\operatorname{dev}\nabla\varphi\rvert_{\mathrm{L}^2}$

(ii) all ranges $R(\mathrm{A}_n) = R(\mathcal{A}_n), \quad R(\mathrm{A}_n^*) = R(\mathcal{A}_n^*)$ are cl in L^2

(iii) all inverse ops $\mathcal{A}_n^{-1}$, $(\mathcal{A}_n^*)^{-1}$ are cont, even cpt

(iv) all Helmholtz decomposition hold, e.g.,

$$\mathrm{H}_1 = R(\mathrm{A}_0) \oplus N_1 \oplus R(\mathrm{A}_1^*) \quad \Leftrightarrow \quad \mathrm{L}^2_{\mathbb{S}} = R(\mathring{\nabla}\nabla) \oplus_{\mathrm{L}^2_{\mathbb{S}}} \mathcal{H}_{\mathrm{D},\mathbb{S}} \oplus_{\mathrm{L}^2_{\mathbb{S}}} R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}),$$

$$\mathrm{H}_2 = R(\mathrm{A}_1) \oplus N_2 \oplus R(\mathrm{A}_2^*) \quad \Leftrightarrow \quad \mathrm{L}^2_{\mathbb{T}} = R(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \oplus_{\mathrm{L}^2_{\mathbb{T}}} \mathcal{H}_{\mathrm{N},\mathbb{T}} \oplus_{\mathrm{L}^2_{\mathbb{T}}} R(\operatorname{dev}\nabla)$$

(v)-(ix) solution theories, variational formulations, functional a posteriori error estimates, div-curl-lemmas, ...

Solving PDEs with Hilbert Complexes

APPENDIX III: Literature

literature (FA-ToolBox, complexes, a posteriori error estimates, ...)

some results of this talk:

- Py: *Solution Theory, Variational Formulations, and Functional a Posteriori Error Estimates for General First Order Systems with Applications to Electro-Magneto-Statics and More*, (NFAO) Numerical Functional Analysis and Optimization, 2020

literature (complexes, Friedrichs type constants, Maxwell constants)

results of this talk:

- Py: *On Constants in Maxwell Inequalities for Bounded and Convex Domains*, (JMS) Journal of Mathematical Sciences, 2015
- Py: *On Maxwell's and Poincare's Constants*, (DCDS) Discrete and Continuous Dynamical Systems, 2015
- Py: *On the Maxwell Constants in 3D*, (M2AS) Mathematical Methods in the Applied Sciences, 2017
- Py: *On the Maxwell and Friedrichs/Poincaré Constants in ND*, (MZ) Mathematische Zeitschrift, 2019

- Py: *. . . some (so far) unpublished results*

literature (complexes, Friedrichs type constants, compact embeddings)

- Weck, N.: *Maxwell's boundary value problems on Riemannian manifolds with nonsmooth boundaries*, (JMA2) Journal of Mathematical Analysis and Applications, 1974 (1972)
- Picard, R.: *An elementary proof for a compact imbedding result in generalized electromagnetic theory*, (MZ) Mathematische Zeitschrift, 1984
- Witsch, K.-J.: *A remark on a compactness result in electromagnetic theory*, (M2AS) Mathematical Methods in the Applied Sciences, 1993

results of this talk:

- Bauer, S., Py, Schomburg, M.: *The Maxwell Compactness Property in Bounded Weak Lipschitz Domains with Mixed Boundary Conditions*, (SIMA) SIAM Journal on Mathematical Analysis, 2016
- Py, Zulehner, W.: *The divDiv-Complex and Applications to Biharmonic Equations*, (AA) Applicable Analysis, 2020
- Py, Zulehner, W.: *The Elasticity Complex*, submitted, 2020

Solving PDEs with Hilbert Complexes

APPENDIX IV: A Posteriori Error Estimates for BEM (Boundary Element Method)

joint work with
Stefan Kurz (Bosch GmbH & TU Darmstadt),
Dirk Praetorius (TU Wien),
Sergey Repin (Steklov Mathematical Institute , St. Petersburg),
Daniel Sebastian (UDE, TU Wien)

functional a posteriori error estimates for BEM

problem: num approx with BEM

$$\Delta u = 0 \quad \text{in } \Omega, \qquad\qquad u|_\Gamma = g \quad \text{on } \Gamma.$$

functional a posteriori error estimates: num approx with FEM

$$\max_{\substack{E \in \mathsf{L}^2(\Omega) \\ \operatorname{div} E = 0}} \Big(2\langle n \cdot E, g - \widetilde{u}|_\Gamma \rangle_{\mathsf{H}^{-1/2}(\Gamma)} - |E|^2_{\mathsf{L}^2(\Omega)} \Big) = |\nabla(u - \widetilde{u})|^2_{\mathsf{L}^2(\Omega)} = \min_{\substack{v \in \mathsf{H}^1(\Omega) \\ v|_\Gamma = g - \widetilde{u}|_\Gamma}} |\nabla v|^2_{\mathsf{L}^2(\Omega)}$$

natural energy norm ($\mathsf{H}^1(\Omega)$-volume norm)

idea: compute upper and lower bounds in a thin boundary layer using FEM

functional a posteriori error estimates for BEM

$$\max_{\substack{E\in \mathrm{L}^2(\Omega)\\ \operatorname{div} E=0}} \Big(2\langle n\cdot E, g-\widetilde{u}|_\Gamma\rangle_{\mathrm{H}^{-1/2}(\Gamma)} - |E|^2_{\mathrm{L}^2(\Omega)}\Big) = |\nabla(u-\widetilde{u})|^2_{\mathrm{L}^2(\Omega)} = \min_{\substack{v\in \mathrm{H}^1(\Omega)\\ v|_\Gamma = g-\widetilde{u}|_\Gamma}} |\nabla v|^2_{\mathrm{L}^2(\Omega)}$$

minimiser of upper bound $\overline{v} = u - \widetilde{u}$: standard Dirichlet-Laplacian

$$\Delta v = 0 \quad \text{in } \Omega, \qquad v|_\Gamma = g - \widetilde{u}|_\Gamma \quad \text{on } \Gamma$$

exact solution is $v = \overline{v}$ $\Rightarrow$ standard FEM on boundary layer for v

maximiser of lower bound $\underline{E} = \nabla\overline{v} = \nabla(u-\widetilde{u})$: Neumann-type-Laplacian

$$\Delta v = 0 \quad \text{in } \Omega, \qquad n\cdot\nabla v|_\Gamma = \langle g - \widetilde{u}|_\Gamma, n\cdot\nabla\widehat{(\,\cdot\,)}|_\Gamma\rangle \quad \text{in } \mathrm{H}^{-1/2}(\Gamma)$$

(here $\widehat{(\,\cdot\,)}$ harmonic extension and $n\cdot\nabla\widehat{(\,\cdot\,)}|_\Gamma$ Dirichlet2Neumann operator)

exact solution is $v = \overline{v}$ and $\nabla v = \underline{E}$ $\Rightarrow$ non-standard FEM on bd layer for E

$\Rightarrow$ saddle point formulation (mixed/dual Laplacian)

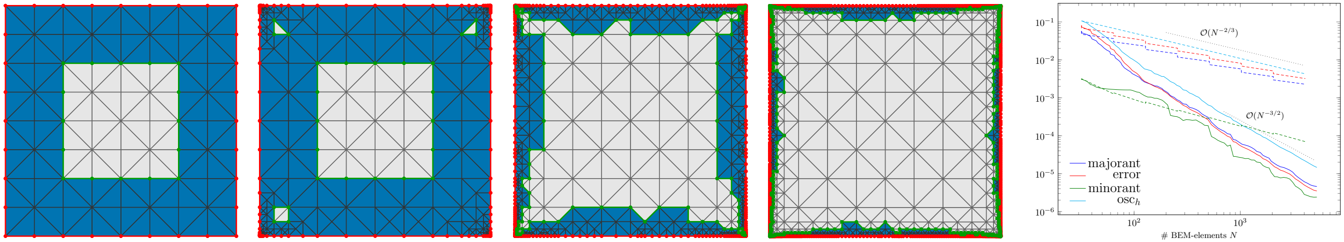
Find $(E, v) \in \mathrm{H}(\operatorname{div},\Omega)\times \mathrm{L}^2(\Omega)$ s.t. for all $(\Phi,\varphi)\in \mathrm{H}(\operatorname{div},\Omega)\times \mathrm{L}^2(\Omega)$

$$\langle E,\Phi\rangle_{\mathrm{L}^2(\Omega)} + \langle \operatorname{div}\Phi, v\rangle_{\mathrm{L}^2(\Omega)} = \langle n\cdot\Phi, g-\widetilde{u}|_\Gamma\rangle_{\mathrm{H}^{-1/2}(\Gamma)},$$
$$\langle \operatorname{div} E,\varphi\rangle_{\mathrm{L}^2(\Omega)} = 0$$

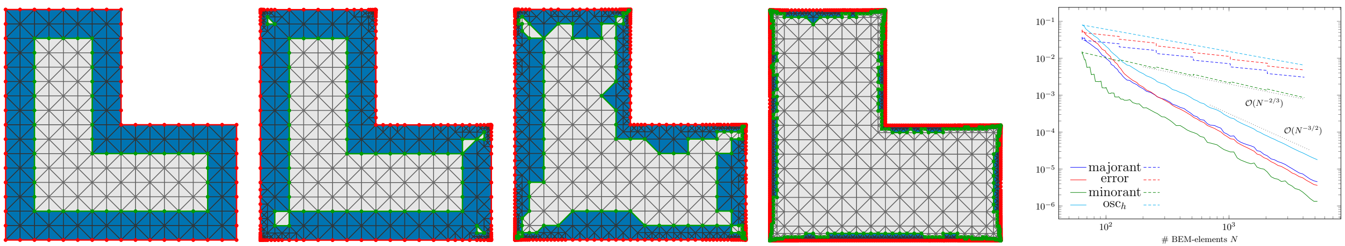
unique sol $(E, v) = (\underline{E}, \overline{v})$

functional a posteriori error estimates for BEM - some pics

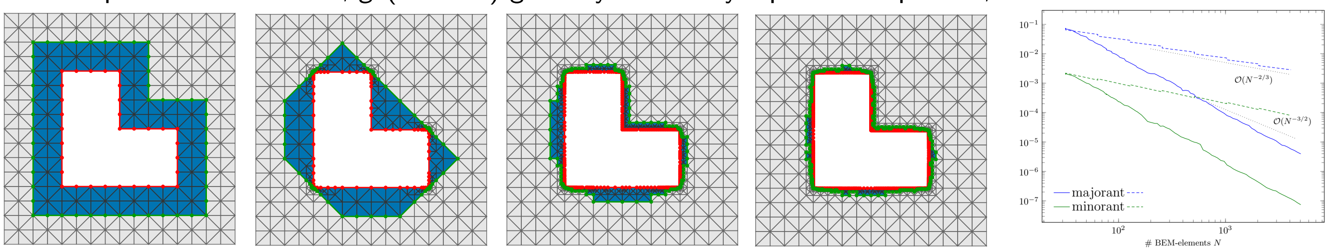
Ω: unit square, $u(x) = cosh(x_1)cos(x_2)$, known smooth solution u



Ω: L-shaped domain, $u(x) = u(r, \varphi) = r^{2/3}\cos(2/3\varphi)$, known non-smooth solution u



Ω: L-shaped exterior domain, g (bd data) given by double-layer potential operator, unknown exact solution u



oscillatory error upper bound exact error lower bound convergence rates

adaptive mesh-ref with Dörfler marking (solid lines) vs. unif mesh-ref (dashed lines)

Solving PDEs with Hilbert Complexes

APPENDIX V: DEC - Discrete Exterior Calculus:
A Discrete Version of the FA-ToolBox
(joint work with the Jyväskylä Group)

https://sites.google.com/jyu.fi/gfd/method/online-time-integrator