A Global div-curl-Lemma for Mixed Boundary Conditions in Weak Lipschitz Domains and a Corresponding Generalized A_0^*-A_1-Lemma in Hilbert Spaces

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Open-Minded

;-)

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classical div-curl-lemma

Let $\Omega \subset \mathbb{R}^3$ be open.

Lemma (classical div-curl-lemma)

Assumptions:

(i) $(E_n), (H_n)$ *bounded in* $\mathrm{L}^2(\Omega)$

(ii) $(\operatorname{rot} E_n)$ *bounded in* $\mathrm{L}^2(\Omega)$

(ii') $(\operatorname{div} H_n)$ *bounded in* $\mathrm{L}^2(\Omega)$

$\Rightarrow \exists\ E, H$ *and subseq st* $E_n \rightharpoonup E$, $\operatorname{rot} E_n \rightharpoonup \operatorname{rot} E$ *and* $H_n \rightharpoonup H$, $\operatorname{div} H_n \rightharpoonup \operatorname{div} H$ *and*

$$\forall\, \varphi \in \mathring{\mathrm{C}}^\infty(\Omega) \qquad \int_\Omega \varphi(E_n \cdot H_n) \to \int_\Omega \varphi(E \cdot H)$$

classical div-curl-lemma is local!

div-curl-lemma

We shall prove:

Let $\Omega \subset \mathbb{R}^3$ be a bounded weak Lipschitz domain with boundary Γ and weak Lipschitz boundary parts Γ_t and $\Gamma_n = \Gamma \setminus \overline{\Gamma_t}$.

Lemma (div-curl-lemma (global version))

Assumptions:

- **(i)** $(E_n), (H_n)$ *bounded in* $\mathrm{L}^2(\Omega)$
- **(ii)** $(\operatorname{rot} E_n)$ *bounded in* $\mathrm{L}^2(\Omega)$
- **(ii')** $(\operatorname{div} H_n)$ *bounded in* $\mathrm{L}^2(\Omega)$
- **(iii)** $\nu \times E_n = 0$ *on* Γ_t
- **(iii')** $\nu \cdot H_n = 0$ *on* Γ_n

$\Rightarrow \exists\ E,\ H$ *and subseq st* $E_n \rightharpoonup E$, $\operatorname{rot} E_n \rightharpoonup \operatorname{rot} E$ *and* $H_n \rightharpoonup H$, $\operatorname{div} H_n \rightharpoonup \operatorname{div} H$ *and*

$$\int_\Omega E_n \cdot H_n \to \int_\Omega E \cdot H$$

Proof.

- generalize and fa-toolbox
- crucial points: complex property and compact embedding □

literature

original papers (local div-curl-lemma):

- Murat, F.: *Compacité par compensation*,
  Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 1978
- Tartar, L.: *Compensated compactness and applications to partial differential equations*,
  Nonlinear analysis and mechanics, Heriot-Watt symposium, 1979

recent papers (global div-curl-lemma, unfortunately H^1-detour):

- Gloria, A., Neukamm, S., Otto, F.: *Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics*,
  (IM) Invent. Math., 2015
- Kozono, H., Yanagisawa, T.: *Global compensated compactness theorem for general differential operators of first order*,
  (ARMA) Arch. Ration. Mech. Anal., 2013
- Schweizer, B.: *On Friedrichs inequality, Helmholtz decomposition, vector potentials, and the div-curl lemma*,
  accepted preprint, 2018

fa-toolbox for linear problems/systems

idea: solve problem with general and simple linear functional analysis ($\Rightarrow$ fa-toolbox) ...

literature: probably very well known for ages, but hard to find ...

Friedrichs, Weyl, Hörmander, Fredholm, von Neumann, Riesz, Banach, ... ?

Why not rediscover, modify, and extend?

A_0^*-A_1-lemma (generalized global div-curl-lemma)

setting:

$$A_0 : D(A_0) \subset H_0 \to H_1$$

$$A_1 : D(A_1) \subset H_1 \to H_2$$

two densely defined and closed linear operators on three Hilbert spaces H_0, H_1, H_2 (possibly and generally unbounded)

Hilbert space adjoints

$$A_0^* : D(A_0^*) \subset H_1 \to H_0$$

$$A_1^* : D(A_1^*) \subset H_2 \to H_1$$

Moreover, complex property

$$\boxed{A_1 A_0 = 0} \quad \Leftrightarrow \quad A_0^* A_1^* = 0$$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$R(A_0) \subset N(A_1) \quad \Leftrightarrow \quad R(A_1^*) \subset N(A_0^*)$$

A_0^*-A_1-lemma (generalized global div-curl-lemma)

We shall prove:

Let $A_0 : D(A_0) \subset H_0 \to H_1$, $A_1 : D(A_1) \subset H_1 \to H_2$ (possibly and generally unbounded) be two densely defined and closed linear operators on three Hilbert spaces H_0, H_1, H_2 with Hilbert space adjoints $A_0^* : D(A_0^*) \subset H_1 \to H_0$, $A_1^* : D(A_1^*) \subset H_2 \to H_1$.
Moreover, let $A_1 A_0 = 0$, i.e. $R(A_0) \subset N(A_1)$. (complex property)

Lemma (A_0^*-A_1-lemma)

Let $D(A_1) \cap D(A_0^*) \hookrightarrow H_1$ be compact, and

(i) *(x_n) bounded in $D(A_1)$,*

(ii) *(y_n) bounded in $D(A_0^*)$.*

$\Rightarrow$ *$\exists\ x \in D(A_1)$, $y \in D(A_0^*)$ and subseq st $x_n \rightharpoonup x$ in $D(A_1)$ and $y_n \rightharpoonup y$ in $D(A_0^*)$ and*

$$\langle x_n, y_n \rangle_{H_1} \to \langle x, y \rangle_{H_1}.$$

Proof.

... blackboard ... or ... next slides ... □

A_0^*-A_1-lemma (generalized global div-curl-lemma)

app to classical general global case

$$\begin{aligned}
A_0 : D(A_0) \subset H_0 \to H_1 \quad &:= \quad \nabla_{\Gamma_t} : D(\nabla_{\Gamma_t}) \subset L^2(\Omega) \to L^2(\Omega)\\
A_1 : D(A_1) \subset H_1 \to H_2 \quad &:= \quad \operatorname{rot}_{\Gamma_t} : D(\operatorname{rot}_{\Gamma_t}) \subset L^2(\Omega) \to L^2(\Omega)\\
A_0^* : D(A_0^*) \subset H_1 \to H_0 \quad &= \quad -\operatorname{div}_{\Gamma_n} : D(\operatorname{div}_{\Gamma_n}) \subset L^2(\Omega) \to L^2(\Omega)\\
A_1^* : D(A_1^*) \subset H_2 \to H_1 \quad &= \quad \operatorname{rot}_{\Gamma_n} : D(\operatorname{rot}_{\Gamma_n}) \subset L^2(\Omega) \to L^2(\Omega)
\end{aligned}$$

complex property: $\boxed{\operatorname{rot}_{\Gamma_t} \nabla_{\Gamma_t} = 0}$

compact embedding

$$D(A_1) \cap D(A_0^*) \hookrightarrow H_1$$

in global div-curl-lemma reads:

$$\begin{aligned}
&D(\operatorname{rot}_{\Gamma_t}) \cap D(\operatorname{div}_{\Gamma_n})\\
&= H_{\Gamma_t}(\operatorname{rot}, \Omega) \cap H_{\Gamma_n}(\operatorname{div}, \Omega)\\
&= \{E \in L^2(\Omega) : \operatorname{rot} E \in L^2(\Omega), \operatorname{div} E \in L^2(\Omega), \nu \times E = 0 \text{ on } \Gamma_t, \nu \cdot E = 0 \text{ on } \Gamma_n\} \hookrightarrow L^2(\Omega)
\end{aligned}$$

is compact

Weck's selection theorem, '72/'74

also Bauer, Costabel, Kuhn, Jochmann, Osterbrink, Py, Picard, Schomburg, Weber, Witsch

A_0^*-A_1-lemma (generalized global div-curl-lemma)

slight generalization

Corollary (A_0^*-A_1-lemma)

Let $R(A_0)$ and $R(A_1)$ be closed and let $N(A_1) \cap N(A_0^*)$ be finite dimensional, and

(i) *let $(x_n) \subset D(A_1)$ be bounded in H_1 with $(A_1 x_n)$ rel. compact in $D(\mathcal{A}_1^*)'$,*

(ii) *let $(y_n) \subset D(A_0^*)$ be bounded in H_1 with $(A_0^* y_n)$ rel. compact in $D(\mathcal{A}_0)'$.*

$\Rightarrow \exists\, x, y \in H_1$ *and subseq st* $x_n \rightharpoonup x$ *and* $y_n \rightharpoonup y$ *in* H_1 *and*

$$\langle x_n, y_n \rangle_{H_1} \to \langle x, y \rangle_{H_1}.$$

Proof.

... very similar ... □

Remark

homogen app: often, e.g., $x_n = \nabla u_n$ *bd with some* $u_n \in H^1(\Omega) + bc$
as well as $\operatorname{rot} x_n = 0$ *and* $\operatorname{div} y_n = f \in H^{-1}(\Omega)$, *even const*

A_0^*-A_1-lemma (proof using fa-toolbox)

Proof.

- use fa-toolbox
- w.l.o.g. (subsequences) $x_n \rightharpoonup x$ in $D(A_1)$ and $y_n \rightharpoonup y$ in $D(A_0^*)$
- ortho Helm type deco $\Rightarrow$ $D(A_1) = R(\mathcal{A}_0) \cap \big(D(A_1) \cap N(A_0^*)\big)$ (complex)

  $$D(A_1) \ni x_n = A_0 z_n + \tilde{x}_n, \qquad z_n \in D(\mathcal{A}_0), \qquad \tilde{x}_n \in D(A_1) \cap N(A_0^*)$$

- $\Rightarrow$ (z_n) is bd in $D(\mathcal{A}_0)$ by ortho and Friedrichs/Poincaré type est, i.e.,

  $$\exists\, c_{A_0} > 0 \qquad \forall\, z \in D(\mathcal{A}_0) \qquad |z|_{H_0} \leq c_{A_0} |A_0 z|_{H_1}$$

- $\Rightarrow$ $(\tilde{x}_n)$ is bd in $D(A_1) \cap N(A_0^*)$ by ortho and $A_1 \tilde{x}_n = A_1 x_n$ (complex)
- $D(A_1) \cap D(A_0^*) \hookrightarrow H_1$ cpt $\Rightarrow$ $D(\mathcal{A}_0) \hookrightarrow H_0$ cpt
- $\Rightarrow$ $\exists$ $z \in D(\mathcal{A}_0)$ and $\tilde{x} \in D(A_1) \cap N(A_0^*)$ st (extract subsequences)
  $z_n \rightharpoonup z$ in $D(A_0)$ and $z_n \to z$ in H_0
  $\tilde{x}_n \rightharpoonup \tilde{x}$ in $D(A_1) \cap N(A_0^*)$ and $\tilde{x}_n \to \tilde{x}$ in H_1
- $x = A_0 z + \tilde{x}$ (ortho Helm type deco for x)
- Finally $\langle x_n, y_n \rangle_{H_1} = \langle A_0 z_n, y_n \rangle_{H_1} + \langle \tilde{x}_n, y_n \rangle_{H_1} = \langle z_n, A_0^* y_n \rangle_{H_0} + \langle \tilde{x}_n, y_n \rangle_{H_1}$
  $\to \langle z, A_0^* y \rangle_{H_0} + \langle \tilde{x}, y \rangle_{H_1} = \langle A_0 z, y \rangle_{H_1} + \langle \tilde{x}, y \rangle_{H_1} = \langle x, y \rangle_{H_1}$
- q.e.d.

□

fa-foolbox $\Rightarrow$ red stuff

A_0^*-A_1-lemma (fa-toolbox, some fundamental results)

\begin{fundamental part of fa-toolbox}

A_0^*-A_1-lemma (fa-toolbox, some fundamental results)

$A : D(A) \subset H_0 \to H_1$ lddc, $A^* : D(A^*) \subset H_1 \to H_0$ Hilbert space adjoint

(A, A^*) dual pair as $(A^*)^* = \overline{A} = A$

A, A^* may not be inj

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$H_1 = N(A^*) \oplus \overline{R(A)} \qquad H_0 = N(A) \oplus \overline{R(A^*)}$$

reduced operators restr to $N(A)^\perp$ and $N(A^*)^\perp$

$$\mathcal{A} := A|_{N(A)^\perp} = A|_{\overline{R(A^*)}} \qquad \mathcal{A}^* := A^*|_{N(A^*)^\perp} = A^*|_{\overline{R(A)}}$$

$\mathcal{A}$, $\mathcal{A}^*$ inj $\quad \Rightarrow \quad \mathcal{A}^{-1}$, $(\mathcal{A}^*)^{-1}$ ex

A_0^*-A_1-lemma (fa-toolbox, some fundamental results)

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1, \quad \mathrm{A}^* : D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0$ lddc $\qquad (\mathrm{A}, \mathrm{A}^*)$ dual pair

$$\mathrm{H}_1 = N(\mathrm{A}^*) \oplus \overline{R(\mathrm{A})} \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

more precisely

$$\mathcal{A} := \mathrm{A}|_{\overline{R(\mathrm{A}^*)}} : D(\mathcal{A}) \subset \overline{R(\mathrm{A}^*)} \to \overline{R(\mathrm{A})}, \qquad D(\mathcal{A}) := D(\mathrm{A}) \cap N(\mathrm{A})^{\perp} = D(\mathrm{A}) \cap \overline{R(\mathrm{A}^*)}$$

$$\mathcal{A}^* := \mathrm{A}^*|_{\overline{R(\mathrm{A})}} : D(\mathcal{A}^*) \subset \overline{R(\mathrm{A})} \to \overline{R(\mathrm{A}^*)}, \quad D(\mathcal{A}^*) := D(\mathrm{A}^*) \cap N(\mathrm{A}^*)^{\perp} = D(\mathrm{A}^*) \cap \overline{R(\mathrm{A})}$$

$(\mathcal{A}, \mathcal{A}^*)$ dual pair and $\mathcal{A}$, $\mathcal{A}^*$ inj $\Rightarrow$

inverse ops exist (and bij)

$$\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A}) \qquad (\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$$

refined decompositions

$$D(\mathrm{A}) = N(\mathrm{A}) \oplus D(\mathcal{A}) \qquad D(\mathrm{A}^*) = N(\mathrm{A}^*) \oplus D(\mathcal{A}^*)$$

$\Rightarrow$

$$R(\mathrm{A}) = R(\mathcal{A}) \qquad R(\mathrm{A}^*) = R(\mathcal{A}^*)$$

A_0^*-A_1-lemma (fa-toolbox, some fundamental results)

closed range theorem & closed graph theorem $\Rightarrow$

Lemma (Friedrichs-Poincaré type est/cl range/cont inv)

The following assertions are equivalent:

(i) $\exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathrm{H}_0} \le c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$

(i*) $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathrm{H}_1} \le c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$

(ii) $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_1.

(ii*) $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_0.

(iii) $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is continuous and bijective.*

(iii*) $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is continuous and bijective.*

In case that one of the latter assertions is true, e.g., (ii), $R(\mathrm{A})$ *is closed, we have*

$$\mathrm{H}_0 = N(\mathrm{A}) \oplus R(\mathrm{A}^*) \qquad \mathrm{H}_1 = N(\mathrm{A}^*) \oplus R(\mathrm{A})$$
$$D(\mathrm{A}) = N(\mathrm{A}) \oplus D(\mathcal{A}) \qquad D(\mathrm{A}^*) = N(\mathrm{A}^*) \oplus D(\mathcal{A}^*)$$
$$D(\mathcal{A}) = D(\mathrm{A}) \cap R(\mathrm{A}^*) \qquad D(\mathcal{A}^*) = D(\mathrm{A}^*) \cap R(\mathrm{A})$$

and $\quad \mathcal{A} : D(\mathcal{A}) \subset R(\mathrm{A}^*) \to R(\mathrm{A}), \quad \mathcal{A}^* : D(\mathcal{A}^*) \subset R(\mathrm{A}) \to R(\mathrm{A}^*).$

A_0^*-A_1-lemma (fa-toolbox, some fundamental results)

recall

(i) $\exists\, c_A \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{H_0} \le c_A |Ax|_{H_1}$

(i*) $\exists\, c_{A^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{H_1} \le c_{A^*} |A^* y|_{H_0}$

'best' consts in **(i)** and **(i*)** equal norms of the inv ops and Rayleigh quotients

$$c_A = |\mathcal{A}^{-1}|_{R(A),R(A^*)} \qquad c_{A^*} = |(\mathcal{A}^*)^{-1}|_{R(A^*),R(A)}$$

$$\frac{1}{c_A} = \inf_{0 \ne x \in D(\mathcal{A})} \frac{|Ax|_{H_1}}{|x|_{H_0}} \qquad \frac{1}{c_{A^*}} = \inf_{0 \ne y \in D(\mathcal{A}^*)} \frac{|A^* y|_{H_0}}{|y|_{H_1}}$$

Lemma (Friedrichs-Poincaré type const)

$c_A = c_{A^*}$

A_0^*-A_1-lemma (fa-toolbox, some fundamental results)

Lemma (cpt emb/cpt inv)

The following assertions are equivalent:

(i) $D(\mathcal{A}) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_0$ *is compact.*

(i*) $D(\mathcal{A}^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$ *is compact.*

(ii) $\mathcal{A}^{-1} : R(\mathrm{A}) \to R(\mathrm{A}^*)$ *is compact.*

(ii*) $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to R(\mathrm{A})$ *is compact.*

Lemma (Friedrichs-Poincaré type est/cl range/cont inv)

$\Downarrow$ $\boxed{D(\mathcal{A}) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_0 \text{ compact}}$

(i) $\exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathrm{H}_0} \le c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$

(i*) $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathrm{H}_1} \le c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$

(ii) $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_1.

(ii*) $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_0.

(iii) $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is continuous and bijective.*

(iii*) $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is continuous and bijective.*

(i)-(iii*) *equi* & *the resp Helm deco hold* & $|\mathcal{A}^{-1}| = c_{\mathrm{A}} = c_{\mathrm{A}^*} = |(\mathcal{A}^*)^{-1}|$

A_0^*-A_1-lemma (fa-toolbox, some fundamental results)

So far no complex...

$A_0 : D(A_0) \subset H_0 \to H_1, \quad A_1 : D(A_1) \subset H_1 \to H_2$ (lddc)
$A_0^* : D(A_0^*) \subset H_1 \to H_0, \quad A_1^* : D(A_1^*) \subset H_2 \to H_1$ (lddc)

general complex ($A_1 A_0 = 0$, i.e., $\quad R(A_0) \subset N(A_1)$ and $R(A_1^*) \subset N(A_0^*)$)

$$\cdots \; \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \; H_0 \; \underset{A_0^*}{\overset{A_0}{\rightleftarrows}} \; H_1 \; \underset{A_1^*}{\overset{A_1}{\rightleftarrows}} \; H_2 \; \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \; \cdots$$

recall Helmholtz deco

$$\begin{array}{rcl} H_1 = & \overline{R(A_0)} \oplus N(A_0^*) & \\ & \cap \qquad \cup & \\ = & N(A_1) \oplus \overline{R(A_1^*)} & \end{array} \qquad \Rightarrow \quad \text{(e.g.)} \; N(A_1) = \overline{R(A_0)} \oplus \big(\underbrace{N(A_1) \cap N(A_0^*)}_{=:K_1} \big)$$

$\Rightarrow$ refined Helmholtz deco

$$H_1 = \overline{R(A_0)} \oplus K_1 \oplus \overline{R(A_1^*)}$$

A_0^*-A_1-lemma (fa-toolbox, some fundamental results)

recall

$$D(\mathrm{A}_1) = D(\mathcal{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \qquad R(\mathrm{A}_1) = R(\mathcal{A}_1) \qquad R(\mathrm{A}_1^*) = R(\mathcal{A}_1^*)$$

$$D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \cap \overline{R(\mathrm{A}_0)} \qquad R(\mathrm{A}_0^*) = R(\mathcal{A}_0^*) \qquad R(\mathrm{A}_0) = R(\mathcal{A}_0)$$

cohomology group $K_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)$

Lemma (Helmholtz deco I)

$$\mathrm{H}_1 = \overline{R(\mathrm{A}_0)} \oplus N(\mathrm{A}_0^*) \qquad \mathrm{H}_1 = \overline{R(\mathrm{A}_1^*)} \oplus N(\mathrm{A}_1)$$

$$D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus N(\mathrm{A}_0^*) \qquad D(\mathrm{A}_1) = D(\mathcal{A}_1) \oplus N(\mathrm{A}_1)$$

$$N(\mathrm{A}_1) = D(\mathcal{A}_0^*) \oplus K_1 \qquad N(\mathrm{A}_0^*) = D(\mathcal{A}_1) \oplus K_1$$

$$D(\mathrm{A}_1) = \overline{R(\mathrm{A}_0)} \oplus \big(D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)\big) \qquad D(\mathrm{A}_0^*) = \overline{R(\mathrm{A}_1^*)} \oplus \big(D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1)\big)$$

Lemma (Helmholtz deco II)

$$\mathrm{H}_1 = \overline{R(\mathrm{A}_0)} \oplus K_1 \oplus \overline{R(\mathrm{A}_1^*)}$$

$$D(\mathrm{A}_1) = \overline{R(\mathrm{A}_0)} \oplus K_1 \oplus D(\mathcal{A}_1)$$

$$D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus K_1 \oplus \overline{R(\mathrm{A}_1^*)}$$

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus K_1 \oplus D(\mathcal{A}_1)$$

A_0^*-A_1-lemma (fa-toolbox, some fundamental results)

$$K_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \qquad D(\mathrm{A}_1) = D(\mathcal{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \qquad D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \cap \overline{R(\mathrm{A}_0)}$$

Lemma (cpt emb II)

The following assertions are equivalent:

(i) $D(\mathcal{A}_0) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_0$, $D(\mathcal{A}_1) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1$, *and* $K_1 \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1$ *are compact.*

(ii) $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1$ *is compact.*

In this case $K_1 < \infty$.

Theorem (fa-toolbox I)

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1 \text{ compact}}$

(i) *all emb cpt, i.e.,* $D(\mathcal{A}_0) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_0$, $D(\mathcal{A}_1) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1$, $D(\mathcal{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1$, $D(\mathcal{A}_1^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_2$ *cpt*

(ii) *cohomology group* K_1 *finite dim*

(iii) *all ranges closed, i.e.,* $R(\mathrm{A}_0) = R(\mathcal{A}_0)$, $R(\mathrm{A}_0^*) = R(\mathcal{A}_0^*)$ *cl,*
$R(\mathrm{A}_1) = R(\mathcal{A}_1)$, $R(\mathrm{A}_1^*) = R(\mathcal{A}_1^*)$ *cl*

(iv) *all Friedrichs-Poincaré type est hold*

(v) *all Hodge-Helmholtz-Weyl type deco I & II hold with closed ranges*

A_0^*-A_1-lemma (fa-toolbox, some fundamental results)

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} H_0 \underset{A_0^*}{\overset{A_0}{\rightleftarrows}} H_1 \underset{A_1^*}{\overset{A_1}{\rightleftarrows}} H_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

Theorem (fa-toolbox I (Friedrichs-Poincaré type est))

$\Downarrow$ $\boxed{D(A_1) \cap D(A_0^*) \hookrightarrow\hookrightarrow H_1 \textit{ compact}}$ $\Rightarrow$ $\exists$ $|\mathcal{A}_i^{-1}| = c_{A_i} = c_{A_i^*} = |(\mathcal{A}_i^*)^{-1}| \in (0, \infty)$

(i) $\forall x \in D(\mathcal{A}_0)$ $|x|_{H_0} \leq c_{A_0} |A_0 x|_{H_1}$

(i*) $\forall y \in D(\mathcal{A}_0^*)$ $|y|_{H_1} \leq c_{A_0} |A_0^* y|_{H_0}$

(ii) $\forall y \in D(\mathcal{A}_1)$ $|y|_{H_1} \leq c_{A_1} |A_1 y|_{H_2}$

(ii*) $\forall z \in D(\mathcal{A}_1^*)$ $|z|_{H_2} \leq c_{A_1} |A_1^* z|_{H_1}$

(iii) $\forall y \in D(A_1) \cap D(A_0^*)$ $|(1 - \pi_{K_1}) y|_{H_1} \leq c_{A_1} |A_1 y|_{H_2} + c_{A_0} |A_0^* y|_{H_0}$

note $\pi_{K_1} y \in K_1$ and $(1 - \pi_{K_1}) y \in K_1^\perp$

Remark

enough $R(A_0)$ and $R(A_1)$ cl

A_0^*-A_1-lemma (fa-toolbox, some fundamental results)

complex $\cdots \overset{\cdots}{\underset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \overset{\mathrm{A}_0}{\underset{\mathrm{A}_0^*}{\rightleftarrows}} \mathrm{H}_1 \overset{\mathrm{A}_1}{\underset{\mathrm{A}_1^*}{\rightleftarrows}} \mathrm{H}_2 \overset{\cdots}{\underset{\cdots}{\rightleftarrows}} \cdots$

Theorem (fa-toolbox I (Helmholtz deco))

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\rightarrow \mathrm{H}_1 \textit{ compact}}$

$$\mathrm{H}_1 = R(\mathrm{A}_0) \oplus N(\mathrm{A}_0^*) \qquad \mathrm{H}_1 = R(\mathrm{A}_1^*) \oplus N(\mathrm{A}_1)$$

$$D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus N(\mathrm{A}_0^*) \qquad D(\mathrm{A}_1) = D(\mathcal{A}_1) \oplus N(\mathrm{A}_1)$$

$$N(\mathrm{A}_1) = D(\mathcal{A}_0^*) \oplus K_1 \qquad N(\mathrm{A}_0^*) = D(\mathcal{A}_1) \oplus K_1$$

$$D(\mathrm{A}_1) = R(\mathrm{A}_0) \oplus \big(D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)\big) \qquad D(\mathrm{A}_0^*) = R(\mathrm{A}_1^*) \oplus \big(D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1)\big)$$

$$\mathrm{H}_1 = R(\mathrm{A}_0) \oplus K_1 \oplus R(\mathrm{A}_1^*)$$

$$D(\mathrm{A}_1) = R(\mathrm{A}_0) \oplus K_1 \oplus D(\mathcal{A}_1)$$

$$D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus K_1 \oplus R(\mathrm{A}_1^*)$$

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus K_1 \oplus D(\mathcal{A}_1)$$

Remark

enough $R(\mathrm{A}_0)$ *and* $R(\mathrm{A}_1)$ *cl*

A_0^*-A_1-lemma (fa-toolbox, some fundamental results)

\end{fundamental part of fa-toolbox}

classical de Rham complex in 3D (∇-rot-div-complex)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot{\cup} \Gamma_n}$

(electro-magneto dynamics, Maxwell's equations)

$$\{0\} \quad \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{-\operatorname{div}}{\overset{\mathring{\nabla}}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{\operatorname{rot}}{\overset{\mathring{\operatorname{rot}}}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{-\nabla}{\overset{\mathring{\operatorname{div}}}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}} \quad \mathbb{R}$$

mixed boundary conditions and inhomogeneous and anisotropic media

$$\{0\} \text{ or } \mathbb{R} \quad \underset{\pi}{\overset{\iota}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{-\operatorname{div}_{\Gamma_n}\varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}} \quad \mathsf{L}^2_{\varepsilon} \quad \underset{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{\iota}{\overset{\pi}{\rightleftarrows}} \quad \mathbb{R} \text{ or } \{0\}$$

classical de Rham complex in 3D (∇-rot-div-complex)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot\cup \Gamma_n}$

(electro-magneto dynamics, Maxwell's equations with mixed boundary conditions)

$$\{0\} \text{ or } \mathbb{R} \; \underset{\pi}{\overset{\iota}{\rightleftarrows}} \; \mathrm{L}^2 \; \underset{-\operatorname{div}_{\Gamma_n}\varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}} \; \mathrm{L}^2_\varepsilon \; \underset{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}} \; \mathrm{L}^2 \; \underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}} \; \mathrm{L}^2 \; \underset{\iota}{\overset{\pi}{\rightleftarrows}} \; \mathbb{R} \text{ or } \{0\}$$

related fos

$$\begin{array}{ll|ll|ll|ll} \nabla_{\Gamma_t} u = A & \text{in } \Omega & \operatorname{rot}_{\Gamma_t} E = J & \text{in } \Omega & \operatorname{div}_{\Gamma_t} H = k & \text{in } \Omega & \pi v = b & \text{in } \Omega \\ \pi u = a & \text{in } \Omega & -\operatorname{div}_{\Gamma_n} \varepsilon E = j & \text{in } \Omega & \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} H = K & \text{in } \Omega & -\nabla_{\Gamma_n} v = B & \text{in } \Omega \end{array}$$

related sos

$$\begin{array}{ll|ll|ll} -\operatorname{div}_{\Gamma_n} \varepsilon \nabla_{\Gamma_t} u = j & \text{in } \Omega & \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} \operatorname{rot}_{\Gamma_t} E = K & \text{in } \Omega & -\nabla_{\Gamma_n} \operatorname{div}_{\Gamma_t} H = B & \text{in } \Omega \\ \pi u = a & \text{in } \Omega & -\operatorname{div}_{\Gamma_n} \varepsilon E = j & \text{in } \Omega & \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} H = K & \text{in } \Omega \end{array}$$

corresponding compact embeddings:

$$D(\nabla_{\Gamma_t}) \cap D(\pi) = D(\nabla_{\Gamma_t}) = \mathrm{H}^1_{\Gamma_t} \hookrightarrow \mathrm{L}^2 \qquad \text{(Rellich's selection theorem)}$$

$$D(\operatorname{rot}_{\Gamma_t}) \cap D(-\operatorname{div}_{\Gamma_n} \varepsilon) = \mathrm{R}_{\Gamma_t} \cap \varepsilon^{-1} \mathrm{D}_{\Gamma_n} \hookrightarrow \mathrm{L}^2_\varepsilon \qquad \text{(Weck's selection theorem, '72/'74)}$$

$$D(\operatorname{div}_{\Gamma_t}) \cap D(\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}) = \mathrm{D}_{\Gamma_t} \cap \mathrm{R}_{\Gamma_n} \hookrightarrow \mathrm{L}^2 \qquad \text{(Weck's selection theorem, '72/'74)}$$

$$D(\nabla_{\Gamma_n}) \cap D(\pi) = D(\nabla_{\Gamma_n}) = \mathrm{H}^1_{\Gamma_n} \hookrightarrow \mathrm{L}^2 \qquad \text{(Rellich's selection theorem)}$$

Weck's selection theorem for weak Lip. dom. and mixed bc: Bauer/Py/Schomburg ('16)

de Rham complex in ND or on Riemannian manifolds (d-complex)

$\Omega \subset \mathbb{R}^N$ bd w. Lip. dom. or Ω Riemannian manifold with cpt cl. and Lip. boundary Γ

(generalized Maxwell equations)

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^{2,0} \underset{-\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \mathsf{L}^{2,1} \underset{-\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \dots \mathsf{L}^{2,q} \underset{-\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \mathsf{L}^{2,q+1} \dots \mathsf{L}^{2,N-1} \underset{-\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \mathsf{L}^{2,N} \underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}} \mathbb{R}$$

de Rham complex in ND or on Riemannian manifolds (d-complex)

$\Omega \subset \mathbb{R}^N$ bd w. Lip. dom. or Ω Riemannian manifold with cpt cl. and Lip. boundary Γ

(generalized Maxwell equations)

$$\{0\} \text{ or } \mathbb{R} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathsf{L}^{2,0} \underset{-\delta^1_{\Gamma_n}}{\overset{\mathrm{d}^0_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^{2,1} \underset{-\delta^2_{\Gamma_n}}{\overset{\mathrm{d}^1_{\Gamma_t}}{\rightleftarrows}} \ldots \mathsf{L}^{2,q} \underset{-\delta^{q+1}_{\Gamma_n}}{\overset{\mathrm{d}^q_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^{2,q+1} \ldots \mathsf{L}^{2,N-1} \underset{-\delta^N_{\Gamma_n}}{\overset{\mathrm{d}^{N-1}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^{2,N} \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathbb{R} \text{ or } \{0\}$$

related fos

$$\mathrm{d}^q_{\Gamma_t} E = F \qquad \text{in } \Omega$$

$$-\delta^q_{\Gamma_n} E = G \qquad \text{in } \Omega$$

related sos

$$-\delta^{q+1}_{\Gamma_n} \mathrm{d}^q_{\Gamma_t} E = F \qquad \text{in } \Omega$$

$$-\delta^q_{\Gamma_n} E = G \qquad \text{in } \Omega$$

includes: EMS rot / div, Laplacian, rot rot, and more...
corresponding compact embeddings:

$$D(\mathrm{d}^q_{\Gamma_t}) \cap D(\delta^q_{\Gamma_n}) \hookrightarrow \mathsf{L}^{2,q} \qquad \text{(Weck's selection theorems, '72/'74)}$$

Weck's selection theorem for Lip. manifolds and mixed bc: Bauer/Py/Schomburg ('17)

elasticity complex in 3D (sym ∇-Rot $\mathrm{Rot}_{\mathbb{S}}^{\top}$-$\mathrm{Div}_{\mathbb{S}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\,\mathrm{Div}_{\mathbb{S}}}{\overset{\mathring{\mathrm{sym}}\,\nabla}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top}}{\overset{\mathrm{Rot}\,\mathring{\mathrm{Rot}}_{\mathbb{S}}^{\top}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{-\,\mathrm{sym}\,\nabla}{\overset{\mathring{\mathrm{Div}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathsf{RM}}}{\overset{\pi_{\mathsf{RM}}}{\rightleftarrows}} \mathsf{RM}$$

elasticity complex in 3D (sym ∇-Rot Rot$_{\mathbb{S}}^{\top}$-Div$_{\mathbb{S}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{Div}_{\mathbb{S}}}{\overset{\operatorname{sym}\mathring{\nabla}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}}{\overset{\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^{\top}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{-\operatorname{sym}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathsf{RM}}}{\overset{\pi_{\mathsf{RM}}}{\rightleftarrows}} \mathsf{RM}$$

related fos (Rot Rot$_{\mathbb{S},\Gamma}^{\top}$, Rot Rot$_{\mathbb{S}}^{\top}$ first order operators!)

$$\begin{array}{llllllll} \operatorname{sym}\nabla_{\Gamma} v = M & \text{in } \Omega & | \quad \operatorname{Rot}\operatorname{Rot}_{\mathbb{S},\Gamma}^{\top} M = F & \text{in } \Omega & | \quad \operatorname{Div}_{\mathbb{S},\Gamma} N = g & \text{in } \Omega & | \quad \pi v = r & \text{in } \Omega \\ \pi v = 0 & \text{in } \Omega & | \quad -\operatorname{Div}_{\mathbb{S}} M = f & \text{in } \Omega & | \quad \operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top} N = G & \text{in } \Omega & | \quad -\operatorname{sym}\nabla v = M & \text{in } \Omega \end{array}$$

related sos (Rot Rot$_{\mathbb{S}}^{\top}$ Rot Rot$_{\mathbb{S},\Gamma}^{\top}$ second order operator!)

$$\begin{array}{llllll} -\operatorname{Div}_{\mathbb{S}}\operatorname{sym}\nabla_{\Gamma} v = f & \text{in } \Omega & | \quad \operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}\operatorname{Rot}\operatorname{Rot}_{\mathbb{S},\Gamma}^{\top} M = G & \text{in } \Omega & | \quad -\operatorname{sym}\nabla\operatorname{Div}_{\mathbb{S},\Gamma} N = M & \text{in } \Omega \\ \pi v = 0 & \text{in } \Omega & | \quad -\operatorname{Div}_{\mathbb{S}} M = f & \text{in } \Omega & | \quad \operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top} N = G & \text{in } \Omega \end{array}$$

corresponding compact embeddings:

$$D(\operatorname{sym}\nabla_{\Gamma}) \cap D(\pi) = D(\nabla_{\Gamma}) = \mathsf{H}^1_{\Gamma} \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem and Korn ineq.)}$$

$$D(\operatorname{Rot}\operatorname{Rot}_{\mathbb{S},\Gamma}^{\top}) \cap D(\operatorname{Div}_{\mathbb{S}}) \hookrightarrow \mathsf{L}^2_{\mathbb{S}} \qquad \text{(new selection theorem)}$$

$$D(\operatorname{Div}_{\mathbb{S},\Gamma}) \cap D(\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}) \hookrightarrow \mathsf{L}^2_{\mathbb{S}} \qquad \text{(new selection theorem)}$$

$$D(\pi) \cap D(\operatorname{sym}\nabla) = D(\nabla) = \mathsf{H}^1 \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem and Korn ineq.)}$$

two new selection theorems for strong Lip. dom.: Py/Schomburg/Zulehner ('18)

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\mathrm{Rot}_{\mathbb{S}}$-$\mathrm{Div}_{\mathbb{T}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \;\underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\nabla\nabla}}{\rightleftarrows}}\; \mathsf{L}^2_{\mathbb{S}} \;\underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}}\; \mathsf{L}^2_{\mathbb{T}} \;\underset{-\operatorname{dev}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota_{\mathsf{RT}}}{\overset{\pi_{\mathsf{RT}}}{\rightleftarrows}}\; \mathsf{RT}$$

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\operatorname{Rot}_{\mathbb{S}}$-$\operatorname{Div}_{\mathbb{T}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\nabla\nabla}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}} \underset{-\operatorname{dev}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathsf{RT}}}{\overset{\pi_{\mathsf{RT}}}{\rightleftarrows}} \mathsf{RT}$$

related fos ($\nabla\nabla_\Gamma$, $\operatorname{div}\operatorname{Div}_{\mathbb{S}}$ first order operators!)

$$\begin{array}{rl|rl|rl|rl} \nabla\nabla_\Gamma u = M & \text{in } \Omega & \operatorname{Rot}_{\mathbb{S},\Gamma} M = F & \text{in } \Omega & \operatorname{Div}_{\mathbb{T},\Gamma} N = g & \text{in } \Omega & \pi v = r & \text{in } \Omega \\ \pi u = 0 & \text{in } \Omega & \operatorname{div}\operatorname{Div}_{\mathbb{S}} M = f & \text{in } \Omega & \operatorname{sym}\operatorname{Rot}_{\mathbb{T}} N = G & \text{in } \Omega & -\operatorname{dev}\nabla v = T & \text{in } \Omega \end{array}$$

related sos ($\operatorname{div}\operatorname{Div}_{\mathbb{S}}\nabla\nabla_\Gamma = \Delta^2_\Gamma$ second order operator!)

$$\begin{array}{rl|rl|rl} \operatorname{div}\operatorname{Div}_{\mathbb{S}}\nabla\nabla_\Gamma u = \Delta^2_\Gamma u = f & \text{in } \Omega & \operatorname{sym}\operatorname{Rot}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma} M = G & \text{in } \Omega & -\operatorname{dev}\nabla\operatorname{Div}_{\mathbb{T},\Gamma} N = T & \text{in } \Omega \\ \pi u = 0 & \text{in } \Omega & \operatorname{div}\operatorname{Div}_{\mathbb{S}} M = f & \text{in } \Omega & \operatorname{sym}\operatorname{Rot}_{\mathbb{T}} N = G & \text{in } \Omega \end{array}$$

corresponding compact embeddings:

$$D(\nabla\nabla_\Gamma) \cap D(\pi) = D(\nabla\nabla_\Gamma) = \mathsf{H}^2_\Gamma \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

$$D(\operatorname{Rot}_{\mathbb{S},\Gamma}) \cap D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \hookrightarrow \mathsf{L}^2_{\mathbb{S}} \qquad \text{(new selection theorem)}$$

$$D(\operatorname{Div}_{\mathbb{T},\Gamma}) \cap D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \hookrightarrow \mathsf{L}^2_{\mathbb{T}} \qquad \text{(new selection theorem)}$$

$$D(\pi) \cap D(\operatorname{dev}\nabla) = D(\operatorname{dev}\nabla) = D(\nabla) = \mathsf{H}^1 \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem and Korn type ineq.)}$$

two new selection theorems for strong Lip. dom. and Korn Type ineq.: Py/Zulehner ('16)

literature

results of this talk (gen global div-curl-lemma, A_0^*-A_1-lemma, fa-toolbox, cpt emb):

- Bauer, S., Py, Schomburg, M.: *The Maxwell Compactness Property in Bounded Weak Lipschitz Domains with Mixed Boundary Conditions*, (SIMA) SIAM Journal on Mathematical Analysis, 2016
- Py: *Solution Theory and Functional A Posteriori Error Estimates for General First Order Systems with Applications to Electro-Magneto-Statics*, (NFAO) Numerical Functional Analysis and Optimization, 2018
- Py: *A Global div-curl-Lemma for Mixed Boundary Conditions in Weak Lipschitz Domains and a Corresponding Generalized* A_0^*-A_1*-Lemma in Hilbert Spaces*, (ANA) Analysis (Munich), 2018

recent papers (global gen div-curl-lemma, similar results):

- Waurick, M.: *A Functional Analytic Perspective to the div-curl Lemma*, (JOP) J. Operator Theory, 2018

(parts of) fa-toolbox used for numerical purposes by:

- Arnold, D., Falk, R., Winther, R.
- Hiptmair, R.
- Kettunen, L.
- Schöberl, J.

literature

some more results of this talk:

- Py: *On Maxwell's and Poincare's Constants*,
(DCDS) Discrete and Continuous Dynamical Systems - Series S, 2015
- Zulehner, W., Py: *On Closed and Exact Grad grad- and div Div-Complexes, Corresponding Compact Embeddings for Tensor Rotations, and a Related Decomposition Result for Biharmonic Problems in 3D*,
submitted, 2016

upcoming books:

- Langer, U., Py, Repin, S. (Eds): *Maxwell's equations. Analysis and numerics*,
Radon Series on Applied Mathematics, De Gruyter, 2018
- Py: *Maxwell's Equations: Hilbert Space Methods for the Theory of Electromagnetism*,
Radon Series on Applied Mathematics, De Gruyter, $\approx$ 2020

(last book: contains all results of this talk and more...)

. . . the world is full of complexes . . . ;)

⇒ relaxing at ...

AANMPDE 11

11th Workshop on Analysis and Advanced Numerical Methods for Partial Differential Equations (not only) for Junior Scientists

https://www.uni-due.de/mathematik/ag-pauly
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https://www.uni-due.de/maxwell

August 6–10 2018, Särkisaari, Finland

organizers: Ulrich Langer, Py, Sergey Repin