LOW-FREQUENCY ASYMPTOTICS FOR TIME-HARMONIC MAXWELL EQUATIONS IN EXTERIOR DOMAINS

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Open-Minded ;-)

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OVERVIEW

CLASSICAL TIME-HARMONIC SCATTERING MAXWELL PROBLEM

time-harmonic Maxwell (electro-magnetic scattering) problem
in $\Omega \subset \mathbb{R}^3$ exterior domain

$$\begin{aligned}
-\operatorname{rot} H_\omega + \mathrm{i}\,\omega\varepsilon E_\omega &= F && \text{in } \Omega && \text{(pde)}\\
\operatorname{rot} E_\omega + \mathrm{i}\,\omega\mu H_\omega &= G && \text{in } \Omega && \text{(pde)}\\
\nu \times E_\omega &= 0 (= \lambda) && \text{on } \partial\Omega && \text{(boundary cond.)}\\
E\,,\ H &= O(r^{-1}) && \text{for } r \to \infty && \text{(decay cond.)}\\
\xi \times E_\omega + H_\omega\,,\ -\xi \times H_\omega + E_\omega &= o(r^{-1}) && \text{for } r \to \infty && \text{(Silver-Müller radiation cond.)}
\end{aligned}$$

here: $0 \neq \omega \in \mathbb{C}$, $r(x) = |x|$, $\xi(x) := x/|x|$

inhom. aniso. media $\varepsilon, \mu \in \mathsf{L}^\infty(\Omega, \mathbb{R}^{3\times 3})$, sym, unif. pos. def.

QUESTION / AIM: low frequency asymptotics?

$$\boxed{\lim_{\omega\to 0} E_\omega, \quad \lim_{\omega \to 0} H_\omega \quad ?}$$

CLASSICAL TIME-HARMONIC SCATTERING MAXWELL PROBLEM

analytical motivation:

- Weck, N. and Witsch, K.-J.: CPDE, (1992)
  Complete low frequency Analysis for the reduced wave Equation with variable coefficients in three dimensions
- Weck, N. and Witsch, K.-J.: M2AS, (1997)
  Generalized linear elasticity in exterior domains – I: radiation problems
- Weck, N. and Witsch, K.-J.: M2AS, (1997)
  Generalized linear elasticity in exterior domains – II: low-frequency asymptotics

analytical/numerical motivation:

- Ammari, H. and Nédélec, J.-C.: SIAM JMA, (2000)
  Low-frequency electromagnetic scattering
- Ammari, H. and Buffa, A. and Nédélec, J.-C.: SIAM JAM, (2000)
  A justification of eddy currents model for the Maxwell equations
  (! cited 49 times in MathSciNet / unfortunately wrong !)

disadvantages of Ammari/Nédélec-papers

- no identification of terms in the expansion by proper boundary value problems
- estimates just in local L^2-norms
- non local boundary conditions due to EtM-operators (DtN-operators)
- comp. supp. F, G; $\varepsilon = \mu = 1$

CLASSICAL TIME-HARMONIC SCATTERING MAXWELL PROBLEM

more compact and proper notation

$$(M-\omega)u_\omega = f \in \mathsf{L}^2_{>1/2}(\Omega)\times\mathsf{L}^2_{>1/2}(\Omega)$$
$$u_\omega \in \mathring{\mathbf{H}}_{<-1/2}(\mathrm{rot};\Omega)\times\mathbf{H}_{<-1/2}(\mathrm{rot};\Omega)$$
$$(S+1)u_\omega \in \mathsf{L}^2_{>-1/2}(\Omega)\times\mathsf{L}^2_{>-1/2}(\Omega)$$

here: $u_\omega := (E_\omega, H_\omega)$, $f := \mathrm{i}\,\Lambda^{-1}(F,G)$, $\Lambda = \begin{bmatrix}\varepsilon & 0\\ 0 & \mu\end{bmatrix}$, $\Lambda^{-1} = \begin{bmatrix}\varepsilon^{-1} & 0\\ 0 & \mu^{-1}\end{bmatrix}$,
$M = \mathrm{i}\,\Lambda^{-1}\,\mathrm{Rot}$, $\mathrm{Rot} := \begin{bmatrix}0 & -\mathrm{rot}\\ \mathrm{rot} & 0\end{bmatrix}$, $S = C_{\mathrm{Rot},r} = \begin{bmatrix}0 & -\xi\times\\ \xi\times & 0\end{bmatrix}$

$M : \mathring{\mathbf{H}}(\mathrm{rot};\Omega)\times\mathbf{H}(\mathrm{rot};\Omega)\subset\mathsf{L}^2(\Omega)\times\mathsf{L}^2(\Omega)\to\mathsf{L}^2(\Omega)\times\mathsf{L}^2(\Omega)$ s.a. unbd. lin. op.

$\Rightarrow$ unique L^2-solutions u_ω for $\omega\in\mathbb{C}\setminus\mathbb{R}$

later: gen. Fredholm alternative for $\omega\in\mathbb{R}\setminus\{0\}$
(Eidus' principle of limiting absorption (1962), a priori estimates)

QUESTION: low frequency asymptotics? $\boxed{\lim_{\mathbb{C}\setminus\{0\}\ni\omega\to 0} u_\omega}$

METHOD: Weck & Witsch, i.e., full ext. dom. and no artificial boundary

GENERALIZED TIME-HARMONIC SCATTERING MAXWELL PROBLEM

gen. time-harmonic Maxwell (electro-magnetic scattering) problem
in $\Omega \subset \mathbb{R}^N$ exterior domain, $0 \neq \omega \in \mathbb{C}$

$$\begin{aligned}
\delta H_\omega + \mathrm{i}\,\omega\varepsilon E_\omega &= F && \text{in } \Omega && \text{(pde)}\\
\mathrm{d}\, E_\omega + \mathrm{i}\,\omega\mu H_\omega &= G && \text{in } \Omega && \text{(pde)}\\
\iota^* E_\omega &= 0 (= \lambda) && \text{on } \partial\Omega && \text{(bc)}\\
E\,,\ H &= O(r^{-1}) && \text{for } r \to \infty && \text{(dc)}\\
\mathrm{d}\ r \wedge E_\omega + H_\omega\,,\ (-1)^{qN} * \mathrm{d}\ r \wedge *H_\omega + E_\omega &= o(r^{-1}) && \text{for } r \to \infty && \text{(gen. Silver-Müller rc)}
\end{aligned}$$

here: E, F q-forms, H, G $(q+1)$-forms
inhom. aniso. media ε, μ (linear transformations) sym, unif. pos. def.

QUESTION / AIM: low frequency asymptotics?

$$\boxed{\lim_{\omega \to 0} E_\omega, \quad \lim_{\omega \to 0} H_\omega \quad ?}$$

GENERALIZED TIME-HARMONIC SCATTERING MAXWELL PROBLEM

time-harmonic Maxwell problem in $\Omega \subset \mathbb{R}^N$ exterior domain
for simplicity $N \geq 3$ odd, frequencies from upper half plane $\omega \in \mathbb{C}_+$

$$(M - \omega)u_\omega = f \in \mathsf{L}^{2,q,q+1}_{>1/2}(\Omega)$$

$$u_\omega \in \overset{\circ}{\mathbf{D}}{}^q_{<-\frac{1}{2}}(\Omega) \times \mathbf{\Delta}^{q+1}_{<-\frac{1}{2}}(\Omega)$$

$$(S + 1)u_\omega \in \mathsf{L}^{2,q,q+1}_{>-1/2}(\Omega)$$

here: $u_\omega := (E_\omega, H_\omega)$, $f := \mathrm{i}\,\Lambda^{-1}(F, G)$, E, F q-forms, H, G $(q+1)$-forms,
$M = \mathrm{i}\,\Lambda^{-1}\begin{bmatrix} 0 & \delta \\ \mathrm{d} & 0 \end{bmatrix}$, $\Lambda = \begin{bmatrix} \varepsilon & 0 \\ 0 & \mu \end{bmatrix}$, $S = \begin{bmatrix} 0 & T \\ R & 0 \end{bmatrix}$, $R := \mathrm{d}\, r\wedge$, $T := \pm * R *$
d ext. deriv., $\delta = \pm * \mathrm{d} *$ co-deriv., $R = C_{\mathrm{d},r}$, $T = C_{\delta,r}$

$M : \overset{\circ}{\mathbf{D}}{}^q(\Omega) \times \mathbf{\Delta}^{q+1}(\Omega) \subset \mathsf{L}^{2,q,q+1}(\Omega) \to \mathsf{L}^{2,q,q+1}(\Omega)$ s.a. unbd. lin. op.

denote sol. op. of time-harmonic prob. by $\mathcal{L}_\omega := (M - \omega)^{-1}$ $(u_\omega = \mathcal{L}_\omega f)$

QUESTION: low frequency asymptotics?

$$\boxed{\lim_{\mathbb{C}_+ \setminus \{0\} \ni \omega \to 0} \mathcal{L}_\omega = ?}$$

(topology: operator norm of polyn. weighted Sobolev spaces)

BOUNDED DOMAIN

time-harmonic Maxwell problem in $\Omega \subset \mathbb{R}^N$ bounded Lipschitz domain

$$(M-\omega)u_\omega = f \in \mathsf{L}^{2,q,q+1}(\Omega)$$

$$u_\omega \in \mathring{\mathbf{D}}^q(\Omega) \times \mathbf{\Delta}^{q+1}(\Omega) =: D(M)$$

Helmholtz deco. $\Rightarrow$ $\mathsf{L}^{2,q,q+1}(\Omega) = N(M) \oplus_\Lambda \overline{R(M)}$

$$M : D(M) \subset \mathsf{L}^{2,q,q+1}(\Omega) \to \mathsf{L}^{2,q,q+1}(\Omega) \quad \text{s.a.},$$

$$\mathcal{M} : D(\mathcal{M}) := D(M) \cap \overline{R(M)} \subset \overline{R(M)} \to \overline{R(M)} \quad \text{s.a.}$$

Maxwell compactness prop., i.e., $D(\mathcal{M}) \hookrightarrow \mathsf{L}^{2,q,q+1}(\Omega)$ comp.

$\Rightarrow$ Maxwell estimate, i.e., $\exists\, c_m > 0 \quad \forall\, u \in D(\mathcal{M}) \quad \|u\|_{\mathsf{L}^{2,q}(\Omega)} \leq c_m \|\mathcal{M}u\|_{\mathsf{L}^{2,q}(\Omega)}$

$\Rightarrow$ $R(M) = R(\mathcal{M})$ closed and $\mathcal{L}_0 := \mathcal{M}^{-1} : R(M) \to D(\mathcal{M})$ cont.

$\Rightarrow$ $\mathcal{L}_0 : R(M) \to R(M)$ comp. (static sol. op. cont./comp.)

standard sol. theory $\Rightarrow$ Fredholm's alternative, especially

$$\sigma_p(\mathcal{M}) = \sigma(\mathcal{M}) = \sigma(M) \setminus \{0\} = \sigma_p(M) \setminus \{0\} = \pm\{\omega_n\}_{n=1}^\infty \subset \mathbb{R} \setminus \{0\}$$

with (ω_n) strongly monotone and unbounded

$\Rightarrow$ sol. op. time-harmonic prob. ($f \mapsto u_\omega = \mathcal{L}_\omega f$) well def. for $0 < |\omega|$ small

$$\mathcal{L}_\omega : \mathsf{L}^{2,q,q+1}(\Omega) \to D(M), \quad \mathcal{L}_\omega : R(M) \to D(\mathcal{M})$$

BOUNDED DOMAIN

time-harmonic Maxwell problem in $\Omega \subset \mathbb{R}^N$ bounded Lipschitz domain

$$(M - \omega)u_\omega = f \in \mathsf{L}^{2,q,q+1}(\Omega)$$
$$u_\omega \in D(M)$$

Helmholtz deco. $\Rightarrow$ $\mathsf{L}^{2,q,q+1}(\Omega) = N(M) \oplus_\Lambda R(M)$ and $D(M) = N(M) \oplus_\Lambda D(\mathcal{M})$

orth.-norm.-projectors $\Pi : \mathsf{L}^{2,q,q+1}(\Omega) \to N(M)$, $1 - \Pi : \mathsf{L}^{2,q,q+1}(\Omega) \to R(M)$

$$\Rightarrow \quad -\omega\Pi u_\omega = \Pi f \quad \text{and} \quad (M-\omega)(1-\Pi)u_\omega = (1-\Pi)f \in \mathsf{L}^{2,q,q+1}(\Omega)$$
$$\Pi u_\omega \in N(M) \qquad\qquad (1-\Pi)u_\omega \in D(\mathcal{M})$$

note: $D(\mathcal{M}) = D(M) \cap R(M) = \big(\overset{\circ}{\mathbf{D}}{}^q(\Omega) \cap \varepsilon^{-1}\,\delta\,\mathbf{\Delta}^{q+1}(\Omega)\big) \times \big(\mathbf{\Delta}^{q+1}(\Omega) \cap \mu^{-1}\,\mathrm{d}\,\overset{\circ}{\mathbf{D}}{}^q(\Omega)\big)$

set $v := (1-\Pi)u_\omega \in D(\mathcal{M}) \subset R(M)$ and $g := (1-\Pi)f \in R(M)$ $\quad \Rightarrow \quad \mathcal{L}_0\, Mv = v$

$\Rightarrow \quad (M-\omega)v = g \quad \Leftrightarrow \quad (1 - \omega\,\mathcal{L}_0)v = \mathcal{L}_0\, g$

$$\overset{\text{Neumann ser.}}{\Leftrightarrow} \qquad v = (1-\omega\,\mathcal{L}_0)^{-1}\,\mathcal{L}_0\, g = \sum_{j=0}^{\infty} \omega^j\,\mathcal{L}_0^j\,\mathcal{L}_0\, g$$

for small $0 < |\omega|$ since $\|\omega\,\mathcal{L}_0\| < 1 \quad \Leftrightarrow \quad |\omega| < 1/\|\mathcal{L}_0\|$ (1st pos. Maxwell ev)

$$\Rightarrow \quad \mathcal{L}_\omega f = u_\omega = \Pi u_\omega + v = -\omega^{-1}\Pi f + \sum_{j=0}^{\infty} \omega^j\,\mathcal{L}_0^{j+1}(1-\Pi)f$$

BOUNDED DOMAIN

$\Rightarrow$ low frequency asymptotics in L^2-operator norm

$$\mathcal{L}_\omega = \underbrace{-\omega^{-1}\Pi}_{\text{trivial part}} + \underbrace{\sum_{j=0}^{\infty} \omega^j \, \mathcal{L}_0{}^{j+1} \, \Pi_{\text{reg}}}_{\text{Neumann series}}, \quad \omega \in \mathbb{C}_+ \setminus \{0\} \text{ small}$$

$\Pi : \mathsf{L}^{2,q,q+1}(\Omega) \to N(M), \quad \Pi_{\text{reg}} := 1 - \Pi : \mathsf{L}^{2,q,q+1}(\Omega) \to R(M)$

$\mathcal{L}_0 : R(M) \to D(M) \cap R(M)$

problems if Ω exterior domain

- this low frequency asymptotic is wrong, even not well defined
- static solution theory needs weighted Poincare estimate!
  $\Rightarrow$ leaving L^2-setting
  e.g., static sol. op. maps unweighted data f to $(1+r)^{-1}$-weighted sol. u_0
- not clear how to define higher powers of $\mathcal{L}_0$?
- careful investigation of static sol. theo. in weighted Sobolev spaces

EXTERIOR DOMAIN

aim: give meaning to Neumann sum in terms of an asymptotic expansion

$$\mathcal{L}_\omega + \omega^{-1}\Pi - \sum_{j=0}^{J-1} \omega^j \, {\mathcal{L}_0}^{j+1} \, \Pi_{\mathrm{reg}} = O\big(|\omega|^J\big) \quad , \quad J \in \mathbb{N}_0, \quad \omega \in \mathbb{C}_+ \setminus \{0\} \text{ small}$$

<u>3 major complications</u>

- growing $J \Rightarrow$ stronger data norms for f and weaker solution norms for $u_\omega = \mathcal{L}_\omega f$
- Π , Π_{reg} indicate need for polyn. weighted Hodge-Helmholtz deco. of

  $$\mathsf{L}_s^{2,q,q+1}(\Omega) = \big(\,\mathsf{Tri}_s^q(\Omega) \dotplus \mathsf{Reg}_s^{q,-1}(\Omega)\big) \cap \mathsf{L}_s^{2,q,q+1}(\Omega)$$

  respecting inhomogeneities Λ (topological direct decomposition)

  $$(N(M) =)\, \mathsf{Tri}_s^q(\Omega) = \Pi \mathsf{L}_s^{2,q,q+1}(\Omega) \subset {}_0\mathring{\mathsf{D}}_t^q(\Omega) \times {}_0\Delta_t^{q+1}(\Omega)$$
  $$\mathsf{Reg}_s^{q,-1}(\Omega) = \Pi_{\mathrm{reg}} \mathsf{L}_s^{2,q,q+1}(\Omega) \subset \Lambda^{-1}\big({}_0\Delta_t^q(\Omega) \times {}_0\mathring{\mathsf{D}}_t^{q+1}(\Omega)\big)$$

  only subspaces of $\mathsf{L}_t^{2,q,q+1}(\Omega)$ with $t \leq s$ and $t < N/2$
  <u>not</u> of $\mathsf{L}_s^{2,q,q+1}(\Omega)$ if $s \geq N/2$
- expansion has to be corrected by special, explicitly computable degenerate op.

EXTERIOR DOMAIN

more precisely: $J \in \mathbb{N}_0$ and $s, -t > 1/2$ as well as $f \in \mathsf{L}^{2,q,q+1}_s(\Omega)$

$\Rightarrow$ main result: asymptotic estimates

$$\Big\| \mathcal{L}_\omega f + \omega^{-1}\Pi f - \sum_{j=0}^{J-1} \omega^j \, \mathcal{L}_0{}^{j+1} \, \Pi_{\mathrm{reg}} f - \omega^{N-1} \sum_{j=0}^{J-N} \omega^j \Gamma_j f \Big\|_{\mathsf{L}^{2,q,q+1}_t(\Omega)} = O\big(|\omega|^J\big) \, \|f\|_{\mathsf{L}^{2,q,q+1}_s(\Omega)}$$

O-symbol always for $\omega \to 0$ and uniformly w.r.t. ω and f

with $\omega \in \mathbb{C}_+ \setminus \{0\}$ and $|\omega| \leq \hat{\omega}$, where $\hat{\omega} > 0$

GENERAL ASSUMPTIONS

- $\Omega \subset \mathbb{R}^N$ exterior domain with Lipschitz boundary
  (Maxwell local compactness property,
  exist. of special forms with bounded supports repl. Dirichlet/Neumann forms)
- $1 \leq q \leq N-2$ and odd space dimensions N (class. $N=3$, $q=1$)
  (even dim., especially $N=2$, OK
  but logarithmic terms due to expansions of Hankel's functions)
- fix radius $r_0 > 0$ with $\mathbb{R}^N \setminus \Omega \subset B_{r_0}$, cut-off function η
- $\varepsilon = \mathrm{Id} + \hat{\varepsilon}$, $\mu = \mathrm{Id} + \hat{\mu}$ ($\Lambda = \mathrm{Id} + \hat{\Lambda}$) τ-C^1-admissible, i.e.,
  linear, real, sym., unif. pos. def. L^∞-transformations with $\hat{\Lambda} \in \mathrm{C}^1$ for $|x| > r_0$
  asymptotically homogeneous, i.e.,
  $\partial^\alpha \hat{\Lambda} = O(r^{-\tau-|\alpha|})$ for all $|\alpha| \leq 1$ with order of decay τ at infinity,
  $\tau > 0$ depending on t, s

DESCRIPTION OF RESULTS

- degenerate correction operators Γ_j by recursion consisting of

$$E^{+}_{\sigma,m},\ H^{+}_{\sigma,n},\quad E^{+,k}_{\sigma,m} =: \mathcal{L}^k_0(E^{+}_{\sigma,m},0),\ H^{+,k}_{\sigma,n} =: \mathcal{L}^k_0(0,H^{+}_{\sigma,n}) \in \mathsf{L}^{2,q,q+1}_{-N/2-\sigma-k}(\Omega)$$

  sol. of hom. static boundary value problems with inhom. at infinity, e.g.,

$$E^{+}_{\sigma,m} \in {}_0\mathring{\mathsf{D}}^q_{\mathrm{loc}}(\Omega) \cap \varepsilon^{-1}\big({}_0\Delta^q_{\mathrm{loc}}(\Omega) \cap \mathring{\mathsf{B}}^q(\Omega)^{\perp}\big)$$
$$E^{+}_{\sigma,m} - {}^{+}\Delta^{q,0}_{\sigma,m} \in \mathsf{L}^{2,q}_{>-\frac{N}{2}}(\Omega)$$

  'harmonic polynomials' ${}^{+}\Delta^{q,k}_{\sigma,m}$ behave like $r^{k+\sigma}$ at infinity ($k,\sigma \geq 0$)

- 'trivial' subspace $\mathrm{Tri}^q_s(\Omega) = \Pi\mathsf{L}^{2,q,q+1}_s(\Omega) \subset {}_0\mathring{\mathsf{D}}^q_t(\Omega) \times {}_0\Delta^{q+1}_t(\Omega)\ \big(\subset N(M)\big)$

$$\mathcal{L}_\omega f = -\omega^{-1} f, \quad f \in \mathrm{Tri}^q_s(\Omega)$$

- two kinds of media $\Lambda = \mathrm{Id} + \hat{\Lambda}$
  1. $\hat{\Lambda}$ comp. supp., results for any J
  2. $\hat{\Lambda}$ 'decays' with $\tau > 0$ at infinity, results for $J \leq \hat{J}$ dep. on τ

DESCRIPTION OF RESULTS

- identify closed subspaces $\mathrm{Reg}_s^{q,J}(\Omega)$ of $\mathrm{Reg}_s^{q,0}(\Omega) \subset \mathrm{L}_s^{2,q,q+1}(\Omega)$, 'spaces of regular convergence', $\Rightarrow$ 'usual' Neumann expansion

$$\Big\| \mathcal{L}_\omega f - \sum_{j=0}^{J-1} \omega^j \mathcal{L}_0{}^{j+1} f \Big\|_{\mathrm{L}_t^{2,q,q+1}(\Omega)} = O\big(|\omega|^J\big) \|f\|_{\mathrm{L}_s^{2,q,q+1}(\Omega)}$$

for $f \in \mathrm{Reg}_s^{q,J}(\Omega)$

- charact. of $\mathrm{Reg}_s^{q,J}(\Omega)$ by orthogonality in L^2 to the spec. grow. st. sol. $E_{\sigma,m}^{+,k}$, $H_{\sigma,n}^{+,k}$
- corrected Neumann expansion

$$\Big\| \mathcal{L}_\omega f - \sum_{j=0}^{J-1} \omega^j \mathcal{L}_0{}^{j+1} f - \omega^{N-1} \sum_{j=0}^{J-N} \omega^j \Gamma_j f \Big\|_{\mathrm{L}_t^{2,q,q+1}(\Omega)} = O\big(|\omega|^J\big) \|f\|_{\mathrm{L}_s^{2,q,q+1}(\Omega)}$$

for $f \in \mathrm{Reg}_s^{q,-1}(\Omega) = \Pi_{\mathrm{reg}} \mathrm{L}_s^{2,q,q+1}(\Omega) \subset \Lambda^{-1}\big({}_0\Delta_t^q(\Omega) \times {}_0\mathring{\mathrm{D}}_t^{q+1}(\Omega)\big)$

- fully corrected Neumann expansion

$$\Big\| \mathcal{L}_\omega f + \omega^{-1} \Pi f - \sum_{j=0}^{J-1} \omega^j \mathcal{L}_0{}^{j+1} \Pi_{\mathrm{reg}} f - \omega^{N-1} \sum_{j=0}^{J-N} \omega^j \Gamma_j f \Big\|_{\mathrm{L}_t^{2,q,q+1}(\Omega)} = O\big(|\omega|^J\big) \|f\|_{\mathrm{L}_s^{2,q,q+1}(\Omega)}$$

for $f \in \mathrm{L}_s^{2,q,q+1}(\Omega) = \big(\mathrm{Tri}_s^q(\Omega) \dotplus \mathrm{Reg}_s^{q,-1}(\Omega)\big) \cap \mathrm{L}_s^{2,q,q+1}(\Omega)$

MAIN RESULT

Theorem (low frequency asymptotics)

Let $J \in \mathbb{N}$ *and* $s \notin \mathbb{I} = (\mathbb{N}_0 + N/2) \cup (1 - N/2 - \mathbb{N}_0)$ *with*

$$s > J + 1/2, \qquad (f)$$

$$t < \min\{N/2 - J - 2, -1/2\}, \qquad (u_\omega)$$

$$\tau > \max\{(N+1)/2, s - t\}. \qquad (\hat{\Lambda})$$

Then for all small enough $\mathbb{C}_+ \setminus \{0\} \ni \omega \to 0$ *the asymptotic expansion*

$$\mathcal{L}_\omega + \omega^{-1}\Pi - \sum_{j=0}^{J-1} \omega^j \, \mathcal{L}_0{}^{j+1} \, \Pi_{\mathrm{reg}} - \omega^{N-1} \sum_{j=0}^{J-N} \omega^j \Gamma_j = O(|\omega|^J)$$

holds in the norm of bounded linear operators from $\mathsf{L}^{2,q,q+1}_s(\Omega)$ *to* $\mathsf{L}^{2,q,q+1}_t(\Omega)$.

Remark *The main theorem holds also for* $J = 0$ *with slightly different* t *and* τ.

TIME-HARMONIC SCATTERING PROBLEM

Solving $(M-\omega)u_\omega = f$?

$$M \quad : \quad \overset{\circ}{\mathbf{D}}{}^q(\Omega) \times \mathbf{\Delta}^{q+1}(\Omega) \subset \mathsf{L}^{2,q,q+1}_\Lambda(\Omega) \longrightarrow \mathsf{L}^{2,q,q+1}_\Lambda(\Omega)$$
$$u \longmapsto \mathrm{i}\,\Lambda^{-1}\begin{bmatrix}0 & \delta\\ \mathrm{d} & 0\end{bmatrix} u$$

M unbd. lin. s.a. $\quad \Rightarrow \quad \sigma(\mathcal{M}) \subset \mathbb{R}$

$\omega \in \mathbb{C}\setminus\mathbb{R} \quad \Rightarrow \quad \mathcal{L}_\omega = (M-\omega)^{-1}$ bounded $\quad \Rightarrow \quad \mathsf{L}^2$-sol. for all $f \in \mathsf{L}^{2,q,q+1}(\Omega)$

solving in $\sigma(\mathcal{M})\setminus\{0\}$ with Eidus' 'limiting absorption principle' (approx. from $\mathbb{C}_+$)

Definition (time-harmonic (scattering) solutions)

Let $\omega \in \mathbb{R}\setminus\{0\}$ and $f \in \mathsf{L}^{2,q,q+1}_{\mathrm{loc}}(\Omega)$. u_ω solves Max(f,ω), iff

$$\begin{array}{lll} \text{(i)} & \forall \quad t < -1/2 & u_\omega \in \overset{\circ}{\mathbf{D}}{}^q_t(\Omega) \times \mathbf{\Delta}^{q+1}_t(\Omega), \\ \text{(ii)} & \exists \quad t > -1/2 & (S+1)u_\omega \in \mathsf{L}^{2,q,q+1}_t(\Omega), \\ \text{(iii)} & & (M-\omega)u_\omega = f. \end{array}$$

TOOLS: a priori estimate, polynomial decay of eigensolutions, decomposition lemma, Helmholtz' equation

TIME-HARMONIC SCATTERING PROBLEM

Theorem (time-harmonic (scattering) solution theory)

Let $\omega \in \mathbb{R} \setminus \{0\}$ and $s > 1/2$, $\tau > 1$.

(i) $\mathrm{Max}(0,\omega) \subset \big(\mathring{\mathbf{D}}_t^q(\Omega) \cap \varepsilon^{-1}\,\delta\,\mathbf{\Delta}_t^{q+1}(\Omega)\big) \times \big(\mathbf{\Delta}_t^{q+1}(\Omega) \cap \mu^{-1}\,\mathrm{d}\,\mathring{\mathbf{D}}_t^q(\Omega)\big)$ *for all* $t \in \mathbb{R}$, *i.e. gen. eigensolutions decay polynomially (and exponentially for* $\Lambda \in \mathrm{C}^2$*), no gen. eigenvalues for* $\Lambda = \mathrm{Id}$*, comp. Helmholtz eq., Rellich's est., princ. uniq. cont.*

(ii) $\dim \mathrm{Max}(0,\omega) < \infty$

(iii) $\sigma_{gen}(M)$ *has no accumulation point in* $\mathbb{R} \setminus \{0\}$

(iv) *Fredholm's Alternative holds:*
$\forall\, f \in \mathrm{L}_s^{2,q,q+1}(\Omega) \quad \exists\, u_\omega$ *solution of* $\mathrm{Max}(f,\omega)$*, iff*

$$\forall \quad v \in \mathrm{Max}(0,\omega) \qquad \langle f, v\rangle_{\mathrm{L}_\Lambda^{2,q,q+1}(\Omega)} = 0$$

The solution u_ω *can be chosen, such that*

$$\forall \quad v \in \mathrm{Max}(0,\omega) \qquad \langle u_\omega, v\rangle_{\mathrm{L}_\Lambda^{2,q,q+1}(\Omega)} = 0.$$

Then u_ω *is uniquely determined.*

(v) *For all* $t < -1/2$ *the solution operator* $\mathcal{L}_\omega$ *maps* $\mathrm{L}_s^{2,q,q+1}(\Omega) \cap \mathrm{Max}(0,\omega)^{\perp_\Lambda}$ *to* $\big(\mathring{\mathbf{D}}_t^q(\Omega) \times \mathbf{\Delta}_t^{q+1}(\Omega)\big) \cap \mathrm{Max}(0,\omega)^{\perp_\Lambda}$ *continuously.*

LOW FREQUENCY TIME-HARMONIC SCATTERING PROBLEM

Theorem (low frequency time-harmonic estimate)

Let $\tau > (N+1)/2$ and $s \in (1/2, N/2)$ as well as $t := s-(N+1)/2 \in (-N/2, -1/2)$.

(i) *$\sigma_{gen}(M)$ does not accumulate in $\mathbb{R}$ (especially not at zero). $\sigma_{gen}(M) \cap \mathbb{C}_+ = \{0\}$ for ω sufficiently small.*

(ii) *$\mathcal{L}_\omega$ is well defined on $\mathsf{L}^{2,q,q+1}_s(\Omega)$ for all $0 \neq \omega \in \mathbb{C}_+$ small enough.*

(iii) $\exists\, c > 0 \quad \forall\, 0 \neq \omega \in \mathbb{C}_+$ *small enough* $\quad \forall\, \Lambda f = \Lambda(F,G) \in \boldsymbol{\Delta}^q_s(\Omega) \times \mathring{\mathbf{D}}^{q+1}_s(\Omega)$

$$\|\mathcal{L}_\omega f\|_{\mathsf{L}^{2,q,q+1}_t(\Omega)} \leq c\Big(\|f\|_{\mathsf{L}^{2,q,q+1}_s(\Omega)} + |\omega|^{-1}\|(\delta\,\varepsilon F, \mathrm{d}\,\mu G)\|_{\mathsf{L}^{2,q-1,q+2}_s(\Omega)}$$
$$+ |\omega|^{-1}\sum_{\ell=1}^{d^q}\big|\langle \varepsilon F, \mathring{b}^q_\ell\rangle_{\mathsf{L}^{2,q}(\Omega)}\big| + |\omega|^{-1}\sum_{\ell=1}^{d^{q+1}}\big|\langle \mu G, b^{q+1}_\ell\rangle_{\mathsf{L}^{2,q+1}(\Omega)}\big|\Big).$$

Especially $\|\mathcal{L}_\omega f\|_{\mathsf{L}^{2,q,q+1}_t(\Omega)} \leq c\|f\|_{\mathsf{L}^{2,q,q+1}_s(\Omega)}$ *holds for*

$$\Lambda f = \Lambda(F,G) \in {}_0\mathbb{\Delta}^q_s(\Omega) \times {}_0\mathring{\mathbb{D}}^{q+1}_s(\Omega) := \big({}_0\Delta^q_s(\Omega) \cap \mathring{\mathsf{B}}^q(\Omega)^\perp\big) \times \big({}_0\mathring{\mathsf{D}}^{q+1}_s(\Omega) \cap \mathsf{B}^{q+1}(\Omega)^\perp\big),$$

i.e., no terms with negative frequency power $|\omega|^{-1}$ occur.

TOOLS: fundamental sol. Helmholtz' eq. (Hankel's function), repr. of sol. for $\Omega = \mathbb{R}^N$ as conv., cutt. tech., indirect arg.

FIRST LOW FREQUENCY ASYMPTOTIC

Theorem (first and simple static solution theory)

Let $\tau > 0$. Then there exists a linear and bounded static solution operator

$$\mathcal{L}_0 : \Lambda^{-1}\big({}_0\mathbb{\Delta}^q(\Omega)\times{}_0\mathring{\mathbb{D}}^{q+1}(\Omega)\big) \to \big(\mathring{\mathsf{D}}^q_{-1}(\Omega)\times\Delta^{q+1}_{-1}(\Omega)\big)\cap\Lambda^{-1}\big({}_0\mathbb{\Delta}^q_{-1}(\Omega)\times{}_0\mathring{\mathbb{D}}^{q+1}_{-1}(\Omega)\big).$$

More precisely: $u = (E, H) = \mathcal{L}_0 f$ for $f = (F, G)$ solves $Mu = f$, i.e., the static system

$$\begin{aligned} &\mathrm{i}\,\mu^{-1}\,\mathrm{d}\,E = G, && \delta\,\varepsilon E = 0, && \varepsilon E \perp \mathring{\mathsf{B}}^q(\Omega),\\ &\mathrm{i}\,\varepsilon^{-1}\,\delta\,H = F, && \mathrm{d}\,\mu H = 0, && \mu H \perp \mathsf{B}^{q+1}(\Omega). \end{aligned}$$

Theorem (first and simple low frequency asymptotics)

Let $\tau > (N+1)/2$ and $s \in (1/2, N/2)$ as well as $t < s - (N+1)/2 \in (-N/2, -1/2)$. Then

$$\boxed{\lim_{\mathbb{C}_+\ni\omega\to 0}\mathcal{L}_\omega = \mathcal{L}_0}$$

in the norm of bounded linear operators

$$\Lambda^{-1}\big({}_0\mathbb{\Delta}^q_s(\Omega)\times{}_0\mathring{\mathbb{D}}^{q+1}_s(\Omega)\big) \longrightarrow \mathring{\mathbf{D}}^q_t(\Omega)\times\mathbf{\Delta}^{q+1}_t(\Omega).$$

EXTENDED STATIC SOLUTION THEORY

Theorem (extended static solution theory)

Let $s \in (1 - N/2, \infty) \setminus \mathbb{I}$ and $\tau > \max\{0, s - N/2\}$, $\tau \geq -s$. Then

$$\begin{array}{rcccc} \mathrm{i}\,\mu^{-1}\,\mathrm{d} & : & \big(\mathring{\mathsf{D}}^{q}_{s-1}(\Omega) \boxplus \eta\,\curlywedge^{q,0,-}_{s-1}\big) \cap \varepsilon^{-1}\,{}_0\mathbb{\Delta}^{q}_{\mathrm{loc}}(\Omega) & \longrightarrow & \mu^{-1}\,{}_0\mathring{\mathbb{D}}^{q+1}_{s}(\Omega) \\ & & E & \longmapsto & \mathrm{i}\,\mu^{-1}\,\mathrm{d}\,E \end{array},$$

$$\begin{array}{rcccc} \mathrm{i}\,\varepsilon^{-1}\,\delta & : & \big(\Delta^{q+1}_{s-1}(\Omega) \boxplus \eta\mathcal{D}^{q+1,0,-}_{s-1}\big) \cap \mu^{-1}\,{}_0\mathring{\mathbb{D}}^{q+1}_{\mathrm{loc}}(\Omega) & \longrightarrow & \varepsilon^{-1}\,{}_0\mathbb{\Delta}^{q}_{s}(\Omega) \\ & & H & \longmapsto & \mathrm{i}\,\varepsilon^{-1}\,\delta\,H \end{array}$$

are topological isomorphisms.

note: $\curlywedge^{q,0,-}_{s-1} = \curlywedge^{q}(\bar{\mathbb{J}}^{q,0}_{s-1})$ finite dim. subspace of $\mathrm{C}^{\infty}(\mathbb{R}^N \setminus \{0\})$

$\eta\,\curlywedge^{q,0,-}_{s-1} \subset \mathrm{L}^{2,q}_{t}(\Omega)$ for $t \leq s-1$, $t < N/2$ and $\eta\,\curlywedge^{q,0,-}_{s-1} \not\subset \mathrm{L}^{2,q}_{s-1}(\Omega)$

same for $\mathcal{D}^{q+1,0,-}_{s-1} = \mathcal{D}^{q+1}(\bar{\mathcal{J}}^{q+1,0}_{s-1})$

consisting of 'neg. tower-forms' of shape $r^{\ell}\,\check{\tau}\,S^{q}_{m,n}$ ($S^{q}_{m,n}$ gen. spherical harmonics)

EXTENDED STATIC SOLUTION THEORY

Corollary (extended static solution theory)

Let $s \in (1 - N/2, \infty) \setminus \mathbb{I}$ *and* $\tau > \max\{0, s - N/2\}$, $\tau \geq -s$. *Then*

$$M : \Big(\big(\mathring{\mathrm{D}}^{q}_{s-1}(\Omega) \times \Delta^{q+1}_{s-1}(\Omega) \big) \boxplus \big(\eta \mathcal{R}^{q,0,-}_{s-1} \times \eta \mathcal{D}^{q+1,0,-}_{s-1} \big) \Big) \cap \Lambda^{-1} \big({}_0\mathbb{\Delta}^{q}_{\mathrm{loc}}(\Omega) \times {}_0\mathring{\mathbb{D}}^{q+1}_{\mathrm{loc}}(\Omega) \big)$$

$$\longrightarrow \Lambda^{-1} \big({}_0\mathbb{\Delta}^{q}_{s}(\Omega) \times {}_0\mathring{\mathbb{D}}^{q+1}_{s}(\Omega) \big)$$

$$u = (E, H) \longmapsto Mu = \mathrm{i}\, \Lambda^{-1} (\delta\, H, \mathrm{d}\, E)$$

is a topological isomorphism with bounded inverse

$$\boxed{\mathcal{L}_0 = M^{-1} : \Lambda^{-1} \big({}_0\mathbb{\Delta}^{q}_{s}(\Omega) \times {}_0\mathring{\mathbb{D}}^{q+1}_{s}(\Omega) \big) \longrightarrow \Lambda^{-1} \big({}_0\mathbb{\Delta}^{q}_{s-1}(\bar{\mathfrak{I}}^{q,0}_{s-1}, \Omega) \times {}_0\mathring{\mathbb{D}}^{q+1}_{s-1}(\bar{\mathfrak{J}}^{q+1,0}_{s-1}, \Omega) \big).}$$

goal: higher powers of $\mathcal{L}_0$ even acting on $\Lambda^{-1} \big({}_0\mathbb{\Delta}^{q}_{s-1}(\mathfrak{I}, \Omega) \times {}_0\mathring{\mathbb{D}}^{q+1}_{s-1}(\mathfrak{J}, \Omega) \big)$

TOWER FORMS

...		...	...	
	$\delta \swarrow$			$\searrow$ d
3. floor	$\pm\Delta^{q-1,3}_{\sigma,m}$			$\pm D^{q+1,3}_{\sigma,m}$
	d $\searrow$			$\swarrow \delta$
2. floor		$\pm D^{q,2}_{\sigma,m}$	$\pm\Delta^{q,2}_{\sigma,m}$	
	$\delta \swarrow$			$\searrow$ d
1. floor	$\pm\Delta^{q-1,1}_{\sigma,m}$			$\pm D^{q+1,1}_{\sigma,m}$
	d $\searrow$			$\swarrow \delta$
ground		$\pm D^{q,0}_{\sigma,m}$ $\cong$	$\pm\Delta^{q,0}_{\sigma,m}$	
	d-tower		δ-tower	

$\pm\Delta^{q,k}_{\sigma,m}, \pm D^{q,k}_{\sigma,m} \in \mathrm{C}^\infty(\mathbb{R}^N \setminus \{0\})$ homogeneous of deg. $k+\sigma$ resp. $k-\sigma-N$

HIGHER POWERS OF THE STATIC SOLUTION OPERATOR

Theorem (higher powers of $\mathcal{L}_0$)

Let $j \in \mathbb{N}$ and $s \in (j - N/2, \infty) \setminus \mathbb{I}$ and $\mathfrak{I}, \mathfrak{J}$ finite index sets as well as $\tau \geq j - 1 - s$, $\tau > \max\{0, s - N/2\}$ and $\tau > s + N/2 + \max\{\mathrm{h}_{\mathfrak{I}}, \mathrm{h}_{\mathfrak{J}}\}$. Then

$$\mathcal{L}_0^j \ : \ \Lambda^{-1}\big({}_0\mathbb{\Delta}_s^q(\mathfrak{I},\Omega) \times {}_0\mathring{\mathbb{D}}_s^{q+1}(\mathfrak{J},\Omega)\big)$$
$$\longrightarrow \Lambda^{-1}\begin{cases} {}_0\mathbb{\Delta}_{s-j}^q(\bar{\mathfrak{I}}_{s-j}^{q,\leq j-1} \cup {}_j\mathfrak{I},\Omega) \times {}_0\mathring{\mathbb{D}}_{s-j}^{q+1}(\bar{\mathfrak{J}}_{s-j}^{q+1,\leq j-1} \cup {}_j\mathfrak{J},\Omega) & \text{, if } j \text{ even} \\ {}_0\mathbb{\Delta}_{s-j}^q(\bar{\mathfrak{I}}_{s-j}^{q,\leq j-1} \cup {}_j\mathfrak{J},\Omega) \times {}_0\mathring{\mathbb{D}}_{s-j}^{q+1}(\bar{\mathfrak{J}}_{s-j}^{q+1,\leq j-1} \cup {}_j\mathfrak{I},\Omega) & \text{, if } j \text{ odd} \end{cases}$$

is a continuous linear operator with range in $\Lambda^{-1}\big({}_0\mathbb{\Delta}_t^q(\Omega) \times {}_0\mathring{\mathbb{D}}_t^{q+1}(\Omega)\big)$ for $t \leq s - j$, $t < N/2 - j + 1$, $t < -j - N/2 - \max\{\mathrm{h}_{\mathfrak{I}}, \mathrm{h}_{\mathfrak{J}}\}$.

SPACES OF REGULAR CONVERGENCE

$$\mathrm{Reg}_s^{q,-1}(\Omega) = \Pi_{\mathrm{reg}} \mathrm{L}_s^{2,q,q+1}(\Omega) \subset \Lambda^{-1}\big({}_0\Delta_t^q(\Omega) \times {}_0\mathring{\mathrm{D}}_t^{q+1}(\Omega)\big)$$

$$\mathrm{Reg}_s^{q,0}(\Omega) := \Lambda^{-1}\big({}_0\mathbb{\Delta}_s^q(\Omega) \times {}_0\mathring{\mathbb{D}}_s^{q+1}(\Omega)\big)$$

$$\mathrm{Reg}_s^{q,j}(\Omega) := \big\{ f \in \mathrm{Reg}_s^{q,0}(\Omega) : \mathcal{L}_0^j f \in \mathrm{L}_{s-j}^{2,q,q+1}(\Omega) \big\}$$

'usual Neumann sum'

Lemma (spaces of regular convergence)

Let $J \in \mathbb{N}_0$ and $s \in (J+1/2, \infty) \setminus \mathbb{I}$ as well as $\tau > \max\big\{(N+1)/2, s - N/2\big\}$. Then for all $0 \neq \omega \in \mathbb{C}_+$ small enough on $\mathrm{Reg}_s^{q,J}(\Omega)$ the resolvent formula

$$\mathcal{L}_\omega - \sum_{j=0}^{J-1} \omega^j \, \mathcal{L}_0^{j+1} = \omega^J \, \mathcal{L}_\omega \, \mathcal{L}_0^J$$

holds. Especially for $s \in (J+1/2, J+N/2) \setminus \mathbb{I}$ and $t = s - J - (N+1)/2$

$$\boxed{\Big\| \mathcal{L}_\omega f - \sum_{j=0}^{J-1} \omega^j \, \mathcal{L}_0^{j+1} f \Big\|_{\mathrm{L}_t^{2,q,q+1}(\Omega)} = O\big(|\omega|^J\big) \big\| f \big\|_{\mathrm{L}_s^{2,q,q+1}(\Omega)}}$$

holds uniformly w.r.t. $f \in \mathrm{Reg}_s^{q,J}(\Omega)$.

aim: characterize $\mathrm{Reg}_s^{q,j}(\Omega)$ by orthogonality constraints

GROWING STATIC SOLUTIONS

again conditions on τ ...

$$E^{+}_{\sigma,m} \in {}_0\mathring{\mathsf{D}}^{q}_{\mathrm{loc}}(\Omega) \cap \varepsilon^{-1}\,{}_0\mathbb{\Delta}^{q}_{\mathrm{loc}}(\Omega)$$
$$E^{+}_{\sigma,m} - {}^{+}\Delta^{q,0}_{\sigma,m} \in \mathsf{L}^{2,q}_{>-\frac{N}{2}}(\Omega)$$

$$H^{+}_{\sigma,m} \in {}_0\Delta^{q+1}_{\mathrm{loc}}(\Omega) \cap \mu^{-1}\,{}_0\mathring{\mathbb{D}}^{q+1}_{\mathrm{loc}}(\Omega)$$
$$H^{+}_{\sigma,m} - {}^{+}D^{q+1,0}_{\sigma,m} \in \mathsf{L}^{2,q+1}_{>-\frac{N}{2}}(\Omega)$$

$$E^{+,k}_{\sigma,m} = \mathcal{L}^{k}_{0}(E^{+}_{\sigma,m}, 0), \quad H^{+,k}_{\sigma,n} = \mathcal{L}^{k}_{0}(0, H^{+}_{\sigma,n}) \in \mathsf{L}^{2,q,q+1}_{-N/2-\sigma-k}(\Omega)$$

${}^{+}\Delta^{q,k}_{\sigma,m}$, ${}^{+}D^{q+1,k}_{\sigma,m}$ behave like $r^{k+\sigma}$, $k, \sigma \geq 0$ at infinity

$$E^{+,k}_{\sigma,m} - \eta({}^{+}\Delta^{q,k}_{\sigma,m}, 0) \in \Lambda^{-1}\Big(\big(\Delta^{q}_{s-k-1}(\Omega) \boxplus \eta\,\mathcal{A}^{q}(\bar{\mathfrak{I}}^{q,\leq k}_{s-k-1})\big) \times \{0\}\Big) \qquad k \text{ even}$$
$$E^{+,k}_{\sigma,m} - \eta(0, {}^{+}D^{q+1,k}_{\sigma,m}) \in \Lambda^{-1}\Big(\{0\} \times \big(\mathring{\mathsf{D}}^{q+1}_{s-k-1}(\Omega) \boxplus \eta\mathcal{D}^{q+1}(\bar{\mathfrak{J}}^{q+1,\leq k}_{s-k-1})\big)\Big) \qquad k \text{ odd}$$

supp $\hat{\Lambda}$ compact, then series rep. of neg. tower-forms of height $\leq k$
(gen. spherical harmonics expansion)

PROJECTION ONTO SPACES OF REGULAR CONVERGENCE

powers $\mathcal{L}_0^j f$ have neg. tower-form parts

$$\langle C_{\Delta,\eta}{}^{\theta} D^{q,k}_{\sigma,m}, {}^{\vartheta} D^{q,\ell}_{\gamma,n}\rangle_{\mathsf{L}^{2,q}(\mathbb{R}^N)} = \langle C_{\Delta,\eta}{}^{\theta} \Delta^{q,k}_{\sigma,m}, {}^{\vartheta} \Delta^{q,\ell}_{\gamma,n}\rangle_{\mathsf{L}^{2,q}(\mathbb{R}^N)} = \delta_{\vartheta\theta,-1}\delta_{k,\ell}\delta_{\sigma,\gamma}\delta_{m,n},$$
$$\langle C_{\Delta,\eta}{}^{\theta} D^{q,k}_{\sigma,m}, {}^{\vartheta} \Delta^{q,\ell}_{\gamma,n}\rangle_{\mathsf{L}^{2,q}(\mathbb{R}^N)} = 0$$

<u>assume:</u> supp $\hat{\Lambda}$ compact $\Rightarrow$

Lemma (orthogonality def. of spaces of regular convergence)

Let $J \in \mathbb{N}$ and $s \in (J+1-N/2, \infty) \setminus \mathbb{I}$ as well as $f \in \mathrm{Reg}^{q,0}_s(\Omega)$.
Then $f \in \mathrm{Reg}^{q,J}_s(\Omega)$, iff

$$\langle f, E^{+,k+1}_{\sigma,m}\rangle_{\mathsf{L}^{2,q,q+1}_{\Lambda}(\Omega)} = \langle f, H^{+,\ell+1}_{\gamma,n}\rangle_{\mathsf{L}^{2,q,q+1}_{\Lambda}(\Omega)} = 0$$

for all $(k,\sigma,m) \in \Theta^{q,J}_s$ and $(\ell,\gamma,n) \in \Theta^{q+1,J}_s$, where

$$\Theta^{q,J}_s := \big\{(k,\sigma,m) \in \mathbb{N}_0^3 : k \le J-1 \wedge \sigma < s - N/2 - k - 1 \wedge 1 \le m \le \mu^q_\sigma\big\}.$$

Especially $\mathrm{Reg}^{q,J}_s(\Omega)$ is a closed subspace of $\mathrm{Reg}^{q,0}_s(\Omega) \subset \mathsf{L}^{2,q,q+1}_s(\Omega)$.

DUAL BASIS OF GROWING TOWERS FORMS

Define

$$e_{\sigma,n}^{\pm,\ell} := M^\ell \eta({}^{\pm}\Delta_{\sigma,n}^{q,1}, 0), \quad h_{\sigma,m}^{\pm,\ell} := M^\ell \eta(0, {}^{\pm}D_{\sigma,m}^{q+1,1}).$$

Then $e_{\sigma,n}^{\pm,\ell}, h_{\sigma,m}^{\pm,\ell} \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^N)$ with $\operatorname{supp} e_{\sigma,n}^{\pm,\ell} = \operatorname{supp} h_{\sigma,m}^{\pm,\ell} = \operatorname{supp} \nabla\eta$ for $\ell \geq 2$ and

$$\langle e_{\gamma,n}^{-,\ell+2}, E_{\sigma,m}^{+,k+1} \rangle_{\mathrm{L}^{2,q,q+1}(\Omega)} = 0,$$

$$\langle h_{\gamma,n}^{-,\ell+2}, E_{\sigma,m}^{+,k+1} \rangle_{\mathrm{L}^{2,q,q+1}(\Omega)} = (-1)^\ell \delta_{k,\ell}\delta_{\sigma,\gamma}\delta_{m,n}.$$

same for $H_{\sigma,m}^{+,k+1}$

Lemma (dual basis of $E_{\sigma,m}^{+,k+1}$ and $H_{\gamma,n}^{+,\ell+1}$)

Let $J \in \mathbb{N}$ and $s \in (J+1-N/2, \infty) \setminus \mathbb{I}$. Then

$$\mathrm{Reg}_s^{q,0}(\Omega) = \mathrm{Reg}_s^{q,J}(\Omega) \dotplus \Upsilon_s^{q,J}, \qquad \Upsilon_s^{q,J} \subset \mathring{\mathrm{C}}^\infty(\mathbb{R}^N),$$

where for $f \in \mathrm{Reg}_s^{q,0}(\Omega)$

$$f_\Upsilon := \sum_{(k,\sigma,m)\in\Theta_s^{q,J}} (-1)^k \langle f, E_{\sigma,m}^{+,k+1} \rangle_{\mathrm{L}^{2,q,q+1}(\Omega)} h_{\sigma,m}^{-,k+2} + \sum_{(k,\sigma,m)\in\Theta_s^{q+1,J}} (-1)^k \langle f, H_{\sigma,m}^{+,k+1} \rangle_{\mathrm{L}^{2,q,q+1}(\Omega)} e_{\sigma,m}^{-,k+2}.$$

with $\Upsilon_s^{q,J} := \mathrm{Lin}\,\{e_{\sigma,m}^{-,k+2}, h_{\gamma,n}^{-,\ell+2} : (k,\sigma,m) \in \Theta_s^{q,J}, (\ell,\gamma,n) \in \Theta_s^{q+1,J}\}$.

PROOF OF LOW FREQUENCY ASYMPTOTICS

step one: proof in the reduced case, this is:

- compactly supported perturbations $\hat{\Lambda}$
- right hand sides from $\mathrm{Reg}_s^{q,0}(\Omega)$
- estimates in local norms

step two: replacing $\mathrm{Reg}_s^{q,0}(\Omega)$ by $\mathrm{L}_s^{2,q,q+1}(\Omega)$ (polynomially weighted Helmholtz decomposition)

step three: replacing local norms by weighted norms

step four: replacing compactly supported perturbations $\hat{\varepsilon}$, $\hat{\mu}$ by asymptotically vanishing perturbations

We only drop the assumption of compactly supported perturbations of the medium in the last step.

STEP ONE

latter lemma $\Rightarrow$

$$\mathrm{Reg}_s^{q,0}(\Omega) = \mathrm{Reg}_s^{q,J}(\Omega) \dotplus \Upsilon_s^{q,J}, \qquad e_{\sigma,m}^{-,k+2}, h_{\sigma,m}^{-,k+2} \Upsilon_s^{q,J} \subset \mathring{\mathrm{C}}^{\infty}(\mathbb{R}^N)$$

- asymptotics clear on $\mathrm{Reg}_s^{q,J}(\Omega)$ (gen. Neumann sum) $\surd$
- asymptotics on $\Upsilon_s^{q,J}$? $\Rightarrow$ asymptotics for $e_{\sigma,m}^{-,k+2}$, $h_{\sigma,m}^{-,k+2}$?

$\mathcal{L}_0^k\, e_{\sigma,m}^{-,k+2} = e_{\sigma,m}^{-,2}$ ($\mathring{\mathrm{C}}^{\infty}(\mathbb{R}^N)$ and right shape) $\Rightarrow$ $e_{\sigma,m}^{-,k+2} \in \mathring{\mathrm{C}}^{\infty}(\mathbb{R}^N) \cap \mathrm{Reg}_s^{q,k}(\Omega)$

$$\Big(\mathcal{L}_\omega - \underbrace{\sum_{j=0}^{J-1} \omega^j\, \mathcal{L}_0^{j+1}}_{= \sum_{j=0}^{k-1} \cdots + \sum_{j=k}^{J-1} \cdots}\Big) e_{\sigma,m}^{-,k+2} = \omega^k\, \mathcal{L}_\omega\, \mathcal{L}_0^k\, e_{\sigma,m}^{-,k+2} - \omega^k \sum_{j=0}^{J-1-k} \omega^j\, \mathcal{L}_0^{j+1+k}\, e_{\sigma,m}^{-,k+2}$$

$$= \omega^k \Big(\mathcal{L}_\omega - \sum_{j=0}^{J-1-k} \omega^j\, \mathcal{L}_0^{j+1}\Big) e_{\sigma,m}^{-,2}$$

same for $h_{\sigma,m}^{-,k+2}$

just unkn. asym. for $\boxed{\Big(\mathcal{L}_\omega - \sum_{j=0}^{J-1-k} \omega^j\, \mathcal{L}_0^{j+1}\Big) e_{\sigma,m}^{-,2}}$ and $\boxed{\Big(\mathcal{L}_\omega - \sum_{j=0}^{J-1-k} \omega^j\, \mathcal{L}_0^{j+1}\Big) h_{\sigma,m}^{-,k+2}}$

STEP ONE

asymptotics for $\boxed{(\mathcal{L}_\omega - \sum_{j=0}^{J-1-k} \omega^j \mathcal{L}_0^{j+1}) e^{-,2}_{\sigma,m}}$ and $\boxed{(\mathcal{L}_\omega - \sum_{j=0}^{J-1-k} \omega^j \mathcal{L}_0^{j+1}) h^{-,k+2}_{\sigma,m}}$?

<u>idea:</u> compare with special radiating solutions of the homo. problem in $\mathbb{R}^N \setminus \{0\}$

$$\begin{aligned}
\mathbb{E}^{1,\omega}_{\sigma,m} &= \beta_\sigma \omega^{\nu_\sigma} r^{1-\frac{N}{2}} H^1_{\nu_\sigma}(\omega\, r)\, \check{\tau}\, T^q_{\sigma,m} \qquad (H^1_{\nu_\sigma} \text{ Hankel's function}) \\
&= \sum_{k=0}^{\infty} (-\mathrm{i}\,\omega)^{2k}\ {}^{-}\Delta^{q,2k+1}_{\sigma,m} + \kappa^{q+1}_\sigma\, \omega^{2\nu_\sigma} \sum_{k=0}^{\infty} (-\mathrm{i}\,\omega)^{2k}\ {}^{+}\Delta^{q,2k+1}_{\sigma,m} \\
\mathbb{H}^{1,\omega}_{\sigma,m} &= \frac{\mathrm{i}}{\omega}\, \mathrm{d}\, \mathbb{E}^{1,\omega}_{\sigma,m} \\
&= \frac{\mathrm{i}}{\omega} \Big(\sum_{k=0}^{\infty} (-\mathrm{i}\,\omega)^{2k}\ {}^{-}D^{q+1,2k}_{\sigma,m} + \kappa^{q+1}_\sigma\, \omega^{2\nu_\sigma} \sum_{k=0}^{\infty} (-\mathrm{i}\,\omega)^{2k}\ {}^{+}D^{q+1,2k}_{\sigma,m} \Big)
\end{aligned}$$

similarly second solution pair $(\mathbb{E}^{2,\omega}_{\sigma,m}, \mathbb{H}^{2,\omega}_{\sigma,m})$

$$(\mathrm{i} \begin{bmatrix} 0 & \delta \\ \mathrm{d} & 0 \end{bmatrix} - \omega)(\mathbb{E}^{j,\omega}_{\sigma,m}, \mathbb{H}^{j,\omega}_{\sigma,m}) = (0,0) \quad \Rightarrow \quad (\Delta + \omega^2)(\mathbb{E}^{j,\omega}_{\sigma,m}, \mathbb{H}^{j,\omega}_{\sigma,m}) = (0,0)$$

(comp.-wise Helmholtz)

STEP ONE

note: $(M-\omega)\eta(\mathbb{E}^{j,\omega}_{\sigma,m},\mathbb{H}^{j,\omega}_{\sigma,m}) = C_{M,\eta}(\mathbb{E}^{j,\omega}_{\sigma,m},\mathbb{H}^{j,\omega}_{\sigma,m})$

comparing

$$\mathcal{L}_\omega\, e^{-,2}_{\sigma,m} \quad \text{with} \quad \mathcal{L}_\omega\, C_{M,\eta}(\mathbb{E}^{1,\omega}_{\sigma,m},\mathbb{H}^{1,\omega}_{\sigma,m}) = \eta(\mathbb{E}^{1,\omega}_{\sigma,m},\mathbb{H}^{1,\omega}_{\sigma,m}),$$

$$\mathcal{L}_\omega\, h^{-,2}_{\sigma,m} \quad \text{with} \quad \mathcal{L}_\omega\, C_{M,\eta}(\mathbb{E}^{2,\omega}_{\sigma,m},\mathbb{H}^{2,\omega}_{\sigma,m}) = \eta(\mathbb{E}^{2,\omega}_{\sigma,m},\mathbb{H}^{2,\omega}_{\sigma,m})$$

and a (really) long, long, long, ... calculation

Theorem (low frequency asymptotics on $\mathrm{Reg}^{q,0}_s(\Omega)$)

Let $J\in\mathbb{N}_0$ and $s\in(J+1/2,\infty)\setminus\mathbb{I}$. Then for all bounded subdomains $\Omega_{\mathrm{b}}\subset\Omega$

$$\Big\|\,\mathcal{L}_\omega f - \sum_{j=0}^{J-1}\omega^j\,\mathcal{L}_0^{j+1} f - \omega^N \sum_{j=0}^{J-1-N}\omega^j\hat{\Gamma}_j f\Big\|_{\mathsf{L}^{2,q,q+1}(\Omega_{\mathrm{b}})} = O\big(|\omega|^J\big)\,\|f\|_{\mathsf{L}^{2,q,q+1}_s(\Omega)}$$

holds uniformly w.r.t. $f\in\mathrm{Reg}^{q,0}_s(\Omega)$ and $0\neq\omega\in\mathbb{C}_+$ small enough.

degenerate correction operators

$$\hat{\Gamma}_j f \in \mathrm{Lin}\{E^{+,k}_{\sigma,m}, H^{+,k}_{\sigma,n} : k+2\sigma\leq j\}$$

with coefficients of shape $\langle f, E^{+,k}_{\sigma,m}\rangle_{\mathsf{L}^{2,q,q+1}(\Omega)}$ and $\langle f, H^{+,k}_{\sigma,m}\rangle_{\mathsf{L}^{2,q,q+1}(\Omega)}$

STEP TWO

Theorem (polynomially weighted Helmholtz decomposition)

conditions on τ ...
For $s > -N/2$ let ${}_\varepsilon\mathbb{L}^{2,q}_s(\Omega) := \mathsf{L}^{2,q}_s(\Omega) \cap {}_\varepsilon\mathcal{H}^q(\Omega)^{\perp_\varepsilon}$.

(i) $-N/2 < s < N/2$:

$$ {}_\varepsilon\mathbb{L}^{2,q}_s(\Omega) = {}_0\mathring{\mathbb{D}}^q_s(\Omega) \dot{+} \varepsilon^{-1}{}_0\mathbb{\Delta}^q_s(\Omega) $$

For $s \geq 0$ the decomposition is $\langle \varepsilon\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^{2,q}(\Omega)}$-orthogonal.

(ii) $s > N/2$:

$$ {}_\varepsilon\mathbb{L}^{2,q}_s(\Omega) = \Big(\big([\mathsf{L}^{2,q}_s(\Omega) \boxplus \eta\bar{\mathcal{D}}^q_s] \cap {}_0\mathring{\mathbb{D}}^q_{<\frac{N}{2}}(\Omega) \big) \oplus_\varepsilon \varepsilon^{-1} \big([\mathsf{L}^{2,q}_s(\Omega) \boxplus \eta\bar{\mathcal{R}}^q_s] \cap {}_0\mathbb{\Delta}^q_{<\frac{N}{2}}(\Omega) \big) \Big) \cap \mathsf{L}^{2,q}_s(\Omega) $$

$$ {}_\varepsilon\mathbb{L}^{2,q}_s(\Omega) = {}_0\mathring{\mathbb{D}}^q_s(\Omega) \dot{+} \varepsilon^{-1}{}_0\mathbb{\Delta}^q_s(\Omega) \dot{+} \Delta_\varepsilon \eta \bar{\mathcal{P}}^q_{s-2} $$

The first two terms in the second decomposition are $\langle \varepsilon\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^{2,q}(\Omega)}$-orthogonal.

$$ \mathsf{L}^{2,q}_s(\Omega) \cap {}_\varepsilon\mathcal{H}^q_{-s}(\Omega)^{\perp_\varepsilon} = {}_0\mathring{\mathbb{D}}^q_s(\Omega) \oplus_\varepsilon \varepsilon^{-1}{}_0\mathbb{\Delta}^q_s(\Omega) $$

(iii) $s < -N/2$:
deco. holds, but loosing directness, larger space of Dirichlet/Neumann forms

STEP TWO

polynomially weighted Helmholtz decomposition for large weights s

$$\mathsf{L}^{2,q,q+1}_s(\Omega) = \big(\mathsf{Tri}^q_s(\Omega) \dotplus \mathsf{Reg}^{q,-1}_s(\Omega)\big) \cap \mathsf{L}^{2,q,q+1}_s(\Omega)$$

with projections Π and $\Pi_{\mathrm{reg}} := (1-\Pi)$ as well as $t \leq s$ and $t < N/2$

$$(N(M) =)\,\mathsf{Tri}^q_s(\Omega) = \Pi\mathsf{L}^{2,q,q+1}_s(\Omega) \subset {}_0\mathring{\mathsf{D}}^q_t(\Omega) \times {}_0\Delta^{q+1}_t(\Omega)$$

$$\mathsf{Reg}^{q,-1}_s(\Omega) = \Pi_{\mathrm{reg}}\mathsf{L}^{2,q,q+1}_s(\Omega) \subset {}_0\Delta^q_t(\Omega) \times {}_0\mathring{\mathsf{D}}^{q+1}_t(\Omega)$$

still: $\operatorname{supp}\hat{\lambda}$ compact

Theorem (low frequency asymptotics on $\mathsf{L}^{2,q,q+1}_s(\Omega)$ in local norms)

Let $J \in \mathbb{N}_0$ and $s \in (J+1/2,\infty) \setminus \mathbb{I}$. Then for all bounded subdomains $\Omega_{\mathrm{b}} \subset \Omega$

$$\Big\| \mathcal{L}_\omega f + \omega^{-1}\Pi f - \sum_{j=0}^{J-1} \omega^j \mathcal{L}_0^{j+1} \Pi_{\mathrm{reg}} f - \omega^{N-1} \sum_{j=0}^{J-N} \omega^j \Gamma_j f \Big\|_{\mathsf{L}^{2,q,q+1}(\Omega_{\mathrm{b}})} = O(|\omega|^J)\, \|f\|_{\mathsf{L}^{2,q,q+1}_s(\Omega)}$$

holds uniformly with respect to $f \in \mathsf{L}^{2,q,q+1}_s(\Omega)$ and $0 \neq \omega \in \mathbb{C}_+$ small enough.

STEPS THREE AND FOUR

- cutting technique $\Rightarrow$ bounded domain and unbounded domain
- comparing with the homogeneous whole space case $\Omega = \mathbb{R}^N$ and $\Lambda = \mathrm{Id}$
  - represent solution by convolution with fundamental solution
  - Taylor expansion of fundamental solution (Hankel's function)

  $\Rightarrow$ low frequency asymptotics in this special case
- low frequency asymptotics in weighted norms $\mathsf{L}_t^{2,q,q+1}(\Omega)$
- approx. of asymptotically homo. media by compactly supported media (convergence in operator norm)

done ■

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