Electro-Magneto Statics and (Much) More by a (Linear) Functional Analysis Toolbox

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Open-Minded ;-)

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Linear Maxwell's Equations in 3D - Electro-Magneto-Dynamics

$\Omega \subset \mathbb{R}^3$ domain (open and connected set) with boundary Γ

$$\begin{aligned}
\partial_t(\varepsilon E) - \operatorname{rot} H + \sigma E &= -J && \text{in } \mathbb{R}_+ \times \Omega && \text{(DE 1.1, Ampére/Maxwell law)}\\
\partial_t(\mu H) + \operatorname{rot} E &= 0 && \text{in } \mathbb{R}_+ \times \Omega && \text{(DE 1.2, Faraday/Maxwell law)}\\
\operatorname{div}(\varepsilon E) &= \rho && \text{in } \mathbb{R}_+ \times \Omega && \text{(DE 2.1, electric Gauß law)}\\
\operatorname{div}(\mu H) &= 0 && \text{in } \mathbb{R}_+ \times \Omega && \text{(DE 2.2, magnetic Gauß law)}\\
n \times E &= 0 && \text{on } \mathbb{R}_+ \times \Gamma && \text{(BC 1, perfect conductor)}\\
n \cdot (\mu H) &= 0 && \text{on } \mathbb{R}_+ \times \Gamma && \text{(BC 2, perfect conductor)}\\
(E, H)(0) &= (E_0, H_0) && \text{in } \Omega && \text{(IC)}
\end{aligned}$$

$$\begin{aligned}
E, H : \mathbb{R}_+ \times \Omega &\to \mathbb{C}^3 && \text{electric resp. magnetic/magnetization field}\\
J : \mathbb{R}_+ \times \Omega &\to \mathbb{C}^3 && \text{electric current density,}\\
\rho : \mathbb{R}_+ \times \Omega &\to \mathbb{C} && \text{charge density}\\
\varepsilon, \mu, \sigma : \mathbb{R}_+ \times \Omega &\to \mathbb{C}^{3\times 3} && \text{permittivity resp. permeability resp. conductivity}
\end{aligned}$$

$$\operatorname{div} F = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3, \qquad \operatorname{rot} F = \operatorname{curl} F = \begin{bmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{bmatrix}$$

Linear Maxwell's Equations in 3D - rewrite as system

$\Omega \subset \mathbb{R}^3$ domain

$$\begin{aligned}
\partial_t(\varepsilon E) - \operatorname{rot} H + \sigma E &= -J && \text{in } \mathbb{R}_+ \times \Omega && (\text{DE 1.1, needed})\\
\partial_t(\mu H) + \operatorname{rot} E &= 0 && \text{in } \mathbb{R}_+ \times \Omega && (\text{DE 1.2, needed})\\
\operatorname{div}(\varepsilon E) &= \rho && \text{in } \mathbb{R}_+ \times \Omega && (\text{DE 2.1, not needed})\\
\operatorname{div}(\mu H) &= 0 && \text{in } \mathbb{R}_+ \times \Omega && (\text{DE 2.2, not needed})\\
n \times E &= 0 && \text{on } \mathbb{R}_+ \times \Gamma && (\text{BC 1, needed})\\
n \cdot (\mu H) &= 0 && \text{on } \mathbb{R}_+ \times \Gamma && (\text{BC 2, not needed})\\
(E, H)(0) &= (E_0, H_0) && \text{in } \Omega && (\text{IC, later needed})
\end{aligned}$$

$$\underbrace{\Big(\partial_t \overbrace{\begin{bmatrix} \varepsilon & 0 \\ 0 & \mu \end{bmatrix}}^{=\Lambda \text{ (bd, sa, } \geq 0)} + \overbrace{\begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix}}^{=\Sigma \text{ (bd)}} + \overbrace{\begin{bmatrix} 0 & -\operatorname{rot} \\ \mathring{\operatorname{rot}} & 0 \end{bmatrix}}^{=M \text{ (unbd, ssa)}}\Big)}_{=\partial_t \Lambda + \Sigma + M \qquad (\operatorname{rot} = \mathring{\operatorname{rot}}^*)} \begin{bmatrix} E \\ H \end{bmatrix} = \begin{bmatrix} -J \\ 0 \end{bmatrix} \qquad \text{in } \mathbb{R} \times \Omega \qquad (\text{DE})$$

Rainer Picard '09 (and earlier) and his Dresden school, e.g., Marcus Waurick '11, ...
$\Rightarrow$ very nice and elegant ("simple") solution theory:
$\partial_t \Lambda + \Sigma + M$ cont inv, i.e., $(\partial_t \Lambda + \Sigma + M)^{-1}$ ex cont op (time-weighted-L^2-sense)
sol theo Hadamard sense + causality

Maxwell's Equations in 3D - Simplifications

$\varepsilon = \mu = \mathrm{id}$, $\sigma = 0 \Rightarrow$

$$\begin{aligned}
\partial_t E - \operatorname{rot} H &= -F = -J && \text{in } \mathbb{R}_+ \times \Omega && \text{(DE 1.1)}\\
\partial_t H + \operatorname{rot} E &= G = 0 && \text{in } \mathbb{R}_+ \times \Omega && \text{(DE 1.2)}\\
\operatorname{div} E &= f = \rho && \text{in } \mathbb{R}_+ \times \Omega && \text{(DE 2.1)}\\
\operatorname{div} H &= g = 0 && \text{in } \mathbb{R}_+ \times \Omega && \text{(DE 2.2)}\\
n \times E &= 0 && \text{on } \mathbb{R}_+ \times \Gamma && \text{(BC 1)}\\
n \cdot H &= 0 && \text{on } \mathbb{R}_+ \times \Gamma && \text{(BC 2)}\\
(E, H)(0) &= (E_0, H_0) && \text{in } \Omega && \text{(IC)}
\end{aligned}$$

Maxwell's Equations in 3D - Static Cases

time indep $\Rightarrow \partial_t(E,H)=0 \Rightarrow$

$$\begin{aligned}
\operatorname{rot} H &= F && \text{in } \Omega && \text{(DE 1.1)}\\
\operatorname{rot} E &= G && \text{in } \Omega && \text{(DE 1.2)}\\
\operatorname{div} E &= f && \text{in } \Omega && \text{(DE 2.1)}\\
\operatorname{div} H &= g && \text{in } \Omega && \text{(DE 2.2)}\\
n \times E &= 0 && \text{on } \Gamma && \text{(BC 1)}\\
n \cdot H &= 0 && \text{on } \Gamma && \text{(BC 2)}
\end{aligned}$$

E, H decoupled $\Rightarrow$

$$\begin{aligned}
\operatorname{rot} E &= G && \text{in } \Omega && \text{(DE 1.2)}\\
\operatorname{div} E &= f && \text{in } \Omega && \text{(DE 2.1)}\\
n \times E &= 0 && \text{on } \Gamma && \text{(BC 1)}
\end{aligned}$$

electro statics

$$\begin{aligned}
\operatorname{rot} H &= F && \text{in } \Omega && \text{(DE 1.1)}\\
\operatorname{div} H &= g && \text{in } \Omega && \text{(DE 2.2)}\\
n \cdot H &= 0 && \text{on } \Gamma && \text{(BC 2)}
\end{aligned}$$

magneto statics

$$\begin{aligned}
\operatorname{rot} E &= J && \text{in } \Omega\\
\operatorname{div} E &= j && \text{in } \Omega\\
n \times E &= 0 && \text{on } \Gamma_t\\
n \cdot E &= 0 && \text{on } \Gamma_n
\end{aligned}$$

model problem: electro-magneto statics (EMS)

Model Problem: Electro-Magneto-Static Maxwell Equations

setting: Hilbert/L^2-based Sobolev spaces
geometry: $\Omega \subset \mathbb{R}^3$ bounded domain with weak Lipschitz boundary $\Gamma = \partial\Omega$
for simplicity: no mixed boundary conditions

$$\operatorname{rot} E = J \quad \text{in } \Omega \tag{1}$$

$$-\operatorname{div} E = j \quad \text{in } \Omega \tag{2}$$

$$n \times E = 0 \quad \text{on } \Gamma \tag{3}$$

non-trivial kernel: $\mathcal{H} = \{H \in L^2 : \operatorname{rot} H = 0, \operatorname{div} H = 0, \nu \times H|_\Gamma = 0\}$
additional condition on Dirichlet/Neumann fields for uniqueness:

$$\pi E = H \in \mathcal{H} \tag{4}$$

well known: (1)-(4) uniquely solvable
by Helmholtz decompositions and Friedrichs/Poincaré/Maxwell type estimates
for certain given right hand sides J, j, H

aim: general theory | FA-ToolBox for linear problems/systems |

FA-ToolBox for linear problems/systems

literature: probably very well known for ages (≥ 80 years), but hard to find ...

Friedrichs, Weyl, Hörmander, Fredholm, von Neumann, Riesz, Banach, ... ?

Why not rediscover?

Underlying Structure of the Model Problem

∇-rot-div-complex (de Rham complex):

$$\{0\} \;\underset{\pi}{\overset{\iota}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\operatorname{div}}{\overset{\mathring{\nabla}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\operatorname{rot}}{\overset{\mathring{\operatorname{rot}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\nabla}{\overset{\mathring{\operatorname{div}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota}{\overset{\pi}{\rightleftarrows}}\; \mathbb{R}$$

unbounded, densely defined, closed, linear operators with adjoints

$$\mathring{\nabla} : \mathring{\mathsf{H}}^1 \subset \mathsf{L}^2 \to \mathsf{L}^2, \qquad -\operatorname{div} = (\mathring{\nabla})^* : \mathsf{D} \subset \mathsf{L}^2 \to \mathsf{L}^2 \qquad \text{sometimes: } \mathsf{D} = H(\operatorname{div})$$

$$\mathring{\operatorname{rot}} : \mathring{\mathsf{R}} \subset \mathsf{L}^2 \to \mathsf{L}^2, \qquad \operatorname{rot} = (\mathring{\operatorname{rot}})^* : \mathsf{R} \subset \mathsf{L}^2 \to \mathsf{L}^2 \qquad \mathsf{R} = H(\operatorname{rot}) = H(\operatorname{curl})$$

$$\mathring{\operatorname{div}} : \mathring{\mathsf{D}} \subset \mathsf{L}^2 \to \mathsf{L}^2, \qquad -\nabla = (\mathring{\operatorname{div}})^* : \mathsf{H}^1 \subset \mathsf{L}^2 \to \mathsf{L}^2 \qquad \mathring{\mathsf{R}} = H_0(\operatorname{rot}) = H_0(\operatorname{curl})$$

complex: 'range $\subset$ kernel' ($\operatorname{rot}\nabla = 0$, $\operatorname{div}\operatorname{rot} = 0$)

$$\mathring{\nabla}\mathring{\mathsf{H}}^1 \subset \mathring{\mathsf{R}}_0, \quad \mathring{\operatorname{rot}}\mathring{\mathsf{R}} \subset \mathring{\mathsf{D}}_0, \quad \mathring{\operatorname{div}}\mathring{\mathsf{D}} \subset N(\pi) = \mathsf{L}^2_\perp = \mathsf{L}^2 \cap \mathbb{R}^\perp$$

crucial: compact embeddings (Rellich's selection theorem & Weck's selection theorems)

$$\mathsf{H}^1 \hookrightarrow \mathsf{L}^2, \qquad \mathring{\mathsf{R}} \cap \mathsf{D} \hookrightarrow \mathsf{L}^2, \qquad \mathsf{R} \cap \mathring{\mathsf{D}} \hookrightarrow \mathsf{L}^2$$

$\Rightarrow$ Helmholtz decompositions, closed ranges, continuous inverses, and Friedrichs/Poincaré/Maxwell type estimates $\checkmark$

Underlying Structure of the Model Problem

$$\mathring{\mathsf{R}} \cap \mathsf{D} \hookrightarrow \mathsf{L}^2, \qquad \mathsf{R} \cap \mathring{\mathsf{D}} \hookrightarrow \mathsf{L}^2$$

Weck's selection theorems: Weck '74 (Habil.), stimulated by Rolf Leis

more literature on Weck's selection theorems:

Weber '80, Picard '84, Costabel '90, Witsch '93, Jochmann '97, Kuhn '99, Picard/Weck/Witsch '01, Bauer/P/Schomburg '16, '17

Abstract Formulation

$$\begin{aligned} \operatorname{rot} E &= J && \text{in } \Omega \\ -\operatorname{div} E &= j && \text{in } \Omega \\ \nu \times E &= 0 && \text{on } \Gamma \\ \pi E &= H \in \mathcal{H} \end{aligned}$$

$$\rightsquigarrow$$

$$\begin{aligned} \mathring{\operatorname{rot}} E &= J \\ -\operatorname{div}_{\Gamma_n} E &= j \\ \pi E &= H \in \mathcal{H} \end{aligned}$$

$$\rightsquigarrow \; (x := E, \quad \mathrm{A}_1 := \mathring{\operatorname{rot}}, \quad \mathrm{A}_1^* = \mathring{\operatorname{rot}}^* = \operatorname{rot}, \quad \mathrm{A}_0 := \mathring{\nabla}, \quad \mathrm{A}_0^* = \mathring{\nabla}^* = -\operatorname{div})$$

$$\begin{aligned} \mathrm{A}_1 x &= f \\ \mathrm{A}_0^* x &= g \\ \pi_i x &= h \in N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \end{aligned}$$

General or Abstract Problem

setting: unbounded, densely defined, closed, linear operators with adjoints

$$\mathrm{A}_i : D(\mathrm{A}_i) \subset \mathsf{H}_i \to \mathsf{H}_{i+1}, \quad \mathrm{A}_i^* : D(\mathrm{A}_i^*) \subset \mathsf{H}_{i+1} \to \mathsf{H}_i, \quad i \in \mathbb{Z}$$

complex: (here $i = 1$)

$$\cdots \rightleftarrows \mathsf{H}_{i-2} \underset{\mathrm{A}_{i-2}^*}{\overset{\mathrm{A}_{i-2}}{\rightleftarrows}} \boxed{\mathsf{H}_{i-1} \underset{\mathrm{A}_{i-1}^*}{\overset{\mathrm{A}_{i-1}}{\rightleftarrows}} \boxed{\mathsf{H}_i} \underset{\mathrm{A}_i^*}{\overset{\mathrm{A}_i}{\rightleftarrows}} \mathsf{H}_{i+1}} \underset{\mathrm{A}_{i+1}^*}{\overset{\mathrm{A}_{i+1}}{\rightleftarrows}} \mathsf{H}_{i+2} \rightleftarrows \cdots$$

complex property: 'range $\subset$ kernel' ($\boxed{\mathrm{A}_i \mathrm{A}_{i-1} = 0} \quad \Leftrightarrow \quad \mathrm{A}_{i-1}^* \mathrm{A}_i^* = 0$)

$$\boxed{R(\mathrm{A}_{i-1}) \subset N(\mathrm{A}_i)} \quad \Leftrightarrow \quad R(\mathrm{A}_i^*) \subset N(\mathrm{A}_{i-1}^*)$$

problem: find $x \in D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)$ s.t.

$$\boxed{\mathrm{A}_i x = f, \quad \mathrm{A}_{i-1}^* x = g, \quad \pi_i x = h,}$$

where $f \in R(\mathrm{A}_i)$, $g \in R(\mathrm{A}_{i-1}^*)$ and $h \in \mathcal{H}_i$ with kernel $N(\mathrm{A}_i) \cap N(\mathrm{A}_{i-1}^*)$

Toolbox

Hodge/Helmholtz/Weyl decompositions:

$$\mathrm{H}_i = N(\mathrm{A}_i) \oplus_{\mathrm{H}_i} \overline{R(\mathrm{A}_i^*)}, \qquad \mathrm{H}_{i+1} = N(\mathrm{A}_i^*) \oplus_{\mathrm{H}_{i+1}} \overline{R(\mathrm{A}_i)}$$

$\Rightarrow$ reduce A_i to $N(\mathrm{A}_i)^\perp = \overline{R(\mathrm{A}_i^*)}$ and A_i^* to $N(\mathrm{A}_i^*)^\perp = \overline{R(\mathrm{A}_i)}$

$\Rightarrow$ injective reduced operators $\mathcal{A}_i$, $\mathcal{A}_i^*$, "same" complex for $\mathcal{A}_i$, $\mathcal{A}_i^*$

$\Rightarrow$ $\mathcal{A}_i^{-1}$, $(\mathcal{A}_i^*)^{-1}$ exist always, but might be unbounded

$\Rightarrow$ crucial lemmas

Toolbox

Lemma (P)

The following assertions are equivalent:

- (i) $\exists\, c_i > 0 \quad \forall\, \varphi \in D(\mathcal{A}_i) \qquad |\varphi|_{\mathrm{H}_i} \le c_i \, |\mathrm{A}_i \varphi|_{\mathrm{H}_{i+1}}$ *(general Friedrichs/Poincaré/*
- (i*) $\exists\, c_i^* > 0 \quad \forall\, \psi \in D(\mathcal{A}_i^*) \quad |\psi|_{\mathrm{H}_{i+1}} \le c_i^* \, |\mathrm{A}_i^* \psi|_{\mathrm{H}_i}$ *Maxwell type estimates)*
- (ii) *The ranges* $R(\mathrm{A}_i) = R(\mathcal{A}_i)$ *are closed.*
- (ii*) *The ranges* $R(\mathrm{A}_i^*) = R(\mathcal{A}_i^*)$ *are closed.*
- (iii) *The inverse operator* $\mathcal{A}_i^{-1} : R(\mathrm{A}_i) \to D(\mathcal{A}_i)$ *is continuos.*
- (iii*) *The inverse operator* $(\mathcal{A}_i^*)^{-1} : R(\mathrm{A}_i^*) \to D(\mathcal{A}_i^*)$ *is continuos.*

Lemma (P)

If c_i, c_i^* *are "best" constants, then* $c_i = c_i^* = |\mathcal{A}_i^{-1}| = |(\mathcal{A}_i^*)^{-1}|$.

Lemma (P)

If $D(\mathcal{A}_i) \hookrightarrow \mathrm{H}_i$ *is compact, then the latter assertions hold and* $\mathcal{A}_i^{-1} : R(\mathrm{A}_i) \to R(\mathrm{A}_i^*)$, $(\mathcal{A}_i^*)^{-1} : R(\mathrm{A}_i^*) \to R(\mathrm{A}_i)$ *are compact.*

proofs: elementary computations and closed range/graph theorem

Toolbox

Lemma (P)

$D(\mathcal{A}_i) \hookrightarrow \mathsf{H}_i$ *compact* $\quad\Leftrightarrow\quad$ $D(\mathcal{A}_i^*) \hookrightarrow \mathsf{H}_{i+1}$ *compact*

Lemma (P)

$$
\begin{aligned}
& D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*) \hookrightarrow \mathsf{H}_i \quad \textit{compact} \\
\Leftrightarrow\quad & D(\mathcal{A}_i) \hookrightarrow \mathsf{H}_i, \quad D(\mathcal{A}_{i-1}^*) \hookrightarrow \mathsf{H}_i, \quad N(\mathrm{A}_i) \cap N(\mathrm{A}_{i-1}^*) \hookrightarrow \mathsf{H}_i \quad \textit{compact} \\
\Leftrightarrow\quad & D(\mathcal{A}_i) \hookrightarrow \mathsf{H}_i, \quad D(\mathcal{A}_{i-1}) \hookrightarrow \mathsf{H}_{i-1}, \quad N(\mathrm{A}_i) \cap N(\mathrm{A}_{i-1}^*) \hookrightarrow \mathsf{H}_i \quad \textit{compact}
\end{aligned}
$$

Lemma (P)

Lots of Helmholtz type decompositions hold, such as

$$
\mathsf{H}_i = \underbrace{\overline{R(\mathrm{A}_{i-1})} \oplus_{\mathsf{H}_i} \big(N(\mathrm{A}_i) \cap N(\mathrm{A}_{i-1}^*)\big)}_{=N(\mathrm{A}_i)} \oplus_{\mathsf{H}_i} \overline{R(\mathrm{A}_i^*)}, \qquad \text{with } \big(N(\mathrm{A}_i) \cap N(\mathrm{A}_{i-1}^*)\big) \oplus_{\mathsf{H}_i} \overline{R(\mathrm{A}_i^*)} = N(\mathrm{A}_{i-1}^*), \qquad D(\mathrm{A}_i) = N(\mathrm{A}_i) \oplus_{\mathsf{H}_i} D(\mathcal{A}_i), \quad \ldots
$$

proofs: elementary computations and different Helmholtz type decompositions

Toolbox

Remark

best world: $D(\mathrm{A}_i) \cap D(\mathrm{A}^*_{i-1}) \hookrightarrow \mathrm{H}_i$ *compact*

usually true for bounded domains	$\Rightarrow$	*full toolbox*
typically not true for unbounded domains	$\Rightarrow$	*toolbox without compactness results*
(locally compact embeddings)		

Solution Theory for Abstract Problem

problem: find $x \in D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)$ st

$$\begin{aligned} \mathrm{A}_i x &= f \\ \mathrm{A}_{i-1}^* x &= g \\ \pi_i x &= h \end{aligned}$$

Theorem (solution theory, P)

unique sol in Hadamard sense (cont dpd on data, ...)

$$\Leftrightarrow \quad f \in R(\mathrm{A}_i), \quad g \in R(\mathrm{A}_{i-1}^*) \quad \text{and} \quad h \in N(\mathrm{A}_i) \cap N(\mathrm{A}_{i-1}^*)$$

Proof.

$$x = x_f + x_g + h, \qquad x_f := \mathcal{A}_i^{-1} f, \quad x_g := (\mathcal{A}_{i-1}^*)^{-1} g$$

□

note: problem is linear and hence decouples

$$\begin{aligned} \mathrm{A}_i x_f &= f \\ \mathrm{A}_{i-1}^* x_f &= 0 \\ \pi_i x_f &= 0 \end{aligned} \qquad\qquad \begin{aligned} \mathrm{A}_i x_g &= 0 \\ \mathrm{A}_{i-1}^* x_g &= g \\ \pi_i x_g &= 0 \end{aligned}$$

Variational Formulations for Abstract Problem

How to find $x \in D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)$, ie, $x = x_f + x_g + h$, ie,
$x_f \in D(\mathcal{A}_i) = D(\mathrm{A}_i) \cap R(\mathrm{A}_i^*)$ and $x_g \in D(\mathrm{A}_{i-1}^*) = D(\mathcal{A}_{i-1}^*) \cap R(\mathrm{A}_{i-1})$ with

$$\begin{aligned} \mathrm{A}_i x &= f & \mathrm{A}_i x_f &= f & \mathrm{A}_i x_g &= 0 \\ \mathrm{A}_{i-1}^* x &= g & \mathrm{A}_{i-1}^* x_f &= 0 & \mathrm{A}_{i-1}^* x_g &= g \\ \pi_i x &= h & \pi_i x_f &= 0 & \pi_i x_g &= 0 \end{aligned}$$

by variational formulations that can be "easily" implemented by numerical methods such as FEM?

formulation 1: test $\mathrm{A}_i x_f = f$ by $\mathrm{A}_i \varphi$ with $\varphi \in D(\mathcal{A}_i)$

$\Rightarrow$ find $x_f \in D(\mathcal{A}_i)$ st

$$\forall\, \varphi \in D(\mathcal{A}_i) \qquad \langle \mathrm{A}_i x_f, \mathrm{A}_i \varphi \rangle_{\mathrm{H}_{i+1}} = \langle f, \mathrm{A}_i \varphi \rangle_{\mathrm{H}_{i+1}}$$

well posed by toolbox, uniq sol by Riesz

note: $R(\mathcal{A}_i) = R(\mathrm{A}_i)$ $\Rightarrow$ var form holds for all $\varphi \in D(\mathrm{A}_i)$

$\Rightarrow$ $\mathrm{A}_i x_f - f \in R(\mathrm{A}_i) \cap R(\mathrm{A}_i)^\perp$ $\Rightarrow$ $\mathrm{A}_i x_f = f$ holds

note: additional condition $x_f \in R(\mathrm{A}_i^*) = N(\mathrm{A}_i)^\perp$ $\Rightarrow$ sadd point form / inf-sup

note: num approx satisfies $\tilde{x}_f \in D(\mathrm{A}_i)$ but $\tilde{x}_f \in R(\mathrm{A}_i^*) = N(\mathrm{A}_i)^\perp$ only weakly

corresponding idea works for $\mathrm{A}_{i-1}^* x_g = g$

Variational Formulations for Abstract Problem

How to find $x \in D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)$, ie, $x = x_f + x_g + h$, ie,
$x_f \in D(\mathcal{A}_i) = D(\mathrm{A}_i) \cap R(\mathrm{A}_i^*)$ and $x_g \in D(\mathrm{A}_{i-1}^*) = D(\mathcal{A}_{i-1}^*) \cap R(\mathrm{A}_{i-1})$ with

$$\begin{aligned} \mathrm{A}_i x &= f & \mathrm{A}_i x_f &= f & \mathrm{A}_i x_g &= 0 \\ \mathrm{A}_{i-1}^* x &= g & \mathrm{A}_{i-1}^* x_f &= 0 & \mathrm{A}_{i-1}^* x_g &= g \\ \pi_i x &= h & \pi_i x_f &= 0 & \pi_i x_g &= 0 \end{aligned}$$

by variational formulations that can be "easily" implemented by numerical methods such as FEM?

formulation 2: $x_f \in R(\mathrm{A}_i^*) \quad \Rightarrow \quad x_f = \mathrm{A}_i^* y_f$ with $y_f = (\mathcal{A}_i^*)^{-1} x_f \in D(\mathcal{A}_i^*)$
test x_f by $\mathrm{A}_i^* \phi$ with $\phi \in D(\mathcal{A}_i^*)$ and use $\mathrm{A}_i x_f = f$
$\Rightarrow$ find $y_f \in D(\mathcal{A}_i^*)$ st

$$\forall\, \phi \in D(\mathcal{A}_i^*) \qquad \langle \mathrm{A}_i^* y_f, \mathrm{A}_i^* \phi \rangle_{\mathrm{H}_i} = \langle x_f, \mathrm{A}_i^* \phi \rangle_{\mathrm{H}_i} = \langle \mathrm{A}_i x_f, \phi \rangle_{\mathrm{H}_{i+1}} = \langle f, \phi \rangle_{\mathrm{H}_{i+1}}$$

well posed by toolbox, uniq sol by Riesz

note: $R(\mathcal{A}_i^*) = R(\mathrm{A}_i^*)$ and $f \in R(\mathrm{A}_i) \quad \Rightarrow \quad$ var form holds for all $\phi \in D(\mathrm{A}_i^*)$
$\Rightarrow \quad x_f := \mathrm{A}_i^* y_f \in D(\mathrm{A}_i)$ and $\mathrm{A}_i x_f = f$ holds
note: additional condition $y_f \in R(\mathrm{A}_i) = N(\mathrm{A}_i^*)^\perp \quad \Rightarrow \quad$ sadd point form / inf-sup
note: num approx satisfies $\tilde{x}_f \in R(\mathrm{A}_i^*)$ but $\tilde{x}_f \in D(\mathrm{A}_i)$ only weakly

corresponding idea works for $\mathrm{A}_{i-1}^* x_g = g$

Solution Theory for Abstract Problem

problem: find $x \in D(\mathrm{A}_i^* \mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)$ st

$$\mathrm{A}_i^* \mathrm{A}_i x = f$$
$$\mathrm{A}_{i-1}^* x = g$$
$$\pi_i x = h$$

$\Rightarrow$ introduce potential $y := \mathrm{A}_i x$

$\Rightarrow$ equiv mixed form: find pair $(x, y) \in \big(D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)\big) \times \big(\underbrace{D(\mathrm{A}_i^*) \cap R(\mathrm{A}_i)}_{=D(\mathcal{A}_i^*)}\big)$ st

$$\mathrm{A}_i x = y, \qquad \mathrm{A}_{i+1} y = 0$$
$$\mathrm{A}_{i-1}^* x = g, \qquad \mathrm{A}_i^* y = f$$
$$\pi_i x = h, \qquad \pi_{i+1} y = 0$$

Theorem (solution theory, P)

unique sol in Hadamard sense (cont dpd on data, ...)
$\Leftrightarrow \quad f \in R(\mathrm{A}_i^*), \quad g \in R(\mathrm{A}_{i-1}^*)$ *and* $h \in N(\mathrm{A}_i) \cap N(\mathrm{A}_{i-1}^*)$

Proof.

$x = x_y + x_g + h, \qquad x_y := \mathcal{A}_i^{-1} y, \quad x_g := (\mathcal{A}_{i-1}^*)^{-1} g, \quad y := (\mathcal{A}_i^*)^{-1} f$ □

Solution Theory for Abstract Problem

Theorem (solution theory, P)

unique sol in Hadamard sense (cont dpd on data, ...)
$\Leftrightarrow \quad f \in R(\mathrm{A}_i^*), \quad g \in R(\mathrm{A}_{i-1}^*) \quad$ *and* $\quad h \in N(\mathrm{A}_i) \cap N(\mathrm{A}_{i-1}^*)$

Proof.

$x = x_y + x_g + h, \qquad x_y := \mathcal{A}_i^{-1} y, \quad x_g := (\mathcal{A}_{i-1}^*)^{-1} g, \quad y := (\mathcal{A}_i^*)^{-1} f$ □

note: problem is linear and hence decouples

$$\begin{aligned} \mathrm{A}_i^* \mathrm{A}_i x &= f \\ \mathrm{A}_{i-1}^* x &= g \\ \pi_i x &= h \end{aligned} \qquad \Leftrightarrow \qquad \begin{aligned} \mathrm{A}_i x &= y, \\ \mathrm{A}_{i-1}^* x &= g, \\ \pi_i x &= h, \end{aligned} \qquad \begin{aligned} \mathrm{A}_{i+1} y &= 0 \\ \mathrm{A}_i^* y &= f \\ \pi_{i+1} y &= 0 \end{aligned}$$

$$\Leftrightarrow \qquad \begin{aligned} \mathrm{A}_i x_y &= y \\ \mathrm{A}_{i-1}^* x_y &= 0 \\ \pi_i x_y &= 0 \end{aligned} \qquad \begin{aligned} \mathrm{A}_i x_g &= 0 \\ \mathrm{A}_{i-1}^* x_g &= g \\ \pi_i x_g &= 0 \end{aligned} \qquad \begin{aligned} \mathrm{A}_{i+1} y &= 0 \\ \mathrm{A}_i^* y &= f \\ \pi_{i+1} y &= 0 \end{aligned}$$

Prototypical FOS: Electro-Magneto-Static Maxwell

$\Omega \subset \mathbb{R}^3$ bounded domain with weak Lipschitz boundary $\Gamma = \partial\Omega$, "$\Gamma = \Gamma_t \dot{\cup} \Gamma_n$"

$$
\begin{aligned}
\operatorname{rot}_{\Gamma_t} E &= J \in \operatorname{rot}_{\Gamma_t} \mathsf{R}_{\Gamma_t} && \text{in } \Omega \\
-\operatorname{div}_{\Gamma_n} \varepsilon E &= j \in \operatorname{div}_{\Gamma_n} \mathsf{D}_{\Gamma_n} = \mathsf{L}^2 \text{ or } \mathsf{L}^2_{\perp} && \text{in } \Omega \\
\nu \times E &= 0 && \text{on } \Gamma_t \\
\nu \cdot \varepsilon E &= 0 && \text{on } \Gamma_n \\
\pi E &= H \in \mathcal{H}_{\mathsf{D},\varepsilon} = \mathsf{R}_{\Gamma_t,0} \cap \varepsilon^{-1} \mathsf{D}_{\Gamma_n,0}
\end{aligned}
$$

$\Rightarrow E \in D(\operatorname{rot}_{\Gamma_t}) \cap D(\operatorname{div}_{\Gamma_n} \varepsilon) = \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1} \mathsf{D}_{\Gamma_n}$

$$
\mathrm{A}_0 := \nabla_{\Gamma_t} : \mathsf{H}^1_{\Gamma_t} \subset \mathsf{L}^2 \to \mathsf{L}^2_{\varepsilon}, \qquad \boxed{\mathrm{A}_1 := \operatorname{rot}_{\Gamma_t}} : \mathsf{R}_{\Gamma_t} \subset \mathsf{L}^2_{\varepsilon} \to \mathsf{L}^2
$$

$$
\boxed{\mathrm{A}_0^* = -\operatorname{div}_{\Gamma_n} \varepsilon} : \varepsilon^{-1} \mathsf{D}_{\Gamma_n} \subset \mathsf{L}^2_{\varepsilon} \to \mathsf{L}^2, \qquad \mathrm{A}_1^* = \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} : \mathsf{R}_{\Gamma_n} \subset \mathsf{L}^2 \to \mathsf{L}^2_{\varepsilon}
$$

Prototypical FOS: Electro-Magneto-Static Maxwell

compact embeddings:

$D(\mathcal{A}_0) \hookrightarrow \mathrm{H}_0 \quad \Leftrightarrow \quad \mathrm{H}^1_{\Gamma_t} \hookrightarrow \mathrm{L}^2$ (Rellich's selection theorem)

$D(\mathcal{A}_1) \hookrightarrow \mathrm{H}_1 \quad \Leftrightarrow \quad \mathrm{R}_{\Gamma_t} \cap \varepsilon^{-1} \operatorname{rot} \mathrm{R}_{\Gamma_n} \subset \mathrm{R}_{\Gamma_t} \cap \varepsilon^{-1} \mathrm{D}_{\Gamma_n} \hookrightarrow \mathrm{L}^2_\varepsilon$ (Weck's selection theorem)

$c_0 = c_{\mathtt{fp}}$ (Friedrichs/Poincaré constant) and $c_1 = c_{\mathtt{m}}$ (Maxwell constant)

$$\forall\, \varphi \in D(\mathcal{A}_0) \quad |\varphi|_{\mathrm{H}_0} \le c_0 |\mathrm{A}_0 \varphi|_{\mathrm{H}_1} \quad \Leftrightarrow \quad \forall\, \varphi \in \mathrm{H}^1_{\Gamma_t} \quad |\varphi|_{\mathrm{L}^2} \le c_{\mathtt{fp}} |\nabla \varphi|_{\mathrm{L}^2_\varepsilon}$$

$$\forall\, \phi \in D(\mathcal{A}_0^*) \quad |\phi|_{\mathrm{H}_1} \le c_0 |\mathrm{A}_0^* \phi|_{\mathrm{H}_0} \quad \Leftrightarrow \quad \forall\, \Phi \in \varepsilon^{-1} \mathrm{D}_{\Gamma_n} \cap \nabla \mathrm{H}^1_{\Gamma_t} \quad |\Phi|_{\mathrm{L}^2_\varepsilon} \le c_{\mathtt{fp}} |\operatorname{div} \varepsilon \Phi|_{\mathrm{L}^2}$$

$$\forall\, \varphi \in D(\mathcal{A}_1) \quad |\varphi|_{\mathrm{H}_1} \le c_1 |\mathrm{A}_1 \varphi|_{\mathrm{H}_2} \quad \Leftrightarrow \quad \forall\, \Phi \in \mathrm{R}_{\Gamma_t} \cap \varepsilon^{-1} \operatorname{rot} \mathrm{R}_{\Gamma_n} \quad |\Phi|_{\mathrm{L}^2_\varepsilon} \le c_{\mathtt{m}} |\operatorname{rot} \Phi|_{\mathrm{L}^2}$$

$$\forall\, \psi \in D(\mathcal{A}_1^*) \quad |\psi|_{\mathrm{H}_2} \le c_1 |\mathrm{A}_1^* \psi|_{\mathrm{H}_1} \quad \Leftrightarrow \quad \forall\, \Psi \in \mathrm{R}_{\Gamma_n} \cap \operatorname{rot} \mathrm{R}_{\Gamma_t} \quad |\Psi|_{\mathrm{L}^2} \le c_{\mathtt{m}} |\operatorname{rot} \Psi|_{\mathrm{L}^2_\varepsilon}$$

Helmholtz decomposition:

$$\mathrm{H}_1 = R(\mathrm{A}_0) \oplus_{\mathrm{H}_1} \big(N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \big) \oplus_{\mathrm{H}_i} R(\mathrm{A}_1^*) \quad \Leftrightarrow \quad \mathrm{L}^2_\varepsilon = \nabla \mathrm{H}^1_{\Gamma_t} \oplus_{\mathrm{L}^2_\varepsilon} \mathcal{H}_{\mathtt{D},\varepsilon} \oplus_{\mathrm{L}^2_\varepsilon} \varepsilon^{-1} \operatorname{rot} \mathrm{R}_{\Gamma_n}$$

Simplest SOS: Dirichlet/Neumann Laplace

$\Omega \subset \mathbb{R}^3$ bounded domain with weak Lipschitz boundary $\Gamma = \partial\Omega$, "$\Gamma = \Gamma_t \dot\cup \Gamma_n$"

$$\begin{aligned} -\operatorname{div}_{\Gamma_n} \varepsilon \nabla_{\Gamma_t} u &= f \in \mathsf{L}^2 \text{ or } \mathsf{L}^2_\perp && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma_t \\ \nu \cdot \varepsilon \nabla u &= 0 && \text{on } \Gamma_n \end{aligned}$$

$$\Leftrightarrow \qquad \begin{aligned} \nabla_{\Gamma_t} u &= E \in \nabla_{\Gamma_t} \mathsf{H}^1_{\Gamma_t} & \operatorname{rot}_{\Gamma_t} E &= 0 \in \operatorname{rot}_{\Gamma_t} \mathsf{R}_{\Gamma_t} && \text{in } \Omega \\ & & -\operatorname{div}_{\Gamma_n} \varepsilon E &= f \in \mathsf{L}^2 \text{ or } \mathsf{L}^2_\perp && \text{in } \Omega \\ u &= 0 & \nu \times E &= 0 && \text{on } \Gamma_t \\ & & \nu \cdot \varepsilon E &= 0 && \text{on } \Gamma_n \\ & & \pi E &= 0 \in \mathcal{H}_{\mathtt{D},\varepsilon} && \end{aligned}$$

$$\Rightarrow \ (u, E) \in D(\nabla_{\Gamma_t}) \times \big(D(\operatorname{div}_{\Gamma_n} \varepsilon) \cap R(\nabla_{\Gamma_t}) \big) = \mathsf{H}^1_{\Gamma_t} \times \big(\varepsilon^{-1} \mathsf{D}_{\Gamma_n} \cap \nabla \mathsf{H}^1_{\Gamma_t} \big)$$

$$\boxed{\mathrm{A}_0 := \nabla_{\Gamma_t}} : \mathsf{H}^1_{\Gamma_t} \subset \mathsf{L}^2 \to \mathsf{L}^2_\varepsilon, \qquad \mathrm{A}_1 := \operatorname{rot}_{\Gamma_t} : \mathsf{R}_{\Gamma_t} \subset \mathsf{L}^2_\varepsilon \to \mathsf{L}^2$$

$$\boxed{\mathrm{A}_0^* = -\operatorname{div}_{\Gamma_n} \varepsilon} : \varepsilon^{-1} \mathsf{D}_{\Gamma_n} \subset \mathsf{L}^2_\varepsilon \to \mathsf{L}^2, \qquad \mathrm{A}_1^* = \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} : \mathsf{R}_{\Gamma_n} \subset \mathsf{L}^2 \to \mathsf{L}^2_\varepsilon$$

More First and Second Order Systems (FOS & SOS)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain

Electro/Magneto-Static Maxwell with mixed boundary conditions
∇-rot-div-complex (symmetry!, de Rham complex):

$$\{0\} \text{ or } \mathbb{R} \; \underset{\pi}{\overset{\iota}{\rightleftarrows}} \; \mathsf{L}^2 \; \underset{-\operatorname{div}_{\Gamma_n} \varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}} \; \mathsf{L}^2_{\varepsilon} \; \underset{\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}} \; \mathsf{L}^2 \; \underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}} \; \mathsf{L}^2 \; \underset{\iota}{\overset{\pi}{\rightleftarrows}} \; \mathbb{R} \text{ or } \{0\}$$

related fos

$$\begin{array}{rl|rl|rl|rl} \nabla_{\Gamma_t} u = A & \text{in } \Omega & \operatorname{rot}_{\Gamma_t} E = J & \text{in } \Omega & \operatorname{div}_{\Gamma_t} H = k & \text{in } \Omega & \pi v = b & \text{in } \Omega \\ \pi u = a & \text{in } \Omega & -\operatorname{div}_{\Gamma_n} \varepsilon E = j & \text{in } \Omega & \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} H = K & \text{in } \Omega & -\nabla_{\Gamma_n} v = B & \text{in } \Omega \end{array}$$

related sos

$$\begin{array}{rl|rl|rl} -\operatorname{div}_{\Gamma_n} \varepsilon \nabla_{\Gamma_t} u = j & \text{in } \Omega & \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} \operatorname{rot}_{\Gamma_t} E = K & \text{in } \Omega & -\nabla_{\Gamma_n} \operatorname{div}_{\Gamma_t} H = B & \text{in } \Omega \\ \pi u = a & \text{in } \Omega & -\operatorname{div}_{\Gamma_n} \varepsilon E = j & \text{in } \Omega & \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} H = K & \text{in } \Omega \end{array}$$

corresponding compact embeddings:

$$D(\nabla_{\Gamma_t}) \cap D(\pi) = D(\nabla_{\Gamma_t}) = \mathsf{H}^1_{\Gamma_t} \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

$$D(\operatorname{rot}_{\Gamma_t}) \cap D(-\operatorname{div}_{\Gamma_n} \varepsilon) = \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1} \mathsf{D}_{\Gamma_n} \hookrightarrow \mathsf{L}^2_{\varepsilon} \qquad \text{(Weck's selection theorem)}$$

$$D(\operatorname{div}_{\Gamma_t}) \cap D(\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}) = \mathsf{D}_{\Gamma_t} \cap \mathsf{R}_{\Gamma_n} \hookrightarrow \mathsf{L}^2 \qquad \text{(Weck's selection theorem)}$$

$$D(\nabla_{\Gamma_n}) \cap D(\pi) = D(\nabla_{\Gamma_n}) = \mathsf{H}^1_{\Gamma_n} \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

Weck's selection theorem for weak Lip. dom. and mixed bc: Bauer/P/Schomburg ('16)

More First and Second Order Systems (FOS & SOS)

$\Omega \subset \mathbb{R}^N$ bd w. Lip. dom. or Ω Riemannian manifold with cpt cl. and Lip. boundary Γ

Generalized Electro/Magneto-Static Maxwell with mixed boundary conditions
d-d-complex (symmetry!, de Rham complex):

$$\{0\} \text{ or } \mathbb{R} \quad \underset{\pi}{\overset{\iota}{\rightleftarrows}} \quad \mathrm{L}^{2,0} \quad \underset{-\,\delta^1_{\Gamma_n}}{\overset{\mathrm{d}^0_{\Gamma_t}}{\rightleftarrows}} \quad \mathrm{L}^{2,1} \quad \underset{-\,\delta^2_{\Gamma_n}}{\overset{\mathrm{d}^1_{\Gamma_t}}{\rightleftarrows}} \quad \ldots \quad \mathrm{L}^{2,q} \quad \underset{-\,\delta^{q+1}_{\Gamma_n}}{\overset{\mathrm{d}^q_{\Gamma_t}}{\rightleftarrows}} \quad \mathrm{L}^{2,q+1} \ldots \mathrm{L}^{2,N-1} \quad \underset{-\,\delta^N_{\Gamma_n}}{\overset{\mathrm{d}^{N-1}_{\Gamma_t}}{\rightleftarrows}} \quad \mathrm{L}^{2,N} \quad \underset{\iota}{\overset{\pi}{\rightleftarrows}} \quad \mathbb{R} \text{ or } \{0\}$$

related fos

$$\mathrm{d}^q_{\Gamma_t} E = F \qquad \text{in } \Omega$$
$$-\,\delta^q_{\Gamma_n} E = G \qquad \text{in } \Omega$$

related sos

$$-\,\delta^{q+1}_{\Gamma_n} \mathrm{d}^q_{\Gamma_t} E = F \qquad \text{in } \Omega$$
$$-\,\delta^q_{\Gamma_n} E = G \qquad \text{in } \Omega$$

includes: EMS rot / div, Laplacian, rot rot, and more...
corresponding compact embeddings:

$$D(\mathrm{d}^q_{\Gamma_t}) \cap D(\delta^q_{\Gamma_n}) \hookrightarrow \mathrm{L}^{2,q} \qquad \text{(Weck's selection theorems)}$$

Weck's selection theorem for Lip. manifolds and mixed bc: Bauer/P/Schomburg ('17)

More First and Second Order Systems (FOS & SOS)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

Elasticity
$\operatorname{sym}\nabla$-$\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}$-$\operatorname{Div}_{\mathbb{S}}$-complex (symmetry!):

$$\{0\} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{Div}_{\mathbb{S}}}{\overset{\operatorname{sym}\nabla_\Gamma}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}}{\overset{\operatorname{Rot}\operatorname{Rot}_{\mathbb{S},\Gamma}^{\top}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{-\operatorname{sym}\nabla}{\overset{\operatorname{Div}_{\mathbb{S},\Gamma}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathsf{RM}$$

related fos ($\operatorname{Rot}\operatorname{Rot}_{\mathbb{S},\Gamma}^{\top}$, $\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}$ first order operators!)

$\operatorname{sym}\nabla_\Gamma v = M$ in Ω	$\operatorname{Rot}\operatorname{Rot}_{\mathbb{S},\Gamma}^{\top} M = F$ in Ω	$\operatorname{Div}_{\mathbb{S},\Gamma} N = g$ in Ω	$\pi v = r$ in Ω
$\pi v = 0$ in Ω	$-\operatorname{Div}_{\mathbb{S}} M = f$ in Ω	$\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top} N = G$ in Ω	$-\operatorname{sym}\nabla v = M$ in Ω

related sos ($\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}\operatorname{Rot}\operatorname{Rot}_{\mathbb{S},\Gamma}^{\top}$ second order operator!)

$-\operatorname{Div}_{\mathbb{S}}\operatorname{sym}\nabla_\Gamma v = f$ in Ω	$\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}\operatorname{Rot}\operatorname{Rot}_{\mathbb{S},\Gamma}^{\top} M = G$ in Ω	$-\operatorname{sym}\nabla\operatorname{Div}_{\mathbb{S},\Gamma} N = M$ in Ω
$\pi v = 0$ in Ω	$-\operatorname{Div}_{\mathbb{S}} M = f$ in Ω	$\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top} N = G$ in Ω

corresponding compact embeddings:

$$D(\operatorname{sym}\nabla_\Gamma)\cap D(\pi) = D(\nabla_\Gamma) = \mathsf{H}^1_\Gamma \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem and Korn ineq.)}$$

$$D(\operatorname{Rot}\operatorname{Rot}_{\mathbb{S},\Gamma}^{\top})\cap D(\operatorname{Div}_{\mathbb{S}}) \hookrightarrow \mathsf{L}^2_{\mathbb{S}} \qquad \text{(new selection theorem)}$$

$$D(\operatorname{Div}_{\mathbb{S},\Gamma})\cap D(\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}) \hookrightarrow \mathsf{L}^2_{\mathbb{S}} \qquad \text{(new selection theorem)}$$

$$D(\pi)\cap D(\operatorname{sym}\nabla) = D(\nabla) = \mathsf{H}^1 \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem and Korn ineq.)}$$

two new selection theorems for strong Lip. dom.: P/Zulehner ('17)

More First and Second Order Systems (FOS & SOS)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

General Relativity or Biharmonic Equation
$\nabla\nabla$-$\mathrm{Rot}_{\mathbb{S}}$-$\mathrm{Div}_{\mathbb{T}}$-complex (no symmetry!):

$$\{0\} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\nabla\nabla_\Gamma}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\operatorname{Rot}_{\mathbb{S},\Gamma}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}} \underset{-\operatorname{dev}\nabla}{\overset{\operatorname{Div}_{\mathbb{T},\Gamma}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathsf{RT}$$

related fos ($\nabla\nabla_\Gamma$, $\operatorname{div}\operatorname{Div}_{\mathbb{S}}$ first order operators!)

$\nabla\nabla_\Gamma u = M$	in Ω	|	$\operatorname{Rot}_{\mathbb{S},\Gamma} M = F$	in Ω	|	$\operatorname{Div}_{\mathbb{T},\Gamma} N = g$	in Ω	|	$\pi v = r$	in Ω
$\pi u = 0$	in Ω	|	$\operatorname{div}\operatorname{Div}_{\mathbb{S}} M = f$	in Ω	|	$\operatorname{sym}\operatorname{Rot}_{\mathbb{T}} N = G$	in Ω	|	$-\operatorname{dev}\nabla v = T$	in Ω

related sos ($\operatorname{div}\operatorname{Div}_{\mathbb{S}}\nabla\nabla_\Gamma = \Delta^2_\Gamma$ second order operator!)

$\operatorname{div}\operatorname{Div}_{\mathbb{S}}\nabla\nabla_\Gamma u = \Delta^2_\Gamma u = f$	in Ω	|	$\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma} M = G$	in Ω	|	$-\operatorname{dev}\nabla\operatorname{Div}_{\mathbb{T},\Gamma} N = T$	in Ω
$\pi u = 0$	in Ω	|	$\operatorname{div}\operatorname{Div}_{\mathbb{S}} M = f$	in Ω	|	$\operatorname{sym}\operatorname{Rot}_{\mathbb{T}} N = G$	in Ω

corresponding compact embeddings:

$$D(\nabla\nabla_\Gamma) \cap D(\pi) = D(\nabla\nabla_\Gamma) = \mathsf{H}^2_\Gamma \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

$$D(\operatorname{Rot}_{\mathbb{S},\Gamma}) \cap D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \hookrightarrow \mathsf{L}^2_{\mathbb{S}} \qquad \text{(new selection theorem)}$$

$$D(\operatorname{Div}_{\mathbb{T},\Gamma}) \cap D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \hookrightarrow \mathsf{L}^2_{\mathbb{T}} \qquad \text{(new selection theorem)}$$

$$D(\pi) \cap D(\operatorname{dev}\nabla) = D(\operatorname{dev}\nabla) = D(\nabla) = \mathsf{H}^1 \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem and Korn type ineq.)}$$

two new selection theorems for strong Lip. dom. and Korn Type ineq.: P/Zulehner ('16)

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