FA-TOOLBOX: SOLVING PDES WITH HILBERT COMPLEXES

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The aim of this talk is to present parts of the so-called *Functional Analysis Toolbox* (FA-ToolBox), a unified and general approach to solve PDEs. *Hilbert Complexes* are of particular interest.

We shall motivate our concept by discussing the well known and prototypical div-curl-system

$$\mathring{\operatorname{curl}} E = F, \qquad \operatorname{div} E = g,$$

arising, e.g., in electro-magneto statics. Employing techniques from linear functional analysis (FA-ToolBox) we develop a comprehensive (and surprisingly simple) solution theory for static problems of the above type. We will introduce the notion of Hilbert complexes

$$H_0 \xrightarrow{\mathrm{A}_0} H_1 \xrightarrow{\mathrm{A}_1} H_2,$$

of densely defined and closed linear operators

$$\mathrm{A}_0 : D(\mathrm{A}_0) \subset H_0 \to H_1, \qquad \mathrm{A}_1 : D(\mathrm{A}_1) \subset H_1 \to H_2,$$

satisfying the so-called *complex property*

$$R(\mathrm{A}_0) \subset N(\mathrm{A}_1).$$

The latter electro static system is then generalised to

$$\mathrm{A}_1 x = f, \qquad \mathrm{A}_0^* x = g.$$

The aim is to provide criteria on the complex such that existence and uniqueness of x can be guaranteed. It will turn out that the crucial property is the *compactness* of the embedding

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow H_1,$$

i.e., in classical terms the compactness of

$$D(\mathring{\operatorname{curl}}) \cap D(\operatorname{div}) \hookrightarrow L^2,$$

the co-called Picard-Weber-Weck selection theorem.

Our general theory is not only applicable to the classical *de Rham complex* involving grad, curl, and div, but also to other important Hilbert complexes, such as the *elasticity complex* or the *biharmonic complex*. Moreover, important results can be proved in this general setting, such as general div-curl-type lemmas and informations about generalised Poincaré/Friedrichs estimates, e.g., for the Maxwell constants.

This talk contains parts of joined work with colleagues from Essen, Linz, and Prag, in particular, with Walter Zulehner (JKU Linz). Some parts are strongly related to the work of Doug Arnold (Minnesota) and Ragnar Winther (Oslo) and their co-authors.

Results of this talk can be found in, e.g., [2, 1, 3, 4, 5, 7, 8, 6].

References

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