Traces for Hilbert Complexes

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Traces for Hilbert Complexes

OVERVIEW and BASIC IDEAS

paper in JFA 2023:

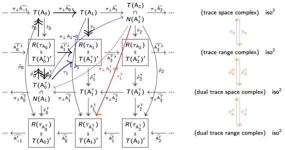
R. Hiptmair, D. Pauly, and E. Schulz: *Traces for Hilbert Complexes*

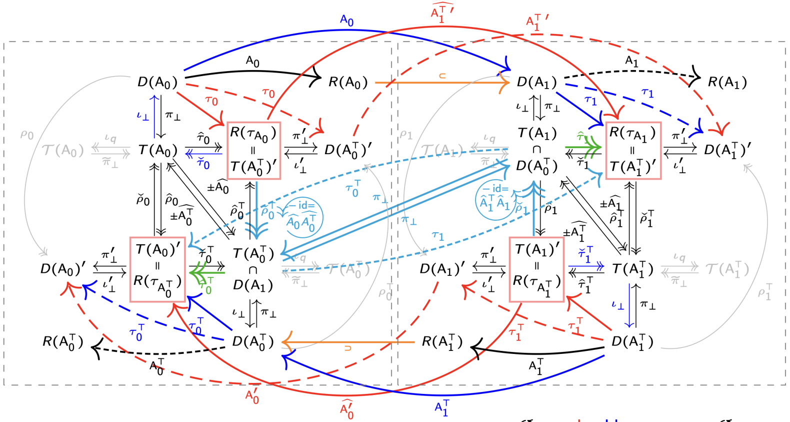
Traces for Hilbert Complexes

Question: Why are traces so complicated?

Question ?: What is $H^{-1/2}(\partial\Omega)$?

... some answers below





$\circlearrowleft$ red = blue = cyan $\circlearrowleft$

Traces

Traces without any regularity of the domain?

Is this even possible?

even better: Traces without domains (and boundaries)?

Traces

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ lddc: lin, dendef, cl

Traces for $D(\mathrm{A})$?

$\Omega \subset \mathbb{R}^N$ Lipschitz:

very classical

$D(\mathrm{A}) = \mathrm{H}^1$ or $\mathrm{W}^{1,p}$, scalar trace $u_{\mathrm{s}} = u|_\Gamma$

classical (we stay in Hilbert spaces)

$D(\mathrm{A}) = \mathrm{H}(\mathrm{curl})$ or $\mathrm{H}(\mathrm{div})$, tan or nor traces $v_{\mathrm{t}} = (\nu \times v \times \nu)\big|_\Gamma$, $v_{\mathrm{n}} = (\nu \cdot v)\big|_\Gamma$

more recent

$$D(\mathrm{A}) = \mathrm{H}(\mathrm{Curl\,Curl}), \mathrm{H}(\mathrm{div\,Div}), \dots$$
$$\dots \mathrm{H}(\mathrm{Curl\,Curl\,Curl}), \mathrm{H}(\mathrm{curl\,Div}), \mathrm{H}(\mathrm{Grad\,curl}) \dots$$

traces?

Traces

$A : D(A) \subset H_0 \to H_1$ ldd c

Traces for $D(A)$?

$\Omega \subset \mathbb{R}^N$ Lipschitz

What if less regularity? What if

- Ω just open / no regularity and $D(A) = H^1(\Omega)$, $H(\mathrm{curl}, \Omega)$, $H(\mathrm{div}, \Omega)$, ...?
- no Ω at all, just $D(A)$?

Traces

$$A : D(A) \subset H_0 \to H_1 \qquad \text{lddc}$$
$$A^* : D(A^*) \subset H_1 \to H_0 \qquad \text{lddc, Hilbert space adjoint}$$

Traces for $D(A)$?

basic idea: integration by parts / extension of adjoints

$$\forall\, x \in D(A) \quad \forall\, y \in D(A^*) \qquad \boxed{\langle y, A\,x\rangle_{H_1} - \langle A^*\,y, x\rangle_{H_0} = 0}$$

think of $A = \mathring{\text{grad}} : D(A) = \mathring{H}^1 \subset L^2 \to L^2$
and $A^* = -\operatorname{div} : D(A^*) = H(\operatorname{div}) \subset L^2 \to L^2$

$$\langle y, \mathring{\text{grad}}\, x\rangle_{L^2} + \langle \operatorname{div} y, x\rangle_{L^2} = 0$$

Traces

$\mathring{A} \subset A$ lddc

$A^* \subset A^\top := \mathring{A}^*$ ($A^\top$ transpose of A) lddc, Hilbert space adjoints

Traces for $D(A)$?

basic idea and setting: integration by parts / extension of adjoints

$$\exists\, x \in D(A) \quad \exists\, y \in D(A^\top) \qquad \boxed{\langle y, A\,x\rangle_{H_1} - \langle A^\top y, x\rangle_{H_0} \neq 0}$$

think of $\mathring{\text{grad}} = \mathring{A} \subset A = \text{grad}$ $\qquad D(\mathring{\text{grad}}) = \mathring{H}^1 \subset D(\text{grad}) = H^1$

and $-\mathring{\text{div}} = \text{grad}^* = A^* \subset A^\top = \mathring{\text{grad}}^* = -\text{div}$ $\qquad D(\mathring{\text{div}}) = \mathring{H}(\text{div}) \subset D(\text{div}) = H(\text{div})$

$$\langle y, \text{grad}\, x\rangle_{L^2(\Omega)} + \langle \text{div}\, y, x\rangle_{L^2(\Omega)} = \underbrace{\langle y_n, x_s\rangle_{\text{“}L^2(\Gamma)\text{''}}} \neq 0$$

$$= \text{“}\int_\Gamma y_n x_s\text{''} = \langle\!\langle y_n, x_s\rangle\!\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}$$

for some $x \in H^1$, $y \in H(\text{div})$

For simplicity of the talk: real Hilbert spaces

Traces

$\mathring{A} \subset A$	lddc
$A^* \subset A^\top = \mathring{A}^*$	lddc, Hilbert space adjoints

Traces for $D(A)$?

basic idea and setting: integration by parts / extension of adjoints

trace $\quad \tau_A : D(A) \to D(A^\top)', \quad x \mapsto \tau_A x, \qquad \tau_A x(y) := \langle y, A\,x\rangle_{H_1} - \langle A^\top y, x\rangle_{H_0}$

$$\boxed{x \in D(A),\ y \in D(A^\top)}$$

dual trace $\quad \tau_{A^\top} : D(A^\top) \to D(A)', \quad y \mapsto \tau_{A^\top} y, \qquad \tau_{A^\top} y(x) := \langle x, A^\top y\rangle_{H_0} - \langle A\,x, y\rangle_{H_1}$

note $\boxed{\tau_{A^\top} y(x) = -\tau_A x(y)}$ $\qquad \big(\tau_{A^\top} y(x) = -\overline{\tau_A x(y)}$ in complex Hilbert spaces$\big)$

Traces

$\mathring{\mathrm{A}} \subset \mathrm{A}$ — lddc

$\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ — lddc, Hilbert space adjoints

$(\mathring{\mathrm{A}}, \mathrm{A}^*)$ pair with boundary conditions
$(\mathrm{A}, \mathrm{A}^\top)$ pair without boundary conditions
$(\mathrm{A}, \mathrm{A}^*)$, $(\mathring{\mathrm{A}}, \mathrm{A}^\top = \mathring{\mathrm{A}}^*)$ dual/adjoint pairs

Traces for $D(\mathrm{A})$?

bounded trace $\tau_{\mathrm{A}} : D(\mathrm{A}) \to D(\mathrm{A}^\top)'$

bounded dual trace $\tau_{\mathrm{A}^\top} : D(\mathrm{A}^\top) \to D(\mathrm{A})'$

or bilinear (sesquilinear) form on $D(\mathrm{A}) \times D(\mathrm{A}^\top)$

$$\langle\langle x, y \rangle\rangle := \tau_{\mathrm{A}} x(y) = -\tau_{\mathrm{A}^\top} y(x) = \langle y, \mathrm{A}\, x \rangle_{\mathrm{H}_1} - \langle \mathrm{A}^\top y, x \rangle_{\mathrm{H}_0}$$

Hilbert Complexes

Why do we need Hilbert Complexes?

Hilbert Complexes

$$\mathrm{H}_0 \xrightarrow{\mathrm{A}_0} \mathrm{H}_1 \xrightarrow{\mathrm{A}_1} \mathrm{H}_2 \qquad \mathrm{A}_\ell \text{ lddc with } R(\mathrm{A}_0) \subset N(\mathrm{A}_1) \quad \text{(complex)}$$

$$\mathrm{H}_0 \xleftarrow{\mathrm{A}_0^*} \mathrm{H}_1 \xleftarrow{\mathrm{A}_1^*} \mathrm{H}_2 \qquad \mathrm{A}_\ell \text{ lddc with } R(\mathrm{A}_1^*) \subset N(\mathrm{A}_0^*) \quad \text{(complex)}$$

Why do we need Hilbert Complexes?

Example: static Maxwell's Equations (de Rham complex)

$$\boxed{\mathring{\operatorname{curl}}\, E = F}$$

What is a good second equation for E?

Physics $\Rightarrow$ $\operatorname{div} E = f$

Just Physics? Or $\exists$ Math justification for div?

Is div uniquely determined?

abstract setting: $\mathrm{A}_1\, x = f \quad \overset{x \text{ unique}}{\Rightarrow} \quad x \in N(\mathrm{A}_1)^\perp = \overline{R(\mathrm{A}_1^*)} \subset N(\mathrm{A}_0^*) \quad \Rightarrow \quad \mathrm{A}_0^*\, x = 0$

Example: $\mathrm{A}_1 = \mathring{\operatorname{curl}} \quad \Rightarrow \quad \mathrm{A}_0 = \mathring{\operatorname{grad}} \quad \Rightarrow \quad \mathrm{A}_0^* = -\operatorname{div} \quad \Rightarrow$ div is Math necessary!

Hilbert Complexes

Traces for Hilbert Complexes

- We give Traces for Hilbert Complexes.
- Hilbert Complexes are necessary for Traces.

Traces for Single Operators

Traces for Single Operators

Traces for Single Operators

$\mathring{A} \subset A$ and $A^* \subset A^\top = \mathring{A}^*$ lddc (Hilbert space adjoints)

Traces for $D(A)$ and $D(A^\top)$ — traces come always in pairs

trace $\quad \tau_A : D(A) \to D(A^\top)'$
$$x \mapsto \tau_A x$$

dual trace $\quad \tau_{A^\top} : D(A^\top) \to D(A)'$
$$y \mapsto \tau_{A^\top} y$$

recall: for $x \in D(A)$ and $y \in D(A^\top)$ (idea: integration by parts / extension of adjoints)

$$\tau_A x(y) = \langle y, A\,x\rangle_{H_1} - \langle A^\top y, x\rangle_{H_0} \qquad \big(= -\tau_{A^\top} y(x) \big)$$

$$\Rightarrow \quad \big|\tau_A x(y)\big| \leq |x|_{D(A)} |y|_{D(A^\top)}$$

note: $\tau_A x(y) = 0 \quad \Leftrightarrow \quad x \in D(\mathring{A}) \vee y \in D(A^*) \quad \Leftrightarrow \quad \tau_{A^\top} y(x) = 0$

$\Rightarrow \quad \tau_A x = 0 \quad \Leftrightarrow \quad x \in D(\mathring{A})$ and $\tau_{A^\top} y = 0 \quad \Leftrightarrow \quad y \in D(A^*)$

Lemma (kernels and boundedness)

$N(\tau_A) = D(\mathring{A})$ *and* $N(\tau_{A^\top}) = D(A^*)$ *and* $\|\tau_A\|, \|\tau_{A^\top}\| \leq 1$

Traces for Single Operators (Adjoints)

$\mathring{A} \subset A$ and $A^* \subset A^\top = \mathring{A}^*$ lddc (Hilbert space adjoints)

primal / dual traces $\tau_A : D(A) \to D(A^\top)'$, $\tau_{A^\top} : D(A^\top) \to D(A)'$

primal / dual adjoint traces $\tau'_A : D(A^\top)'' \to D(A)'$, $\tau'_{A^\top} : D(A)'' \to D(A^\top)'$

note: Hilbert spaces H self-dual (Riesz) and reflexive here: $H = D(A) \vee D(A^\top)$

$\Rightarrow$ isometric isomorphisms $\rho_H : H \to H'$, $\iota_d : H \to H''$,

$x \mapsto \rho_H x := \langle\,\cdot\,, x\rangle_H$ $x \mapsto \iota_d x$ with $\iota_d x(x') := x'x$

for all $x \in H$ and all $x' \in H'$

note for $x \in D(A)$ and $y \in D(A^\top)$:

$$(\tau'_A \iota_d y)(x) = (\iota_d y)(\tau_A x) = \tau_A x(y) = -\tau_{A^\top} y(x)$$

Theorem (adjoints)

$\tau'_A \iota_d = -\tau_{A^\top}$ *and* $\tau'_{A^\top} \iota_d = -\tau_A$

note: $\overline{R(\tau_{A^\top})} = \overline{R(\tau'_A)} = N(\tau_A)^\circ = D(\mathring{A})^\circ$

Remark

$\overline{R(\tau_A)} = D(A^*)^\circ$ and $\overline{R(\tau_{A^\top})} = D(\mathring{A})^\circ$

Traces for Single Operators (Riesz Isometric Isometries)

$\mathring{A} \subset A$ and $A^* \subset A^\top = \mathring{A}^*$ lddc (Hilbert space adjoints)

recall Riesz iso[2] $\rho_{D(A^\top)} : D(A^\top) \to D(A^\top)'$,

$y \mapsto \langle \cdot, y \rangle_{D(A^\top)}$

define domain trace $\sigma_A := \rho^{-1}_{D(A^\top)} \tau_A : D(A) \to D(A^\top)$, $\tau_A : D(A) \to D(A^\top)'$

note: $\tau_A = \rho_{D(A^\top)} \sigma_A$

Let $x \in D(A)$.

What is / solves

$$\check{y} := \sigma_A x := \rho^{-1}_{D(A^\top)} \tau_A x \in D(A^\top)\,?$$

Traces for Single Operators (Riesz Isometric Isometries)

$\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ lddc (Hilbert space adjoints)

Let $x \in D(\mathrm{A})$.

What is $D(\mathrm{A}^\top) \ni \check{y} = \sigma_{\mathrm{A}} x = \rho^{-1}_{D(\mathrm{A}^\top)} \tau_{\mathrm{A}} x$, i.e., $\rho_{D(\mathrm{A}^\top)} \check{y} = \tau_{\mathrm{A}} x \in D(\mathrm{A}^\top)'$?

for all $y \in D(\mathrm{A}^\top)$

$$\underbrace{\langle y, \check{y} \rangle_{\mathrm{H}_1} + \langle \mathrm{A}^\top y, \mathrm{A}^\top \check{y} \rangle_{\mathrm{H}_0}}_{= \langle y, \check{y} \rangle_{D(\mathrm{A}^\top)}} = \rho_{D(\mathrm{A}^\top)} \check{y}(y) = \tau_{\mathrm{A}} x(y) = \langle y, \mathrm{A}\, x \rangle_{\mathrm{H}_1} - \langle \mathrm{A}^\top y, x \rangle_{\mathrm{H}_0}$$

$$\Rightarrow \quad \langle \mathrm{A}^\top y, \mathrm{A}^\top \check{y} + x \rangle_{\mathrm{H}_0} = \langle y, \mathrm{A}\, x - \check{y} \rangle_{\mathrm{H}_1}$$

$$\Rightarrow \quad \mathrm{A}^\top \check{y} + x \in D(\mathrm{A}^{\top *} = \mathring{\mathrm{A}}) \text{ and } \mathring{\mathrm{A}}(\mathrm{A}^\top \check{y} + x) = \mathrm{A}\, x - \check{y}$$

$$x \in D(\mathrm{A}) \supset D(\mathring{\mathrm{A}}) \quad \Rightarrow \quad \check{x} := \mathrm{A}^\top \check{y} \in D(\mathrm{A}) \quad \Rightarrow \quad (\mathrm{A}\,\mathrm{A}^\top + 1)\check{y} = 0$$

$$\Rightarrow \quad \check{y} \in N(\mathrm{A}\,\mathrm{A}^\top + 1) \text{ and } \check{x} \in N(\mathrm{A}^\top \mathrm{A} + 1) \text{ and } \check{x} + x \in D(\mathring{\mathrm{A}})$$

note: $\tau_{\mathrm{A}}(\check{x} + x) = \tau_{\mathrm{A}}(\mathrm{A}^\top \check{y} + x) = 0$
and "formally" $(\mathring{\mathrm{A}}\,\mathrm{A}^\top \check{y} + 1)\check{y} = (\mathrm{A} - \mathring{\mathrm{A}})x$ and $\check{y} = (\mathring{\mathrm{A}}\,\mathrm{A}^\top \check{y} + 1)^{-1}(\mathrm{A} - \mathring{\mathrm{A}})x$
are "boundary terms" as $(\mathrm{A} - \mathring{\mathrm{A}})x = 0$ for $x \in D(\mathring{\mathrm{A}})$

Traces for Single Operators (Riesz Isometric Isometries)

$\mathring{A} \subset A$ and $A^* \subset A^\top = \mathring{A}^*$ lddc (Hilbert space adjoints)

Lemma (extension / right inverse)

Let $x \in D(A)$ and let $\check{x} := A^\top \check{y}$ *and* $\check{y} = \sigma_A x = \rho^{-1}_{D(A^\top)} \tau_A x$. *Then:*

$$(\check{x}, \check{y}) \in N(A^\top A + 1) \times N(A A^\top + 1) \quad \text{and} \quad \check{x} + x \in D(\mathring{A}) = N(\tau_A)$$

$$\Rightarrow \quad \boxed{\tau_A A^\top \check{y} = \tau_A \check{x} = -\tau_A x}$$

$\varphi := -\tau_A x \quad \Rightarrow \quad \tau_A A^\top \check{y} = \tau_A \check{x} = \varphi$ with $\check{x} := A^\top \check{y}$ and $\check{y} := -\rho^{-1}_{D(A^\top)} \varphi \quad \Rightarrow$

Corollary (extension / right inverse)

$-\tau_A A^\top \rho^{-1}_{D(A^\top)} = \mathrm{id}_{R(\tau_A)}$ *and* $-A^\top \rho^{-1}_{D(A^\top)}$ *right inverse of* τ_A *on* $R(\tau_A)$

note: $A \check{x} = A A^\top \check{y} = -\check{y} \quad \Rightarrow \quad |\check{x}|_{D(A)} = \sqrt{|\check{x}|^2_{H_0} + |A \check{x}|^2_{H_1}} = |\check{y}|_{D(A^\top)}$ and

Corollary (extension / right inverse (S form skw sym))

$$\left(S^2 - 1\right) \begin{bmatrix} \check{x} \\ \check{y} \end{bmatrix} = \left(S - 1\right) \begin{bmatrix} \check{x} \\ \check{y} \end{bmatrix} = 0, \qquad S^2 = -\begin{bmatrix} A^\top A & 0 \\ 0 & A A^\top \end{bmatrix}, \quad S := \begin{bmatrix} 0 & A^\top \\ -A & 0 \end{bmatrix}$$

Traces for Single Operators (Riesz Isometric Isometries)

$\mathring{A} \subset A$ and $A^* \subset A^\top = \mathring{A}^*$ lddc (Hilbert space adjoints)

Definition (extensions / right inverses)

Let $\phi \in R(\tau_A)$ and $\psi \in R(\tau_{A^\top})$. We call:

- $\check{\phi} = -\rho^{-1}_{D(A^\top)}\phi \in N(A\,A^\top + 1)$ harm Neumann ext of ϕ since $\tau_A\,A^\top\,\check{\phi} = \phi$
- $\check{\check{\phi}} = A^\top\,\check{\phi} = -A^\top\,\rho^{-1}_{D(A^\top)}\phi \in N(A^\top A + 1)$ harm Dirichlet ext of ϕ since $\tau_A\check{\check{\phi}} = \phi$
- $\check{\psi} = -\rho^{-1}_{D(A)}\psi \in N(A^\top A + 1)$ harm Neumann ext of ψ since $\tau_{A^\top}\,A\,\check{\psi} = \psi$
- $\check{\check{\psi}} = A\,\check{\psi} = -A\,\rho^{-1}_{D(A)}\psi \in N(A\,A^\top + 1)$ harm Dirichlet ext of ψ since $\tau_{A^\top}\check{\check{\psi}} = \psi$

Corollary (extension / right inverse)

$-\tau_A\,A^\top\,\rho^{-1}_{D(A^\top)} = \mathrm{id}_{R(\tau_A)}$ *and* $\check{\tau}_A := -A^\top\,\rho^{-1}_{D(A^\top)}$ *right inverse of* τ_A *on* $R(\tau_A)$

$-\tau_{A^\top}\,A\,\rho^{-1}_{D(A)} = \mathrm{id}_{R(\tau_{A^\top})}$ *and* $\check{\tau}_{A^\top} := -A\,\rho^{-1}_{D(A)}$ *right inverse of* $\tau_{A^\top}$ *on* $R(\tau_{A^\top})$

Corollary (extension / right inverse (S form skw sym))

$$\big(S - 1\big)\begin{bmatrix}\check{\check{\phi}}\\ \check{\phi}\end{bmatrix} = \big(-S - 1\big)\begin{bmatrix}\check{\check{\psi}}\\ \check{\psi}\end{bmatrix} = 0, \qquad S^2 = -\begin{bmatrix}A^\top A & 0\\ 0 & A\,A^\top\end{bmatrix}, \qquad -S = \begin{bmatrix}0 & -A^\top\\ A & 0\end{bmatrix}$$

Traces for Single Operators $\mathring{A} \subset A$ and $A^* \subset A^\top = \mathring{A}^*$ (lddc)

recall trace / dual trace $\tau_A : D(A) \to D(A^\top)'$, $\tau_{A^\top} : D(A^\top) \to D(A)'$

with $-\tau_{A^\top} y(x) = \tau_A x(y) = \langle y, A\, x\rangle_{H_1} - \langle A^\top y, x\rangle_{H_0}$
extensions $\Rightarrow$

Theorem (kernels, ranges = annihilators)

- $N(\tau_A) = D(\mathring{A})$
- $R(\tau_A) = D(A^*)^\circ = \{\Phi \in D(A^\top)' : D(A^*) \subset N(\Phi)\}$
- $N(\tau_{A^\top}) = D(A^*)$
- $R(\tau_{A^\top}) = D(\mathring{A})^\circ = \{\Phi \in D(A)' : D(\mathring{A}) \subset N(\Phi)\}$

In particular, the kernels and ranges are closed.

Definition and Lemma (trace spaces)

- $T(A) := D(\mathring{A})^{\perp_{D(A)}} = N(A^\top A + 1) \cong D(\tau_A)/N(\tau_A) = D(A)/D(\mathring{A}) =: \mathcal{T}(A)$
- $T(A^\top) := D(A^*)^{\perp_{D(A^\top)}} = N(A\, A^\top + 1) \cong D(\tau_{A^\top})/N(\tau_{A^\top}) = D(A^\top)/D(A^*) =: \mathcal{T}(A^\top)$

note: $T(A) \subset R(A^\top) \subset N(\mathring{A})^{\perp_{H_0}} = N(\mathring{A})^{\perp_{D(A)}}$ and $T(A^\top) \subset R(A) \subset N(A^*)^{\perp_{H_1}} = N(A^*)^{\perp_{D(A^\top)}}$
note: *Hilbert* (orthogonal complements) and *Banach* (quotients) space structures.
note:

$$\iota_q : D(A) \to \mathcal{T}(A), \quad x \mapsto [x] \qquad \iota_q : D(A^\top) \to \mathcal{T}(A^\top), \quad y \mapsto [y]$$

Traces for Single Operators $\mathring{A} \subset A$ and $A^* \subset A^\top = \mathring{A}^*$ (lddc)

some proofs . . .

Traces for Single Operators $\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ (lddc)

proods of **isometries**

- $x \in D(\mathrm{A}) \quad \Rightarrow \quad \iota_q x = [x] \in \mathcal{T}(\mathrm{A})$ and $x = x_\perp - x_0$

$$|\iota_q x|_{\mathcal{T}(\mathrm{A})} = \inf_{\xi \in D(\mathring{\mathrm{A}})} |x - \xi|_{D(\mathrm{A})} = \inf_{\xi \in D(\mathring{\mathrm{A}})} \sqrt{|x_\perp|^2_{D(\mathrm{A})} + |x_0 - \xi|^2_{D(\mathrm{A})}} = |x_\perp|_{D(\mathrm{A})}$$

$$x \in T(\mathrm{A}) \quad \Rightarrow \quad \boxed{|\iota_q x|_{\mathcal{T}(\mathrm{A})} = |x|_{D(\mathrm{A})}}$$

- $\phi \in R(\tau_\mathrm{A}) \quad \Rightarrow \quad \check{\tau}_\mathrm{A}\phi = -\mathrm{A}^\top \rho^{-1}_{D(\mathrm{A}^\top)}\phi \in T(\mathrm{A}) = N(\mathrm{A}^\top \mathrm{A} + 1)$ and $\mathrm{A}^\top \mathrm{A}\, \check{\tau}_\mathrm{A}\phi = -\check{\tau}_\mathrm{A}\phi$
and $\rho^{-1}_{D(\mathrm{A}^\top)}\phi \in T(\mathrm{A}^\top) = N(\mathrm{A}\,\mathrm{A}^\top + 1)$ and $\mathrm{A}\,\check{\tau}_\mathrm{A}\phi = \rho^{-1}_{D(\mathrm{A}^\top)}\phi$

$$|\check{\tau}_\mathrm{A}\phi|_{D(\mathrm{A})} = \sqrt{|\check{\tau}_\mathrm{A}\phi|^2_{\mathrm{H}_0} + |\mathrm{A}\,\check{\tau}_\mathrm{A}\phi|^2_{\mathrm{H}_1}} = |\mathrm{A}\,\check{\tau}_\mathrm{A}\phi|_{D(\mathrm{A}^\top)} = |\phi|_{D(\mathrm{A}^\top)'}$$

$$x \in D(\mathrm{A}) \quad \Rightarrow \quad |\tau_\mathrm{A} x|_{D(\mathrm{A})'} = |\check{\tau}_\mathrm{A}\tau_\mathrm{A} x|_{D(\mathrm{A})} = |x_\perp|_{D(\mathrm{A})}$$

$$x \in T(\mathrm{A}) \quad \Rightarrow \quad \boxed{|\hat{\tau}_\mathrm{A} x|_{D(\mathrm{A})'} = |\tau_\mathrm{A} x|_{D(\mathrm{A})'} = |x|_{D(\mathrm{A})}}$$

- $x \in N(\mathrm{A}^\top \mathrm{A} + 1) \quad \Rightarrow \quad \mathrm{A}\,x \in N(\mathrm{A}\,\mathrm{A}^\top + 1) \quad \Rightarrow \quad \mathrm{A}^\top \mathrm{A}\,x = -x \in N(\mathrm{A}^\top \mathrm{A} + 1)$
$\Rightarrow \quad \mathrm{A} : N(\mathrm{A}^\top \mathrm{A} + 1) \to N(\mathrm{A}\,\mathrm{A}^\top + 1)$ iso^2 since:
inj: $\mathrm{A}\,x = 0 \quad \Rightarrow \quad x = 0$
surj: $y \in N(\mathrm{A}\,\mathrm{A}^\top + 1) \quad \Rightarrow \quad -x := \mathrm{A}^\top y \in N(\mathrm{A}^\top \mathrm{A} + 1)$ and $\mathrm{A}\,x = -\mathrm{A}\,\mathrm{A}^\top y = y$
iso: $\boxed{|\mathrm{A}\,x|_{D(\mathrm{A}^\top)} = \sqrt{|\mathrm{A}\,x|^2_{\mathrm{H}_0} + |\mathrm{A}^\top \mathrm{A}\,x|^2_{\mathrm{H}_1}} = |x|_{D(\mathrm{A})}}$

Traces for Single Operators $\mathring{A} \subset A$ and $A^* \subset A^\top = \mathring{A}^*$ (Iddc)

proof of $\boxed{R(\tau_{A^\top}) = D(\mathring{A})^\circ \subset D(A)'}$

- $\psi \in R(\tau_{A^\top}) \quad \Rightarrow \quad \psi(x) = \tau_{A^\top} y(x) = 0$ for $x \in D(\mathring{A}) \quad \Rightarrow \quad \psi \in D(\mathring{A})^\circ \subset D(A)'$
- $\psi \in D(\mathring{A})^\circ \subset D(A)' \quad \Rightarrow \quad \psi = \rho_{D(A)} \check{x}$

$$\Rightarrow \quad \forall\, x \in D(A) \quad \langle x, \check{x} \rangle_{D(A)} = \psi(x) \quad \text{and} \quad \forall\, x \in D(\mathring{A}) \quad \langle x, \check{x} \rangle_{D(A)} = 0$$

$$\Rightarrow \quad \forall\, x \in D(\mathring{A}) \quad \langle x, \check{x} \rangle_{H_0} + \langle A x, A \check{x} \rangle_{H_1} = 0$$

$$\Rightarrow \quad \check{y} := A \check{x} \in D(\mathring{A}^* = A^\top) \text{ and } A^\top A \check{x} = -\check{x}$$

$$\Rightarrow \quad \forall\, x \in D(A) \quad \psi(x) = \underbrace{\langle x, \check{x} \rangle_{H_0}}_{= -\langle x, A^\top A \check{x} \rangle_{H_0}} + \langle A x, A \check{x} \rangle_{H_1} = -\tau_{A^\top} \check{y}(x)$$

$$\Rightarrow \quad \psi = -\tau_{A^\top} \check{y} \in R(\tau_{A^\top})$$

note: $(\check{x}, \check{y}) \in N(A^\top A + 1) \times N(A A^\top + 1) = T(A) \times T(A^\top)$

proof of $\boxed{R(\tau_{A^\top}) = \rho_{D(A)} T(A)}$

$$\psi \in R(\tau_{A^\top}) = D(\mathring{A})^\circ \subset D(A)'$$

$$\Leftrightarrow \quad \rho_{D(A)}^{-1} \psi \in D(A) \quad \text{and} \quad \forall\, x \in D(\mathring{A}) \quad \langle x, \rho_{D(A)}^{-1} \psi \rangle_{D(A)} = \psi(x) = 0$$

$$\Leftrightarrow \quad \rho_{D(A)}^{-1} \psi \in D(\mathring{A})^{\perp_{D(A)}} = T(A)$$

$$\Leftrightarrow \quad \psi \in \rho_{D(A)} T(A)$$

Traces for Single Operators $\mathring{A} \subset A$ and $A^* \subset A^\top = \mathring{A}^*$ (Iddc)

one more characterisation of the range:

$$\boxed{R(\tau_{A^\top}) = T(A)'} \quad \text{and} \quad \hat{\tau}_{A^\top} : T(A^\top) \to T(A)' \text{ iso}^2$$

- $R(\tau_{A^\top}) \subset D(A)' \subset T(A)'$ (as $T(A) \subset D(A)$ share the same norms)
- $\phi \in T(A)' \quad \Rightarrow \quad \widetilde{\phi} := \phi \circ \pi_\perp \in D(A)'$
  with $\pi_\perp : D(A) \to D(A)$ orth proj onto $T(A)$ in $D(A) = D(\mathring{A}) \oplus_{D(A)} T(A)$
  $\Rightarrow \quad \widetilde{\phi}|_{D(\mathring{A})} = 0 \quad \Rightarrow \quad D(\mathring{A}) \subset N(\widetilde{\phi}) \quad \Rightarrow \quad \widetilde{\phi} \in D(\mathring{A})^\circ = R(\tau_{A^\top})$
  $x \in T(A) \quad \Rightarrow \quad \phi x = \widetilde{\phi} x \quad \Rightarrow \quad \phi = \widetilde{\phi} \in R(\tau_{A^\top})$
- $$|\hat{\tau}_{A^\top} y|_{T(A)'} = \sup_{\substack{x \in T(A), \\ |x|_{D(A)}=1}} |\hat{\tau}_{A^\top} y(x)| = \sup_{\substack{x \in D(A), \\ |x|_{D(A)}=1}} |\hat{\tau}_{A^\top} y(x)| = |\hat{\tau}_{A^\top} y|_{D(A)'} = |y|_{D(A^\top)} = |y|_{T(A^\top)}$$

Traces for Single Operators $\mathring{A} \subset A$ and $A^* \subset A^\top = \mathring{A}^*$ (lddc)

. . . some proofs, end.

Traces for Single Operators $\mathring{A} \subset A$ and $A^* \subset A^\top = \mathring{A}^*$ (Iddc)

$\Rightarrow$ well defined reduced traces

$$\hat{\tau}_A := \tau_A|_{T(A)} : T(A) \to R(\tau_A), \qquad \hat{\tau}_{A^\top} := \tau_{A^\top}|_{T(A^\top)} : T(A^\top) \to R(\tau_{A^\top})$$

recall: $T(A) = D(\mathring{A})^{\perp_{D(A)}} = N(A^\top A + 1)$ and $T(A^\top) = D(A^*)^{\perp_{D(A^\top)}} = N(A A^\top + 1)$

Lemma (ranges)

$$R(\hat{\tau}_A) = R(\tau_A) = D(A^*)^\circ = \rho_{D(A^\top)} T(A^\top) = T(A^\top)'$$
$$R(\hat{\tau}_{A^\top}) = R(\tau_{A^\top}) = D(\mathring{A})^\circ = \rho_{D(A)} T(A) = T(A)'$$

Theorem (trace isometries)

The reduced operators are isometric isomorphisms.

Remark

Continuity of traces and extensions for free! (no ass on $R(A)$ or domains Ω)

Traces for Single Operators $\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ (lddc)

Remark (trace /Riesz isometric isomorphisms $\twoheadrightarrow$)

$$\tau_{\mathrm{A}} : D(\mathrm{A}) \to R(\tau_{\mathrm{A}}) \subset D(\mathrm{A}^\top)', \qquad \rho_{\mathrm{A}} := \rho_{D(\mathrm{A})} : D(\mathrm{A}) \twoheadrightarrow D(\mathrm{A})'$$

$$\tau_{\mathrm{A}^\top} : D(\mathrm{A}^\top) \to R(\tau_{\mathrm{A}^\top}) \subset D(\mathrm{A})', \qquad \rho_{\mathrm{A}^\top} := \rho_{D(\mathrm{A}^\top)} : D(\mathrm{A}^\top) \twoheadrightarrow D(\mathrm{A}^\top)'$$

$$\widehat{\tau}_{\mathrm{A}} = \tau_{\mathrm{A}}|_{T(\mathrm{A})} : T(\mathrm{A}) \twoheadrightarrow R(\tau_{\mathrm{A}}) = T(\mathrm{A}^\top)', \qquad \widehat{\rho}_{\mathrm{A}} := \rho_{\mathrm{A}}|_{T(\mathrm{A})} : T(\mathrm{A}) \twoheadrightarrow T(\mathrm{A})'$$

$$\widehat{\tau}_{\mathrm{A}^\top} = \tau_{\mathrm{A}^\top}|_{T(\mathrm{A}^\top)} : T(\mathrm{A}^\top) \twoheadrightarrow R(\tau_{\mathrm{A}^\top}) = T(\mathrm{A})' \qquad \widehat{\rho}_{\mathrm{A}^\top} := \rho_{\mathrm{A}^\top}|_{T(\mathrm{A}^\top)} : T(\mathrm{A}^\top) \twoheadrightarrow T(\mathrm{A}^\top)'$$

Lemma (trace /Riesz isometric isomorphisms $\twoheadrightarrow$)

$$R(\tau_{\mathrm{A}}) = R(\widehat{\tau}_{\mathrm{A}}) = R(\widehat{\rho}_{\mathrm{A}^\top}) = T(\mathrm{A}^\top)', \qquad \widehat{\tau}_{\mathrm{A}} : T(\mathrm{A}) \twoheadrightarrow T(\mathrm{A}^\top)', \qquad \widehat{\rho}_{\mathrm{A}^\top} : T(\mathrm{A}^\top) \twoheadrightarrow T(\mathrm{A}^\top)'$$

$$R(\tau_{\mathrm{A}^\top}) = R(\widehat{\tau}_{\mathrm{A}^\top}) = R(\widehat{\rho}_{\mathrm{A}}) = T(\mathrm{A})', \qquad \widehat{\tau}_{\mathrm{A}^\top} : T(\mathrm{A}^\top) \twoheadrightarrow T(\mathrm{A})', \qquad \widehat{\rho}_{\mathrm{A}} : T(\mathrm{A}) \twoheadrightarrow T(\mathrm{A})'$$

Definition (inverses of trace /Riesz isometric isomorphisms $\twoheadrightarrow$)

$$\check{\tau}_{\mathrm{A}} := \widehat{\tau}_{\mathrm{A}}^{-1} : T(\mathrm{A}^\top)' \twoheadrightarrow T(\mathrm{A}), \qquad \check{\rho}_{\mathrm{A}^\top} := \widehat{\rho}_{\mathrm{A}^\top}^{-1} : T(\mathrm{A}^\top)' \twoheadrightarrow T(\mathrm{A}^\top)$$

$$\check{\tau}_{\mathrm{A}^\top} := \widehat{\tau}_{\mathrm{A}^\top}^{-1} : T(\mathrm{A})' \twoheadrightarrow T(\mathrm{A}^\top), \qquad \check{\rho}_{\mathrm{A}} := \widehat{\rho}_{\mathrm{A}}^{-1} : T(\mathrm{A})' \twoheadrightarrow T(\mathrm{A})$$

Traces for Single Operators $\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ (Iddc)

Theorem (trace /Riesz isometric isomorphisms $\twoheadrightarrow$)

$$T(\mathrm{A})' \cong_{\hat{\rho}_\mathrm{A}} \boxed{T(\mathrm{A}) \;_{\hat{\tau}_\mathrm{A}}\!\cong T(\mathrm{A}^\top)'}$$

$$T(\mathrm{A}^\top)' \cong_{\hat{\rho}_{\mathrm{A}^\top}} \boxed{T(\mathrm{A}^\top) \;_{\hat{\tau}_{\mathrm{A}^\top}}\!\cong T(\mathrm{A})'}$$

bilinear (sesquilinear) form on $T(\mathrm{A}) \times T(\mathrm{A}^\top)$ or $D(\mathrm{A}) \times D(\mathrm{A}^\top)$

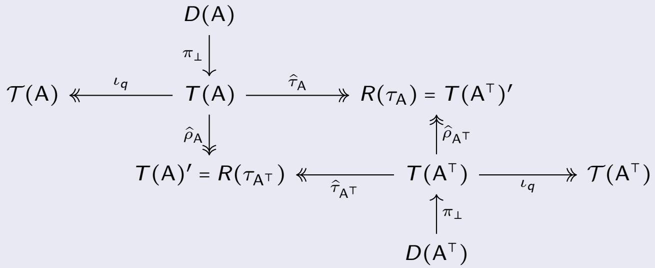
$$\langle\!\langle x, y \rangle\!\rangle := \tau_\mathrm{A} x(y) = -\tau_{\mathrm{A}^\top} y(x) = \langle \mathrm{A}\, x, y \rangle_{\mathrm{H}_1} - \langle x, \mathrm{A}^\top y \rangle_{\mathrm{H}_0}$$

Corollary ("integration by parts")

$$\langle \mathrm{A}\, x, y \rangle_{\mathrm{H}_1} = \langle x, \mathrm{A}^\top y \rangle_{\mathrm{H}_0} + \langle\!\langle x, y \rangle\!\rangle$$

Traces for Single Operators $\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ (Iddc)

Isometric Isomorphisms ($\twoheadrightarrow$)



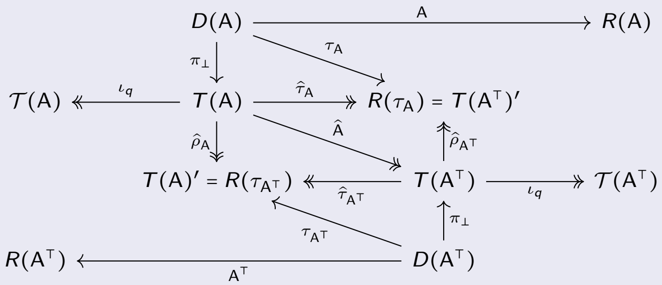
here: $D(\mathrm{A}) \ni x = x_0 + x_\perp \in D(\mathring{\mathrm{A}}) \oplus_{D(\mathrm{A})} T(\mathrm{A})$
and $[x_\perp] = [x]$ and $\tau_{\mathrm{A}} x_\perp = \tau_{\mathrm{A}} x$ with orthogonal projections

$$\pi_0 : D(\mathrm{A}) \to D(\mathrm{A}), \quad x \mapsto \pi_0 x = x_0 \in D(\mathring{\mathrm{A}})$$

$$\pi_\perp : D(\mathrm{A}) \to D(\mathrm{A}), \quad x \mapsto \pi_\perp x = x_\perp \in T(\mathrm{A})$$

Traces for Single Operators $\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ (Iddc)

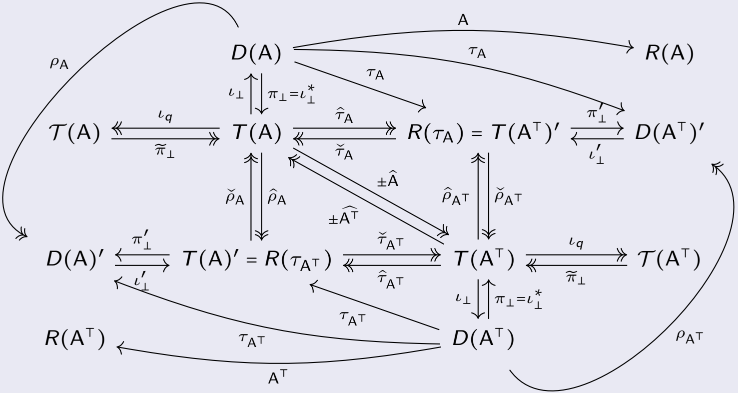
Isometric Isomorphisms ($\twoheadrightarrow$)



$\widehat{\mathrm{A}} := \mathrm{A}|_{T(\mathrm{A})}$

Traces for Single Operators $\mathring{A} \subset A$ and $A^* \subset A^\top = \mathring{A}^*$ (Iddc)

Isometric Isomorphisms ($\twoheadrightarrow$)



"on $T(A) = N(A^\top A + 1)$ and $T(A^\top) = N(A A^\top + 1)$":

$$\check{\tau}_A = -\widehat{A}^\top \check{\rho}_{A^\top} \qquad \widehat{\tau}_A = \widehat{\rho}_{A^\top} \widehat{A} \qquad \widehat{A}^{-1} = -\widehat{A}^\top \qquad \widehat{A} = A|_{T(A)}$$

$$\check{\tau}_{A^\top} = -\widehat{A} \check{\rho}_A \qquad \widehat{\tau}_{A^\top} = \widehat{\rho}_A \widehat{A}^\top \qquad (\widehat{A}^\top)^{-1} = -\widehat{A} \qquad \widehat{A}^\top = A^\top|_{T(A^\top)}$$

Traces for Single Operators $\mathring{A} \subset A$ and $A^* \subset A^\top = \mathring{A}^*$ (lddc)

Theorem (kernels and ranges of traces / isometric isomorphisms)

- $N(\tau_A) = D(\mathring{A})$
- $R(\tau_A) = R(\widehat{\tau}_A) = D(A^*)^\circ = R(\widehat{\rho}_{A^\top}) = T(A^\top)'$
- $T(A) = D(\mathring{A})^{\perp_{D(A)}} = N(A^\top A + 1)$
- $N(\tau_{A^\top}) = D(A^*)$
- $R(\tau_{A^\top}) = R(\widehat{\tau}_{A^\top}) = D(\mathring{A})^\circ = R(\widehat{\rho}_A) = T(A)'$
- $T(A^\top) = D(A^*)^{\perp_{D(A^\top)}} = N(A A^\top + 1)$

note:

- elements of the trace spaces / kernels $N(A^\top A + 1)$ and $N(A A^\top + 1)$ are “smooth”
- regularity is never a problem $\Rightarrow$ regularity not a good term
- integrability is the problem

Traces for Hilbert Complexes

Traces for Hilbert Complexes

Traces for Hilbert Complexes

Traces for Hilbert Complexes

$$\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathsf{H}_0 \underset{\mathring{\mathsf{A}}_0^{\top}=\mathring{\mathsf{A}}_0^*}{\overset{\mathring{\mathsf{A}}_0}{\rightleftarrows}} \mathsf{H}_1 \underset{\mathring{\mathsf{A}}_1^{\top}=\mathring{\mathsf{A}}_1^*}{\overset{\mathring{\mathsf{A}}_1}{\rightleftarrows}} \mathsf{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$$

$$\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathsf{H}_0 \underset{\mathsf{A}_0^*}{\overset{\mathsf{A}_0}{\rightleftarrows}} \mathsf{H}_1 \underset{\mathsf{A}_1^*}{\overset{\mathsf{A}_1}{\rightleftarrows}} \mathsf{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$$

Traces for Hilbert Complexes $\mathring{A}_\ell \subset A_\ell$ and $R(\dots) \subset N(\dots)$

setting: $\ell \in \{0, 1\}$

- $\mathring{A}_\ell \subset A_\ell$ and $A_\ell^* \subset A_\ell^\top = \mathring{A}_\ell^*$ (lddc)
- $R(\mathring{A}_\ell) \subset N(\mathring{A}_{\ell+1})$, $R(A_\ell) \subset N(A_{\ell+1})$ (prim HilComs)
- $R(A_{\ell+1}^*) \subset N(A_\ell^*)$, $R(A_{\ell+1}^\top) \subset N(A_\ell^\top)$ (dual HilComs)

Hilbert complex $\Rightarrow$

$$
\begin{array}{ccccccc}
R(\mathring{A}_0) \subset \overline{R(\mathring{A}_0)} \subset N(\mathring{A}_1) \subset D(\mathring{A}_1), & & R(A_1^\top) \subset \overline{R(A_1^\top)} \subset N(A_0^\top) \subset D(A_0^\top) \\
\cap \quad \cap \quad \cap \quad \cap & & \cup \quad \cup \quad \cup \quad \cup \\
R(A_0) \subset \overline{R(A_0)} \subset N(A_1) \subset D(A_1), & & R(A_1^*) \subset \overline{R(A_1^*)} \subset N(A_0^*) \subset D(A_0^*)
\end{array}
$$

consider (Banach space) adjoints

$A_0 : D(A_0) \to D(A_1)$	$A_1^\top : D(A_1^\top) \to D(A_0^\top)$	(vol diff ops)
${A_1^\top}' : D(A_0^\top)' \to D(A_1^\top)'$	$A_0' : D(A_1)' \to D(A_0)'$	(surf diff ops)

Traces for Hilbert Complexes $\mathring{A}_\ell \subset A_\ell$ and $R(\ldots) \subset N(\ldots)$

$A_0 : D(A_0) \to D(A_1)$	$A_1^\top : D(A_1^\top) \to D(A_0^\top)$	(vol diff ops)
$\tau_{A_0} : D(A_0) \to D(A_0^\top)'$	$\tau_{A_1^\top} : D(A_1^\top) \to D(A_1)'$	(trace ops)
$A_1^\top{}' : D(A_0^\top)' \to D(A_1^\top)'$	$A_0' : D(A_1)' \to D(A_0)'$	(surf diff ops)

observation:

$$T(A_1) \subset R(A_1^\top) \subset \overline{R(A_1^\top)} \subset N(A_0^\top) \subset D(A_0^\top)$$
$$T(A_0^\top) \subset R(A_0) \subset \overline{R(A_0)} \subset N(A_1) \subset D(A_1)$$

Remark

$T(A_0^\top) \subset N(A_1)$ and $T(A_1) \subset N(A_0^\top) \Rightarrow A_0^\top, A_1, \tau_{A_0^\top}, \tau_{A_1}$ well def on $T(A_0^\top)$ and $T(A_1)$

Traces for Hilbert Complexes $\mathring{A}_\ell \subset A_\ell$ and $R(\dots) \subset N(\dots)$

$$A_0 : D(A_0) \to D(A_1) \qquad A_1^\top : D(A_1^\top) \to D(A_0^\top) \qquad \text{(vol diff ops)}$$

$$\tau_{A_0} : D(A_0) \to D(A_0^\top)' \qquad \tau_{A_1^\top} : D(A_1^\top) \to D(A_1)' \qquad \text{(trace ops)}$$

$$A_1^{\top\,\prime} : D(A_0^\top)' \to D(A_1^\top)' \qquad A_0' : D(A_1)' \to D(A_0)' \qquad \text{(surf diff ops)}$$

computation: $x \in D(A_0)$ and $z \in D(A_1^\top)$

$$(\tau_{A_1} A_0 x)(z) = \underbrace{\langle z, A_1 A_0 x\rangle_{H_2}}_{=0} - \langle A_1^\top z, A_0 x\rangle_{H_1} = \underbrace{\langle A_0^\top A_1^\top z, x\rangle_{H_1}}_{=0} - \langle A_1^\top z, A_0 x\rangle_{H_1}$$

$$= -\tau_{A_0} x(A_1^\top z) = -A_1^{\top\,\prime}(\tau_{A_0} x)(z)$$

Theorem (surface differential operators / commutators with traces)

$$\tau_{A_1} A_0 = -A_1^{\top\,\prime} \tau_{A_0}$$

Corollary (surface differential operators for domain traces $\tau^{\mathrm{d}}_{A_n} := \rho^{-1}_{A_n^\top} \tau_{A_n}$)

$$\tau^{\mathrm{d}}_{A_1} A_0 = \rho^{-1}_{A_1^\top} \tau_{A_1} A_0 = -\rho^{-1}_{A_1^\top} A_1^{\top\,\prime} \rho_{A_0^\top} \rho^{-1}_{A_0^\top} \tau_{A_0} = -A_1^{\top\,\star} \rho^{-1}_{A_0^\top} \tau_{A_0} = -A_1^{\top\,\star} \tau^{\mathrm{d}}_{A_0}$$

Banach space adjoints vs (domain) Hilbert space adjoints $(A_1^{\top\,\star} \neq A_1^{\top\,*} = \mathring{A}_1)$

Traces for Hilbert Complexes $\mathring{\mathrm{A}}_\ell \subset \mathrm{A}_\ell$ and $R(\dots) \subset N(\dots)$

$\mathrm{A}_0 : D(\mathrm{A}_0) \to D(\mathrm{A}_1)$	$\mathrm{A}_1^\top : D(\mathrm{A}_1^\top) \to D(\mathrm{A}_0^\top)$	(vol diff ops)
$\tau_{\mathrm{A}_0} : D(\mathrm{A}_0) \to D(\mathrm{A}_0^\top)'$	$\tau_{\mathrm{A}_1^\top} : D(\mathrm{A}_1^\top) \to D(\mathrm{A}_1)'$	(trace ops)
$\mathrm{A}_1^\top{}' : D(\mathrm{A}_0^\top)' \to D(\mathrm{A}_1^\top)'$	$\mathrm{A}_0' : D(\mathrm{A}_1)' \to D(\mathrm{A}_0)'$	(surf diff ops)

Theorem (surface differential operators / commutators with traces)

- $\boxed{\tau_{\mathrm{A}_1} \mathrm{A}_0 = -\mathrm{A}_1^\top{}' \tau_{\mathrm{A}_0}}$ *and* $\tau_{\mathrm{A}_1} \widehat{\mathrm{A}_0} = -\widehat{\mathrm{A}_1^\top{}'} \widehat{\tau}_{\mathrm{A}_0}$
- $\mathrm{A}_1^\top{}' |_{R(\tau_{\mathrm{A}_0})} =: \widehat{\mathrm{A}_1^\top{}'} = -\tau_{\mathrm{A}_1} \widehat{\mathrm{A}_0} \check{\tau}_{\mathrm{A}_0} = \boxed{\boxed{-\tau_{\mathrm{A}_1} \widehat{\mathrm{A}_0} \check{\tau}_{\mathrm{A}_0} = \widehat{\mathrm{A}_1^\top{}'} : R(\tau_{\mathrm{A}_0}) \to R(\tau_{\mathrm{A}_1})}}$
- $\boxed{\tau_{\mathrm{A}_0^\top} \mathrm{A}_1^\top = -\mathrm{A}_0' \tau_{\mathrm{A}_1^\top}}$ *and* $\tau_{\mathrm{A}_0^\top} \widehat{\mathrm{A}_1^\top} = -\widehat{\mathrm{A}_0'} \widehat{\tau}_{\mathrm{A}_1^\top}$
- $\mathrm{A}_0' |_{R(\tau_{\mathrm{A}_1^\top})} =: \widehat{\mathrm{A}_0'} = -\tau_{\mathrm{A}_0^\top} \widehat{\mathrm{A}_1^\top} \check{\tau}_{\mathrm{A}_1^\top} = \boxed{\boxed{-\tau_{\mathrm{A}_0^\top} \widehat{\mathrm{A}_1^\top} \check{\tau}_{\mathrm{A}_1^\top} = \widehat{\mathrm{A}_0'} : R(\tau_{\mathrm{A}_1^\top}) \to R(\tau_{\mathrm{A}_0^\top})}}$

note:

$$\mathrm{A}_0' \tau_{\mathrm{A}_1^\top} = -\mathrm{A}_0' \tau_{\mathrm{A}_1}{}' \iota_d = -(\tau_{\mathrm{A}_1} \mathrm{A}_0)' \iota_d = (\mathrm{A}_1^\top{}' \tau_{\mathrm{A}_0})' \iota_d = \tau_{\mathrm{A}_0}' \mathrm{A}_1^\top{}'' \iota_d = \tau_{\mathrm{A}_0}' \iota_d \mathrm{A}_1^\top = -\tau_{\mathrm{A}_0^\top} \mathrm{A}_1^\top$$

with $\tau_{\mathrm{A}}' \iota_d = -\tau_{\mathrm{A}^\top}$ and $\mathrm{A}'' \iota_d = \iota_d \mathrm{A}$

since $(\mathrm{A}'' \iota_d x)(z) = (\iota_d x)(\mathrm{A}' z) = \mathrm{A}' z(x) = z(\mathrm{A} x) = (\iota_d \mathrm{A} x)(z)$

Traces for Hilbert Complexes $\mathring{\mathrm{A}}_\ell \subset \mathrm{A}_\ell$ and $R(\dots) \subset N(\dots)$

note: $\check{\tau}_{\mathrm{A}_0} = -\widehat{\mathrm{A}_0^\top}\check{\rho}_{\mathrm{A}_0^\top}$ and $\check{\tau}_{\mathrm{A}_0} = -\widehat{\mathrm{A}_0^\top}\check{\rho}_{\mathrm{A}_0^\top}$

more formulas

- $\widehat{\mathrm{A}_1^\top}' = -\tau_{\mathrm{A}_1}\widehat{\mathrm{A}_0}\check{\tau}_{\mathrm{A}_0} = \tau_{\mathrm{A}_1}\widehat{\mathrm{A}_0}\widehat{\mathrm{A}_0^\top}\check{\rho}_{\mathrm{A}_0^\top} = -\tau_{\mathrm{A}_1}\check{\rho}_{\mathrm{A}_0^\top} : R(\tau_{\mathrm{A}_0}) \to R(\tau_{\mathrm{A}_1})$
- $\hat{\rho}_{\mathrm{A}_0}\check{\tau}_{\mathrm{A}_0} = -\hat{\rho}_{\mathrm{A}_0}\widehat{\mathrm{A}_0^\top}\check{\rho}_{\mathrm{A}_0^\top} = -\hat{\tau}_{\mathrm{A}_0^\top}\check{\rho}_{\mathrm{A}_0^\top} : R(\tau_{\mathrm{A}_0}) \to R(\tau_{\mathrm{A}_0^\top})$
- $\pi_\perp = \check{\tau}_{\mathrm{A}_0}\tau_{\mathrm{A}_0} : D(\mathrm{A}_0) \to T(\mathrm{A}_0)$ as $\check{\tau}_{\mathrm{A}_0}\underbrace{\tau_{\mathrm{A}_0}\check{\tau}_{\mathrm{A}_0}}_{=\tau_{\mathrm{A}_0}\iota_\perp\check{\tau}_{\mathrm{A}_0}=\mathrm{id}_{R(\tau_{\mathrm{A}_0})}}\tau_{\mathrm{A}_0} = \check{\tau}_{\mathrm{A}_0}\tau_{\mathrm{A}_0}$
- actually $\pi_\perp = \iota_\perp\iota_\perp^* : D(\mathrm{A}_0) \to D(\mathrm{A}_0)$ and $R(\pi_\perp) = T(\mathrm{A}_0)$
- $R(\mathring{\mathrm{A}}_0) \subset N(\mathring{\mathrm{A}}_1) \subset D(\mathring{\mathrm{A}}_1)$
- $\pi_\perp\widehat{\mathrm{A}_0} = \check{\tau}_{\mathrm{A}_1}\tau_{\mathrm{A}_1}\widehat{\mathrm{A}_0}\check{\tau}_{\mathrm{A}_0}\hat{\tau}_{\mathrm{A}_0} = -\check{\tau}_{\mathrm{A}_1}\widehat{\mathrm{A}_1^\top}'\hat{\tau}_{\mathrm{A}_0}$
- $\pi_\perp' = \iota_{R(\tau_{\mathrm{A}_0^\top})} : R(\tau_{\mathrm{A}_0^\top}) = T(\mathrm{A}_0)' \to D(\mathrm{A}_0)'$ as

  $\pi_\perp' = \tau_{\mathrm{A}_0}'\check{\tau}_{\mathrm{A}_0}' = -\tau_{\mathrm{A}_0^\top}\iota_d^{-1}(\hat{\tau}_{\mathrm{A}_0}')^{-1} = \tau_{\mathrm{A}_0^\top}\iota_d^{-1}(\hat{\tau}_{\mathrm{A}_0^\top}\iota_d^{-1})^{-1} = \tau_{\mathrm{A}_0^\top}\check{\tau}_{\mathrm{A}_0^\top} = \mathrm{id}_{R(\tau_{\mathrm{A}_0^\top})}$ or $\iota_{R(\tau_{\mathrm{A}_0^\top})}$

  note: $\tau_{\mathrm{A}_0^\top}\check{\tau}_{\mathrm{A}_0^\top} = \iota_{R(\tau_{\mathrm{A}_0^\top})}$ and $\hat{\tau}_{\mathrm{A}_0^\top}\check{\tau}_{\mathrm{A}_0^\top} = \mathrm{id}_{R(\tau_{\mathrm{A}_0^\top})}$
- $(\pi_\perp\widehat{\mathrm{A}_0})' = (\pi_\perp\,\mathrm{A}_0\,\iota_\perp)' = \iota_\perp'\,\mathrm{A}_0'\,\pi_\perp' = \iota_\perp'\,\mathrm{A}_0'\,\iota_{R(\tau_{\mathrm{A}_0})} = \iota_\perp'\widehat{\mathrm{A}_0'} = \widehat{\mathrm{A}_0'}$

Traces for Hilbert Complexes $\mathring{\mathrm{A}}_\ell \subset \mathrm{A}_\ell$ and $R(\dots) \subset N(\dots)$

$$\pi_\perp = \check{\tau}_{\mathrm{A}_0}\tau_{\mathrm{A}_0} : D(\mathrm{A}_0) \to T(\mathrm{A}_0)$$

more formulas and trace complexes

- $\mathrm{A}_1^\top{}' : D(\mathrm{A}_0^\top)' \to D(\mathrm{A}_1^\top)'$ with domain complex

$$\cdots \xrightarrow{\cdots} D(\mathrm{A}_0^\top)' \xrightarrow{\mathrm{A}_1^\top{}'} D(\mathrm{A}_1^\top)' \xrightarrow{\mathrm{A}_2^\top{}'} D(\mathrm{A}_2^\top)' \xrightarrow{\cdots} \cdots$$

since $\mathrm{A}_2^\top{}'\mathrm{A}_1^\top{}' = (\mathrm{A}_1^\top\,\mathrm{A}_2^\top)' = 0$

- $\widehat{\mathrm{A}_1^\top{}'} = -\tau_{\mathrm{A}_1}\widehat{\mathrm{A}}_0\check{\tau}_{\mathrm{A}_0} : R(\tau_{\mathrm{A}_0}) \to R(\tau_{\mathrm{A}_1})$ with domain complex

$$\cdots \xrightarrow{\cdots} R(\tau_{\mathrm{A}_0}) \xrightarrow{\widehat{\mathrm{A}_1^\top{}'}} R(\tau_{\mathrm{A}_1}) \xrightarrow{\widehat{\mathrm{A}_2^\top{}'}} R(\tau_{\mathrm{A}_2}) \xrightarrow{\cdots} \cdots$$

since $\widehat{\mathrm{A}_2^\top{}'}\widehat{\mathrm{A}_1^\top{}'} = (\widehat{\mathrm{A}_2^\top{}'\mathrm{A}_1^\top{}'}) = 0$

- $\pi_\perp\widehat{\mathrm{A}}_0 = -\check{\tau}_{\mathrm{A}_1}\widehat{\mathrm{A}_1^\top{}'}\hat{\tau}_{\mathrm{A}_0} : T(\mathrm{A}_0) \to T(\mathrm{A}_1)$ with domain complex

$$\cdots \xrightarrow{\cdots} T(\mathrm{A}_0) \xrightarrow{\pi_\perp\widehat{\mathrm{A}}_0} T(\mathrm{A}_1) \xrightarrow{\pi_\perp\widehat{\mathrm{A}}_1} T(\mathrm{A}_2) \xrightarrow{\cdots} \cdots$$

since $\pi_\perp\widehat{\mathrm{A}}_1\pi_\perp\widehat{\mathrm{A}}_0 = \check{\tau}_{\mathrm{A}_2}\underbrace{\tau_{\mathrm{A}_2}\widehat{\mathrm{A}}_1\check{\tau}_{\mathrm{A}_1}}_{=-\widehat{\mathrm{A}_2^\top{}'}}\underbrace{\tau_{\mathrm{A}_1}\widehat{\mathrm{A}}_0\check{\tau}_{\mathrm{A}_0}}_{=\widehat{\mathrm{A}_1^\top{}'}}\hat{\tau}_{\mathrm{A}_0} = 0$

or $\pi_\perp\widehat{\mathrm{A}}_1\pi_\perp\widehat{\mathrm{A}}_0 = \check{\tau}_{\mathrm{A}_2}\widehat{\mathrm{A}_2^\top{}'}\underbrace{\hat{\tau}_{\mathrm{A}_1}\check{\tau}_{\mathrm{A}_1}}_{=\mathrm{id}|_{R(\tau_{\mathrm{A}_1})}}\widehat{\mathrm{A}_1^\top{}'}\hat{\tau}_{\mathrm{A}_0} = 0$

Traces for Hilbert Complexes $\mathring{A}_\ell \subset A_\ell$ and $R(\dots) \subset N(\dots)$

note: $\tau_A x(y) = 0 \quad \Leftrightarrow \quad x \in D(\mathring{A}) \vee y \in D(A^*) \quad \Leftrightarrow \quad \tau_{A^\top} y(x) = 0$

more formulas

- $\widehat{A_1^\top}' = -\tau_{A_1}\widehat{A_0}\check{\tau}_{A_0} = \tau_{A_1}\widehat{A_0}\widehat{A_0^\top}\check{\rho}_{A_0^\top} = -\tau_{A_1}\check{\rho}_{A_0^\top} : R(\tau_{A_0}) \to R(\tau_{A_1})$
- $\pi_\perp\widehat{A_0} = -\check{\tau}_{A_1}\widehat{A_1^\top}'\hat{\tau}_{A_0}$ and $\pi_\perp = \check{\tau}_{A_1}\tau_{A_1}$ and $\pi'_\perp = \tau'_{A_1}\check{\tau}'_{A_1} = \tau_{A_1^\top}\check{\tau}_{A_1^\top} = \iota_{R(\tau_{A_1^\top})}$
- $x \in D(A_0),\ z \in D(A_1^\top) \quad \Rightarrow \quad$ integration by parts "on trace domains"

$$\langle\langle A_0 x, z\rangle\rangle_1 = \tau_{A_1}(A_0 x)(z) = \langle z, A_1 A_0 x\rangle_{H_2} - \langle A_1^\top z, A_0 x\rangle_{H_1}$$
$$= -\langle A_1^\top z, A_0 x\rangle_{H_1} + \langle A_0^\top A_1^\top z, x\rangle_{H_0} = -\tau_{A_0}(x)(A_1^\top z) = -\langle\langle x, A_1^\top z\rangle\rangle_0$$

or simply: $\boxed{\tau_{A_1} A_0 = -A_1^{\top\prime}\tau_{A_0}}$

- $x \in T(A_0),\ z \in T(A_1^\top) \quad \Rightarrow \quad$ integration by parts "on trace spaces"

$$\langle\langle \pi_\perp\widehat{A_0}x, z\rangle\rangle_1 = \hat{\tau}_{A_1}(\pi_\perp\widehat{A_0}x)(z) = \tau_{A_1}(A_0\,\iota_\perp x)(\iota_\perp z)$$
$$= -\tau_{A_0}(\iota_\perp x)(A_1^\top\,\iota_\perp z) = -\hat{\tau}_{A_0}(x)(\pi_\perp\widehat{A_1^\top}z) = -\langle\langle x, \pi_\perp\widehat{A_1^\top}z\rangle\rangle_0$$

as $(1-\pi_\perp)A_0 x \in D(\mathring{A}_0)$ and $(1-\pi_\perp)A_1^\top z \in D(A_0^*)$

or simply: $\boxed{\hat{\tau}_{A_1}\pi_\perp\widehat{A_0} = \tau_{A_1}\widehat{A_0} = -\widehat{A_1^\top}'\hat{\tau}_{A_0} = -\widehat{A_1^\top}'\pi'_\perp\hat{\tau}_{A_0}}$ as $\pi'_\perp = \hat{\tau}_{A_0}\check{\tau}_{A_0} = \mathrm{id}_{R(\tau_{A_0})}$

Traces for Hilbert Complexes $\mathring{A}_\ell \subset A_\ell$ and $R(\dots) \subset N(\dots)$

Theorem (integration by parts ...)

- *... on domains:* $x \in D(A)$, $y \in D(A^\top)$ *or* $x \in T(A)$, $y \in T(A^\top)$ $\Rightarrow$

$$\langle A\,x, y\rangle_{H_1} = \langle x, A^\top y\rangle_{H_0} + \langle\!\langle x, y\rangle\!\rangle$$

- *... on trace domains* $\boxed{\tau_{A_1} A_0 = -A_1^\top{}'\tau_{A_0}}$*:* $x \in D(A_0)$, $z \in D(A_1^\top)$ $\Rightarrow$

$$\langle\!\langle A_0\,x, z\rangle\!\rangle_1 = \tau_{A_1}(A_0\,x)(z) = -\tau_{A_0}(x)(A_1^\top z) = -\langle\!\langle x, A_1^\top z\rangle\!\rangle_0$$

- *... on trace spaces* $\boxed{\hat\tau_{A_1}\pi_\perp\widehat{A_0} = -\widehat{A_1^\top}{}'\pi_\perp'\hat\tau_{A_0}}$*:* $x \in T(A_0)$, $z \in T(A_1^\top)$ $\Rightarrow$

$$\langle\!\langle \pi_\perp\widehat{A_0}x, z\rangle\!\rangle_1 = \hat\tau_{A_1}(\pi_\perp\widehat{A_0}x)(z) = -\hat\tau_{A_0}(x)(\pi_\perp\widehat{A_1^\top}z) = -\langle\!\langle x, \pi_\perp\widehat{A_1^\top}z\rangle\!\rangle_0$$

- $\langle\!\langle x, y\rangle\!\rangle = \tau_A x(y) = -\tau_{A^\top} y(x) = \langle A\,x, y\rangle_{H_1} - \langle x, A^\top y\rangle_{H_0}$
- $\widehat{A_1^\top}{}' = -\tau_{A_1}\widehat{A_0}\check\tau_{A_0} : R(\tau_{A_0}) \to R(\tau_{A_1})$
- $\pi_\perp\widehat{A_0} = -\check\tau_{A_1}\widehat{A_1^\top}{}'\hat\tau_{A_0} : T(A_0) \to T(A_1)$ and $\pi_\perp = \check\tau_{A_1}\tau_{A_1}$, $\pi_\perp' = \tau_{A_1^\top}\check\tau_{A_1^\top} = \iota_{R(\tau_{A_1^\top})}$

Traces for Hilbert Complexes $\mathring{A}_\ell \subset A_\ell$ and $R(\dots) \subset N(\dots)$

$$\pi'_\perp = \widehat{\tau}_{A_0} \check{\tau}_{A_0} = \mathrm{id}_{R(\tau_{A_0})} \quad \text{and} \quad \pi'_\perp = \widehat{\tau}_{A_1^\top} \check{\tau}_{A_1^\top} = \mathrm{id}_{R(\tau_{A_1^\top})}$$

integration by parts on trace ranges

. . . on trace spaces $\boxed{\pi_\perp \widehat{A_0} = -\check{\tau}_{A_1} \widehat{A_1^\top}{}' \pi'_\perp \widehat{\tau}_{A_0} = -\check{\tau}_{A_1} \widehat{A_1^\top}{}' \widehat{\tau}_{A_0}}$ on $T(A_0)$

. . . on trace spaces $\boxed{\pi_\perp \widehat{A_1^\top} = -\check{\tau}_{A_0^\top} \widehat{A_0'} \pi'_\perp \widehat{\tau}_{A_1^\top} = -\check{\tau}_{A_0^\top} \widehat{A_0'} \widehat{\tau}_{A_1^\top}}$ on $T(A_1^\top)$ $\Rightarrow$

$$\langle\!\langle \check{\tau}_{A_1} \widehat{A_1^\top}{}' \widehat{\tau}_{A_0} x, z \rangle\!\rangle_1 = -\langle\!\langle \pi_\perp \widehat{A_0} x, z \rangle\!\rangle_1 = \langle\!\langle x, \pi_\perp \widehat{A_1^\top} z \rangle\!\rangle_0 = -\langle\!\langle x, \check{\tau}_{A_0^\top} \widehat{A_0'} \widehat{\tau}_{A_1^\top} z \rangle\!\rangle_0$$

Definition

$$\forall\, \phi \in R(\tau_{A_1}) \quad \forall\, \psi \in R(\tau_{A_1^\top}) \qquad \langle\!\langle\!\langle \phi, \psi \rangle\!\rangle\!\rangle_1 := \langle\!\langle \check{\tau}_{A_1} \phi, \check{\tau}_{A_1^\top} \psi \rangle\!\rangle_1$$

$\Rightarrow \quad \forall\, \varphi \in R(\tau_{A_0}) \quad \forall\, \psi \in R(\tau_{A_1^\top})$

$$\langle\!\langle\!\langle \widehat{A_1^\top}{}' \varphi, \psi \rangle\!\rangle\!\rangle_1 = \langle\!\langle \check{\tau}_{A_1} \widehat{A_1^\top}{}' \widehat{\tau}_{A_0} \check{\tau}_{A_0} \varphi, \check{\tau}_{A_1^\top} \psi \rangle\!\rangle_1$$

$$= -\langle\!\langle \check{\tau}_{A_0} \varphi, \check{\tau}_{A_0^\top} \widehat{A_0'} \widehat{\tau}_{A_1^\top} \check{\tau}_{A_1^\top} \psi \rangle\!\rangle_0 = -\langle\!\langle\!\langle \varphi, \widehat{A_0'} \psi \rangle\!\rangle\!\rangle_0$$

Traces for Hilbert Complexes $\mathring{\mathrm{A}}_\ell \subset \mathrm{A}_\ell$ and $R(\dots) \subset N(\dots)$

$$\pi_\perp \widehat{\mathrm{A}_0} = -\check{\tau}_{\mathrm{A}_1} \widehat{\mathrm{A}_1^\top}{}' \pi_\perp' \widehat{\tau}_{\mathrm{A}_0} = -\check{\tau}_{\mathrm{A}_1} \widehat{\mathrm{A}_1^\top}{}' \widehat{\tau}_{\mathrm{A}_0} \qquad \pi_\perp \widehat{\mathrm{A}_1^\top} = -\check{\tau}_{\mathrm{A}_0^\top} \widehat{\mathrm{A}_0'} \pi_\perp' \widehat{\tau}_{\mathrm{A}_1^\top} = -\check{\tau}_{\mathrm{A}_0^\top} \widehat{\mathrm{A}_0'} \widehat{\tau}_{\mathrm{A}_1^\top}$$

$$\Rightarrow \quad \widehat{\mathrm{A}_1^\top}{}' = (\pi_\perp \widehat{\mathrm{A}_1^\top})' = -\widehat{\tau}'_{\mathrm{A}_1^\top} (\widehat{\mathrm{A}_0'})' \check{\tau}'_{\mathrm{A}_0^\top} = -\widehat{\tau}_{\mathrm{A}_1} \iota_d^{-1} (\widehat{\mathrm{A}_0'})' \iota_d \check{\tau}_{\mathrm{A}_0} = -\widehat{\tau}_{\mathrm{A}_1} \iota_d^{-1} (\pi_\perp \widehat{\mathrm{A}_0})'' \iota_d \check{\tau}_{\mathrm{A}_0}$$

Theorem (integration by parts ...)

- *... on domains:* $x \in D(\mathrm{A})$, $y \in D(\mathrm{A}^\top)$ *or* $x \in T(\mathrm{A})$, $y \in T(\mathrm{A}^\top)$ $\Rightarrow$

$$\langle \mathrm{A}\, x, y \rangle_{\mathrm{H}_1} = \langle x, \mathrm{A}^\top y \rangle_{\mathrm{H}_0} + \langle\langle x, y \rangle\rangle$$

- *... on trace domains* $\boxed{\tau_{\mathrm{A}_1} \mathrm{A}_0 = -\mathrm{A}_1^\top{}' \tau_{\mathrm{A}_0}}$: $x \in D(\mathrm{A}_0)$, $z \in D(\mathrm{A}_1^\top)$ $\Rightarrow$

$$\langle\langle \mathrm{A}_0\, x, z \rangle\rangle_1 = \tau_{\mathrm{A}_1}(\mathrm{A}_0\, x)(z) = -\tau_{\mathrm{A}_0}(x)(\mathrm{A}_1^\top z) = -\langle\langle x, \mathrm{A}_1^\top z \rangle\rangle_0$$

- *... on trace spaces* $\boxed{\widehat{\tau}_{\mathrm{A}_1} \pi_\perp \widehat{\mathrm{A}_0} = -\widehat{\mathrm{A}_1^\top}{}' \pi_\perp' \widehat{\tau}_{\mathrm{A}_0}}$: $x \in T(\mathrm{A}_0)$, $z \in T(\mathrm{A}_1^\top)$ $\Rightarrow$

$$\langle\langle \pi_\perp \widehat{\mathrm{A}_0} x, z \rangle\rangle_1 = \widehat{\tau}_{\mathrm{A}_1}(\pi_\perp \widehat{\mathrm{A}_0} x)(z) = -\widehat{\tau}_{\mathrm{A}_0}(x)(\pi_\perp \widehat{\mathrm{A}_1^\top} z) = -\langle\langle x, \pi_\perp \widehat{\mathrm{A}_1^\top} z \rangle\rangle_0$$

- *... on trace ranges* $\boxed{\widehat{\mathrm{A}_1^\top}{}' = -\widehat{\tau}_{\mathrm{A}_1} \iota_d^{-1} (\widehat{\mathrm{A}_0'})' \iota_d \check{\tau}_{\mathrm{A}_0}}$: $\varphi \in R(\tau_{\mathrm{A}_0})$, $\psi \in R(\tau_{\mathrm{A}_1^\top})$ $\Rightarrow$

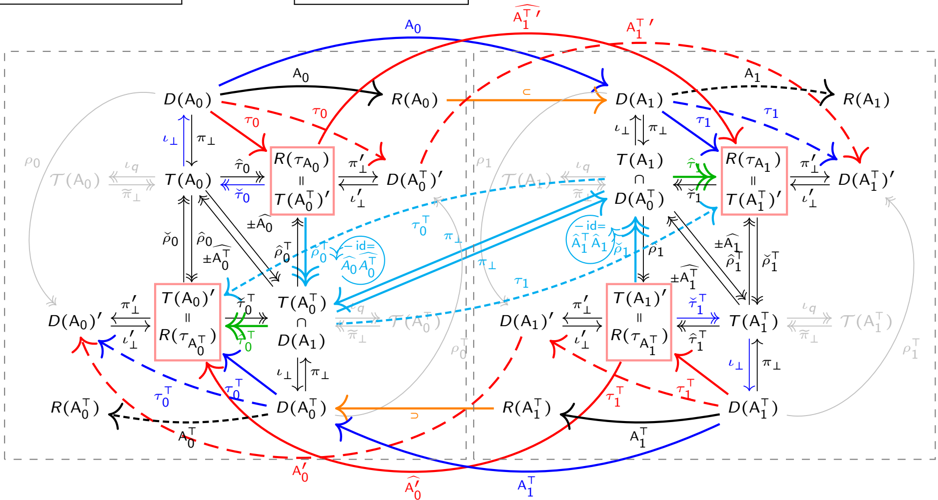
$$\langle\langle\langle \widehat{\mathrm{A}_1^\top}{}' \varphi, \psi \rangle\rangle\rangle_1 = \langle\langle \check{\tau}_{\mathrm{A}_1} \widehat{\mathrm{A}_1^\top}{}' \varphi, \check{\tau}_{\mathrm{A}_1^\top} \psi \rangle\rangle_1 = -\langle\langle \check{\tau}_{\mathrm{A}_0} \varphi, \check{\tau}_{\mathrm{A}_0^\top} \widehat{\mathrm{A}_0'} \psi \rangle\rangle_0 = -\langle\langle\langle \varphi, \widehat{\mathrm{A}_0'} \psi \rangle\rangle\rangle_0$$

Traces for Hilbert Complexes $\mathring{\mathrm{A}}_\ell \subset \mathrm{A}_\ell$ and $R(\dots) \subset N(\dots)$

$\mathrm{A}_0 : D(\mathrm{A}_0) \to D(\mathrm{A}_1)$	(vol diff ops)	$\mathrm{A}_1^\top : D(\mathrm{A}_1^\top) \to D(\mathrm{A}_0^\top)$
$\mathrm{A}_1^{\top\,\prime} : D(\mathrm{A}_0^\top)' \to D(\mathrm{A}_1^\top)'$	(surf diff ops)	$\mathrm{A}_0' : D(\mathrm{A}_1)' \to D(\mathrm{A}_0)'$

!!! CRAZY !!!

simple idea $\Rightarrow$ amazing complexity



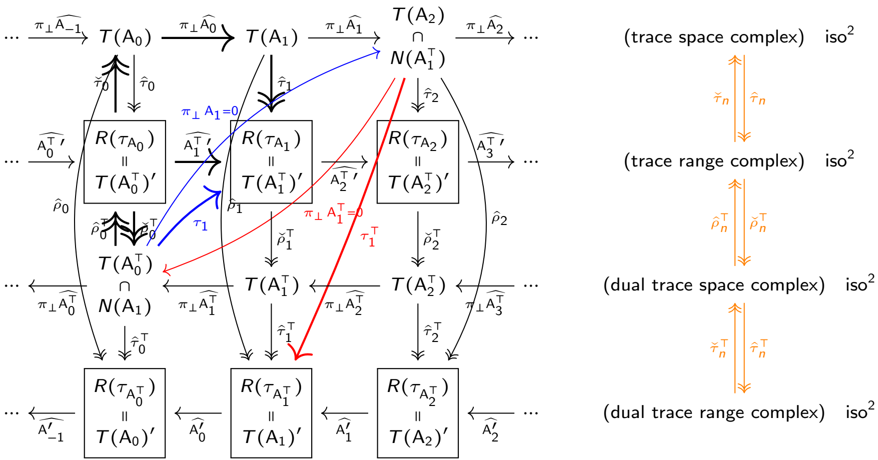
$\circlearrowleft$ red = blue = cyan $\circlearrowleft$

Traces for Hilbert Complexes $\mathring{A}_\ell \subset A_\ell$ and $R(\dots) \subset N(\dots)$

trace spaces and operators

$$\widehat{A_1^\top}' = -\hat{\tau}_1 \pi_\perp \widehat{A_0} \check{\tau}_0 = -\tau_1 \widehat{A_0} \check{\tau}_0 = \tau_1 \widehat{A_0} \widehat{A_0^\top} \check{\rho}_0^\top = -\tau_1 \check{\rho}_0^\top : R(\tau_{A_0}) \to R(\tau_{A_1})$$

$$A_1 |_{T(A_0^\top)} = 0 = \underbrace{\pi_\perp \widehat{A}_1 \pi_\perp \widehat{A_0}}_{=0} \check{\tau}_0 \hat{\rho}_0^\top : T(A_0^\top) \to T(A_2)$$



Regular Subspaces and Duals

"Regular Subspaces" and Duals

Regular Subspaces and Duals

setting

assume (for all A_n and $\mathrm{A}_n^\top$)

- $\mathrm{H}_n^+ \subset D(\mathrm{A}_n) \subset \mathrm{H}_n$ (bd dense embs of reg subsps)
- $D(\mathrm{A}_{n+1}) = \mathrm{H}_{n+1}^+ + \mathrm{A}_n \mathrm{H}_n^+$ (with bd reg deco ops)
- $\mathrm{H}_n^+(\mathrm{A}_n) := \{x \in \mathrm{H}_n^+ : \mathrm{A}_n x \in \mathrm{H}_{n+1}^+\} \subset \mathrm{H}_n^+ \subset D(\mathrm{A}_n)$ (bd dense embs)
- $\mathring{\mathrm{H}}_n^- := (\mathrm{H}_n^+)'$ (duals)
- $\mathrm{H}_{n+1}^+ \subset D(\mathrm{A}_{n+1}) \cap D(\mathrm{A}_n^\top) \subset \mathrm{H}_{n+1}$ (bd dense embs of reg subsps)

Regular Subspaces and Duals

$A_0' : D(A_1)' \to D(A_0)'$

- $H_1^+ \subset D(A_1) \subset H_1$ (bd dense embs of reg subsps)
- $D(A_1) = H_1^+ + A_0 H_0^+$ (bd reg deco ops)
- $H_0^+(A_0) = \{x \in H_0^+ : A_0 x \in H_1^+\} \subset H_0^+ \subset D(A_0)$ (bd dense embs)
- $\mathring{H}_0^- = H_0^{+\,\prime}$ (duals)
- $H_1^+ \subset D(A_1) \cap D(A_0^\top) \subset H_1$ (bd dense embs of reg subsps)

$H_0^+(A_0) \subset H_0^+ \subset D(A_0) \subset H_0$ and $H_0' \subset D(A_0)' \subset \mathring{H}_0^- \subset H_0^+(A_0)'$

$\Rightarrow$ extend A_0' to $\mathring{H}_1^-$ by

$$A_0' : \mathring{H}_1^- \to H_0^+(A_0)'$$

$$\forall\, \psi \in \mathring{H}_1^- \quad \forall\, x \in H_0^+(A_0) \qquad A_0'\,\psi(x) := \psi(A_0\, x)$$

note: $\big|A_0'\,\psi(x)\big| \le |\psi|_{\mathring{H}_1^-} |A_0\, x|_{H_1^+} \le |\psi|_{\mathring{H}_1^-} |x|_{H_0^+(A_0)}$

Regular Subspaces and Duals

$\mathrm{A}_0' : D(\mathrm{A}_1)' \to D(\mathrm{A}_0)'$ and $\mathrm{A}_0' : \mathring{\mathrm{H}}_1^- \to \mathrm{H}_0^+(\mathrm{A}_0)'$

extend $\mathrm{A}_0'\psi$ to $\widetilde{\mathrm{A}_0'}\psi := \widetilde{\mathrm{A}_0'\psi} \in \mathring{\mathrm{H}}_0^-$ by bd dense emb $\mathrm{H}_0^+(\mathrm{A}_0) \subset \mathrm{H}_0^+$

$\Rightarrow \quad \widetilde{\mathrm{A}_0'} : \mathring{\mathrm{H}}_1^- \to \mathring{\mathrm{H}}_0^-$

$\Rightarrow \quad \mathring{\mathrm{H}}_1^-(\mathrm{A}_0') := \{\psi \in \mathring{\mathrm{H}}_1^- : \mathrm{A}_0'\psi = \widetilde{\mathrm{A}_0'}\psi \in \mathring{\mathrm{H}}_0^-\}$

- $D(\mathrm{A}_1) = \mathrm{H}_1^+ + \mathrm{A}_0\,\mathrm{H}_0^+$ (bd reg deco ops)
- $\mathrm{H}_0^+(\mathrm{A}_0) = \{x \in \mathrm{H}_0^+ : \mathrm{A}_0\,x \in \mathrm{H}_1^+\} \subset \mathrm{H}_0^+ \subset D(\mathrm{A}_0) \subset \mathrm{H}_0$ (bd dense embs)
- $\mathrm{H}_1' \subset D(\mathrm{A}_1)' \overset{?}{\subset} \mathring{\mathrm{H}}_1^-(\mathrm{A}_0') = \{\psi \in \mathring{\mathrm{H}}_1^- : \mathrm{A}_0'\psi \in \mathring{\mathrm{H}}_0^-\} \subset \mathring{\mathrm{H}}_1^- = \mathrm{H}_1^{+\,\prime} \subset \mathrm{H}_1^+(\mathrm{A}_1)'$

?: $\mathrm{H}_1^+ \subset D(\mathrm{A}_1) \quad \Rightarrow \quad \psi \in D(\mathrm{A}_1)' \subset \mathring{\mathrm{H}}_1^-$ and $\mathrm{A}_0'\psi \in D(\mathrm{A}_0)' \subset \mathring{\mathrm{H}}_0^-$

$\supset$: $\psi \in \mathring{\mathrm{H}}_1^-(\mathrm{A}_0')$ and $D(\mathrm{A}_1) \ni y = y_1 + \mathrm{A}_0\,y_0 \in \mathrm{H}_1^+ + \mathrm{A}_0\,\mathrm{H}_0^+$

$$\Rightarrow \quad \psi y := \psi y_1 + (\mathrm{A}_0'\psi)y_0 \quad \text{and} \quad |\psi y| \le c|\psi|_{\mathring{\mathrm{H}}_1^-(\mathrm{A}_0')}|y|_{D(\mathrm{A}_1)}$$

$$\Rightarrow \quad \psi \in D(\mathrm{A}_1)'$$

$\Rightarrow$

- $\mathrm{H}_1' \subset \boxed{D(\mathrm{A}_1)' = \mathring{\mathrm{H}}_1^-(\mathrm{A}_0')} \subset \mathring{\mathrm{H}}_1^- \subset \mathrm{H}_1^+(\mathrm{A}_1)'$

Characterisation of Dual Spaces by Regular Subspaces

Characterisation of Dual Spaces by “Regular Subspaces”

Characterisation of Dual Spaces by Regular Subspaces

Theorem (Characterisation of Dual Spaces by Regular Subspaces)

$$D(\mathrm{A}_1)' = \mathring{\mathrm{H}}_1^-(\mathrm{A}_0') = \{\psi \in \mathring{\mathrm{H}}_1^- : \mathrm{A}_0' \psi \in \mathring{\mathrm{H}}_0^-\}$$
$$D(\mathrm{A}_0^\top)' = \mathring{\mathrm{H}}_1^-(\mathrm{A}_1^{\top\prime}) = \{\psi \in \mathring{\mathrm{H}}_1^- : \mathrm{A}_1^{\top\prime} \psi \in \mathring{\mathrm{H}}_2^-\}$$

with equivalent norms by bounded inverse theorem. Dual results:

$$D(\mathring{\mathrm{A}}_1)' = \mathrm{H}_1^-(\mathring{\mathrm{A}}_0') = \{\psi \in \mathrm{H}_1^- : \mathring{\mathrm{A}}_0' \psi \in \mathrm{H}_0^-\}$$
$$D(\mathrm{A}_0^*)' = \mathrm{H}_1^-(\mathrm{A}_1^{*\prime}) = \{\psi \in \mathrm{H}_1^- : \mathrm{A}_1^{*\prime} \psi \in \mathrm{H}_2^-\}$$

modifications: $\mathring{\mathrm{A}}_0' : D(\mathring{\mathrm{A}}_1)' \to D(\mathring{\mathrm{A}}_0)'$, $\quad \mathrm{A}_1^{*\prime} : D(\mathrm{A}_0^*)' \to D(\mathrm{A}_1^*)'$

- $\mathring{\mathrm{H}}_1^+ \subset D(\mathring{\mathrm{A}}_1) \subset \mathrm{H}_1$, $\quad \overset{*}{\mathrm{H}}{}_1^+ \subset D(\mathrm{A}_0^*) \subset \mathrm{H}_1$ (bd dense embs of reg subsps)
- $D(\mathring{\mathrm{A}}_1) = \mathring{\mathrm{H}}_1^+ + \mathring{\mathrm{A}}_0 \mathring{\mathrm{H}}_0^+$, $\quad D(\mathrm{A}_0^*) = \overset{*}{\mathrm{H}}{}_1^+ + \mathrm{A}_1^* \overset{*}{\mathrm{H}}{}_2^+$ (bd reg deco ops)
- $\mathring{\mathrm{H}}_0^+(\mathring{\mathrm{A}}_0) = \dots$, $\quad \overset{*}{\mathrm{H}}{}_1^+(\mathrm{A}_0^*) = \dots$ (bd dense embs)
- $\mathrm{H}_0^- = \mathring{\mathrm{H}}_0^{+\prime}$ (duals)

Characterisation of Trace Ranges by Regular Subspaces

Characterisation of Trace Ranges by “Regular Subspaces”

Characterisation of Trace Ranges by Regular Subspaces

recall traces: $\tau_{\mathrm{A}_0} : D(\mathrm{A}_0) \to D(\mathrm{A}_0^\top)'$, $\qquad \tau_{\mathrm{A}_1^\top} : D(\mathrm{A}_1^\top) \to D(\mathrm{A}_1)'$

- $N(\tau_{\mathrm{A}_1^\top}) = D(\mathrm{A}_1^*)$
- $N(\tau_{\mathrm{A}_0}) = D(\mathring{\mathrm{A}}_0)$
- $R(\tau_{\mathrm{A}_1^\top}) = D(\mathring{\mathrm{A}}_1)^\circ = \big\{\psi \in D(\mathrm{A}_1)' : \psi|_{D(\mathring{\mathrm{A}}_1)} = 0\big\}$
- $R(\tau_{\mathrm{A}_0}) = D(\mathrm{A}_0^*)^\circ = \big\{\psi \in D(\mathrm{A}_0^\top)' : \psi|_{D(\mathrm{A}_0^*)} = 0\big\}$

density of $\quad \mathring{\mathrm{H}}_1^+ \subset D(\mathring{\mathrm{A}}_1)$ and $\overset{*}{\mathrm{H}}{}_1^+ \subset D(\mathrm{A}_0^*) \quad \Rightarrow$

- $R(\tau_{\mathrm{A}_1^\top}) = \mathring{\mathrm{H}}_1^{+\circ}$ as closed subspace of $D(\mathrm{A}_1)'$
- $R(\tau_{\mathrm{A}_0}) = \overset{*}{\mathrm{H}}{}_1^{+\circ}$ as closed subspace of $D(\mathrm{A}_0^\top)'$

$\Rightarrow$ more detailed

Theorem (Characterisation of Trace Ranges by Regular Subspaces)

$$R(\tau_{\mathrm{A}_1^\top}) = D(\mathrm{A}_1)' \cap D(\mathring{\mathrm{A}}_1)^\circ = \mathring{\mathrm{H}}_1^-(\mathrm{A}_0') \cap \mathring{\mathrm{H}}_1^{+\circ} = \{\psi \in \mathring{\mathrm{H}}_1^- : \mathrm{A}_0'\psi \in \mathring{\mathrm{H}}_0^- \ \wedge\ \psi|_{\mathring{\mathrm{H}}_1^+} = 0\}$$

$$R(\tau_{\mathrm{A}_0}) = D(\mathrm{A}_0^\top)' \cap D(\mathrm{A}_0^*)^\circ = \mathring{\mathrm{H}}_1^-(\mathrm{A}_1^{\top\prime}) \cap \overset{*}{\mathrm{H}}{}_1^{+\circ} = \{\psi \in \mathring{\mathrm{H}}_1^- : \mathrm{A}_1^{\top\prime}\psi \in \mathring{\mathrm{H}}_2^- \ \wedge\ \psi|_{\overset{*}{\mathrm{H}}{}_1^+} = 0\}$$

with equivalent norms.

Trace Hilbert Complexes

Hilbert Complexes of Traces and Trace Spaces

Trace Hilbert Complexes

two interpretations

- $R(\tau_{A_0}) = \overset{*}{H}{}_1^{+\circ}$ as closed subspace of $D(A_0^\top)'$, note: $D(A_0^\top)' \subset \mathring{H}_1^-$
- $\overset{*}{H}{}_1^{+\circ}$ as closed subspace of $\mathring{H}_1^-$
- $\widehat{A_1^\top{}'} = A_1^\top{}' : R(\tau_{A_0}) \to R(\tau_{A_1})$, $D(A_1^\top{}') = R(\tau_{A_0})$

$\Rightarrow$ $\psi \in R(\tau_{A_0}) \subset \overset{*}{H}{}_1^{+\circ} \subset \mathring{H}_1^-$ $\Rightarrow$ $A_1^\top{}'\psi \in R(\tau_{A_1}) \subset \overset{*}{H}{}_2^{+\circ} \subset \mathring{H}_2^-$

$$\Rightarrow \quad \begin{aligned} R(\tau_{A_0}) &= D(A_0^\top)' \cap D(A_0^*)^\circ \\ &= \mathring{H}_1^-(A_1^\top{}') \cap \overset{*}{H}{}_1^{+\circ} \\ &= \{\psi \in \mathring{H}_1^- : A_1^\top{}'\psi \in \mathring{H}_2^- \ \wedge\ \psi|_{\overset{*}{H}{}_1^+} = 0\} \\ &= \{\psi \in \overset{*}{H}{}_1^{+\circ} : A_1^\top{}'\psi \in \overset{*}{H}{}_2^{+\circ}\} =: \overset{*}{H}{}_1^{+\circ}(A_1^\top{}') \end{aligned}$$

$\Rightarrow$ different unbounded versions of “surface differential operators”

Trace Hilbert Complexes

different unbd versions of surf diff ops

$$D({\rm A}_{n+1}^{\top}{}') = R(\tau_{{\rm A}_n}) \quad \text{and} \quad D({\rm A}_n') = R(\tau_{{\rm A}_{n+1}^{\top}})$$

$$\cdots \xrightarrow{\cdots} D({\rm A}_0^{\top})' \xrightarrow{{\rm A}_1^{\top}{}'} D({\rm A}_1^{\top})' \xrightarrow{{\rm A}_2^{\top}{}'} D({\rm A}_2^{\top})' \xrightarrow{\cdots} \cdots$$

$$\cdots \xleftarrow[\cdots]{} D({\rm A}_1)' \xleftarrow[{\rm A}_1']{} D({\rm A}_2)' \xleftarrow[{\rm A}_2']{} D({\rm A}_3)' \xleftarrow[\cdots]{} \cdots$$

$$\cdots \xrightarrow{\cdots} \overset{*}{\rm H}{}_1^{+\circ} \xrightarrow{{\rm A}_1^{\top}{}'} \overset{*}{\rm H}{}_2^{+\circ} \xrightarrow{{\rm A}_2^{\top}{}'} \overset{*}{\rm H}{}_3^{+\circ} \xrightarrow{\cdots} \cdots$$

$$\cdots \xleftarrow[\cdots]{} \mathring{\rm H}{}_1^{+\circ} \xleftarrow[{\rm A}_1']{} \mathring{\rm H}{}_2^{+\circ} \xleftarrow[{\rm A}_2']{} \mathring{\rm H}{}_3^{+\circ} \xleftarrow[\cdots]{} \cdots$$

$$\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathring{\rm H}{}_1^{-} \underset{{\rm A}_1'}{\overset{{\rm A}_1^{\top}{}'}{\rightleftarrows}} \mathring{\rm H}{}_2^{-} \underset{{\rm A}_2'}{\overset{{\rm A}_2^{\top}{}'}{\rightleftarrows}} \mathring{\rm H}{}_3^{-} \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$$

Compact Embeddings for Trace Hilbert Complexes

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Boundary Value Problems on Trace Hilbert Complexes

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