On the Gradgrad and divDiv Complexes, and a Related Decomposition Result for Biharmonic Problems in 3D: Part 2

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Open-Minded :-)

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Kernel of div Div

Walter's central question

$$\boxed{N(\operatorname{div}\operatorname{Div}) = ?}$$

more precisely: kernel

$$N(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) = ?$$

of

$$\operatorname{div}\operatorname{Div}_{\mathbb{S}} : D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \subset \mathsf{L}^2(\Omega,\mathbb{S}) \longrightarrow \mathsf{L}^2(\Omega,\mathbb{R}), \qquad \mathsf{M} \longmapsto \operatorname{div}\operatorname{Div}\mathsf{M}$$

with Ω (open) $\subset \mathbb{R}^3$ and

$$D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) := \{\mathsf{M} \in \mathsf{L}^2(\Omega,\mathbb{S}) \,:\, \operatorname{div}\operatorname{Div} \in \mathsf{L}^2(\Omega,\mathbb{R})\}$$

note in general

$$\operatorname{Div}\mathsf{M} \in \mathsf{H}^{-1}(\Omega,\mathbb{R}^3)$$

$$\Rightarrow \qquad \operatorname{div}\operatorname{Div}_{\mathbb{S}} \neq (\operatorname{div})(\operatorname{Div}_{\mathbb{S}})$$

with

$$D\big((\operatorname{div})(\operatorname{Div}_{\mathbb{S}})\big) := \{\mathsf{M} \in \mathsf{L}^2(\Omega,\mathbb{S}) \,:\, \operatorname{Div}\mathsf{M} \in \mathsf{L}^2(\Omega,\mathbb{R}^3) \text{ and } \operatorname{div}\operatorname{Div}\mathsf{M} \in \mathsf{L}^2(\Omega,\mathbb{R})\}$$

grad-rot-div Complexes

sequence or complex

$$\xrightarrow{0} D(\mathring{\text{grad}}) \xrightarrow{\mathring{\text{grad}}} D(\mathring{\text{rot}}) \xrightarrow{\mathring{\text{rot}}} D(\mathring{\text{div}}) \xrightarrow{\mathring{\text{div}}} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathbb{R}}} \mathbb{R}$$

with dual or adjoint sequence or complex

$$\xleftarrow{0} \mathsf{L}^2(\Omega) \xleftarrow{-\text{div}} D(\text{div}) \xleftarrow{\text{rot}} D(\text{rot}) \xleftarrow{-\text{grad}} D(\text{grad}) \xleftarrow{\iota_{\mathbb{R}}} \mathbb{R}$$

unbounded, densely defined, closed, linear operators with adjoints

$$\mathring{\text{grad}} : D(\mathring{\text{grad}}) \subset \mathsf{L}^2 \to \mathsf{L}^2, \qquad (\mathring{\text{grad}})^* = -\text{div} : D(\text{div}) \subset \mathsf{L}^2 \to \mathsf{L}^2,$$

$$\mathring{\text{rot}} : D(\mathring{\text{rot}}) \subset \mathsf{L}^2 \to \mathsf{L}^2, \qquad (\mathring{\text{rot}})^* = \text{rot} : D(\text{rot}) \subset \mathsf{L}^2 \to \mathsf{L}^2,$$

$$\mathring{\text{div}} : D(\mathring{\text{div}}) \subset \mathsf{L}^2 \to \mathsf{L}^2, \qquad (\mathring{\text{div}})^* = -\text{grad} : D(\text{grad}) \subset \mathsf{L}^2 \to \mathsf{L}^2$$

spaces

$$D(\mathring{\text{grad}}) = \mathring{\mathsf{H}}^1 = H^1_0 = H_0(\text{grad}), \qquad D(\text{div}) = \mathsf{D} = H(\text{div}),$$

$$D(\mathring{\text{rot}}) = \mathring{\mathsf{R}} = H_0(\text{rot}) = H_0(\text{curl}), \qquad D(\text{rot}) = \mathsf{R} = H(\text{rot}) = H(\text{curl}),$$

$$D(\mathring{\text{div}}) = \mathring{\mathsf{D}} = H_0(\text{div}), \qquad D(\text{grad}) = \mathsf{H}^1 = H^1 = H(\text{grad})$$

grad-rot-div Complexes

sequence or complex

$$\xrightarrow{0} D(\mathring{\text{grad}}) \xrightarrow{\mathring{\text{grad}}} D(\mathring{\text{rot}}) \xrightarrow{\mathring{\text{rot}}} D(\mathring{\text{div}}) \xrightarrow{\mathring{\text{div}}} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathbb{R}}} \mathbb{R}$$

with dual or adjoint sequence or complex

$$\xleftarrow{0} \mathsf{L}^2(\Omega) \xleftarrow{-\text{div}} D(\text{div}) \xleftarrow{\text{rot}} D(\text{rot}) \xleftarrow{-\text{grad}} D(\text{grad}) \xleftarrow{\iota_{\mathbb{R}}} \mathbb{R}$$

complex property: 'range $\subset$ kernel' ($\text{rot}\,\text{grad} = 0$, $\text{div}\,\text{rot} = 0$)

$$R(0) = \{0\} = N(\mathring{\text{grad}}), \quad R(\mathring{\text{grad}}) \subset N(\mathring{\text{rot}}), \quad R(\mathring{\text{rot}}) \subset N(\mathring{\text{div}}), \quad R(\mathring{\text{div}}) = N(\pi_{\mathbb{R}}) = \mathbb{R}^{\perp},$$
$$R(\iota_{\mathbb{R}}) = \mathbb{R} = N(\text{grad}), \quad R(-\text{grad}) \subset N(\text{rot}), \quad R(\text{rot}) \subset N(-\text{div}), \quad R(-\text{div}) = N(0) = \mathsf{L}^2(\Omega)$$

complex closed $\Leftrightarrow$ all ranges are closed
complex exact $\Leftrightarrow$ all cohomology groups are trivial, i.e., if

$$\mathcal{H}_D := N(\mathring{\text{rot}}) \ominus R(\mathring{\text{grad}}) = N(\text{div}) \ominus R(\text{rot}) = N(\mathring{\text{rot}}) \cap N(\text{div}) = \{0\} \qquad \text{(Dirichlet fields)},$$
$$\mathcal{H}_N := N(\mathring{\text{div}}) \ominus R(\mathring{\text{rot}}) = N(\text{rot}) \ominus R(\text{grad}) = N(\mathring{\text{div}}) \cap N(\text{rot}) = \{0\} \qquad \text{(Neumann fields)}$$

dimension depends only on topology, Betti numbers

grad-rot-div Complexes

sequence or complex:

$$\xrightarrow{0} D(\mathring{\text{grad}}) \xrightarrow{\mathring{\text{grad}}} D(\mathring{\text{rot}}) \xrightarrow{\mathring{\text{rot}}} D(\mathring{\text{div}}) \xrightarrow{\mathring{\text{div}}} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathbb{R}}} \mathbb{R}$$

with dual or adjoint sequence or complex

$$\xleftarrow{0} \mathsf{L}^2(\Omega) \xleftarrow{-\,\text{div}} D(\text{div}) \xleftarrow{\text{rot}} D(\text{rot}) \xleftarrow{-\,\text{grad}} D(\text{grad}) \xleftarrow{\iota_{\mathbb{R}}} \mathbb{R}$$

'crucial': compact embeddings (for, e.g., Ω bounded and strong Lipschitz)

$$D(\text{grad}) \hookrightarrow\!\!\!\!\rightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)},$$

$$D(\mathring{\text{rot}}) \cap D(\text{div}) \hookrightarrow\!\!\!\!\rightarrow \mathsf{L}^2 \qquad \text{(Weck's selection theorem, Weck '74)},$$

$$D(\text{rot}) \cap D(\mathring{\text{div}}) \hookrightarrow\!\!\!\!\rightarrow \mathsf{L}^2 \qquad \text{(Weck's selection theorem)}$$

$\Rightarrow$ closed complexes and finite cohomology groups
$\Rightarrow$ Helmholtz decompositions
$\Rightarrow$ Friedrichs/Poincaré type estimates
$\Rightarrow$ continuous and compact inverses of reduced operators
$\Rightarrow$ ... all from general fa-toolbox $\surd$

General Complexes

setting: unbounded, densely defined, closed, linear operators with adjoints

$$\mathrm{A}_i : D(\mathrm{A}_i) \subset \mathsf{H}_i \to \mathsf{H}_{i+1}, \quad \mathrm{A}_i^* : D(\mathrm{A}_i^*) \subset \mathsf{H}_{i+1} \to \mathsf{H}_i, \quad i \in \mathbb{Z}$$

sequence or complex with adjoint:

$$\xrightarrow{\cdots} D(\mathrm{A}_{i-1}) \xrightarrow{\mathrm{A}_{i-1}} D(\mathrm{A}_i) \xrightarrow{\mathrm{A}_i} D(\mathrm{A}_{i+1}) \xrightarrow{\mathrm{A}_{i+1}} \ldots$$

$$\ldots \xleftarrow{\mathrm{A}_{i-1}^*} D(\mathrm{A}_{i-1}^*) \xleftarrow{\mathrm{A}_i^*} D(\mathrm{A}_i^*) \xleftarrow{\mathrm{A}_{i+1}^*} D(\mathrm{A}_{i+1}^*) \xleftarrow{\cdots}$$

complex: 'range $\subset$ kernel', i.e., $\mathrm{A}_i \mathrm{A}_{i-1} = 0$, $\mathrm{A}_{i-1}^* \mathrm{A}_i^* = 0$, i.e.,

$$R(\mathrm{A}_{i-1}) \subset N(\mathrm{A}_i), \quad R(\mathrm{A}_i^*) \subset N(\mathrm{A}_{i-1}^*)$$

related problem: find $x \in D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)$ s.t.

$$\boxed{\mathrm{A}_i x = f, \quad \mathrm{A}_{i-1}^* x = g, \quad \pi_i x = h,}$$

where $f \in R(\mathrm{A}_i)$, $g \in R(\mathrm{A}_{i-1}^*)$ and $h \in \mathcal{H}_i$
with kernel/cohomology group $\mathcal{H}_i := N(\mathrm{A}_i) \cap N(\mathrm{A}_{i-1}^*)$

Reduced Operators and Compact Embeddings

Hodge/Helmholtz/Weyl decompositions:

$$\mathsf{H}_i = N(\mathrm{A}_i) \oplus_{\mathsf{H}_i} \overline{R(\mathrm{A}_i^*)}, \qquad \mathsf{H}_{i+1} = N(\mathrm{A}_i^*) \oplus_{\mathsf{H}_{i+1}} \overline{R(\mathrm{A}_i)}$$

$\Rightarrow$ reduced (injective) operators

$$\mathcal{A}_i : D(\mathcal{A}_i) := D(\mathrm{A}_i) \cap \overline{R(\mathrm{A}_i^*)} \subset \overline{R(\mathrm{A}_i^*)} \to \overline{R(\mathrm{A}_i)}, \qquad \big(\mathrm{A}_i : D(\mathrm{A}_i) \subset \mathsf{H}_i \to \mathsf{H}_{i+1}\big)$$
$$\mathcal{A}_i^* : D(\mathcal{A}_i^*) := D(\mathrm{A}_i^*) \cap \overline{R(\mathrm{A}_i)} \subset \overline{R(\mathrm{A}_i)} \to \overline{R(\mathrm{A}_i^*)}, \qquad \big(\mathrm{A}_i^* : D(\mathrm{A}_i^*) \subset \mathsf{H}_{i+1} \to \mathsf{H}_i\big)$$

$\Rightarrow$ $\mathcal{A}_i^{-1}$, $(\mathcal{A}_i^*)^{-1}$ exist, exact sequence for $\mathcal{A}_i$, $\mathcal{A}_i^*$ $\quad\surd$
crucial: compact embeddings

$$D(\mathcal{A}_i) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_i \quad \Leftrightarrow \quad D(\mathcal{A}_i^*) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_{i+1}$$

note

$$D(\mathcal{A}_{i-1}) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_{i-1} \quad \wedge \quad D(\mathcal{A}_i) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_i \quad \wedge \quad \mathcal{H}_i \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_i \quad \Leftrightarrow \quad D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_i$$

$\Rightarrow$
- (general) Poincaré type estimates (Poincaré, Friedrichs, Maxwell, ...)
- closed ranges
- continuous and compact invers operators
- Helmholtz decompositions

Friedrichs/Poincaré Type Estimates

compact embedding $D(\mathcal{A}_i) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_i \Rightarrow$

- $\forall\, \varphi \in D(\mathcal{A}_i) \quad |\varphi|_{\mathsf{H}_i} \leq c_{\mathrm{A}_i} |\mathrm{A}_i \varphi|_{\mathsf{H}_{i+1}}$
- $\forall\, \psi \in D(\mathcal{A}_i^*) \quad |\psi|_{\mathsf{H}_{i+1}} \leq c_{\mathrm{A}_i^*} |\mathrm{A}_i^* \psi|_{\mathsf{H}_i}$
- $R(\mathrm{A}_i) = R(\mathcal{A}_i)$, $R(\mathrm{A}_i^*) = R(\mathcal{A}_i^*)$ closed

$\Rightarrow$ reduced operators

$$\mathcal{A}_i : D(\mathcal{A}_i) := D(\mathrm{A}_i) \cap R(\mathrm{A}_i^*) \subset R(\mathrm{A}_i^*) \to R(\mathrm{A}_i), \qquad \big(\mathrm{A}_i : D(\mathrm{A}_i) \subset \mathsf{H}_i \to \mathsf{H}_{i+1}\big)$$
$$\mathcal{A}_i^* : D(\mathcal{A}_i^*) := D(\mathrm{A}_i^*) \cap R(\mathrm{A}_i) \subset R(\mathrm{A}_i) \to R(\mathrm{A}_i^*), \qquad \big(\mathrm{A}_i^* : D(\mathrm{A}_i^*) \subset \mathsf{H}_{i+1} \to \mathsf{H}_i\big)$$

- $\mathcal{A}_i^{-1} : R(\mathrm{A}_i) \to D(\mathcal{A}_i)$ cont., $\mathcal{A}_i^{-1} : R(\mathrm{A}_i) \to R(\mathcal{A}_i^*)$ cpt., $\|\mathcal{A}_i^{-1}\| = c_{\mathrm{A}_i}$
- $(\mathcal{A}_i^*)^{-1} : R(\mathrm{A}_i^*) \to D(\mathcal{A}_i^*)$ cont., $(\mathcal{A}_i^*)^{-1} : R(\mathrm{A}_i^*) \to R(\mathcal{A}_i)$ cpt., $\|(\mathcal{A}_i^*)^{-1}\| = c_{\mathrm{A}_i^*}$

note: 'best' constants c_{A_i} and $c_{\mathrm{A}_i^*}$ satisfy

$$\frac{1}{c_{\mathrm{A}_i}} = \inf_{0 \neq \varphi \in D(\mathcal{A}_i)} \frac{|\mathrm{A}_i \varphi|_{\mathsf{H}_{i+1}}}{|\varphi|_{\mathsf{H}_i}} = \inf_{0 \neq \psi \in D(\mathcal{A}_i^*)} \frac{|\mathrm{A}_i^* \psi|_{\mathsf{H}_i}}{|\psi|_{\mathsf{H}_{i+1}}} = \frac{1}{c_{\mathrm{A}_i^*}} \qquad \Rightarrow \qquad \boxed{c_i := c_{\mathrm{A}_i} = c_{\mathrm{A}_i^*}}$$

Friedrichs/Poincaré Type Constants

recall: compact embedding $D(\mathcal{A}_i) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_i \Rightarrow$

- $\forall\, \varphi \in D(\mathcal{A}_i) \quad |\varphi|_{\mathrm{H}_i} \leq c_{\mathrm{A}_i} |\mathrm{A}_i \varphi|_{\mathrm{H}_{i+1}}$, with best cons. $\dfrac{1}{c_{\mathrm{A}_i}} = \inf\limits_{0 \neq \varphi \in D(\mathcal{A}_i)} \dfrac{|\mathrm{A}_i \varphi|_{\mathrm{H}_{i+1}}}{|\varphi|_{\mathrm{H}_i}}$
- $\forall\, \psi \in D(\mathcal{A}_i^*) \quad |\psi|_{\mathrm{H}_{i+1}} \leq c_{\mathrm{A}_i^*} |\mathrm{A}_i^* \psi|_{\mathrm{H}_i}$, with best const. $\dfrac{1}{c_{\mathrm{A}_i^*}} = \inf\limits_{0 \neq \psi \in D(\mathcal{A}_i^*)} \dfrac{|\mathrm{A}_i^* \psi|_{\mathrm{H}_i}}{|\psi|_{\mathrm{H}_{i+1}}}$

Lemma

$c_i := c_{\mathrm{A}_i} = c_{\mathrm{A}_i^*}$

Proof.

$\varphi \in D(\mathcal{A}_i) = D(\mathrm{A}_i) \cap R(\mathrm{A}_i^*)$, $R(\mathrm{A}_i^*) = R(\mathcal{A}_i^*)$ closed

$\Rightarrow \quad \varphi = \mathrm{A}_i^* \psi$ with $\psi \in D(\mathcal{A}_i^*)$

$\Rightarrow \quad |\varphi|_{\mathrm{H}_i}^2 = \langle \varphi, \mathrm{A}_i^* \psi \rangle_{\mathrm{H}_i} = \langle \mathrm{A}_i \varphi, \psi \rangle_{\mathrm{H}_{i+1}} \leq |\mathrm{A}_i \varphi|_{\mathrm{H}_{i+1}} |\psi|_{\mathrm{H}_{i+1}} \leq c_{\mathrm{A}_i^*} |\mathrm{A}_i \varphi|_{\mathrm{H}_{i+1}} |\underbrace{\mathrm{A}_i^* \psi}_{=\varphi}|_{\mathrm{H}_i}$

$\Rightarrow \quad |\varphi|_{\mathrm{H}_i} \leq c_{\mathrm{A}_i^*} |\mathrm{A}_i \varphi|_{\mathrm{H}_{i+1}} \quad \Rightarrow \quad c_{\mathrm{A}_i} \leq c_{\mathrm{A}_i^*} \quad \overset{\text{symmetry}}{\Rightarrow} \quad c_{\mathrm{A}_i^*} \leq c_{\mathrm{A}_i}$ □

Helmholtz Type Decomposition

$$\begin{aligned} \mathsf{H}_i &= N(\mathrm{A}_i) \oplus_{\mathsf{H}_i} R(\mathrm{A}_i^*), & \mathsf{H}_i &= R(\mathrm{A}_{i-1}) \oplus_{\mathsf{H}_i} N(\mathrm{A}_{i-1}^*) \\ D(\mathrm{A}_i) &= N(\mathrm{A}_i) \oplus_{\mathsf{H}_i} D(\mathcal{A}_i), & D(\mathrm{A}_{i-1}^*) &= D(\mathcal{A}_{i-1}^*) \oplus_{\mathsf{H}_i} N(\mathrm{A}_{i-1}^*) \end{aligned}$$

exact sequence: $R(\mathrm{A}_{i-1}) \subset N(\mathrm{A}_i)$, $R(\mathrm{A}_i^*) \subset N(\mathrm{A}_{i-1}^*) \Rightarrow$

$$N(\mathrm{A}_{i-1}^*) = \mathcal{H}_i \oplus_{\mathsf{H}_i} R(\mathrm{A}_i^*), \quad N(\mathrm{A}_i) = R(\mathrm{A}_{i-1}) \oplus_{\mathsf{H}_i} \mathcal{H}_i, \quad \mathcal{H}_i = N(\mathrm{A}_i) \cap N(\mathrm{A}_{i-1}^*)$$

$\Rightarrow$ refined Helmholtz decomposition

$$\begin{aligned} \mathsf{H}_i &= R(\mathrm{A}_{i-1}) \oplus_{\mathsf{H}_i} \mathcal{H}_i \oplus_{\mathsf{H}_i} R(\mathrm{A}_i^*) \\ D(\mathrm{A}_i) &= R(\mathrm{A}_{i-1}) \oplus_{\mathsf{H}_i} \mathcal{H}_i \oplus_{\mathsf{H}_i} D(\mathcal{A}_i) \\ D(\mathrm{A}_{i-1}^*) &= D(\mathcal{A}_{i-1}^*) \oplus_{\mathsf{H}_i} \mathcal{H}_i \oplus_{\mathsf{H}_i} R(\mathrm{A}_i^*) \end{aligned}$$

with orthonormal projectors

$$\begin{aligned} &\pi_{\mathrm{A}_{i-1}} : \mathsf{H}_i \to R(\mathrm{A}_{i-1}), && \forall\, \psi \in D(\mathrm{A}_{i-1}^*) && \pi_{\mathrm{A}_{i-1}}\psi \in D(\mathcal{A}_{i-1}^*) && \wedge && \mathrm{A}_{i-1}^*\pi_{\mathrm{A}_{i-1}}\psi = \mathrm{A}_{i-1}^*\psi \\ &\pi_{\mathrm{A}_i^*} : \mathsf{H}_i \to R(\mathrm{A}_i^*), && \forall\, \varphi \in D(\mathrm{A}_i) && \pi_{\mathrm{A}_i^*}\varphi \in D(\mathcal{A}_i) && \wedge && \mathrm{A}_i\pi_{\mathrm{A}_i^*}\varphi = \mathrm{A}_i\varphi \\ &\pi_i : \mathsf{H}_i \to \mathcal{H}_i \end{aligned}$$

Solution Theory

problem: find $x \in D(\mathrm{A}_i) \cap D(\mathrm{A}^*_{i-1})$ s.t.

$$\begin{aligned} \mathrm{A}_i x &= f \\ \mathrm{A}^*_{i-1} x &= g \\ \pi_i x &= h \end{aligned}$$

Theorem (solution theory)

unique solution (dpd. cont. on data) $\Leftrightarrow$ $f \in R(\mathrm{A}_i)$, $g \in R(\mathrm{A}^*_{i-1})$ *and* $h \in \mathcal{H}_i$

Proof.

$x = \mathcal{A}_i^{-1} f + (\mathcal{A}^*_{i-1})^{-1} g + h$ □

Variations (Saddle Point) Formulations

unique solution $x = \mathcal{A}_i^{-1} f + (\mathcal{A}_{i-1}^*)^{-1} g + h \in D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)$ of

$$\boxed{\mathrm{A}_i x = f, \quad \mathrm{A}_{i-1}^* x = g, \quad \pi_i x = h}$$

can be found by variational techniques (Lax-Milgram)

- for $\mathcal{A}_i^{-1} f$ we solve $\mathrm{A}_i \mathrm{A}_i^* \psi = f$: find $\psi \in D(\mathcal{A}_i^*)$ with

$$\forall\, \varphi \in D(\mathcal{A}_i^*) \qquad \langle \mathrm{A}_i^* \psi, \mathrm{A}_i^* \varphi \rangle_{\mathrm{H}_i} = \langle f, \varphi \rangle_{\mathrm{H}_{i+1}} \tag{1}$$

$f \in R(\mathrm{A}_i) \Rightarrow$ (1) holds for all $\varphi \in D(\mathrm{A}_i^*)$
$\Rightarrow x_{\mathrm{A}_i} := \mathrm{A}_i^* \psi \in D(\mathcal{A}_i)$ and $\mathrm{A}_i x_{\mathrm{A}_i} = f$
$\Rightarrow x_{\mathrm{A}_i} = \mathcal{A}_i^{-1} f \in D(\mathcal{A}_i)$ and $|x_{\mathrm{A}_i}|_{\mathrm{H}_i} \leq c_i |f|_{\mathrm{H}_{i+1}}$
note: $D(\mathcal{A}_i^*) = D(\mathrm{A}_i^*) \cap R(\mathrm{A}_i)$ and $R(\mathrm{A}_i) = N(\mathrm{A}_i^*)^{\perp_{\mathrm{H}_{i+1}}}$
$\Rightarrow$ (1) is equivalent to the saddle point problem: find $\psi \in D(\mathrm{A}_i^*)$ with

$$\begin{aligned} &\forall\, \varphi \in D(\mathrm{A}_i^*) && \langle \mathrm{A}_i^* \psi, \mathrm{A}_i^* \varphi \rangle_{\mathrm{H}_i} = \langle f, \varphi \rangle_{\mathrm{H}_{i+1}}, \\ &\forall\, \phi \in N(\mathrm{A}_i^*) = R(\mathrm{A}_{i+1}^*) \oplus_{\mathrm{H}_{i+1}} \mathcal{H}_{i+1} && \langle \psi, \phi \rangle_{\mathrm{H}_{i+1}} = 0 \end{aligned}$$

Variations (Saddle Point) Formulations

unique solution $x = \mathcal{A}_i^{-1} f + (\mathcal{A}_{i-1}^*)^{-1} g + h \in D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)$ of

$$\boxed{\mathrm{A}_i x = f, \quad \mathrm{A}_{i-1}^* x = g, \quad \pi_i x = h}$$

can be found by variational techniques (Lax-Milgram)

- for $(\mathcal{A}_{i-1}^*)^{-1} g$ we solve $\mathrm{A}_{i-1}^* \mathrm{A}_{i-1} \psi = f$: find $\psi \in D(\mathcal{A}_{i-1})$ with

$$\forall\, \varphi \in D(\mathcal{A}_{i-1}) \qquad \langle \mathrm{A}_{i-1}\psi, \mathrm{A}_{i-1}\varphi\rangle_{\mathrm{H}_i} = \langle g, \varphi\rangle_{\mathrm{H}_{i-1}} \tag{2}$$

$g \in R(\mathrm{A}_{i-1}^*) \Rightarrow$ (2) holds for all $\varphi \in D(\mathrm{A}_{i-1})$
$\Rightarrow x_{\mathrm{A}_{i-1}^*} := \mathrm{A}_{i-1}\psi \in D(\mathcal{A}_{i-1}^*)$ and $\mathrm{A}_{i-1}^* x_{\mathrm{A}_{i-1}^*} = g$
$\Rightarrow x_{\mathrm{A}_{i-1}^*} = (\mathcal{A}_{i-1}^*)^{-1} g \in D(\mathcal{A}_{i-1}^*)$ and $|x_{\mathrm{A}_{i-1}^*}|_{\mathrm{H}_i} \leq c_{i-1} |g|_{\mathrm{H}_{i-1}}$
note: $D(\mathcal{A}_{i-1}) = D(\mathrm{A}_{i-1}) \cap R(\mathrm{A}_{i-1}^*)$ and $R(\mathrm{A}_{i-1}^*) = N(\mathrm{A}_{i-1})^{\perp_{\mathrm{H}_{i-1}}}$
$\Rightarrow$ (2) is equivalent to the saddle point problem: find $\psi \in D(\mathrm{A}_{i-1})$ with

$$\forall\, \varphi \in D(\mathrm{A}_{i-1}) \qquad \langle \mathrm{A}_{i-1}\psi, \mathrm{A}_{i-1}\varphi\rangle_{\mathrm{H}_i} = \langle g, \varphi\rangle_{\mathrm{H}_{i-1}},$$

$$\forall\, \phi \in N(\mathrm{A}_{i-1}) = R(\mathrm{A}_{i-2}) \oplus_{\mathrm{H}_i} \mathcal{H}_{i-1} \qquad \langle \psi, \phi\rangle_{\mathrm{H}_{i-1}} = 0$$

Grad grad and div Div Complexes

complex

$$\xrightarrow{0} D(\mathring{\operatorname{Grad}}\operatorname{grad}) \xrightarrow{\mathring{\operatorname{Grad}}\operatorname{grad}} D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \xrightarrow{\mathring{\operatorname{Rot}}_{\mathbb{S}}} D(\mathring{\operatorname{Div}}_{\mathbb{T}}) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathsf{RT}_0}} \mathsf{RT}_0$$

with dual or adjoint complex

$$\xleftarrow{0} \mathsf{L}^2(\Omega) \xleftarrow{\operatorname{div}\operatorname{Div}_{\mathbb{S}}} D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \xleftarrow{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}} D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \xleftarrow{-\operatorname{dev}\operatorname{Grad}} D(\operatorname{dev}\operatorname{Grad}) \xleftarrow{\iota_{\mathsf{RT}_0}} \mathsf{RT}_0$$

unbounded, densely defined, closed, linear operators with adjoints

$$\begin{aligned}
\mathring{\operatorname{Grad}}\operatorname{grad} : D(\mathring{\operatorname{Grad}}\operatorname{grad}) \subset \mathsf{L}^2_{\mathbb{R}} \to \mathsf{L}^2_{\mathbb{S}}, &\quad (\mathring{\operatorname{Grad}}\operatorname{grad})^* = \operatorname{div}\operatorname{Div}_{\mathbb{S}} : D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \subset \mathsf{L}^2_{\mathbb{S}} \to \mathsf{L}^2_{\mathbb{R}}, \\
\mathring{\operatorname{Rot}}_{\mathbb{S}} : D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \subset \mathsf{L}^2_{\mathbb{S}} \to \mathsf{L}^2_{\mathbb{T}}, &\quad (\mathring{\operatorname{Rot}}_{\mathbb{S}})^* = \operatorname{sym}\operatorname{Rot}_{\mathbb{T}} : D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \subset \mathsf{L}^2_{\mathbb{T}} \to \mathsf{L}^2_{\mathbb{S}}, \\
\mathring{\operatorname{Div}}_{\mathbb{T}} : D(\mathring{\operatorname{Div}}_{\mathbb{T}}) \subset \mathsf{L}^2_{\mathbb{T}} \to \mathsf{L}^2_{\mathbb{R}^3}, &\quad (\mathring{\operatorname{Div}}_{\mathbb{T}})^* = -\operatorname{dev}\operatorname{Grad} : D(\operatorname{dev}\operatorname{Grad}) \subset \mathsf{L}^2_{\mathbb{R}^3} \to \mathsf{L}^2_{\mathbb{T}}
\end{aligned}$$

setting in natural Hilbert spaces
one can show

$$D(\mathring{\operatorname{Grad}}\operatorname{grad}) = \mathring{\mathsf{H}}^2, \quad D(\operatorname{dev}\operatorname{Grad}) = \mathsf{H}^1$$

Results for Grad grad and div Div Complexes (triv. top.)

complex and adjoint complex

$$\xrightarrow{0} D(\mathring{\operatorname{Grad}}\operatorname{grad}) \xrightarrow{\mathring{\operatorname{Grad}}\operatorname{grad}} D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \xrightarrow{\mathring{\operatorname{Rot}}_{\mathbb{S}}} D(\mathring{\operatorname{Div}}_{\mathbb{T}}) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathsf{RT}_0}} \mathsf{RT}_0$$

$$\xleftarrow{0} \mathsf{L}^2(\Omega) \xleftarrow{\operatorname{div}\operatorname{Div}_{\mathbb{S}}} D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \xleftarrow{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}} D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \xleftarrow{-\operatorname{dev}\operatorname{Grad}} D(\operatorname{dev}\operatorname{Grad}) \xleftarrow{\iota_{\mathsf{RT}_0}} \mathsf{RT}_0$$

general assumption: $\Omega \subset \mathbb{R}^3$ $(\mathbb{R}^n)$ bounded strong Lipschitz domain

Lemma

Let Ω be additionally topologically trivial. Then:

- $N(\mathring{\operatorname{Rot}}_{\mathbb{S}}) = R(\mathring{\operatorname{Grad}}\operatorname{grad}) = \operatorname{Grad}\operatorname{grad}\mathring{\mathsf{H}}^2$
- $N(\mathring{\operatorname{Div}}_{\mathbb{T}}) = R(\mathring{\operatorname{Rot}}_{\mathbb{S}}) = \operatorname{Rot}\mathring{\mathsf{H}}^1_{\mathbb{S}}$
- $\mathsf{RT}_0^{\perp_{\mathsf{L}^2}} = R(\mathring{\operatorname{Div}}_{\mathbb{T}}) = \operatorname{Div}\mathring{\mathsf{H}}^1_{\mathbb{T}}$
- $N(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) = R(\operatorname{dev}\operatorname{Grad}) = \operatorname{dev}\operatorname{Grad}\mathsf{H}^1$
- $N(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) = R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) = \operatorname{sym}\operatorname{Rot}\mathsf{H}^1_{\mathbb{T}}$ *(original question)*
- $\mathsf{L}^2 = R(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) = \operatorname{div}\operatorname{Div}\mathsf{H}^2_{\mathbb{S}}$

Especially, all ranges are closed and admit regular potentials.

$\Rightarrow$ both complexes are closed

Results for Grad grad and div Div Complexes (triv. top.)

complex and adjoint complex

$$\xrightarrow{0} D(\mathring{\operatorname{Grad}}\operatorname{grad}) \xrightarrow{\mathring{\operatorname{Grad}}\operatorname{grad}} D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \xrightarrow{\mathring{\operatorname{Rot}}_{\mathbb{S}}} D(\mathring{\operatorname{Div}}_{\mathbb{T}}) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathsf{RT}_0}} \mathsf{RT}_0$$

$$\xleftarrow{0} \mathsf{L}^2(\Omega) \xleftarrow{\operatorname{div}\operatorname{Div}_{\mathbb{S}}} D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \xleftarrow{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}} D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \xleftarrow{-\operatorname{dev}\operatorname{Grad}} D(\operatorname{dev}\operatorname{Grad}) \xleftarrow{\iota_{\mathsf{RT}_0}} \mathsf{RT}_0$$

general assumption: $\Omega \subset \mathbb{R}^3$ $(\mathbb{R}^n)$ bounded strong Lipschitz domain

Lemma

Let Ω be additionally topologically trivial. Then all cohomology groups are trivial, i.e.,

$$\mathcal{H}^D_{\mathbb{S}} := N(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \cap N(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) = N(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \ominus R(\mathring{\operatorname{Grad}}\operatorname{grad}) = N(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \ominus R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) = \{0\}$$

$$\mathcal{H}^N_{\mathbb{T}} := N(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap N(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) = N(\mathring{\operatorname{Div}}_{\mathbb{T}}) \ominus R(\mathring{\operatorname{Rot}}_{\mathbb{S}}) = N(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \ominus R(\operatorname{dev}\operatorname{Grad}) = \{0\}$$

(The symmetric Dirichlet fields and trace-free Neumann fields vanish.) Moreover, the following Helmholtz type decompositions hold:

$$\begin{aligned}
&\mathsf{L}^2(\Omega,\mathbb{S}) = N(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \oplus_{\mathsf{L}^2(\Omega,\mathbb{S})} N(\operatorname{div}\operatorname{Div}_{\mathbb{S}}), && \mathsf{L}^2 = R(\operatorname{div}\operatorname{Div}_{\mathbb{S}})\\
&N(\mathring{\operatorname{Rot}}_{\mathbb{S}}) = R(\mathring{\operatorname{Grad}}\operatorname{grad}), && N(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) = R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}),\\
&\mathsf{L}^2(\Omega,\mathbb{T}) = N(\mathring{\operatorname{Div}}_{\mathbb{T}}) \oplus_{\mathsf{L}^2(\Omega,\mathbb{T})} N(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}), && \mathsf{RT}_0^{\perp_{\mathsf{L}^2}} = R(\mathring{\operatorname{Div}}_{\mathbb{T}}),\\
&N(\mathring{\operatorname{Div}}_{\mathbb{T}}) = R(\mathring{\operatorname{Rot}}_{\mathbb{S}}), && N(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) = R(\operatorname{dev}\operatorname{Grad}).
\end{aligned}$$

$\Rightarrow$ both complexes are closed and exact

Results for Grad grad and div Div Complexes (gen. top.)

complex and adjoint complex

$$\xrightarrow{0} D(\mathring{\operatorname{Grad}\operatorname{grad}}) \xrightarrow{\mathring{\operatorname{Grad}\operatorname{grad}}} D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \xrightarrow{\mathring{\operatorname{Rot}}_{\mathbb{S}}} D(\mathring{\operatorname{Div}}_{\mathbb{T}}) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}} \mathrm{L}^2(\Omega) \xrightarrow{\pi_{\mathrm{RT}_0}} \mathrm{RT}_0$$

$$\xleftarrow{0} \mathrm{L}^2(\Omega) \xleftarrow{\operatorname{div}\operatorname{Div}_{\mathbb{S}}} D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \xleftarrow{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}} D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \xleftarrow{-\operatorname{dev}\operatorname{Grad}} D(\operatorname{dev}\operatorname{Grad}) \xleftarrow{\iota_{\mathrm{RT}_0}} \mathrm{RT}_0$$

general assumption: $\Omega \subset \mathbb{R}^3$ $(\mathbb{R}^n)$ bounded strong Lipschitz domain

Lemma

The following embeddings are compact:

$$D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \cap D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \hookrightarrow\!\!\!\!\rightarrow \mathrm{L}^2(\Omega,\mathbb{S}), \qquad D(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \hookrightarrow\!\!\!\!\rightarrow \mathrm{L}^2(\Omega,\mathbb{T})$$

proof: partition of unity, results for triv. top. (HD and reg. pot.) and

Lemma

It holds directly and topologically

$$\{M \in \mathrm{L}^2(\Omega,\mathbb{S}) \,:\, \operatorname{div}\operatorname{Div} M \in \mathrm{H}^{-1}(\Omega)\} = \mathring{\mathrm{H}}^1 \cdot \mathrm{Id} \dotplus N(\operatorname{div}\operatorname{Div}_{\mathbb{S}}).$$

Results for Grad grad and div Div Complexes (gen. top.)

complex and adjoint complex

$$\xrightarrow{0} D(\mathring{\operatorname{Grad}}\operatorname{grad}) \xrightarrow{\mathring{\operatorname{Grad}}\operatorname{grad}} D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \xrightarrow{\mathring{\operatorname{Rot}}_{\mathbb{S}}} D(\mathring{\operatorname{Div}}_{\mathbb{T}}) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathsf{RT}_0}} \mathsf{RT}_0$$

$$\xleftarrow{0} \mathsf{L}^2(\Omega) \xleftarrow{\operatorname{div}\operatorname{Div}_{\mathbb{S}}} D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \xleftarrow{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}} D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \xleftarrow{-\operatorname{dev}\operatorname{Grad}} D(\operatorname{dev}\operatorname{Grad}) \xleftarrow{\iota_{\mathsf{RT}_0}} \mathsf{RT}_0$$

general assumption: $\Omega \subset \mathbb{R}^3$ $(\mathbb{R}^n)$ bounded strong Lipschitz domain

Lemma

The cohomology groups $\mathcal{H}^D_{\mathbb{S}}$, $\mathcal{H}^N_{\mathbb{T}}$ (symmetric Dirichlet fields and trace-free Neumann fields) are finite dimensional. Moreover, the following Helmholtz type decompositions hold:

$$\mathsf{L}^2(\Omega,\mathbb{S}) = R(\mathring{\operatorname{Grad}}\operatorname{grad}) \oplus_{\mathsf{L}^2(\Omega,\mathbb{S})} \mathcal{H}^D_{\mathbb{S}} \oplus_{\mathsf{L}^2(\Omega,\mathbb{S})} R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}),$$

$$N(\mathring{\operatorname{Rot}}_{\mathbb{S}}) = R(\mathring{\operatorname{Grad}}\operatorname{grad}) \oplus_{\mathsf{L}^2(\Omega,\mathbb{S})} \mathcal{H}^D_{\mathbb{S}}, \qquad N(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) = R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \oplus_{\mathsf{L}^2(\Omega,\mathbb{S})} \mathcal{H}^D_{\mathbb{S}},$$

$$\mathsf{L}^2(\Omega,\mathbb{T}) = R(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \oplus_{\mathsf{L}^2(\Omega,\mathbb{T})} \mathcal{H}^N_{\mathbb{T}} \oplus_{\mathsf{L}^2(\Omega,\mathbb{T})} R(\operatorname{dev}\operatorname{Grad}),$$

$$N(\mathring{\operatorname{Div}}_{\mathbb{T}}) = R(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \oplus_{\mathsf{L}^2(\Omega,\mathbb{T})} \mathcal{H}^N_{\mathbb{T}}, \qquad N(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) = R(\operatorname{dev}\operatorname{Grad}) \oplus_{\mathsf{L}^2(\Omega,\mathbb{T})} \mathcal{H}^N_{\mathbb{T}}.$$

Especially, all ranges are closed.

$\Rightarrow$ both complexes are closed

Results for Grad grad and div Div Complexes (gen. top.)

complex and adjoint complex

$$\xrightarrow{0} D(\mathring{\operatorname{Grad}}\operatorname{grad}) \xrightarrow{\mathring{\operatorname{Grad}}\operatorname{grad}} D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \xrightarrow{\mathring{\operatorname{Rot}}_{\mathbb{S}}} D(\mathring{\operatorname{Div}}_{\mathbb{T}}) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}} \mathrm{L}^2(\Omega) \xrightarrow{\pi_{\mathrm{RT}_0}} \mathrm{RT}_0$$

$$\xleftarrow{0} \mathrm{L}^2(\Omega) \xleftarrow{\operatorname{div}\operatorname{Div}_{\mathbb{S}}} D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \xleftarrow{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}} D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \xleftarrow{-\operatorname{dev}\operatorname{Grad}} D(\operatorname{dev}\operatorname{Grad}) \xleftarrow{\iota_{\mathrm{RT}_0}} \mathrm{RT}_0$$

general assumption: $\Omega \subset \mathbb{R}^3$ $(\mathbb{R}^n)$ bounded strong Lipschitz domain

Lemma

The following Friedrichs/Poincaré type estimates hold: $\exists\ c_{\mathrm{Gg}}, c_{\mathrm{D}}, c_{\mathrm{R}} > 0$ *s.t.*

$$\begin{aligned}
&\forall\, u \in \mathring{\mathrm{H}}^2 && |u|_{\mathrm{L}^2(\Omega)} \le c_{\mathrm{Gg}} |\operatorname{Grad}\operatorname{grad} u|_{\mathrm{L}^2(\Omega)},\\
&\forall\, M \in D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \cap R(\mathring{\operatorname{Grad}}\operatorname{grad}) && |M|_{\mathrm{L}^2(\Omega)} \le c_{\mathrm{Gg}} |\operatorname{div}\operatorname{Div} M|_{\mathrm{L}^2(\Omega)},\\
&\forall\, N \in D(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap R(\operatorname{dev}\operatorname{Grad}) && |N|_{\mathrm{L}^2(\Omega)} \le c_{\mathrm{D}} |\operatorname{Div} N|_{\mathrm{L}^2(\Omega)},\\
&\forall\, E \in \mathrm{H}^1 \cap \mathrm{RT}_0^{\perp_{\mathrm{L}^2}} && |E|_{\mathrm{L}^2(\Omega)} \le c_{\mathrm{D}} |\operatorname{dev}\operatorname{Grad} E|_{\mathrm{L}^2(\Omega)},\\
&\forall\, M \in D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \cap R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) && |M|_{\mathrm{L}^2(\Omega)} \le c_{\mathrm{R}} |\operatorname{Rot} M|_{\mathrm{L}^2(\Omega)},\\
&\forall\, N \in D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \cap R(\mathring{\operatorname{Rot}}_{\mathbb{S}}) && |N|_{\mathrm{L}^2(\Omega)} \le c_{\mathrm{R}} |\operatorname{sym}\operatorname{Rot} N|_{\mathrm{L}^2(\Omega)}
\end{aligned}$$

note: same constants!

Results for Grad grad and div Div Complexes (gen. top.)

complex and adjoint complex

$$\xrightarrow{0} D(\mathring{\operatorname{Grad}}\operatorname{grad}) \xrightarrow{\mathring{\operatorname{Grad}}\operatorname{grad}} D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \xrightarrow{\mathring{\operatorname{Rot}}_{\mathbb{S}}} D(\mathring{\operatorname{Div}}_{\mathbb{T}}) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathsf{RT}_0}} \mathsf{RT}_0$$

$$\xleftarrow{0} \mathsf{L}^2(\Omega) \xleftarrow{\operatorname{div}\operatorname{Div}_{\mathbb{S}}} D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \xleftarrow{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}} D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \xleftarrow{-\operatorname{dev}\operatorname{Grad}} D(\operatorname{dev}\operatorname{Grad}) \xleftarrow{\iota_{\mathsf{RT}_0}} \mathsf{RT}_0$$

general assumption: $\Omega \subset \mathbb{R}^3$ $(\mathbb{R}^n)$ bounded strong Lipschitz domain

Lemma

The respective reduced operators have continuous resp. compact inverses, e.g.,

$$(\mathring{\operatorname{Rot}}_{\mathbb{S}})^{-1} : R(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \xrightarrow{cont.} D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \cap R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \overset{cpt.}{\hookrightarrow} R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}),$$

$$(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}})^{-1} : R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \xrightarrow{cont.} D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \cap R(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \overset{cpt.}{\hookrightarrow} R(\mathring{\operatorname{Rot}}_{\mathbb{S}}),$$

$$\vdots$$

Results for Grad grad and div Div Complexes (triv. top.)

complex and adjoint complex

$$\xrightarrow{\;0\;} D(\mathring{\operatorname{Grad}}\operatorname{grad}) \xrightarrow{\mathring{\operatorname{Grad}}\operatorname{grad}} D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \xrightarrow{\mathring{\operatorname{Rot}}_{\mathbb{S}}} D(\mathring{\operatorname{Div}}_{\mathbb{T}}) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathsf{RT}_0}} \mathsf{RT}_0$$

$$\xleftarrow{\;0\;} \mathsf{L}^2(\Omega) \xleftarrow{\operatorname{div}\operatorname{Div}_{\mathbb{S}}} D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \xleftarrow{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}} D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \xleftarrow{-\operatorname{dev}\operatorname{Grad}} D(\operatorname{dev}\operatorname{Grad}) \xleftarrow{\iota_{\mathsf{RT}_0}} \mathsf{RT}_0$$

general assumption: $\Omega \subset \mathbb{R}^3$ $(\mathbb{R}^n)$ bounded strong Lipschitz domain

Lemma

Let Ω be additionally topologically trivial. Then the following regular decompositions hold:

$$D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) = \mathring{\mathsf{H}}^1_{\mathbb{S}} + N(\mathring{\operatorname{Rot}}_{\mathbb{S}}), \qquad N(\mathring{\operatorname{Rot}}_{\mathbb{S}}) = R(\mathring{\operatorname{Grad}}\operatorname{grad}),$$

$$D(\mathring{\operatorname{Div}}_{\mathbb{T}}) = \mathring{\mathsf{H}}^1_{\mathbb{T}} + N(\mathring{\operatorname{Div}}_{\mathbb{T}}), \qquad N(\mathring{\operatorname{Div}}_{\mathbb{T}}) = R(\mathring{\operatorname{Rot}}_{\mathbb{S}}),$$

$$D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) = \mathsf{H}^1_{\mathbb{T}} + N(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}), \qquad N(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) = R(\operatorname{dev}\operatorname{Grad}),$$

$$\vdots$$

extension to general domains ...

Results for Rot Rot$^\top$ or Elasticity Complexes

complex

$$\xrightarrow{0} D(\operatorname{sym}\mathring{\operatorname{Grad}}) \xrightarrow{\operatorname{sym}\mathring{\operatorname{Grad}}} D(\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^\top) \xrightarrow{\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^\top} D(\mathring{\operatorname{Div}}_{\mathbb{S}}) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{S}}} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathsf{RM}}} \mathsf{RM}$$

adjoint complex ...

general assumption: $\Omega \subset \mathbb{R}^3$ $(\mathbb{R}^n)$ bounded strong Lipschitz domain

Lemma

Similar results hold.