On the Grad grad and div Div Complexes, and a Related Decomposition Result for Biharmonic Problems in 3D: Part 1

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Motivation

primal formulation

find $u \in W = \mathrm{H}_0^2(\Omega)$ such that

$$\Delta^2 u = f \quad \text{in } W^* = \mathrm{H}^{-2}(\Omega)$$

decomposition into a sequence of second-order problems?

observe that

$$\Delta^2 u = -\operatorname{div} \operatorname{Div} \mathbf{M} \quad \text{with} \quad \mathbf{M} = -\operatorname{Grad} \operatorname{grad} u$$

mixed formulation

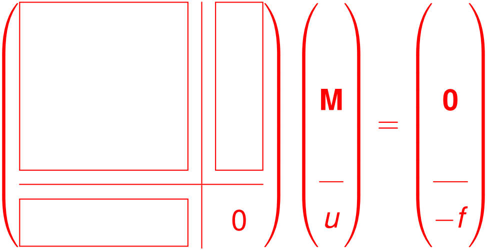
find $\mathbf{M} \in \mathbf{V} \subset \mathbf{L}^2(\Omega, \mathbb{S})$ and $u \in Q \subset \mathrm{L}^2(\Omega)$ such that

$$\begin{aligned} \mathbf{M} \qquad & + \operatorname{Grad} \operatorname{grad} u = 0 && \text{in } \mathbf{V}^* \\ \operatorname{div} \operatorname{Div} \mathbf{M} \qquad & = -f && \text{in } Q^* \end{aligned}$$

block-triangular system for

$$\mathbf{M} \in \mathbf{V} \quad \text{and} \quad u \in Q$$

of the form

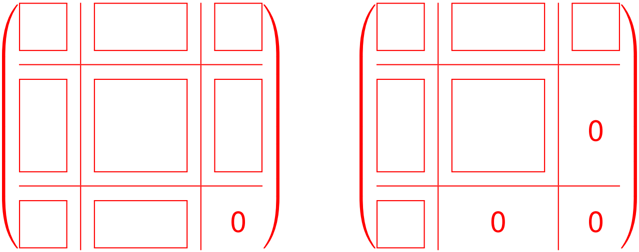


Motivation

decomposition

$$\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2 \qquad \text{with} \qquad \mathbf{V}_2 = \ker \operatorname{div} \operatorname{Div}$$

leads to



Outline

Mixed variational formulation for $f \in \mathrm{H}^{-2}(\Omega)$

Find $\mathbf{M} \in \mathbf{L}^2(\Omega, \mathbb{S})$ and $u \in H_0^2(\Omega)$ such that

$$\begin{aligned} \mathbf{M} \quad & + (\operatorname{div}\operatorname{Div})^* u = 0 && \text{in } \mathbf{L}^2(\Omega, \mathbb{S})^* \\ \operatorname{div}\operatorname{Div}\mathbf{M} \quad & = -f && \text{in } \mathrm{H}^{-2}(\Omega) \end{aligned}$$

Mixed variational formulation for $f \in \mathrm{L}^2(\Omega)$

Find $\mathbf{M} \in \mathbf{H}(\operatorname{div}\operatorname{Div}; \Omega, \mathbb{S})$ and $u \in L^2(\Omega)$ such that

$$\begin{aligned} \mathbf{M} \quad & + (\operatorname{div}\operatorname{Div})^* u = 0 && \text{in } \mathbf{H}(\operatorname{div}\operatorname{Div}; \Omega, \mathbb{S})^* \\ \operatorname{div}\operatorname{Div}\mathbf{M} \quad & = -f && \text{in } \mathrm{L}^2(\Omega)^* \end{aligned}$$

The space $\mathbf{V}$

New mixed variational formulation for $f \in \mathrm{H}^{-1}(\Omega)$

Find $\mathbf{M} \in \mathbf{V}$ and $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} \mathbf{M} \quad &+ (\operatorname{div} \operatorname{Div})^* u = 0 \quad &&\text{in } \mathbf{V}^* \\ \operatorname{div} \operatorname{Div} \mathbf{M} \quad &= -f \quad &&\text{in } \mathrm{H}^{-1}(\Omega) \end{aligned}$$

with

$$\mathbf{V} = \{\mathbf{N} \in \mathbf{L}^2(\Omega, \mathbb{S}) : \operatorname{div} \operatorname{Div} \mathbf{N} \in \mathrm{H}^{-1}(\Omega)\} \equiv \mathbf{H}^{0,-1}(\operatorname{div} \operatorname{Div}; \Omega, \mathbb{S})$$

norm

$$\|\mathbf{N}\|_{\mathbf{V}} = \left(\|\mathbf{N}\|^2_{\mathbf{L}^2(\Omega)} + \|\operatorname{div} \operatorname{Div} \mathbf{N}\|^2_{\mathrm{H}^{-1}(\Omega)} \right)^{1/2}$$

$$\mathbf{H}^1(\Omega, \mathbb{S}) \subset \mathbf{H}^{0,-1}(\operatorname{div} \operatorname{Div}; \Omega, \mathbb{S}) \subset \mathbf{L}^2(\Omega, \mathbb{S})$$

Bernardi/Girault/Maday (1992), Z. (2015), Pechstein/Schöberl (2011)

New mixed variational formulation for $f \in \mathrm{H}^{-1}(\Omega)$

Find $\mathbf{M} \in \mathbf{V} = \mathbf{H}^{0,-1}(\operatorname{div} \operatorname{Div}; \Omega, \mathbb{S})$ and $u \in Q = \mathrm{H}^1_0(\Omega)$ such that

$$\begin{aligned} \langle \mathbf{M}, \mathbf{N} \rangle \quad &+ \langle \operatorname{div} \operatorname{Div} \mathbf{N}, u \rangle = 0 && \text{for all } \mathbf{N} \in \mathbf{V} \\ \langle \operatorname{div} \operatorname{Div} \mathbf{M}, v \rangle \quad && = -\langle f, v \rangle && \text{for all } v \in Q \end{aligned}$$

Theorem

The mixed problem is well-posed in $\mathbf{V} \times Q$, *equipped with the norms*

$$\|\mathbf{N}\|_{\mathbf{V}} = \left(\|\mathbf{N}\|^2_{\mathbf{L}^2(\Omega)} + \|\operatorname{div} \operatorname{Div} \mathbf{N}\|^2_{\mathrm{H}^{-1}(\Omega)} \right)^{1/2}, \quad \|v\|_Q = \|v\|_{\mathrm{H}^1_0(\Omega)}.$$

The space $\mathbf{V}$

Theorem

If $(\mathbf{M}, u) \in \mathbf{V} \times Q = \mathbf{H}^{0,-1}(\operatorname{div}\operatorname{Div}; \Omega, \mathbb{S}) \times \mathrm{H}_0^1(\Omega)$ *is the solution to the mixed problem*

$$\begin{aligned} \mathbf{M} \quad & + (\operatorname{div}\operatorname{Div})^* u = 0 \quad && \text{in } \mathbf{V}^* \\ \operatorname{div}\operatorname{Div}\mathbf{M} \quad & \qquad = -f \quad && \text{in } Q^* \end{aligned}$$

then $u \in W = \mathrm{H}_0^2(\Omega)$ *and* u *is the solution to the primal problem*

$$\Delta^2 u = f \quad \text{in } W^*$$

Vice versa, if $u \in W = H_0^2(\Omega)$ *is the solution to the primal problem, then* $\mathbf{M} = -\operatorname{Grad}\operatorname{grad} u \in \mathbf{V}$ *and* $(\mathbf{M}, u)$ *is the solution to the mixed problem.*

Krendl/Rafetseder/Z. (2014,2016)

A regular decomposition of $\mathbf{H}^{0,-1}(\operatorname{div}\operatorname{Div};\Omega,\mathbb{S})$

Theorem

For each $\mathbf{M} \in \mathbf{H}^{0,-1}(\operatorname{div}\operatorname{Div};\Omega,\mathbb{S})$ *there is a unique decomposition*

$$\mathbf{M} = p\,\mathbf{I} + \mathbf{M}_0 \quad \textit{with} \quad \mathbf{I} \quad \textit{identity matrix in } \mathbb{R}^d$$

with $p \in H_0^1(\Omega)$ *and* $\mathbf{M}_0 \in \ker \operatorname{div}\operatorname{Div}$.

The function $p \in H_0^1(\Omega)$ *is the unique solution of the Poisson problem*

$$\langle \operatorname{grad} p, \operatorname{grad} v\rangle = -\langle \operatorname{div}\operatorname{Div}\mathbf{M}, v\rangle \quad \textit{for all } v \in H_0^1(\Omega)$$

Moreover,

$$\underline{c}\,\left(\|p\|^2_{H_0^1(\Omega)} + \|\mathbf{M}_0\|^2_{\mathbf{L}^2(\Omega)}\right) \le \|\mathbf{M}\|^2_{\mathbf{V}} \le \overline{c}\,\left(\|p\|^2_{H_0^1(\Omega)} + \|\mathbf{M}_0\|^2_{\mathbf{L}^2(\Omega)}\right)$$

Decomposition of the biharmonic problem

Using

$$\mathbf{M} = p\,\mathbf{I} + \mathbf{M}_0, \quad \mathbf{N} = q\,\mathbf{I} + \mathbf{N}_0$$

the variational problem

$$\begin{aligned}
\langle \mathbf{M}, \mathbf{N} \rangle \quad &+ \langle \operatorname{div} \operatorname{Div} \mathbf{N}, u \rangle = 0 && \text{for all } \mathbf{N} \in \mathbf{V} \\
\langle \operatorname{div} \operatorname{Div} \mathbf{M}, v \rangle \quad & \qquad = -\langle f, v \rangle && \text{for all } v \in Q
\end{aligned}$$

can be rewritten as follows:

Find $p \in H_0^1(\Omega)$, $\mathbf{M}_0 \in \ker \operatorname{div} \operatorname{Div} \subset \mathbf{L}^2(\Omega, \mathbb{S})$, $u \in H_0^1(\Omega)$ such that

$$\begin{aligned}
d\,\langle p, q \rangle \quad &+ \langle q, \operatorname{tr} \mathbf{M}_0 \rangle - \langle \operatorname{grad} u, \operatorname{grad} q \rangle &= 0 \\
\langle p, \operatorname{tr} \mathbf{N}_0 \rangle \quad &+ \langle \mathbf{M}_0, \mathbf{N}_0 \rangle &= 0 \\
-\langle \operatorname{grad} p, \operatorname{grad} v \rangle \quad & &= -\langle f, v \rangle
\end{aligned}$$

for all $q \in H_0^1(\Omega)$, $\mathbf{N}_0 \in \ker \operatorname{div} \operatorname{Div}$, and $v \in H_0^1(\Omega)$.

Kernel of $\operatorname{div}\operatorname{Div}$ in 2D

Let Ω be topologically simple.

- Let $\mathbf{M} \in \mathbf{L}^2(\Omega; \mathbb{S})$ with $\operatorname{div}\operatorname{Div}\mathbf{M} = 0$. Then

$$\mathbf{M} = \operatorname{sym}\operatorname{Curl} E, \quad E \in H^1(\Omega)^2 \quad \text{with} \quad \operatorname{Curl} E = \begin{bmatrix} \partial_2 E_1 & -\partial_1 E_1 \\ \partial_2 E_2 & -\partial_1 E_2 \end{bmatrix}$$

- Let $E \in L^2(\Omega)^2$ with $\operatorname{sym}\operatorname{Curl} E = 0$. Then

$$E \in RT_0 = \{ax + b: \ a \in \mathbb{R}, \ b \in \mathbb{R}^2\}$$

Beirão da Veiga/Niiranen/Stenberg(2007), Huang/Huang/Xu (2011)

Decomposition of the biharmonic problem in 2D

Then the variational problem

$$\begin{aligned} d\,\langle p, q\rangle \quad & + \langle q, \operatorname{tr}\mathbf{M}_0\rangle - \langle \operatorname{grad} u, \operatorname{grad} q\rangle &= 0 \\ \langle p, \operatorname{tr}\mathbf{N}_0\rangle \quad & + \langle \mathbf{M}_0, \mathbf{N}_0\rangle &= 0 \\ -\langle \operatorname{grad} p, \operatorname{grad} v\rangle \quad & &= -\langle f, v\rangle \end{aligned}$$

can be rewritten as follows:

Find $p \in H_0^1(\Omega)$, $E \in \mathrm{H}^1(\Omega)^2$, and $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} 2\,\langle p, q\rangle \quad & + \langle q, \operatorname{curl} E\rangle \quad - \langle \operatorname{grad} u, \operatorname{grad} q\rangle &= 0 \\ \langle p, \operatorname{curl} F\rangle \quad & + \langle \operatorname{sym}\operatorname{Curl} E, \operatorname{sym}\operatorname{Curl} F\rangle &= 0 \\ \langle \operatorname{grad} p, \operatorname{grad} v\rangle \quad & &= -\langle f, v\rangle \end{aligned}$$

for all $q \in H_0^1(\Omega)$, $F \in \mathrm{H}^1(\Omega)^2$, and $v \in H_0^1(\Omega)$.

Decomposition of biharmonic problems in 2D

Since

$$\operatorname{sym}\operatorname{Curl} E = (\operatorname{div} V)\,\mathbf{I} - \operatorname{sym}\operatorname{Grad} V \quad \text{with} \quad V = \begin{bmatrix} -E_2 \\ E_1 \end{bmatrix},$$

the variational problem can be rewritten as follows:

Find $p \in H_0^1(\Omega)$, $V \in \mathrm{H}^1(\Omega)^2$, and $u \in H_0^1(\Omega)$ such that

$$\begin{aligned}
&2\,\langle p, q\rangle && + \langle q, \operatorname{div} V\rangle && - \langle \operatorname{grad} u, \operatorname{grad} q\rangle &&= 0\\
&\langle p, \operatorname{div} W\rangle && + \langle \operatorname{sym}\operatorname{Grad} V, \operatorname{sym}\operatorname{Grad} W\rangle && &&= 0\\
&-\langle \operatorname{grad} p, \operatorname{grad} v\rangle && && &&= -\langle f, v\rangle
\end{aligned}$$

for all $q \in H_0^1(\Omega)$, $W \in \mathrm{H}^1(\Omega)^2$, and $v \in H_0^1(\Omega)$.

Decomposition of biharmonic problems in 2D

decomposition (in strong form):

1. **Poisson problem** with Dirichlet boundary conditions for p

$$-\Delta p = f \quad \text{in } \Omega, \qquad p = 0 \quad \text{on } \Gamma$$

2. **pure traction problem** with Poisson ratio 0 for V

$$-\operatorname{Div}\left(\operatorname{sym}\operatorname{Grad} V\right) = \operatorname{grad} p \quad \text{in } \Omega, \qquad \left(\operatorname{sym}\operatorname{Grad} V\right) n = 0 \quad \text{on } \Gamma$$

3. **Poisson problem** with Dirichlet boundary conditions for u

$$-\Delta u = 2\,p + \operatorname{div} V \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma$$

Kernel of $\operatorname{div}\operatorname{Div}$ in 3D

Theorem

Let Ω be topologically simple.

- *Let* $\mathbf{M} \in \mathbf{L}^2(\Omega;\mathbb{S})$ *with* $\operatorname{div}\operatorname{Div}\mathbf{M} = 0$. *Then*

  $$\mathbf{M} = \operatorname{sym}\operatorname{Curl}\mathbf{E}$$

  with

  $$\mathbf{E} \in \mathbf{H}^1(\Omega,\mathbb{T}) \quad \textit{or} \quad \mathbf{E} \in \mathbf{H}(\operatorname{sym}\operatorname{Curl};\Omega,\mathbb{T})$$

  and, vice versa, ...

- *Let* $\mathbf{E} \in \mathbf{L}^2(\Omega,\mathbb{T})$ *with* $\operatorname{sym}\operatorname{Curl}\mathbf{E} = 0$. *Then*

  $$\mathbf{E} = \operatorname{dev}\operatorname{Grad} V \quad \textit{with} \quad V \in H^1(\Omega)^3$$

  and, vice versa, ...

Kernel of $\operatorname{div}\operatorname{Div}$ in 3D

Theorem (Cont.)

- *Let* $V \in \mathrm{L}^2(\Omega)^3$ *with* $\operatorname{dev}\operatorname{Grad} V = 0$. *Then*

$$V \in \mathrm{RT}_0 = \{ax + b\colon\ a \in \mathbb{R},\ b \in \mathbb{R}^3\}$$

and, vice versa, ...

- *The potential* $\mathbf{E}$ *is uniquely determined in*

$$\mathbf{H}(\operatorname{sym}\operatorname{Curl}; \Omega, \mathbb{T}) \cap \mathbf{H}_0(\operatorname{Div}; \Omega, \mathbb{T})$$

with

$$\operatorname{Div}\mathbf{E} = 0 \quad \text{in } \Omega.$$

Quenneville-Bélair (2015), Pauly/Z. (2016)

Decomposition of the biharmonic problem in 3D

Then the variational problem

$$\begin{aligned} d\,\langle p, q\rangle \qquad & + \langle q, \operatorname{tr} \mathbf{M}_0\rangle - \langle \operatorname{grad} u, \operatorname{grad} q\rangle &= 0 \\ \langle p, \operatorname{tr} \mathbf{N}_0\rangle \qquad & + \langle \mathbf{M}_0, \mathbf{N}_0\rangle &= 0 \\ -\langle \operatorname{grad} p, \operatorname{grad} v\rangle & &= -\langle f, v\rangle \end{aligned}$$

can be rewritten as follows:

Find $p \in H_0^1(\Omega)$, $\mathbf{E} \in \mathbf{H}(\operatorname{sym}\operatorname{Curl}, \Omega, \mathbb{T})$, and $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} 3\,\langle p, q\rangle \qquad & + \langle q, \operatorname{tr}\operatorname{sym}\operatorname{Curl}\mathbf{E})\rangle \qquad - \langle \operatorname{grad} u, \operatorname{grad} q\rangle &= 0 \\ \langle p, \operatorname{tr}\operatorname{sym}\operatorname{Curl}\mathbf{F}\rangle & + \langle \operatorname{sym}\operatorname{Curl}\mathbf{E}, \operatorname{sym}\operatorname{Curl}\mathbf{F}\rangle &= 0 \\ -\langle \operatorname{grad} p, \operatorname{grad} v\rangle & &= -\langle f, v\rangle \end{aligned}$$

for all $q \in H_0^1(\Omega)$, $\boldsymbol{\Psi} \in \mathbf{H}(\operatorname{sym}\operatorname{Curl}, \Omega, \mathbb{T})$, and $v \in H_0^1(\Omega)$.

The Hellan-Herrmann-Johnson method

$k \in \mathbb{N}$, P_k polynomials of total degree $\leq k$.

approximation space for $\mathbf{M}$

$$\mathbf{V}_h = \{\mathbf{N} \in \mathbf{L}^2(\Omega, \mathbb{S}) \colon \mathbf{N}|_T \in P_{k-1} \text{ for all } T \in \mathcal{T}_h, \text{ and } \mathbf{N}_{nn} \text{ is continuous across all } e \in \mathcal{E}_h\}.$$

approximation space for u

$$Q_h = \mathcal{S}_{h,0} = \mathcal{S}_h \cap H_0^1(\Omega)$$

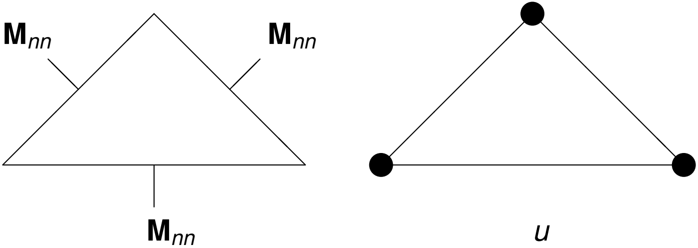
with the standard finite element spaces

$$\mathcal{S}_h = \{v \in C(\overline{\Omega}) \colon v|_T \in P_k \text{ for all } T \in \mathcal{T}_h\}$$

Discretization

Hellan-Herrmann-Johnson element of order $k = 1$

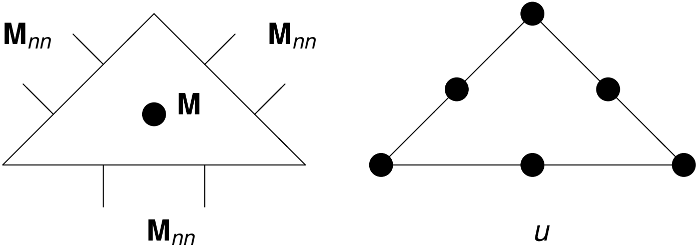
degrees of freedom



Discretization

Hellan-Herrmann-Johnson element of order $k = 2$

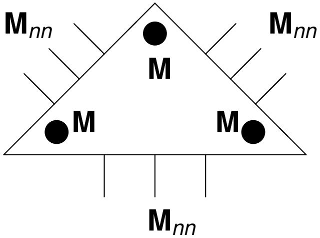
degrees of freedom

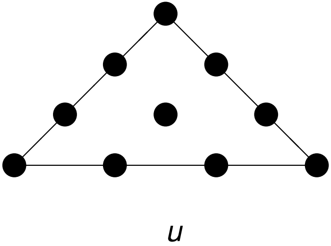


Discretization

Hellan-Herrmann-Johnson element of order $k = 3$

degrees of freedom





The HHJ method

Observe that

$$\mathbf{V}_h \not\subset \mathbf{V} = \mathbf{H}^{0,-1}(\operatorname{div}\operatorname{Div};\Omega,\mathbb{S})$$

Hellan-Herrmann-Johnson (HHJ) method:

Find $\mathbf{M}_h \in \mathbf{V}_h$ and $u_h \in Q_h$ such that

$$\begin{aligned}
\langle \mathbf{M}_h, \mathbf{N}_h\rangle \quad &+ \langle \operatorname{div}\operatorname{Div}_h \mathbf{N}_h, u_h\rangle = 0 && \text{for all } \mathbf{N}_h \in \mathbf{V}_h \\
\langle \operatorname{div}\operatorname{Div}_h \mathbf{M}_h, v_h\rangle \quad & \qquad = -\langle f, v_h\rangle && \text{for all } v_h \in Q_h
\end{aligned}$$

where

$$\langle \operatorname{div}\operatorname{Div}_h \mathbf{N}, v\rangle = \sum_{T\in\mathcal{T}_h} \left\{ \int_T \mathbf{N} : \operatorname{grad}^2 v \, dx - \int_{\partial T} \mathbf{N}_{nn}\, \partial_n v \, ds \right\}$$

The HHJ method

interpolation operator

$$\mathbf{\Pi}_h \colon \boldsymbol{W} \subset \mathbf{V} \longrightarrow \mathbf{V}_h$$

given by the conditions

$$\int_e ((\mathbf{\Pi}_h \mathbf{N})_{nn} - \mathbf{N}_{nn})\, q \, ds = 0, \quad \text{for all } q \in P_{k-1},\ e \in \mathcal{E}_h,$$

$$\int_T (\mathbf{\Pi}_h \mathbf{N} - \mathbf{N})\, q \, dx = 0, \quad \text{for all } q \in P_{k-2},\ T \in \mathcal{T}_h$$

Brezzi/Raviart (1977)

Theorem

For each $\mathbf{M}_h \in \mathbf{V}_h$ *there is a unique decomposition*

$$\mathbf{M}_h = \mathbf{\Pi}_h(p_h \, \mathbf{I}) + \operatorname{sym} \operatorname{Curl} E_h$$

with $p_h \in \mathcal{S}_{h,0}$ *and* $E_h \in (\mathcal{S}_h)^2$.

The HHJ method

With the representation

$$\mathbf{M}_h = \mathbf{\Pi}_h(p_h\,\mathbf{I}) + \operatorname{sym}\operatorname{Curl} E_h \quad \text{and} \quad \mathbf{N}_h = \mathbf{\Pi}_h(q_h\,\mathbf{I}) + \operatorname{sym}\operatorname{Curl}\psi_h$$

the HHJ method reads:

Find $p_h \in \mathcal{S}_{h,0}$, $E_h \in \mathcal{S}_h^2$, and $u_h \in \mathcal{S}_{h,0}$ such that

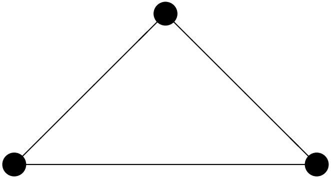
$$\begin{aligned}
&\langle \mathbf{\Pi}_h(p_h\,\mathbf{I}), \mathbf{\Pi}_h(q_h\,\mathbf{I})\rangle + \langle \operatorname{sym}\operatorname{Curl} E_h, \mathbf{\Pi}_h(q_h\,\mathbf{I})\rangle - \langle \operatorname{grad} u_h, \operatorname{grad} q_h\rangle &&= 0\\
&\langle \mathbf{\Pi}_h(p_h\,\mathbf{I}), \operatorname{sym}\operatorname{Curl} F_h\rangle + \langle \operatorname{sym}\operatorname{Curl} E_h, \operatorname{sym}\operatorname{Curl} F_h\rangle &&= 0\\
&-\langle \operatorname{grad} p_h, \operatorname{grad} v_h\rangle &&= -\langle f, v_h\rangle
\end{aligned}$$

for all $q_h \in \mathcal{S}_{h,0}$, $F_h \in \mathcal{S}_h^2$, and $v_h \in \mathcal{S}_{h,0}$.

The HHJ method

Hellan-Herrmann-Johnson element of order $k = 1$

all degrees of freedom for **M** and u are collocated

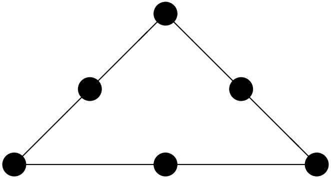


$p, \ E, \ u$

The HHJ method

Hellan-Herrmann-Johnson element of order $k = 2$

all degrees of freedom for $\mathbf{M}$ and u are collocated

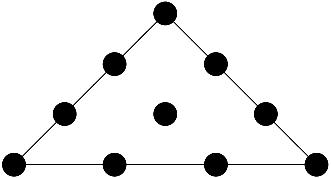


$p, \; E, \; u$

The HHJ method

Hellan-Herrmann-Johnson element of order $k = 3$

all degrees of freedom for $\mathbf{M}$ and u are collocated



$p, \; E, \; u$

A conforming variant of the HHJ method

Removing the interpolation operator $\mathbf{\Pi}_h$ in

$$\begin{aligned} \langle \mathbf{\Pi}_h(p_h \mathbf{I}), \mathbf{\Pi}_h(q_h \mathbf{I})\rangle &+ \langle \operatorname{sym}\operatorname{Curl} E_h, \mathbf{\Pi}_h(q_h \mathbf{I})\rangle &- \langle \operatorname{grad} u_h, \operatorname{grad} q_h\rangle &= 0 \\ \langle \mathbf{\Pi}_h(p_h \mathbf{I}), \operatorname{sym}\operatorname{Curl} F_h\rangle &+ \langle \operatorname{sym}\operatorname{Curl} E_h, \operatorname{sym}\operatorname{Curl} F_h\rangle & &= 0 \\ -\langle \operatorname{grad} p_h, \operatorname{grad} v_h\rangle & & &= -\langle f, v_h\rangle \end{aligned}$$

leads to the following conforming variant:

Find $p_h \in \mathcal{S}_{h,0}$, $E_h \in \mathcal{S}_h^2$, and $u_h \in \mathcal{S}_{h,0}$ such that

$$\begin{aligned} \langle p_h \mathbf{I}, q_h \mathbf{I}\rangle &+ \langle \operatorname{sym}\operatorname{Curl} E_h, q_h \mathbf{I}\rangle &- \langle \operatorname{grad} u_h, \operatorname{grad} q_h\rangle &= 0 \\ \langle p_h \mathbf{I}, \operatorname{sym}\operatorname{Curl} F_h\rangle &+ \langle \operatorname{sym}\operatorname{Curl} E_h, \operatorname{sym}\operatorname{Curl} F_h\rangle & &= 0 \\ -\langle \operatorname{grad} p_h, \operatorname{grad} v_h\rangle & & &= -\langle f, v_h\rangle \end{aligned}$$

for all $q_h \in \mathcal{S}_{h,0}$, $F_h \in \mathcal{S}_h^2$, and $v_h \in \mathcal{S}_{h,0}$.

Conclusion, extension and outlook

- biharmonic problems can be decomposed in three (consecutively to solve) second-order problems

- extension to more general fourth-order problems of the form

$$\operatorname{div}\operatorname{Div}(\mathcal{C}\,\operatorname{Grad}\operatorname{grad} u) - \operatorname{div}(C\,\operatorname{grad} u) + c\,u = f$$

  leads to the construction of optimal preconditioners.

- work in progress: finite element spaces for $\mathbf{H}(\operatorname{sym}\operatorname{Curl};\Omega,\mathbb{T})$