

NON-STANDARD PARTIAL INTEGRATION:
IMPLICATIONS TO MAXWELL AND KORN INEQUALITIES
– OR –
HOW ONE CANNOT APPLY THE CLOSED GRAPH THEOREM!

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Open-Minded :-)

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OVERVIEW

MAXWELL INEQUALITIES

TWO MAXWELL INEQUALITIES

PROOFS

KORN'S FIRST INEQUALITIES

STANDARD HOMOGENEOUS SCALAR BOUNDARY CONDITIONS

NON-STANDARD HOMOGENEOUS TANGENTIAL OR NORMAL BOUNDARY CONDITIONS

REFERENCES

DISTURBING CONSEQUENCES FOR VILLANI'S WORK (FIELDS MEDAL)

CITATIONS

TWO MAXWELL INEQUALITIES

$\Omega \subset \mathbb{R}^3$ bounded, weak Lipschitz (even weaker possible)

$$\Rightarrow \mathring{R}(\Omega) \cap \text{rot } R(\Omega) \hookrightarrow L^2(\Omega) \quad \Leftrightarrow \quad R(\Omega) \cap \text{rot } \mathring{R}(\Omega) \hookrightarrow L^2(\Omega)$$

↓

Maxwell estimates:

$$\exists \mathring{c}_m > 0 \quad \forall E \in \mathring{R}(\Omega) \cap \text{rot } R(\Omega) \quad |E|_{L^2(\Omega)} \leq \mathring{c}_m |\text{rot } E|_{L^2(\Omega)}$$

$$\exists c_m > 0 \quad \forall H \in R(\Omega) \cap \text{rot } \mathring{R}(\Omega) \quad |H|_{L^2(\Omega)} \leq c_m |\text{rot } H|_{L^2(\Omega)}$$

note: best constants

$$\frac{1}{\mathring{c}_m} = \inf_{0 \neq E \in \mathring{R}(\Omega) \cap \text{rot } R(\Omega)} \frac{|\text{rot } E|_{L^2(\Omega)}}{|E|_{L^2(\Omega)}}, \quad \frac{1}{c_m} = \inf_{0 \neq H \in R(\Omega) \cap \text{rot } \mathring{R}(\Omega)} \frac{|\text{rot } H|_{L^2(\Omega)}}{|H|_{L^2(\Omega)}}$$

Theorem

(i) $\mathring{c}_m = c_m$

(ii) Ω convex $\Rightarrow c_m \leq c_p$

PROOF OF MAXWELL INEQUALITIES

step one: two lin., cl., dens. def. op. and their reduced op.

$$\begin{aligned} A : D(A) \subset X &\rightarrow Y, & \mathcal{A} : D(\mathcal{A}) := D(A) \cap R(A^*) &\subset R(A^*) \rightarrow R(A), \\ A^* : D(A^*) \subset Y &\rightarrow X, & \mathcal{A}^* : D(\mathcal{A}^*) := D(A^*) \cap R(A) &\subset R(A) \rightarrow R(A^*) \end{aligned}$$

crucial assumption: $D(\mathcal{A}) \hookrightarrow X \Leftrightarrow D(\mathcal{A}^*) \hookrightarrow Y$

↓

gen. Poincaré estimates:

$$\begin{aligned} \exists c_A > 0 & & \forall x \in D(\mathcal{A}) & & |x| \leq c_A |Ax| \\ \exists c_{A^*} > 0 & & \forall y \in D(\mathcal{A}^*) & & |y| \leq c_{A^*} |A^*y| \end{aligned}$$

note: best constants

$$\frac{1}{c_A} = \inf_{0 \neq x \in D(\mathcal{A})} \frac{|Ax|}{|x|}, \quad \frac{1}{c_{A^*}} = \inf_{0 \neq y \in D(\mathcal{A}^*)} \frac{|A^*y|}{|y|}$$

Theorem

$$c_A = c_{A^*}$$

PROOF OF MAXWELL INEQUALITIES

step two: two lin., cl., den. def. op. and their reduced op.

$$\begin{aligned} A : D(A) \subset X \rightarrow Y, & & \mathcal{A} : D(\mathcal{A}) := D(A) \cap R(A^*) \subset R(A^*) \rightarrow R(A), \\ A^* : D(A^*) \subset Y \rightarrow X, & & \mathcal{A}^* : D(\mathcal{A}^*) := D(A^*) \cap R(A) \subset R(A) \rightarrow R(A^*) \end{aligned}$$

choose

$$\begin{aligned} A := \overset{\circ}{\text{rot}} : \overset{\circ}{\mathring{R}}(\Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega), & & \overset{\circ}{\text{rot}} : \overset{\circ}{\mathring{R}}(\Omega) \cap \text{rot } R(\Omega) \subset \text{rot } R(\Omega) \rightarrow \text{rot } \overset{\circ}{\mathring{R}}(\Omega), \\ \text{rot} = \overset{\circ}{\text{rot}}^* : R(\Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega), & & \text{rot} = \overset{\circ}{\text{rot}}^* : R(\Omega) \cap \text{rot } \overset{\circ}{\mathring{R}}(\Omega) \subset \text{rot } \overset{\circ}{\mathring{R}}(\Omega) \rightarrow \text{rot } R(\Omega) \end{aligned}$$

$$\text{crucial assumption: } \overset{\circ}{\mathring{R}}(\Omega) \cap \text{rot } R(\Omega) \hookrightarrow L^2(\Omega) \quad (\Leftrightarrow \quad R(\Omega) \cap \text{rot } \overset{\circ}{\mathring{R}}(\Omega) \hookrightarrow L^2(\Omega))$$

↓

gen. Poincaré estimates (Maxwell estimates):

$$\begin{aligned} \exists \overset{\circ}{c}_m > 0 & \quad \forall E \in \overset{\circ}{\mathring{R}}(\Omega) \cap \text{rot } R(\Omega) & \quad |E|_{L^2(\Omega)} \leq \overset{\circ}{c}_m |\text{rot } E|_{L^2(\Omega)} \\ \exists c_m > 0 & \quad \forall H \in R(\Omega) \cap \text{rot } \overset{\circ}{\mathring{R}}(\Omega) & \quad |H|_{L^2(\Omega)} \leq c_m |\text{rot } H|_{L^2(\Omega)} \end{aligned}$$

Theorem

$$\overset{\circ}{c}_m = c_m$$

PROOF OF MAXWELL INEQUALITIES

step three:

Proposition (integration by parts (Grisvard's book and older...))

Let $\Omega \subset \mathbb{R}^3$ be piecewise C^2 . Then for all $E \in C^\infty(\bar{\Omega})$

$$\begin{aligned} & |\operatorname{div} E|_{L^2(\Omega)}^2 + |\operatorname{rot} E|_{L^2(\Omega)}^2 - |\nabla E|_{L^2(\Omega)}^2 \\ &= \int_{\Gamma_1} \underbrace{(\operatorname{div} \nu |E_n|^2 + ((\nabla \nu) E_t) \cdot E_t)}_{\text{curvature, sign!}} + \int_{\Gamma_1} \underbrace{(E_n \operatorname{div}_\Gamma E_t - E_t \cdot \nabla_\Gamma E_n)}_{\text{boundary conditions, no sign!}}. \end{aligned}$$

approx. convex Ω from inside by convex and smooth $(\Omega_n) \Rightarrow$

Corollary (Gaffney's inequality)

Let $\Omega \subset \mathbb{R}^3$ be convex and $E \in \mathring{R}(\Omega) \cap D(\Omega)$ or $E \in R(\Omega) \cap \mathring{D}(\Omega)$. Then $E \in H^1(\Omega)$ and

$$|\operatorname{rot} E|_{L^2(\Omega)}^2 + |\operatorname{div} E|_{L^2(\Omega)}^2 - |\nabla E|_{L^2(\Omega)}^2 \geq 0.$$

PROOF OF MAXWELL INEQUALITIES

step four:

$$\text{(Poincaré)} \quad \exists c_p > 0 \quad \forall u \in H^1(\Omega) \cap \mathbb{R}^\perp \quad |u|_{L^2(\Omega)} \leq c_p |\nabla u|_{L^2(\Omega)}$$

Let Ω be convex and $E \in R(\Omega) \cap \mathring{D}_0(\Omega)$.

Note $\mathring{D}_0(\Omega) = \text{rot } \mathring{R}(\Omega)$.

Cor. (Gaffney) $\Rightarrow E \in H^1(\Omega)$ and $E = \text{rot } H$ with $H \in \mathring{R}(\Omega)$.

$$\Rightarrow E \in H^1(\Omega) \cap (\mathbb{R}^3)^\perp \cap \mathring{D}_0(\Omega)$$

↓

$$|E|_{L^2(\Omega)} \leq c_p |\nabla E|_{L^2(\Omega)} \leq c_p |\text{rot } E|_{L^2(\Omega)}$$

↓

$$c_m \leq c_p$$

□

MATRICES

Let $A \in \mathbb{R}^{N \times N}$.

$$\begin{matrix} \text{sym} \\ \text{skw} \end{matrix} A := \frac{1}{2}(A \pm A^T), \quad \text{id}_A := \frac{\text{tr } A}{N} \text{id}, \quad \text{tr } A := A \cdot \text{id}, \quad \text{dev } A := A - \text{id}_A$$

(pointwise orthogonality) \Rightarrow

$$|A|^2 = |\text{dev } A|^2 + \frac{1}{N} |\text{tr } A|^2, \quad |A|^2 = |\text{sym } A|^2 + |\text{skw } A|^2, \quad |\text{sym } A|^2 = |\text{dev sym } A|^2 + \frac{1}{N} |\text{tr } A|^2$$

$$\Rightarrow |\text{dev } A|, N^{-1/2} |\text{tr } A|, |\text{sym } A|, |\text{skw } A| \leq |A|$$

$\Omega \subset \mathbb{R}^N$ and $A := \nabla v := J_v^T$ for $v \in H^1(\Omega)$ \Rightarrow (pointwise)

$$\begin{aligned} |\text{skw } \nabla v|^2 &= \frac{1}{2} |\text{rot } v|^2, \quad \text{tr } \nabla v = \text{div } v, \\ |\nabla v|^2 &= |\text{dev sym } \nabla v|^2 + \frac{1}{N} |\text{div } v|^2 + \frac{1}{2} |\text{rot } v|^2 \end{aligned} \quad (1)$$

Moreover

$$|\nabla v|^2 = |\text{rot } v|^2 + \langle \nabla v, (\nabla v)^T \rangle$$

$$\text{since } 2|\text{skw } \nabla v|^2 = \frac{1}{2} |\nabla v - (\nabla v)^T|^2 = |\nabla v|^2 - \langle \nabla v, (\nabla v)^T \rangle.$$

KORN'S FIRST INEQUALITY: STANDARD BOUNDARY CONDITIONS

Lemma (Korn's first inequality: \mathring{H}^1 -version)

Let Ω be an open subset of \mathbb{R}^N with $2 \leq N \in \mathbb{N}$. Then for all $v \in \mathring{H}^1(\Omega)$

$$|\nabla v|_{L^2(\Omega)}^2 = 2|\operatorname{dev sym} \nabla v|_{L^2(\Omega)}^2 + \frac{2-N}{N}|\operatorname{div} v|_{L^2(\Omega)}^2 \leq 2|\operatorname{dev sym} \nabla v|_{L^2(\Omega)}^2$$

and equality holds if and only if $\operatorname{div} v = 0$ or $N = 2$.

Proof.

note: $-\Delta = \operatorname{rot}^* \operatorname{rot} - \nabla \operatorname{div}$ (vector Laplacian)

$$\Rightarrow \forall v \in \mathring{C}^\infty(\Omega) \quad |\nabla v|_{L^2(\Omega)}^2 = |\operatorname{rot} v|_{L^2(\Omega)}^2 + |\operatorname{div} v|_{L^2(\Omega)}^2 \quad (\text{Gaffney's equality}) \quad (2)$$

(2) extends to all $v \in \mathring{H}^1(\Omega)$ by continuity. Then

$$|\nabla v|_{L^2(\Omega)}^2 = |\operatorname{dev sym} \nabla v|_{L^2(\Omega)}^2 + \frac{1}{2}|\nabla v|_{L^2(\Omega)}^2 + \frac{2-N}{2N}|\operatorname{div} v|_{L^2(\Omega)}^2$$

follows by (1), i.e., $|\nabla v|^2 = |\operatorname{dev sym} \nabla v|^2 + \frac{1}{N}|\operatorname{div} v|^2 + \frac{1}{2}|\operatorname{rot} v|^2$, and (2). \square

KORN'S FIRST INEQUALITY: TANGENTIAL/NORMAL BOUNDARY CONDITIONS

main result:

Theorem (Korn's first inequality: tangential/normal version)

Let $\Omega \subset \mathbb{R}^N$ be piecewise C^2 -concave and $v \in \overset{\circ}{H}_{t,n}^1(\Omega)$. Then Korn's first inequality

$$|\nabla v|_{L^2(\Omega)} \leq \sqrt{2} |\operatorname{dev sym} \nabla v|_{L^2(\Omega)}$$

holds. If Ω is a polyhedron, even

$$|\nabla v|_{L^2(\Omega)}^2 = 2 |\operatorname{dev sym} \nabla v|_{L^2(\Omega)}^2 + \frac{2-N}{N} |\operatorname{div} v|_{L^2(\Omega)}^2 \leq 2 |\operatorname{dev sym} \nabla v|_{L^2(\Omega)}^2$$

is true and equality holds if and only if $\operatorname{div} v = 0$ or $N = 2$.

KORN'S FIRST INEQUALITY: TANGENTIAL/NORMAL BOUNDARY CONDITIONS

tools:

Proposition (integration by parts (Grisvard's book and older...))

Let $\Omega \subset \mathbb{R}^N$ be piecewise C^2 . Then

$$\begin{aligned} |\operatorname{div} v|_{L^2(\Omega)}^2 + |\operatorname{rot} v|_{L^2(\Omega)}^2 - |\nabla v|_{L^2(\Omega)}^2 &= \int_{\Gamma_1} (\operatorname{div} \nu |v_n|^2 + ((\nabla \nu) v_t) \cdot v_t) \\ &\quad + \int_{\Gamma_1} (v_n \operatorname{div}_\Gamma v_t - v_t \cdot \nabla_\Gamma v_n), \\ |\operatorname{div} v|_{L^2(\Omega)}^2 + |\operatorname{rot} v|_{L^2(\Omega)}^2 - |\nabla v|_{L^2(\Omega)}^2 &= \int_{\Gamma_1} (\operatorname{div} \nu |v_n|^2 + ((\nabla \nu) v_t) \cdot v_t). \end{aligned}$$

holds for all $v \in \mathring{C}^\infty(\bar{\Omega})$ resp. $v \in \mathring{C}_{t,n}^\infty(\Omega)$.

Corollary (Gaffney's inequalities)

Let $\Omega \subset \mathbb{R}^N$ be piecewise C^2 and $v \in \mathring{H}_{t,n}^1(\Omega)$. Then

$$|\operatorname{rot} v|_{L^2(\Omega)}^2 + |\operatorname{div} v|_{L^2(\Omega)}^2 - |\nabla v|_{L^2(\Omega)}^2 \begin{cases} \leq 0 & , \text{ if } \Omega \text{ is piecewise } C^2\text{-concave,} \\ = 0 & , \text{ if } \Omega \text{ is a polyhedron,} \\ \geq 0 & , \text{ if } \Omega \text{ is piecewise } C^2\text{-convex.} \end{cases}$$

KORN'S FIRST INEQUALITY: TANGENTIAL/NORMAL BOUNDARY CONDITIONS

Proof.

(1), i.e., $|\nabla v|^2 = |\operatorname{dev sym} \nabla v|^2 + \frac{1}{N} |\operatorname{div} v|^2 + \frac{1}{2} |\operatorname{rot} v|^2$, and the corollary \Rightarrow

$$|\nabla v|_{L^2(\Omega)}^2 \leq |\operatorname{dev sym} \nabla v|_{L^2(\Omega)}^2 + \frac{1}{2} |\nabla v|_{L^2(\Omega)}^2 + \frac{2-N}{2N} |\operatorname{div} v|_{L^2(\Omega)}^2,$$

\Rightarrow first estimate

Ω polyhedron \Rightarrow equality holds

□

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On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation

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*On a variant of Korn's inequality arising in statistical mechanics.
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 - page 607
 - page 608
 - page 609
 - Proposition 5
 - (end of) Theorem 3 (continued)
 - page 609 (closed graph theorem)

- ▶ Desvillettes, L. and Villani, C.: Invent. Math., (2005)
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 - page 306