Low-Frequency Asymptotics for Time-Harmonic Maxwell Equations in Exterior Domains and Comparison to Eddy-Current Approximations

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POems, CNRS, Palaiseau, 2018
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March 22, 2018

OVERVIEW

CLASSICAL TIME-HARMONIC SCATTERING MAXWELL PROBLEM

time-harmonic Maxwell (electro-magnetic scattering) problem
in $\Omega \subset \mathbb{R}^3$ exterior domain

$$
\begin{aligned}
\sigma E_\omega - \operatorname{rot} H_\omega + \mathrm{i}\,\omega\varepsilon E_\omega &= F && \text{in } \Omega && \text{(pde)}\\
\mathring{\operatorname{rot}} E_\omega + \mathrm{i}\,\omega\mu H_\omega &= G && \text{in } \Omega && \text{(pde)}\\
\nu \times E_\omega &= 0 \quad (=\lambda) && \text{on } \partial\Omega && \text{(boundary cond.)}\\
E_\omega\,,\ H_\omega &= O(r^{-1}) && \text{for } r \to \infty && \text{(decay cond.)}\\
\xi \times E_\omega + H_\omega\,, \quad -\xi \times H_\omega + E_\omega &= o(r^{-1}) && \text{for } r \to \infty && \text{(Silver-Müller radiation cond.)}
\end{aligned}
$$

here: $0 \neq \omega \in \mathbb{C}\,,\quad r(x) = |x|\,,\quad \xi(x) := x/|x|$

inhom. aniso. media $\varepsilon, \mu \in \mathsf{L}^\infty(\Omega, \mathbb{R}^{3\times 3})$, sym, unif. pos. def., supp σ compact

for simplicity (in the beginning) $\sigma = 0$

QUESTION / AIM: low frequency asymptotics?

$$\boxed{\lim_{\omega \to 0} E_\omega\,, \qquad \lim_{\omega \to 0} H_\omega \quad ?}$$

CLASSICAL TIME-HARMONIC SCATTERING MAXWELL PROBLEM

analytical motivation:

- Weck, N. and Witsch, K.-J.: CPDE, (1992)
  Complete low frequency Analysis for the reduced wave Equation with variable coefficients in three dimensions
- Weck, N. and Witsch, K.-J.: M2AS, (1997)
  Generalized linear elasticity in exterior domains – I: radiation problems
- Weck, N. and Witsch, K.-J.: M2AS, (1997)
  Generalized linear elasticity in exterior domains – II: low-frequency asymptotics

analytical/numerical motivation:

- Ammari, H. and Buffa, A. and Nédélec, J.-C.: SIAM JAM, (2000)
  A justification of eddy currents model for the Maxwell equations
  (! cited 64 times in MathSciNet / unfortunately wrong !)
- Ammari, H. and Nédélec, J.-C.: SIAM JMA, (2000)
  Low-frequency electromagnetic scattering (sol. theo. by fundamental solution, asymptotic expansion simply by Taylor series of the fundamental solution, non-local bc, not very satisfying)

disadvantages of Ammari/Nédélec-papers

- no identification of terms in the expansion by proper boundary value problems
- estimates just in local L^2-norms
- non local boundary conditions due to EtM-operators (DtN-operators)
- comp. supp. F, G; $\varepsilon = \mu = 1$

CLASSICAL TIME-HARMONIC SCATTERING MAXWELL PROBLEM

more compact and proper notation

$$(M-\omega)u_\omega = f \in \mathsf{L}^2_{>1/2}(\Omega)\times\mathsf{L}^2_{>1/2}(\Omega)$$
$$u_\omega \in \overset{\circ}{\mathbf{H}}_{<-1/2}(\mathrm{rot};\Omega)\times\mathbf{H}_{<-1/2}(\mathrm{rot};\Omega)$$
$$(S+1)u_\omega \in \mathsf{L}^2_{>-1/2}(\Omega)\times\mathsf{L}^2_{>-1/2}(\Omega)$$

here: $u_\omega := (E_\omega, H_\omega)$, $\quad f := \mathrm{i}\,\Lambda^{-1}(F,G)$, $\quad \Lambda = \begin{bmatrix}\varepsilon & 0\\ 0 & \mu\end{bmatrix}$, $\quad \Lambda^{-1} = \begin{bmatrix}\varepsilon^{-1} & 0\\ 0 & \mu^{-1}\end{bmatrix}$,
$M = \mathrm{i}\,\Lambda^{-1}\,\mathrm{Rot}$, $\quad \mathrm{Rot} := \begin{bmatrix}0 & -\mathrm{rot}\\ \overset{\circ}{\mathrm{rot}} & 0\end{bmatrix}$, $\quad S = C_{\mathrm{Rot},r} = \begin{bmatrix}0 & -\xi\times\\ \xi\times & 0\end{bmatrix}$

$M : \overset{\circ}{\mathbf{H}}(\mathrm{rot};\Omega)\times\mathbf{H}(\mathrm{rot};\Omega)\subset\mathsf{L}^2(\Omega)\times\mathsf{L}^2(\Omega)\to\mathsf{L}^2(\Omega)\times\mathsf{L}^2(\Omega)$ s.a. unbd. lin. op.

$\Rightarrow$ unique L^2-solutions u_ω for $\omega\in\mathbb{C}\setminus\mathbb{R}$

later: gen. Fredholm alternative for $\omega\in\mathbb{R}\setminus\{0\}$
(Eidus' principle of limiting absorption (1962), a priori estimates)

QUESTION: low frequency asymptotics? $\boxed{\lim_{\mathbb{C}\setminus\{0\}\ni\omega\to 0} u_\omega}$

METHOD: Weck & Witsch, i.e., full ext. dom. and no artificial boundary

GENERALIZED TIME-HARMONIC SCATTERING MAXWELL PROBLEM

gen. time-harmonic Maxwell (electro-magnetic scattering) problem
in $\Omega \subset \mathbb{R}^N$ exterior domain, $0 \neq \omega \in \mathbb{C}$

$$\delta H_\omega + \mathrm{i}\,\omega\varepsilon E_\omega = F \qquad \text{in } \Omega \qquad \text{(pde)}$$

$$\mathring{\mathrm{d}} E_\omega + \mathrm{i}\,\omega\mu H_\omega = G \qquad \text{in } \Omega \qquad \text{(pde)}$$

$$\iota^* E_\omega = 0 \quad (= \lambda) \qquad \text{on } \partial\Omega \qquad \text{(bc)}$$

$$E_\omega\,,\ H_\omega = O(r^{-1}) \qquad \text{for } r \to \infty \qquad \text{(dc)}$$

$$\mathrm{d}\ r \wedge E_\omega + H_\omega\,, \quad (-1)^{qN} * \mathrm{d}\ r \wedge * H_\omega + E_\omega = o(r^{-1}) \qquad \text{for } r \to \infty \qquad \text{(gen. Silver-Müller rc)}$$

here: E, F q-forms, H, G $(q+1)$-forms
inhom. aniso. media ε, μ (linear transformations) sym, unif. pos. def.

QUESTION / AIM: low frequency asymptotics?

$$\boxed{\lim_{\omega \to 0} E_\omega\,, \quad \lim_{\omega \to 0} H_\omega \quad ?}$$

GENERALIZED TIME-HARMONIC SCATTERING MAXWELL PROBLEM

time-harmonic Maxwell problem in $\Omega \subset \mathbb{R}^N$ exterior domain
for simplicity $N \geq 3$ odd, frequencies from upper half plane $\omega \in \mathbb{C}_+$

$$(M-\omega)u_\omega = f \in \mathsf{L}^{2,q,q+1}_{>1/2}(\Omega)$$

$$u_\omega \in \mathring{\mathbf{D}}^q_{<-\frac{1}{2}}(\Omega) \times \mathbf{\Delta}^{q+1}_{<-\frac{1}{2}}(\Omega)$$

$$(S+1)u_\omega \in \mathsf{L}^{2,q,q+1}_{>-1/2}(\Omega)$$

here: $u_\omega := (E_\omega, H_\omega)$, $f := \mathrm{i}\,\Lambda^{-1}(F,G)$, E, F q-forms, H, G $(q+1)$-forms,
$M = \mathrm{i}\,\Lambda^{-1}\begin{bmatrix}0 & \delta \\ \mathring{\mathrm{d}} & 0\end{bmatrix}$, $\Lambda = \begin{bmatrix}\varepsilon & 0 \\ 0 & \mu\end{bmatrix}$, $S = \begin{bmatrix}0 & T \\ R & 0\end{bmatrix}$, $R := \mathrm{d}\, r\wedge$, $T := \pm * R *$
d ext. deriv., $\delta = \pm * \mathrm{d} *$ co-deriv., $R = C_{\mathrm{d},r}$, $T = C_{\delta,r}$

$M : \mathring{\mathbf{D}}^q(\Omega) \times \mathbf{\Delta}^{q+1}(\Omega) \subset \mathsf{L}^{2,q,q+1}(\Omega) \to \mathsf{L}^{2,q,q+1}(\Omega)$ s.a. unbd. lin. op.

denote sol. op. of time-harmonic prob. by $\mathcal{L}_\omega := (M-\omega)^{-1}$ $(u_\omega = \mathcal{L}_\omega f)$

QUESTION: low frequency asymptotics?

$$\boxed{\lim_{\mathbb{C}_+\setminus\{0\}\ni\omega\to 0} \mathcal{L}_\omega = ?}$$

(topology: operator norm of polyn. weighted Sobolev spaces)

BOUNDED DOMAIN

time-harmonic Maxwell problem in $\Omega \subset \mathbb{R}^N$ bounded Lipschitz domain

$$(M-\omega)u_\omega = f \in \mathsf{L}^{2,q,q+1}(\Omega)$$

$$u_\omega \in \mathring{\mathbf{D}}^q(\Omega) \times \mathbf{\Delta}^{q+1}(\Omega) =: D(M)$$

Helmholtz deco. $\Rightarrow$ $\mathsf{L}^{2,q,q+1}(\Omega) = N(M) \oplus_\Lambda \overline{R(M)}$

$$M : D(M) \subset \mathsf{L}^{2,q,q+1}(\Omega) \to \mathsf{L}^{2,q,q+1}(\Omega) \quad \text{s.a.},$$

$$\mathcal{M} : D(\mathcal{M}) := D(M) \cap \overline{R(M)} \subset \overline{R(M)} \to \overline{R(M)} \quad \text{s.a. (red. op.)}$$

Weck's sel. theo./Maxwell compactness prop., i.e., $D(\mathcal{M}) \hookrightarrow \mathsf{L}^{2,q,q+1}(\Omega)$ comp.

$\Rightarrow$ Maxwell estimate, i.e., $\exists\, c_m > 0 \quad \forall\, u \in D(\mathcal{M}) \quad \|u\|_{\mathsf{L}^{2,q}(\Omega)} \leq c_m \|\mathcal{M}u\|_{\mathsf{L}^{2,q}(\Omega)}$

$\Leftrightarrow$ $R(M) = R(\mathcal{M})$ closed $\Leftrightarrow$ $\mathcal{L}_0 := \mathcal{M}^{-1} : R(M) \to D(\mathcal{M})$ cont.

$\Rightarrow$ $\mathcal{L}_0 : R(M) \to R(M)$ comp. (static sol. op. cont./comp.)

standard sol. theory $\Rightarrow$ Fredholm's alternative, especially

$$\sigma_p(\mathcal{M}) = \sigma(\mathcal{M}) = \sigma(M) \setminus \{0\} = \sigma_p(M) \setminus \{0\} = \{\pm\omega_n\}_{n=1}^\infty \subset \mathbb{R} \setminus \{0\}$$

with $(\omega_n) \subset (0, +\infty)$ strictly monotone increasing with $\omega_n \nearrow +\infty$

$\Rightarrow$ sol. op. time-harmonic prob. ($f \mapsto u_\omega = \mathcal{L}_\omega f$) well def. for $0 < |\omega|$ small

$$\mathcal{L}_\omega : \mathsf{L}^{2,q,q+1}(\Omega) \to D(M), \quad \mathcal{L}_\omega : R(M) \to D(\mathcal{M})$$

BOUNDED DOMAIN

time-harmonic Maxwell problem in $\Omega \subset \mathbb{R}^N$ bounded Lipschitz domain

$$(M - \omega)u_\omega = f \in \mathsf{L}^{2,q,q+1}(\Omega)$$
$$u_\omega \in D(M)$$

Helmholtz deco. $\Rightarrow$ $\mathsf{L}^{2,q,q+1}(\Omega) = N(M) \oplus_\Lambda R(M)$ and $D(M) = N(M) \oplus_\Lambda D(\mathcal{M})$

orth.-norm.-projectors $\Pi : \mathsf{L}^{2,q,q+1}(\Omega) \to N(M)$, $1 - \Pi : \mathsf{L}^{2,q,q+1}(\Omega) \to R(M)$

$$\Rightarrow \quad -\omega\Pi u_\omega = \Pi f \qquad \text{and} \qquad (M-\omega)(1-\Pi)u_\omega = (1-\Pi)f \in R(M)$$
$$\Pi u_\omega \in N(M) \qquad\qquad (1-\Pi)u_\omega \in D(\mathcal{M})$$

note: $D(\mathcal{M}) = D(M) \cap R(M) = \big(\mathring{\mathbf{D}}^q(\Omega) \cap \varepsilon^{-1}\,\delta\,\mathbf{\Delta}^{q+1}(\Omega)\big) \times \big(\mathbf{\Delta}^{q+1}(\Omega) \cap \mu^{-1}\,\mathrm{d}\,\mathring{\mathbf{D}}^q(\Omega)\big)$

set $v := (1-\Pi)u_\omega \in D(\mathcal{M}) \subset R(M)$ and $g := (1-\Pi)f \in R(M)$ $\quad\Rightarrow\quad \mathcal{L}_0\, Mv = v$

$\Rightarrow \quad (M-\omega)v = g \quad \Leftrightarrow \quad (1 - \omega\,\mathcal{L}_0)v = \mathcal{L}_0\, g$

$$\overset{\text{Neumann ser.}}{\Leftrightarrow} \qquad v = (1-\omega\,\mathcal{L}_0)^{-1}\,\mathcal{L}_0\, g = \sum_{j=0}^{\infty} \omega^j\,\mathcal{L}_0{}^j\,\mathcal{L}_0\, g$$

for small $0 < |\omega|$ since $\|\omega\,\mathcal{L}_0\| < 1 \quad \Leftrightarrow \quad |\omega| < 1/\|\mathcal{L}_0\|$ (1st pos. Maxwell ev)

$$\Rightarrow \quad \mathcal{L}_\omega f = u_\omega = \Pi u_\omega + v = -\omega^{-1}\Pi f + \sum_{j=0}^{\infty} \omega^j\,\mathcal{L}_0{}^{j+1}(1-\Pi)f$$

BOUNDED DOMAIN

$\Rightarrow$ low frequency asymptotics in L^2-operator norm

$$\mathcal{L}_\omega = \underbrace{-\omega^{-1}\Pi}_{\text{trivial part}} + \underbrace{\sum_{j=0}^{\infty} \omega^j \, \mathcal{L}_0^{j+1} \, \Pi_{\text{reg}}}_{\text{Neumann series}}, \quad \omega \in \mathbb{C}_+ \setminus \{0\} \text{ small}$$

$\Pi : \mathsf{L}^{2,q,q+1}(\Omega) \to N(M), \quad \Pi_{\text{reg}} := 1 - \Pi : \mathsf{L}^{2,q,q+1}(\Omega) \to R(M)$

$\mathcal{L}_0 : R(M) \to D(M) \cap R(M)$

problems if Ω exterior domain

- this low frequency asymptotic is wrong, even not well defined
- static solution theory needs weighted Poincare estimate!
  $\Rightarrow$ leaving L^2-setting
  e.g., static sol. op. maps unweighted data f to $(1+r)^{-1}$-weighted sol. u_0
- not clear how to define higher powers of $\mathcal{L}_0$?
- careful investigation of static sol. theo. in weighted Sobolev spaces

EXTERIOR DOMAIN

aim: give meaning to Neumann sum in terms of an asymptotic expansion

$$\mathcal{L}_\omega + \omega^{-1}\Pi - \sum_{j=0}^{J-1} \omega^j \, {\mathcal{L}_0}^{j+1} \, \Pi_{\text{reg}} = O\big(|\omega|^J\big) \quad , \quad J \in \mathbb{N}_0, \quad \omega \in \mathbb{C}_+ \setminus \{0\} \text{ small}$$

3 major complications

- growing $J \Rightarrow$ stronger data norms for f and weaker solution norms for $u_\omega = \mathcal{L}_\omega f$
- Π , Π_{reg} indicate need for polyn. weighted Hodge-Helmholtz deco. of

  $$\mathsf{L}^{2,q,q+1}_s(\Omega) = \big(\operatorname{Tri}^q_s(\Omega) \dotplus \operatorname{Reg}^{q,-1}_s(\Omega)\big) \cap \mathsf{L}^{2,q,q+1}_s(\Omega)$$

  respecting inhomogeneities Λ (topological direct decomposition)

  $$(N(M) =) \operatorname{Tri}^q_s(\Omega) = \Pi \mathsf{L}^{2,q,q+1}_s(\Omega) \subset {}_0\mathring{\mathsf{D}}^q_t(\Omega) \times {}_0\Delta^{q+1}_t(\Omega)$$
  $$\operatorname{Reg}^{q,-1}_s(\Omega) = \Pi_{\text{reg}} \mathsf{L}^{2,q,q+1}_s(\Omega) \subset \Lambda^{-1}\big({}_0\Delta^q_t(\Omega) \times {}_0\mathring{\mathsf{D}}^{q+1}_t(\Omega)\big)$$

  only subspaces of $\mathsf{L}^{2,q,q+1}_t(\Omega)$ with $t \leq s$ and $t < N/2$
  not of $\mathsf{L}^{2,q,q+1}_s(\Omega)$ if $s \geq N/2$
- expansion has to be corrected by special, explicitly computable degenerate op.

EXTERIOR DOMAIN

more precisely: $J \in \mathbb{N}_0$ and $s, -t > 1/2$ as well as $f \in \mathsf{L}_s^{2,q,q+1}(\Omega)$

$\Rightarrow$ main result: asymptotic estimates

$$\Big\| \mathcal{L}_\omega f + \omega^{-1}\Pi f - \sum_{j=0}^{J-1} \omega^j \mathcal{L}_0{}^{j+1} \Pi_{\mathrm{reg}} f - \omega^{N-1} \sum_{j=0}^{J-N} \omega^j \Gamma_j f \Big\|_{\mathsf{L}_t^{2,q,q+1}(\Omega)} = O\big(|\omega|^J\big) \, \|f\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)}$$

O-symbol always for $\omega \to 0$ and uniformly w.r.t. ω and f

with $\omega \in \mathbb{C}_+ \setminus \{0\}$ and $|\omega| \leq \hat{\omega}$, where $\hat{\omega} > 0$

GENERAL ASSUMPTIONS

- $\Omega \subset \mathbb{R}^N$ exterior domain with Lipschitz boundary
  (Maxwell local compactness property,
  exist. of special forms with bounded supports repl. Dirichlet/Neumann forms)
- $1 \leq q \leq N-2$ and odd space dimensions N (class. $N=3$, $q=1$)
  (even dim., especially $N=2$, OK
  but logarithmic terms due to expansions of Hankel's functions)
- fix radius $r_0 > 0$ with $\mathbb{R}^N \setminus \Omega \subset B_{r_0}$, cut-off function η
- $\varepsilon = \mathrm{Id} + \hat{\varepsilon}$, $\mu = \mathrm{Id} + \hat{\mu}$ ($\Lambda = \mathrm{Id} + \hat{\Lambda}$) τ-C^1-admissible, i.e.,
  linear, real, sym., unif. pos. def. L^∞-transformations with $\hat{\Lambda} \in \mathrm{C}^1$ for $|x| > r_0$
  asymptotically homogeneous, i.e.,
  $\partial^\alpha \hat{\Lambda} = O(r^{-\tau-|\alpha|})$ for all $|\alpha| \leq 1$ with order of decay τ at infinity,
  $\tau > 0$ depending on t, s

DESCRIPTION OF RESULTS

- degenerate correction operators Γ_j by recursion consisting of

$$E^{+}_{\sigma,m},\ H^{+}_{\sigma,n},\quad E^{+,k}_{\sigma,m} =: \mathcal{L}^k_0(E^{+}_{\sigma,m},0),\ H^{+,k}_{\sigma,n} =: \mathcal{L}^k_0(0,H^{+}_{\sigma,n}) \in \mathsf{L}^{2,q,q+1}_{-N/2-\sigma-k}(\Omega)$$

  sol. of hom. static boundary value problems with inhom. at infinity, e.g.,

$$E^{+}_{\sigma,m} \in {}_0\mathring{\mathsf{D}}^q_{\mathrm{loc}}(\Omega) \cap \varepsilon^{-1}\big({}_0\Delta^q_{\mathrm{loc}}(\Omega) \cap \mathring{\mathsf{B}}^q(\Omega)^{\perp}\big)$$

$$E^{+}_{\sigma,m} - {}^{+}\Delta^{q,0}_{\sigma,m} \in \mathsf{L}^{2,q}_{>-\frac{N}{2}}(\Omega)$$

  'harmonic polynomials' ${}^{+}\Delta^{q,k}_{\sigma,m}$ behave like $r^{k+\sigma}$ at infinity ($k, \sigma \geq 0$)

- 'trivial' subspace $\mathsf{Tri}^q_s(\Omega) = \Pi\mathsf{L}^{2,q,q+1}_s(\Omega) \subset {}_0\mathring{\mathsf{D}}^q_t(\Omega) \times {}_0\Delta^{q+1}_t(\Omega)\ \big(\subset N(M)\big)$

$$\mathcal{L}_\omega f = -\omega^{-1} f, \quad f \in \mathsf{Tri}^q_s(\Omega)$$

- two kinds of media $\Lambda = \mathrm{Id} + \hat{\Lambda}$
  1. $\hat{\Lambda}$ comp. supp., results for any J
  2. $\hat{\Lambda}$ 'decays' with $\tau > 0$ at infinity, results for $J \leq \hat{J}$ dep. on τ

DESCRIPTION OF RESULTS

- identify closed subspaces $\mathrm{Reg}_s^{q,J}(\Omega)$ of $\mathrm{Reg}_s^{q,0}(\Omega) \subset \mathrm{L}_s^{2,q,q+1}(\Omega)$, 'spaces of regular convergence', $\Rightarrow$ 'usual' Neumann expansion

$$\Big\| \mathcal{L}_\omega f - \sum_{j=0}^{J-1} \omega^j \mathcal{L}_0{}^{j+1} f \Big\|_{\mathrm{L}_t^{2,q,q+1}(\Omega)} = O\big(|\omega|^J\big) \|f\|_{\mathrm{L}_s^{2,q,q+1}(\Omega)}$$

for $f \in \mathrm{Reg}_s^{q,J}(\Omega)$

- charact. of $\mathrm{Reg}_s^{q,J}(\Omega)$ by orthogonality in L^2 to the spec. grow. st. sol. $E_{\sigma,m}^{+,k}$, $H_{\sigma,n}^{+,k}$
- corrected Neumann expansion

$$\Big\| \mathcal{L}_\omega f - \sum_{j=0}^{J-1} \omega^j \mathcal{L}_0{}^{j+1} f - \omega^{N-1} \sum_{j=0}^{J-N} \omega^j \Gamma_j f \Big\|_{\mathrm{L}_t^{2,q,q+1}(\Omega)} = O\big(|\omega|^J\big) \|f\|_{\mathrm{L}_s^{2,q,q+1}(\Omega)}$$

for $f \in \mathrm{Reg}_s^{q,-1}(\Omega) = \Pi_{\mathrm{reg}} \mathrm{L}_s^{2,q,q+1}(\Omega) \subset \Lambda^{-1}\big({}_0\Delta_t^q(\Omega) \times {}_0\mathring{\mathrm{D}}_t^{q+1}(\Omega)\big)$

- fully corrected Neumann expansion

$$\Big\| \mathcal{L}_\omega f + \omega^{-1}\Pi f - \sum_{j=0}^{J-1} \omega^j \mathcal{L}_0{}^{j+1} \Pi_{\mathrm{reg}} f - \omega^{N-1} \sum_{j=0}^{J-N} \omega^j \Gamma_j f \Big\|_{\mathrm{L}_t^{2,q,q+1}(\Omega)} = O\big(|\omega|^J\big) \|f\|_{\mathrm{L}_s^{2,q,q+1}(\Omega)}$$

for $f \in \mathrm{L}_s^{2,q,q+1}(\Omega) = \big(\mathrm{Tri}_s^q(\Omega) \dotplus \mathrm{Reg}_s^{q,-1}(\Omega)\big) \cap \mathrm{L}_s^{2,q,q+1}(\Omega)$

MAIN RESULT

Theorem (low frequency asymptotics)

Let $J \in \mathbb{N}$ *and* $s \notin \mathbb{I} = (\mathbb{N}_0 + N/2) \cup (1 - N/2 - \mathbb{N}_0)$ *with*

$$s > J + 1/2, \qquad (f)$$

$$t < \min\{N/2 - J - 2, -1/2\}, \qquad (u_\omega)$$

$$\tau > \max\big\{(N+1)/2, s - t\big\}. \qquad (\hat{\Lambda})$$

Then for all small enough $\mathbb{C}_+ \setminus \{0\} \ni \omega \to 0$ *the asymptotic expansion*

$$\mathcal{L}_\omega + \omega^{-1}\Pi - \sum_{j=0}^{J-1} \omega^j \, \mathcal{L}_0{}^{j+1} \, \Pi_{\text{reg}} - \omega^{N-1} \sum_{j=0}^{J-N} \omega^j \Gamma_j = O\big(|\omega|^J\big)$$

holds in the norm of bounded linear operators from $\mathsf{L}_s^{2,q,q+1}(\Omega)$ *to* $\mathsf{L}_t^{2,q,q+1}(\Omega)$.

Remark *The main theorem holds also for* $J = 0$ *with slightly different* t *and* τ.

TIME-HARMONIC SCATTERING PROBLEM

Solving $(M-\omega)u_\omega = f$?

$$M \quad : \quad \overset{\circ}{\mathbf{D}}{}^q(\Omega) \times \mathbf{\Delta}^{q+1}(\Omega) \subset \mathsf{L}^{2,q,q+1}_\Lambda(\Omega) \longrightarrow \mathsf{L}^{2,q,q+1}_\Lambda(\Omega)$$
$$u \longmapsto \mathrm{i}\,\Lambda^{-1}\begin{bmatrix} 0 & \delta \\ \mathrm{d} & 0 \end{bmatrix} u$$

M unbd. lin. s.a. $\quad\Rightarrow\quad \sigma(\mathcal{M}) \subset \mathbb{R}$

$\omega \in \mathbb{C}\setminus\mathbb{R} \quad\Rightarrow\quad \mathcal{L}_\omega = (M-\omega)^{-1}$ bounded $\quad\Rightarrow\quad$ L^2-sol. for all $f \in \mathsf{L}^{2,q,q+1}(\Omega)$

solving in $\sigma(\mathcal{M})\setminus\{0\}$ with Eidus' 'limiting absorption principle' (approx. from $\mathbb{C}_+$)

Definition (time-harmonic (scattering) solutions)

Let $\omega \in \mathbb{R}\setminus\{0\}$ and $f \in \mathsf{L}^{2,q,q+1}_{\mathrm{loc}}(\Omega)$. u_ω solves $\mathrm{Max}(f,\omega)$, iff

(i) $\quad \forall \quad t < -1/2 \qquad u_\omega \in \overset{\circ}{\mathbf{D}}{}^q_t(\Omega) \times \mathbf{\Delta}^{q+1}_t(\Omega)$,

(ii) $\quad \exists \quad t > -1/2 \qquad (S+1)u_\omega \in \mathsf{L}^{2,q,q+1}_t(\Omega)$,

(iii) $\qquad (M-\omega)u_\omega = f$.

TOOLS: a priori estimate, polynomial decay of eigensolutions, decomposition lemma, Helmholtz' equation

TIME-HARMONIC SCATTERING PROBLEM

Theorem (time-harmonic (scattering) solution theory)

Let $\omega \in \mathbb{R} \setminus \{0\}$ *and* $s > 1/2$, $\tau > 1$.

(i) $\mathrm{Max}(0,\omega) \subset \big(\mathring{\mathbf{D}}_t^q(\Omega) \cap \varepsilon^{-1}\,\delta\,\mathbf{\Delta}_t^{q+1}(\Omega)\big) \times \big(\mathbf{\Delta}_t^{q+1}(\Omega) \cap \mu^{-1}\,\mathrm{d}\,\mathring{\mathbf{D}}_t^q(\Omega)\big)$ *for all* $t \in \mathbb{R}$, *i.e. gen. eigensolutions decay polynomially (and exponentially for* $\Lambda \in \mathrm{C}^2$*), no gen. eigenvalues for* $\Lambda = \mathrm{Id}$*, comp. Helmholtz eq., Rellich's est., princ. uniq. cont.*

(ii) $\dim \mathrm{Max}(0,\omega) < \infty$

(iii) $\sigma_{gen}(M)$ *has no accumulation point in* $\mathbb{R} \setminus \{0\}$

(iv) *Fredholm's Alternative holds:* $\forall\, f \in \mathrm{L}_s^{2,q,q+1}(\Omega) \quad \exists\, u_\omega$ *solution of* $\mathrm{Max}(f,\omega)$*, iff*

$$\forall \quad v \in \mathrm{Max}(0,\omega) \qquad \langle f, v\rangle_{\mathrm{L}_\Lambda^{2,q,q+1}(\Omega)} = 0$$

The solution u_ω *can be chosen, such that*

$$\forall \quad v \in \mathrm{Max}(0,\omega) \qquad \langle u_\omega, v\rangle_{\mathrm{L}_\Lambda^{2,q,q+1}(\Omega)} = 0.$$

Then u_ω *is uniquely determined.*

(v) *For all* $t < -1/2$ *the solution operator* $\mathcal{L}_\omega$ *maps* $\mathrm{L}_s^{2,q,q+1}(\Omega) \cap \mathrm{Max}(0,\omega)^{\perp_\Lambda}$ *to* $\big(\mathring{\mathbf{D}}_t^q(\Omega) \times \mathbf{\Delta}_t^{q+1}(\Omega)\big) \cap \mathrm{Max}(0,\omega)^{\perp_\Lambda}$ *continuously.*

LOW FREQUENCY TIME-HARMONIC SCATTERING PROBLEM

Theorem (low frequency time-harmonic estimate)

Let $\tau > (N+1)/2$ and $s \in (1/2, N/2)$ as well as $t := s - (N+1)/2 \in (-N/2, -1/2)$.

(i) *$\sigma_{gen}(M)$ does not accumulate in $\mathbb{R}$ (especially not at zero). $\sigma_{gen}(M) \cap \mathbb{C}_+ = \{0\}$ for ω sufficiently small.*

(ii) *$\mathcal{L}_\omega$ is well defined on $\mathsf{L}_s^{2,q,q+1}(\Omega)$ for all $0 \neq \omega \in \mathbb{C}_+$ small enough.*

(iii) $\exists\, c > 0 \quad \forall\, 0 \neq \omega \in \mathbb{C}_+$ *small enough* $\quad \forall\, \Lambda f = \Lambda(F, G) \in \boldsymbol{\Delta}_s^q(\Omega) \times \mathring{\mathbf{D}}_s^{q+1}(\Omega)$

$$\|\mathcal{L}_\omega f\|_{\mathsf{L}_t^{2,q,q+1}(\Omega)} \leq c\Big(\|f\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)} + |\omega|^{-1}\|(\delta\,\varepsilon F, \mathrm{d}\,\mu G)\|_{\mathsf{L}_s^{2,q-1,q+2}(\Omega)} + |\omega|^{-1}\sum_{\ell=1}^{d^q}\big|\langle\varepsilon F, \mathring{b}_\ell^q\rangle_{\mathsf{L}^{2,q}(\Omega)}\big| + |\omega|^{-1}\sum_{\ell=1}^{d^{q+1}}\big|\langle\mu G, b_\ell^{q+1}\rangle_{\mathsf{L}^{2,q+1}(\Omega)}\big|\Big).$$

Especially $\|\mathcal{L}_\omega f\|_{\mathsf{L}_t^{2,q,q+1}(\Omega)} \leq c\|f\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)}$ *holds for*

$$\Lambda f = \Lambda(F, G) \in {}_0\mathbb{\Delta}_s^q(\Omega) \times {}_0\mathring{\mathbb{D}}_s^{q+1}(\Omega) := \big({}_0\Delta_s^q(\Omega) \cap \mathring{\mathsf{B}}^q(\Omega)^\perp\big) \times \big({}_0\mathring{\mathsf{D}}_s^{q+1}(\Omega) \cap \mathsf{B}^{q+1}(\Omega)^\perp\big),$$

i.e., no terms with negative frequency power $|\omega|^{-1}$ occur.

TOOLS: fundamental sol. Helmholtz' eq. (Hankel's function), repr. of sol. for $\Omega = \mathbb{R}^N$ as conv., cutt. tech., indirect arg.

FIRST LOW FREQUENCY ASYMPTOTIC

Theorem (first and simple static solution theory)

Let $\tau > 0$. Then there exists a linear and bounded static solution operator

$$\mathcal{L}_0 : \Lambda^{-1}\big({}_0\mathbb{\Delta}^q(\Omega) \times {}_0\mathring{\mathbb{D}}^{q+1}(\Omega)\big) \to \big(\mathring{\mathsf{D}}^q_{-1}(\Omega) \times \Delta^{q+1}_{-1}(\Omega)\big) \cap \Lambda^{-1}\big({}_0\mathbb{\Delta}^q_{-1}(\Omega) \times {}_0\mathring{\mathbb{D}}^{q+1}_{-1}(\Omega)\big).$$

More precisely: $u = (E, H) = \mathcal{L}_0 f$ for $f = (F, G)$ solves $Mu = f$, i.e., the static system

$$\begin{aligned} &\mathrm{i}\,\mu^{-1}\,\mathrm{d}\,E = G, && \delta\,\varepsilon E = 0, && \varepsilon E \perp \mathring{\mathsf{B}}^q(\Omega),\\ &\mathrm{i}\,\varepsilon^{-1}\,\delta\,H = F, && \mathrm{d}\,\mu H = 0, && \mu H \perp \mathsf{B}^{q+1}(\Omega). \end{aligned}$$

Theorem (first and simple low frequency asymptotics)

Let $\tau > (N+1)/2$ and $s \in (1/2, N/2)$ as well as $t < s - (N+1)/2 \in (-N/2, -1/2)$. Then

$$\boxed{\lim_{\mathbb{C}_+ \ni \omega \to 0} \mathcal{L}_\omega = \mathcal{L}_0}$$

in the norm of bounded linear operators

$$\Lambda^{-1}\big({}_0\mathbb{\Delta}^q_s(\Omega) \times {}_0\mathring{\mathbb{D}}^{q+1}_s(\Omega)\big) \longrightarrow \mathring{\mathbf{D}}^q_t(\Omega) \times \mathbf{\Delta}^{q+1}_t(\Omega).$$

EXTENDED STATIC SOLUTION THEORY

Theorem (extended static solution theory)
Let $s \in (1 - N/2, \infty) \setminus \mathbb{I}$ and $\tau > \max\{0, s - N/2\}$, $\tau \geq -s$. Then

$$\begin{array}{rcccl} \mathrm{i}\,\mu^{-1}\,\mathrm{d} & : & \big(\mathring{\mathrm{D}}^{q}_{s-1}(\Omega) \boxplus \eta \curlywedge^{q,0,-}_{s-1}\big) \cap \varepsilon^{-1}{}_{0}\mathbb{\Delta}^{q}_{\mathrm{loc}}(\Omega) & \longrightarrow & \mu^{-1}{}_{0}\mathring{\mathbb{D}}^{q+1}_{s}(\Omega) \\ & & E & \longmapsto & \mathrm{i}\,\mu^{-1}\,\mathrm{d}\,E \end{array},$$

$$\begin{array}{rcccl} \mathrm{i}\,\varepsilon^{-1}\,\delta & : & \big(\Delta^{q+1}_{s-1}(\Omega) \boxplus \eta \mathcal{D}^{q+1,0,-}_{s-1}\big) \cap \mu^{-1}{}_{0}\mathring{\mathbb{D}}^{q+1}_{\mathrm{loc}}(\Omega) & \longrightarrow & \varepsilon^{-1}{}_{0}\mathbb{\Delta}^{q}_{s}(\Omega) \\ & & H & \longmapsto & \mathrm{i}\,\varepsilon^{-1}\,\delta\,H \end{array}$$

are topological isomorphisms.

note: $\curlywedge^{q,0,-}_{s-1} = \curlywedge^{q}(\bar{\mathfrak{I}}^{q,0}_{s-1})$ finite dim. subspace of $\mathrm{C}^{\infty}\big(\mathbb{R}^{N} \setminus \{0\}\big)$

$\eta \curlywedge^{q,0,-}_{s-1} \subset \mathrm{L}^{2,q}_{t}(\Omega)$ for $t \leq s - 1$, $t < N/2$ and $\eta \curlywedge^{q,0,-}_{s-1} \not\subset \mathrm{L}^{2,q}_{s-1}(\Omega)$

same for $\mathcal{D}^{q+1,0,-}_{s-1} = \mathcal{D}^{q+1}(\bar{\mathfrak{J}}^{q+1,0}_{s-1})$

consisting of 'neg. tower-forms' of shape $r^{\ell}\,\check{\tau}\,S^{q}_{m,n}$ ($S^{q}_{m,n}$ gen. spherical harmonics)

EXTENDED STATIC SOLUTION THEORY

Corollary (extended static solution theory)

Let $s \in (1 - N/2, \infty) \setminus \mathbb{I}$ and $\tau > \max\{0, s - N/2\}$, $\tau \geq -s$. Then

$$M : \Big(\big(\mathring{\mathrm{D}}^q_{s-1}(\Omega) \times \Delta^{q+1}_{s-1}(\Omega) \big) \boxplus \big(\eta \mathcal{R}^{q,0,-}_{s-1} \times \eta \mathcal{D}^{q+1,0,-}_{s-1} \big) \Big) \cap \Lambda^{-1} \big({}_0\mathbb{\Delta}^q_{\mathrm{loc}}(\Omega) \times {}_0\mathring{\mathbb{D}}^{q+1}_{\mathrm{loc}}(\Omega) \big)$$

$$\longrightarrow \Lambda^{-1} \big({}_0\mathbb{\Delta}^q_s(\Omega) \times {}_0\mathring{\mathbb{D}}^{q+1}_s(\Omega) \big)$$

$$u = (E, H) \longmapsto Mu = \mathrm{i}\, \Lambda^{-1}(\delta H, \mathrm{d} E)$$

is a topological isomorphism with bounded inverse

$$\boxed{\mathcal{L}_0 = M^{-1} : \Lambda^{-1} \big({}_0\mathbb{\Delta}^q_s(\Omega) \times {}_0\mathring{\mathbb{D}}^{q+1}_s(\Omega) \big) \longrightarrow \Lambda^{-1} \big({}_0\mathbb{\Delta}^q_{s-1}(\bar{\mathcal{I}}^{q,0}_{s-1}, \Omega) \times {}_0\mathring{\mathbb{D}}^{q+1}_{s-1}(\bar{\mathcal{J}}^{q+1,0}_{s-1}, \Omega) \big).}$$

<u>goal:</u> higher powers of $\mathcal{L}_0$ even acting on $\Lambda^{-1} \big({}_0\mathbb{\Delta}^q_{s-1}(\mathcal{I}, \Omega) \times {}_0\mathring{\mathbb{D}}^{q+1}_{s-1}(\mathcal{J}, \Omega) \big)$

TOWER FORMS

...			...		...		
		$\delta \swarrow$				$\searrow \mathrm{d}$	
3. floor	$\pm\Delta^{q-1,3}_{\sigma,m}$						$\pm D^{q+1,3}_{\sigma,m}$
		$\mathrm{d} \searrow$				$\swarrow \delta$	
2. floor			$\pm D^{q,2}_{\sigma,m}$		$\pm\Delta^{q,2}_{\sigma,m}$		
		$\delta \swarrow$				$\searrow \mathrm{d}$	
1. floor	$\pm\Delta^{q-1,1}_{\sigma,m}$						$\pm D^{q+1,1}_{\sigma,m}$
		$\mathrm{d} \searrow$				$\swarrow \delta$	
ground			$\pm D^{q,0}_{\sigma,m}$	$\cong$	$\pm\Delta^{q,0}_{\sigma,m}$		
	d-tower				δ-tower		

$\pm\Delta^{q,k}_{\sigma,m}, \pm D^{q,k}_{\sigma,m} \in \mathrm{C}^\infty(\mathbb{R}^N \setminus \{0\})$ homogeneous of deg. $k+\sigma$ resp. $k-\sigma-N$

HIGHER POWERS OF THE STATIC SOLUTION OPERATOR

Theorem (higher powers of $\mathcal{L}_0$)

Let $j \in \mathbb{N}$ and $s \in (j - N/2, \infty) \setminus \mathbb{I}$ and $\mathcal{I}, \mathcal{J}$ finite index sets as well as $\tau \geq j - 1 - s$, $\tau > \max\{0, s - N/2\}$ and $\tau > s + N/2 + \max\{\mathrm{h}_{\mathcal{I}}, \mathrm{h}_{\mathcal{J}}\}$. Then

$$\mathcal{L}_0^j \;:\; \Lambda^{-1}\big({}_0\mathbb{\Delta}_s^q(\mathcal{I}, \Omega) \times {}_0\mathring{\mathbb{D}}_s^{q+1}(\mathcal{J}, \Omega)\big)$$

$$\longrightarrow \Lambda^{-1} \begin{cases} {}_0\mathbb{\Delta}_{s-j}^q(\bar{\mathcal{I}}_{s-j}^{q,\leq j-1} \cup {}_j\mathcal{I}, \Omega) \times {}_0\mathring{\mathbb{D}}_{s-j}^{q+1}(\bar{\mathcal{J}}_{s-j}^{q+1,\leq j-1} \cup {}_j\mathcal{J}, \Omega) & \text{, if } j \text{ even} \\ {}_0\mathbb{\Delta}_{s-j}^q(\bar{\mathcal{I}}_{s-j}^{q,\leq j-1} \cup {}_j\mathcal{J}, \Omega) \times {}_0\mathring{\mathbb{D}}_{s-j}^{q+1}(\bar{\mathcal{J}}_{s-j}^{q+1,\leq j-1} \cup {}_j\mathcal{I}, \Omega) & \text{, if } j \text{ odd} \end{cases}$$

is a continuous linear operator with range in $\Lambda^{-1}\big({}_0\mathbb{\Delta}_t^q(\Omega) \times {}_0\mathring{\mathbb{D}}_t^{q+1}(\Omega)\big)$ for $t \leq s - j$, $t < N/2 - j + 1$, $t < -j - N/2 - \max\{\mathrm{h}_{\mathcal{I}}, \mathrm{h}_{\mathcal{J}}\}$.

SPACES OF REGULAR CONVERGENCE

$$\mathrm{Reg}_s^{q,-1}(\Omega) = \Pi_{\mathrm{reg}} \mathsf{L}_s^{2,q,q+1}(\Omega) \subset \Lambda^{-1}\big({}_0\Delta_t^q(\Omega) \times {}_0\mathring{\mathsf{D}}_t^{q+1}(\Omega)\big)$$

$$\mathrm{Reg}_s^{q,0}(\Omega) := \Lambda^{-1}\big({}_0\mathbb{\Delta}_s^q(\Omega) \times {}_0\mathring{\mathbb{D}}_s^{q+1}(\Omega)\big)$$

$$\mathrm{Reg}_s^{q,j}(\Omega) := \big\{ f \in \mathrm{Reg}_s^{q,0}(\Omega) : \mathcal{L}_0^j f \in \mathsf{L}_{s-j}^{2,q,q+1}(\Omega) \big\}$$

'usual Neumann sum'

Lemma (spaces of regular convergence)

Let $J \in \mathbb{N}_0$ and $s \in (J + 1/2, \infty) \setminus \mathbb{I}$ as well as $\tau > \max\big\{(N+1)/2, s - N/2\big\}$. Then for all $0 \neq \omega \in \mathbb{C}_+$ small enough on $\mathrm{Reg}_s^{q,J}(\Omega)$ the resolvent formula

$$\mathcal{L}_\omega - \sum_{j=0}^{J-1} \omega^j \, \mathcal{L}_0^{j+1} = \omega^J \, \mathcal{L}_\omega \, \mathcal{L}_0^J$$

holds. Especially for $s \in (J + 1/2, J + N/2) \setminus \mathbb{I}$ and $t = s - J - (N+1)/2$

$$\boxed{\Big\| \, \mathcal{L}_\omega \, f - \sum_{j=0}^{J-1} \omega^j \, \mathcal{L}_0^{j+1} \, f \Big\|_{\mathsf{L}_t^{2,q,q+1}(\Omega)} = O\big(|\omega|^J\big) \, \|f\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)}}$$

holds uniformly w.r.t. $f \in \mathrm{Reg}_s^{q,J}(\Omega)$.

aim: characterize $\mathrm{Reg}_s^{q,j}(\Omega)$ by orthogonality constraints

GROWING STATIC SOLUTIONS

again conditions on τ ...

$$E^{+}_{\sigma,m} \in {}_0\overset{\circ}{\mathsf{D}}{}^q_{\mathrm{loc}}(\Omega) \cap \varepsilon^{-1}\,{}_0\mathbb{\Delta}^q_{\mathrm{loc}}(\Omega)$$
$$E^{+}_{\sigma,m} - {}^{+}\Delta^{q,0}_{\sigma,m} \in \mathsf{L}^{2,q}_{>-\frac{N}{2}}(\Omega)$$

$$H^{+}_{\sigma,m} \in {}_0\Delta^{q+1}_{\mathrm{loc}}(\Omega) \cap \mu^{-1}\,{}_0\overset{\circ}{\mathbb{D}}{}^{q+1}_{\mathrm{loc}}(\Omega)$$
$$H^{+}_{\sigma,m} - {}^{+}D^{q+1,0}_{\sigma,m} \in \mathsf{L}^{2,q+1}_{>-\frac{N}{2}}(\Omega)$$

$$E^{+,k}_{\sigma,m} = \mathcal{L}^k_0(E^{+}_{\sigma,m},0), \quad H^{+,k}_{\sigma,n} = \mathcal{L}^k_0(0,H^{+}_{\sigma,n}) \in \mathsf{L}^{2,q,q+1}_{-N/2-\sigma-k}(\Omega)$$

${}^{+}\Delta^{q,k}_{\sigma,m}$, ${}^{+}D^{q+1,k}_{\sigma,m}$ behave like $r^{k+\sigma}$, $k,\sigma \geq 0$ at infinity

$$E^{+,k}_{\sigma,m} - \eta({}^{+}\Delta^{q,k}_{\sigma,m},0) \in \Lambda^{-1}\Big(\big(\Delta^q_{s-k-1}(\Omega) \boxplus \eta\,\mathcal{A}^q(\bar{\mathfrak{I}}^{q,\leq k}_{s-k-1})\big) \times \{0\}\Big) \qquad k \text{ even}$$
$$E^{+,k}_{\sigma,m} - \eta(0,{}^{+}D^{q+1,k}_{\sigma,m}) \in \Lambda^{-1}\Big(\{0\} \times \big(\overset{\circ}{\mathsf{D}}{}^{q+1}_{s-k-1}(\Omega) \boxplus \eta\,\mathcal{D}^{q+1}(\bar{\mathfrak{J}}^{q+1,\leq k}_{s-k-1})\big)\Big) \qquad k \text{ odd}$$

supp $\hat{\Lambda}$ compact, then series rep. of neg. tower-forms of height $\leq k$
(gen. spherical harmonics expansion)

PROJECTION ONTO SPACES OF REGULAR CONVERGENCE

powers $\mathcal{L}_0^j f$ have neg. tower-form parts

$$\langle C_{\Delta,\eta}{}^{\theta} D^{q,k}_{\sigma,m}, {}^{\vartheta} D^{q,\ell}_{\gamma,n}\rangle_{\mathsf{L}^{2,q}(\mathbb{R}^N)} = \langle C_{\Delta,\eta}{}^{\theta} \Delta^{q,k}_{\sigma,m}, {}^{\vartheta} \Delta^{q,\ell}_{\gamma,n}\rangle_{\mathsf{L}^{2,q}(\mathbb{R}^N)} = \delta_{\vartheta\theta,-1}\delta_{k,\ell}\delta_{\sigma,\gamma}\delta_{m,n},$$
$$\langle C_{\Delta,\eta}{}^{\theta} D^{q,k}_{\sigma,m}, {}^{\vartheta} \Delta^{q,\ell}_{\gamma,n}\rangle_{\mathsf{L}^{2,q}(\mathbb{R}^N)} = 0$$

assume: supp $\hat{\Lambda}$ compact $\quad \Rightarrow$

Lemma (orthogonality def. of spaces of regular convergence)

Let $J \in \mathbb{N}$ and $s \in (J+1-N/2, \infty) \setminus \mathbb{I}$ as well as $f \in \mathrm{Reg}^{q,0}_s(\Omega)$.
Then $f \in \mathrm{Reg}^{q,J}_s(\Omega)$, iff

$$\langle f, E^{+,k+1}_{\sigma,m}\rangle_{\mathsf{L}^{2,q,q+1}_{\Lambda}(\Omega)} = \langle f, H^{+,\ell+1}_{\gamma,n}\rangle_{\mathsf{L}^{2,q,q+1}_{\Lambda}(\Omega)} = 0$$

for all $(k,\sigma,m) \in \Theta^{q,J}_s$ and $(\ell,\gamma,n) \in \Theta^{q+1,J}_s$, where

$$\Theta^{q,J}_s := \big\{(k,\sigma,m) \in \mathbb{N}_0^3 : k \le J-1 \,\wedge\, \sigma < s - N/2 - k - 1 \,\wedge\, 1 \le m \le \mu^q_\sigma\big\}.$$

Especially $\mathrm{Reg}^{q,J}_s(\Omega)$ is a closed subspace of $\mathrm{Reg}^{q,0}_s(\Omega) \subset \mathsf{L}^{2,q,q+1}_s(\Omega)$.

DUAL BASIS OF GROWING TOWERS FORMS

Define

$$e^{\pm,\ell}_{\sigma,n} := M^\ell \eta(\pm\Delta^{q,1}_{\sigma,n}, 0), \quad h^{\pm,\ell}_{\sigma,m} := M^\ell \eta(0, \pm D^{q+1,1}_{\sigma,m}).$$

Then $e^{\pm,\ell}_{\sigma,n}, h^{\pm,\ell}_{\sigma,m} \in \mathring{C}^\infty(\mathbb{R}^N)$ with $\operatorname{supp} e^{\pm,\ell}_{\sigma,n} = \operatorname{supp} h^{\pm,\ell}_{\sigma,m} = \operatorname{supp} \nabla\eta$ for $\ell \geq 2$ and

$$\langle e^{-,\ell+2}_{\gamma,n}, E^{+,k+1}_{\sigma,m}\rangle_{L^{2,q,q+1}(\Omega)} = 0,$$

$$\langle h^{-,\ell+2}_{\gamma,n}, E^{+,k+1}_{\sigma,m}\rangle_{L^{2,q,q+1}(\Omega)} = (-1)^\ell \delta_{k,\ell}\delta_{\sigma,\gamma}\delta_{m,n}.$$

same for $H^{+,k+1}_{\sigma,m}$

Lemma (dual basis of $E^{+,k+1}_{\sigma,m}$ and $H^{+,\ell+1}_{\gamma,n}$)

Let $J \in \mathbb{N}$ and $s \in (J+1-N/2, \infty) \setminus \mathbb{I}$. Then

$$\operatorname{Reg}^{q,0}_s(\Omega) = \operatorname{Reg}^{q,J}_s(\Omega) \dotplus \Upsilon^{q,J}_s, \qquad \Upsilon^{q,J}_s \subset \mathring{C}^\infty(\mathbb{R}^N),$$

where for $f \in \operatorname{Reg}^{q,0}_s(\Omega)$

$$f_\Upsilon := \sum_{(k,\sigma,m)\in\Theta^{q,J}_s} (-1)^k \langle f, E^{+,k+1}_{\sigma,m}\rangle_{L^{2,q,q+1}(\Omega)} h^{-,k+2}_{\sigma,m} + \sum_{(k,\sigma,m)\in\Theta^{q+1,J}_s} (-1)^k \langle f, H^{+,k+1}_{\sigma,m}\rangle_{L^{2,q,q+1}(\Omega)} e^{-,k+2}_{\sigma,m}.$$

with $\Upsilon^{q,J}_s := \operatorname{Lin}\{e^{-,k+2}_{\sigma,m}, h^{-,\ell+2}_{\gamma,n} : (k,\sigma,m) \in \Theta^{q,J}_s, (\ell,\gamma,n) \in \Theta^{q+1,J}_s\}$.

PROOF OF LOW FREQUENCY ASYMPTOTICS

step one: proof in the reduced case, this is:

- compactly supported perturbations $\hat{\Lambda}$
- right hand sides from $\mathrm{Reg}_s^{q,0}(\Omega)$
- estimates in local norms

step two: replacing $\mathrm{Reg}_s^{q,0}(\Omega)$ by $\mathrm{L}_s^{2,q,q+1}(\Omega)$ (polynomially weighted Helmholtz decomposition)

step three: replacing local norms by weighted norms

step four: replacing compactly supported perturbations $\hat{\varepsilon}$, $\hat{\mu}$ by asymptotically vanishing perturbations

We only drop the assumption of compactly supported perturbations of the medium in the last step.

STEP ONE

latter lemma $\Rightarrow$

$$\mathrm{Reg}_s^{q,0}(\Omega) = \mathrm{Reg}_s^{q,J}(\Omega) \dotplus \Upsilon_s^{q,J}, \qquad e_{\sigma,m}^{-,k+2}, h_{\sigma,m}^{-,k+2} \Upsilon_s^{q,J} \subset \mathring{\mathrm{C}}^\infty(\mathbb{R}^N)$$

- asymptotics clear on $\mathrm{Reg}_s^{q,J}(\Omega)$ (gen. Neumann sum) $\surd$
- asymptotics on $\Upsilon_s^{q,J}$? $\quad\Rightarrow\quad$ asymptotics for $e_{\sigma,m}^{-,k+2}$, $h_{\sigma,m}^{-,k+2}$?

$\mathcal{L}_0^k\, e_{\sigma,m}^{-,k+2} = e_{\sigma,m}^{-,2}$ ($\mathring{\mathrm{C}}^\infty(\mathbb{R}^N)$ and right shape) $\quad\Rightarrow\quad e_{\sigma,m}^{-,k+2} \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^N) \cap \mathrm{Reg}_s^{q,k}(\Omega)$

$$\Big(\mathcal{L}_\omega - \underbrace{\sum_{j=0}^{J-1} \omega^j\, \mathcal{L}_0^{j+1}}_{=\sum_{j=0}^{k-1}\cdots + \sum_{j=k}^{J-1}\cdots}\Big) e_{\sigma,m}^{-,k+2} = \omega^k\, \mathcal{L}_\omega\, \mathcal{L}_0^k\, e_{\sigma,m}^{-,k+2} - \omega^k \sum_{j=0}^{J-1-k} \omega^j\, \mathcal{L}_0^{j+1+k}\, e_{\sigma,m}^{-,k+2}$$

$$= \omega^k \Big(\mathcal{L}_\omega - \sum_{j=0}^{J-1-k} \omega^j\, \mathcal{L}_0^{j+1}\Big) e_{\sigma,m}^{-,2}$$

same for $h_{\sigma,m}^{-,k+2}$

just unkn. asym. for $\boxed{\Big(\mathcal{L}_\omega - \sum_{j=0}^{J-1-k} \omega^j\, \mathcal{L}_0^{j+1}\Big) e_{\sigma,m}^{-,2}}$ and $\boxed{\Big(\mathcal{L}_\omega - \sum_{j=0}^{J-1-k} \omega^j\, \mathcal{L}_0^{j+1}\Big) h_{\sigma,m}^{-,k+2}}$

STEP ONE

asymptotics for $\boxed{(\mathcal{L}_\omega - \sum_{j=0}^{J-1-k} \omega^j \, \mathcal{L}_0^{j+1}) \, e^{-,2}_{\sigma,m}}$ and $\boxed{(\mathcal{L}_\omega - \sum_{j=0}^{J-1-k} \omega^j \, \mathcal{L}_0^{j+1}) \, h^{-,k+2}_{\sigma,m}}$?

<u>idea:</u> compare with special radiating solutions of the homo. problem in $\mathbb{R}^N \setminus \{0\}$

$$\begin{aligned}
\mathbb{E}^{1,\omega}_{\sigma,m} &= \beta_\sigma \omega^{\nu_\sigma} r^{1-\frac{N}{2}} H^1_{\nu_\sigma}(\omega\, r)\, \check{\tau}\, T^q_{\sigma,m} \qquad (H^1_{\nu_\sigma} \text{ Hankel's function})\\
&= \sum_{k=0}^{\infty} (-\mathrm{i}\,\omega)^{2k} \; {}^{-}\Delta^{q,2k+1}_{\sigma,m} + \kappa^{q+1}_\sigma \, \omega^{2\nu_\sigma} \sum_{k=0}^{\infty} (-\mathrm{i}\,\omega)^{2k} \; {}^{+}\Delta^{q,2k+1}_{\sigma,m}\\
\mathbb{H}^{1,\omega}_{\sigma,m} &= \frac{\mathrm{i}}{\omega} \, \mathrm{d}\, \mathbb{E}^{1,\omega}_{\sigma,m}\\
&= \frac{\mathrm{i}}{\omega} \Big(\sum_{k=0}^{\infty} (-\mathrm{i}\,\omega)^{2k} \; {}^{-}D^{q+1,2k}_{\sigma,m} + \kappa^{q+1}_\sigma \, \omega^{2\nu_\sigma} \sum_{k=0}^{\infty} (-\mathrm{i}\,\omega)^{2k} \; {}^{+}D^{q+1,2k}_{\sigma,m} \Big)
\end{aligned}$$

similarly second solution pair $(\mathbb{E}^{2,\omega}_{\sigma,m}, \mathbb{H}^{2,\omega}_{\sigma,m})$

$$(\mathrm{i} \begin{bmatrix} 0 & \delta \\ \mathrm{d} & 0 \end{bmatrix} - \omega)(\mathbb{E}^{j,\omega}_{\sigma,m}, \mathbb{H}^{j,\omega}_{\sigma,m}) = (0,0) \quad \Rightarrow \quad (\Delta + \omega^2)(\mathbb{E}^{j,\omega}_{\sigma,m}, \mathbb{H}^{j,\omega}_{\sigma,m}) = (0,0)$$

(comp.-wise Helmholtz)

STEP ONE

note: $(M-\omega)\eta(\mathbb{E}^{j,\omega}_{\sigma,m},\mathbb{H}^{j,\omega}_{\sigma,m}) = C_{M,\eta}(\mathbb{E}^{j,\omega}_{\sigma,m},\mathbb{H}^{j,\omega}_{\sigma,m})$

comparing

$$\begin{aligned} &\mathcal{L}_\omega\, e^{-,2}_{\sigma,m} && \text{with} && \mathcal{L}_\omega\, C_{M,\eta}(\mathbb{E}^{1,\omega}_{\sigma,m},\mathbb{H}^{1,\omega}_{\sigma,m}) = \eta(\mathbb{E}^{1,\omega}_{\sigma,m},\mathbb{H}^{1,\omega}_{\sigma,m}),\\ &\mathcal{L}_\omega\, h^{-,2}_{\sigma,m} && \text{with} && \mathcal{L}_\omega\, C_{M,\eta}(\mathbb{E}^{2,\omega}_{\sigma,m},\mathbb{H}^{2,\omega}_{\sigma,m}) = \eta(\mathbb{E}^{2,\omega}_{\sigma,m},\mathbb{H}^{2,\omega}_{\sigma,m}) \end{aligned}$$

and a (really) long, long, long, ... calculation

Theorem (low frequency asymptotics on $\mathrm{Reg}^{q,0}_s(\Omega)$)

Let $J \in \mathbb{N}_0$ and $s \in (J+1/2,\infty)\setminus\mathbb{I}$. Then for all bounded subdomains $\Omega_b \subset \Omega$

$$\Big\| \mathcal{L}_\omega f - \sum_{j=0}^{J-1} \omega^j \mathcal{L}_0^{j+1} f - \omega^N \sum_{j=0}^{J-1-N} \omega^j \hat{\Gamma}_j f \Big\|_{\mathsf{L}^{2,q,q+1}(\Omega_b)} = O\big(|\omega|^J\big)\, \|f\|_{\mathsf{L}^{2,q,q+1}_s(\Omega)}$$

holds uniformly w.r.t. $f \in \mathrm{Reg}^{q,0}_s(\Omega)$ and $0 \neq \omega \in \mathbb{C}_+$ small enough.

degenerate correction operators

$$\hat{\Gamma}_j f \in \mathrm{Lin}\{E^{+,k}_{\sigma,m}, H^{+,k}_{\sigma,n} : k + 2\sigma \le j\}$$

with coefficients of shape $\langle f, E^{+,k}_{\sigma,m}\rangle_{\mathsf{L}^{2,q,q+1}(\Omega)}$ and $\langle f, H^{+,k}_{\sigma,m}\rangle_{\mathsf{L}^{2,q,q+1}(\Omega)}$

STEP TWO

Theorem (polynomially weighted Helmholtz decomposition)

conditions on τ ...
For $s > -N/2$ let ${}_\varepsilon\mathbb{L}^{2,q}_s(\Omega) := \mathsf{L}^{2,q}_s(\Omega) \cap {}_\varepsilon\mathcal{H}^q(\Omega)^{\perp_\varepsilon}$.

(i) $-N/2 < s < N/2$:

$$ {}_\varepsilon\mathbb{L}^{2,q}_s(\Omega) = {}_0\mathring{\mathbb{D}}^q_s(\Omega) \dotplus \varepsilon^{-1}{}_0\mathbb{\Delta}^q_s(\Omega) $$

For $s \geq 0$ the decomposition is $\langle\varepsilon\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^{2,q}(\Omega)}$-orthogonal.

(ii) $s > N/2$:

$$ {}_\varepsilon\mathbb{L}^{2,q}_s(\Omega) = \Big(\big([\mathsf{L}^{2,q}_s(\Omega) \boxplus \eta\bar{\mathcal{D}}^q_s] \cap {}_0\mathring{\mathbb{D}}^q_{<\frac{N}{2}}(\Omega)\big) \oplus_\varepsilon \varepsilon^{-1}\big([\mathsf{L}^{2,q}_s(\Omega) \boxplus \eta\bar{\mathcal{\lambda}}^q_s] \cap {}_0\mathbb{\Delta}^q_{<\frac{N}{2}}(\Omega)\big)\Big) \cap \mathsf{L}^{2,q}_s(\Omega) $$

$$ {}_\varepsilon\mathbb{L}^{2,q}_s(\Omega) = {}_0\mathring{\mathbb{D}}^q_s(\Omega) \dotplus \varepsilon^{-1}{}_0\mathbb{\Delta}^q_s(\Omega) \dotplus \Delta_\varepsilon\eta\bar{\mathcal{P}}^q_{s-2} $$

The first two terms in the second decomposition are $\langle\varepsilon\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^{2,q}(\Omega)}$-orthogonal.

$$ \mathsf{L}^{2,q}_s(\Omega) \cap {}_\varepsilon\mathcal{H}^q_{-s}(\Omega)^{\perp_\varepsilon} = {}_0\mathring{\mathbb{D}}^q_s(\Omega) \oplus_\varepsilon \varepsilon^{-1}{}_0\mathbb{\Delta}^q_s(\Omega) $$

(iii) $s < -N/2$:
deco. holds, but loosing directness, larger space of Dirichlet/Neumann forms

STEP TWO

polynomially weighted Helmholtz decomposition for large weights s

$$\mathsf{L}_s^{2,q,q+1}(\Omega) = \big(\mathsf{Tri}_s^q(\Omega) \dot{+} \mathsf{Reg}_s^{q,-1}(\Omega)\big) \cap \mathsf{L}_s^{2,q,q+1}(\Omega)$$

with projections Π and $\Pi_{\mathrm{reg}} := (1 - \Pi)$ as well as $t \leq s$ and $t < N/2$

$$(N(M) =)\, \mathsf{Tri}_s^q(\Omega) = \Pi \mathsf{L}_s^{2,q,q+1}(\Omega) \subset {}_0\mathring{\mathsf{D}}_t^q(\Omega) \times {}_0\Delta_t^{q+1}(\Omega)$$

$$\mathsf{Reg}_s^{q,-1}(\Omega) = \Pi_{\mathrm{reg}} \mathsf{L}_s^{2,q,q+1}(\Omega) \subset {}_0\Delta_t^q(\Omega) \times {}_0\mathring{\mathsf{D}}_t^{q+1}(\Omega)$$

still: $\operatorname{supp} \hat{\lambda}$ compact

Theorem (low frequency asymptotics on $\mathsf{L}_s^{2,q,q+1}(\Omega)$ in local norms)

Let $J \in \mathbb{N}_0$ and $s \in (J + 1/2, \infty) \setminus \mathbb{I}$. Then for all bounded subdomains $\Omega_{\mathrm{b}} \subset \Omega$

$$\Big\| \mathcal{L}_\omega f + \omega^{-1}\Pi f - \sum_{j=0}^{J-1} \omega^j \mathcal{L}_0^{j+1} \Pi_{\mathrm{reg}} f - \omega^{N-1} \sum_{j=0}^{J-N} \omega^j \Gamma_j f \Big\|_{\mathsf{L}^{2,q,q+1}(\Omega_{\mathrm{b}})} = O\big(|\omega|^J\big) \|f\|_{\mathsf{L}_s^{2,q,q+1}(\Omega)}$$

holds uniformly with respect to $f \in \mathsf{L}_s^{2,q,q+1}(\Omega)$ and $0 \neq \omega \in \mathbb{C}_+$ small enough.

STEPS THREE AND FOUR

- cutting technique $\Rightarrow$ bounded domain and unbounded domain
- comparing with the homogeneous whole space case $\Omega = \mathbb{R}^N$ and $\Lambda = \mathrm{Id}$
  - represent solution by convolution with fundamental solution
  - Taylor expansion of fundamental solution (Hankel's function)

  $\Rightarrow$ low frequency asymptotics in this special case
- low frequency asymptotics in weighted norms $\mathrm{L}_t^{2,q,q+1}(\Omega)$
- approx. of asymptotically homo. media by compactly supported media (convergence in operator norm)

done ■

CLASSICAL TIME-HARMONIC SCATTERING MAXWELL PROBLEM AND EDDY-CURRENT APPROXIMATION

time-harmonic Maxwell (electro-magnetic scattering) problem in $\mathbb{R}^3$ (now with σ!),
$\Omega_{\text{ext}} \subset \mathbb{R}^3$ exterior domain
with complement $\Omega_{\text{int}} := \mathbb{R}^3 \setminus \overline{\Omega}_{\text{ext}} \subset \mathbb{R}^3$ bounded domain
giving the support of σ, i.e., $\operatorname{supp}\sigma = \overline{\Omega}_{\text{int}}$ compact

$$\begin{aligned}
\sigma E_\omega - \operatorname{rot} H_\omega + \mathrm{i}\,\omega\varepsilon E_\omega &= F && \text{in } \mathbb{R}^3 && \text{(pde)}\\
\operatorname{rot} E_\omega + \mathrm{i}\,\omega\mu H_\omega &= G && \text{in } \mathbb{R}^3 && \text{(pde)}\\
E_\omega\,,\ H_\omega &\in \mathbf{H}_{<-1/2}(\operatorname{rot};\mathbb{R}^3) && && \text{(decay cond.)}\\
\xi \times E_\omega + H_\omega\,,\quad -\xi \times H_\omega + E_\omega &\in \mathsf{L}^2_{>-1/2}(\mathbb{R}^3) && && \text{(Silver-Müller radiation cond.)}
\end{aligned}$$

as before: $0 \neq \omega \in \mathbb{C}$, $\varepsilon, \mu \in \mathsf{L}^\infty(\Omega, \mathbb{R}^{3\times 3})$, sym, unif. pos. def.,

Remark/Theorems Solution theories and asymptotics hold as before with more or less obvious changes.

Remark As before, generalization to differential forms is straight forward.
Let's stay here with the classical case of vector analysis.

CLASSICAL TIME-HARMONIC SCATTERING MAXWELL PROBLEM AND EDDY-CURRENT APPROXIMATION

time-harmonic Maxwell (electro-magnetic scattering) problem in $\mathbb{R}^3$ (with σ!),
$\Omega_{\text{ext}} \subset \mathbb{R}^3$ exterior domain, $\Omega_{\text{int}} := \mathbb{R}^3 \setminus \overline{\Omega}_{\text{ext}} \subset \mathbb{R}^3$ bounded domain, $\operatorname{supp}\sigma = \overline{\Omega}_{\text{int}}$

$$\begin{aligned}
\sigma E_\omega - \operatorname{rot} H_\omega + \mathrm{i}\,\omega\varepsilon E_\omega &= F && \text{in } \mathbb{R}^3 && \text{(pde)}\\
\operatorname{rot} E_\omega + \mathrm{i}\,\omega\mu H_\omega &= G && \text{in } \mathbb{R}^3 && \text{(pde)}\\
E_\omega\,,\ H_\omega &\in \mathbf{H}_{<-1/2}(\operatorname{rot};\mathbb{R}^3) && && \text{(decay cond.)}\\
\xi \times E_\omega + H_\omega\,,\quad -\xi \times H_\omega + E_\omega &\in \mathsf{L}^2_{>-1/2}(\mathbb{R}^3) && && \text{(Silver-Müller radiation cond.)}
\end{aligned}$$

time-harmonic eddy-current Maxwell problem in $\mathbb{R}^3$, (no radiation condition!)

$$\begin{aligned}
\sigma E_\omega^{\text{ec}} - \operatorname{rot} H_\omega^{\text{ec}} &= F && \text{in } \mathbb{R}^3 && \text{(pde)}\\
\operatorname{rot} E_\omega^{\text{ec}} + \mathrm{i}\,\omega\mu H_\omega^{\text{ec}} &= G && \text{in } \mathbb{R}^3 && \text{(pde)}\\
\operatorname{div} \varepsilon E_\omega^{\text{ec}}|_{\Omega_{\text{ext}}} &= 0 && \text{in } \Omega_{\text{ext}} && \text{(pde)}\\
\varepsilon E_\omega^{\text{ec}}|_{\Omega_{\text{ext}}} &\perp_{\mathsf{L}^2(\Omega_{\text{ext}})} \mathring{\mathrm{B}}(\Omega_{\text{ext}}) && && \text{(cohomology or kernel condition)}\\
E_\omega^{\text{ec}}\,,\ H_\omega^{\text{ec}} &\in \mathbf{H}_{-1}(\operatorname{rot};\mathbb{R}^3) && && \text{(decay cond.)}
\end{aligned}$$

KNOWN: low freq. asympt. for (E_ω, H_ω), i.e., $\boxed{\lim_{\omega\to 0} E_\omega = \surd, \quad \lim_{\omega\to 0} H_\omega = \surd}$

QUESTIONS / AIMS: low freq. asympt. for $(E_\omega^{\text{ec}}, H_\omega^{\text{ec}})$ and $(E_\omega - E_\omega^{\text{ec}}, H_\omega - H_\omega^{\text{ec}})$, i.e.,

$$\boxed{\lim_{\omega\to 0} E_\omega^{\text{ec}}, \quad \lim_{\omega\to 0} H_\omega^{\text{ec}} \quad \lim_{\omega\to 0} E_\omega - E_\omega^{\text{ec}} \quad \lim_{\omega\to 0} H_\omega - H_\omega^{\text{ec}} \quad ?}$$

CLASSICAL TIME-HARMONIC SCATTERING MAXWELL PROBLEM AND EDDY-CURRENT APPROXIMATION

as before, more compact and proper notation

time-harmonic Maxwell (electro-magnetic scattering) problem in $\mathbb{R}^3$ (with σ!)

$$(M - \omega + \Sigma)u_\omega = f \in \mathsf{L}^2_{>1/2}(\mathbb{R}^3) \times \mathsf{L}^2_{>1/2}(\mathbb{R}^3)$$

$$u_\omega \in \mathbf{H}_{<-1/2}(\mathrm{rot};\mathbb{R}^3) \times \mathbf{H}_{<-1/2}(\mathrm{rot};\mathbb{R}^3)$$

$$(S+1)u_\omega \in \mathsf{L}^2_{>-1/2}(\mathbb{R}^3) \times \mathsf{L}^2_{>-1/2}(\mathbb{R}^3)$$

here: $\Sigma = \mathrm{i}\,\Lambda^{-1}\begin{bmatrix}\sigma & 0\\ 0 & \mu\end{bmatrix}$ compact perturbation of s.a. unbd. lin. op. M

$\Rightarrow$ sol. theo's for time-harm. and stat. prob. and asym. more or less the same

sol. op's: $\mathcal{L}_{\sigma,\omega}, \quad \mathcal{L}_{\sigma,0}$

time-harmonic eddy-current Maxwell problem in $\mathbb{R}^3$,

$$(M - \omega P + \Sigma)u^{\mathrm{ec}}_\omega = f \in \left(\mathsf{L}^2_1(\mathbb{R}^3) \cap \mathbf{H}(\mathrm{div}\,0;\Omega_{\mathrm{ext}}) \cap \mathring{\mathsf{B}}(\Omega_{\mathrm{ext}})^{\perp_{\mathsf{L}^2(\Omega_{\mathrm{ext}})}}\right) \times \mathsf{L}^2(\mathbb{R}^3)$$

$$\mathrm{div}\,\varepsilon E^{\mathrm{ec}}_\omega|_{\Omega_{\mathrm{ext}}} = 0$$

$$\varepsilon E^{\mathrm{ec}}_\omega|_{\Omega_{\mathrm{ext}}} \perp_{\mathsf{L}^2(\Omega_{\mathrm{ext}})} \mathring{\mathsf{B}}(\Omega_{\mathrm{ext}})$$

$$u^{\mathrm{ec}}_\omega \in \mathbf{H}_{-1}(\mathrm{rot};\mathbb{R}^3) \times \mathbf{H}(\mathrm{rot};\mathbb{R}^3)$$

here: projector P with $P(E,H) := (0,H)$

sol. op's: $\mathcal{L}^{\mathrm{ec}}_{\sigma,\omega}$ and again (as before) $\mathcal{L}^{\mathrm{ec}}_{\sigma,0} = \mathcal{L}_{\sigma,0}$

LOW FREQUENCY ASYMPTOTICS FOR THE EDDY-CURRENT APPROXIMATION

KNOWN: low freq. asympt. of $\mathcal{L}_{\sigma,\omega}$

QUESTIONS / AIMS: low freq. asympt. of $\mathcal{L}^{\text{ec}}_{\sigma,\omega}$ and $\mathcal{L}_{\sigma,\omega} - \mathcal{L}^{\text{ec}}_{\sigma,\omega}$, i.e.,

$$\boxed{\lim_{\mathbb{C}\setminus\{0\}\ni\omega\to 0} \mathcal{L}^{\text{ec}}_{\sigma,\omega} \quad \text{and} \quad \lim_{\mathbb{C}\setminus\{0\}\ni\omega\to 0} \mathcal{L}_{\sigma,\omega} - \mathcal{L}^{\text{ec}}_{\sigma,\omega} \quad ?}$$

ANSWER: The first asym. is trivial (Neumann series) and hence the second as well as the asym. of $\mathcal{L}_{\sigma,\omega}$ is already known!

NOTE:

- asym. of $\mathcal{L}_{\sigma,\omega}$ very complicated
- asym. of $\mathcal{L}^{\text{ec}}_{\sigma,\omega}$ very simple (Neumann series as in the bounded domain case!)

LOW FREQUENCY ASYMPTOTICS FOR THE EDDY-CURRENT APPROXIMATION

sol. op. for the eddy-current prob.

$$\begin{aligned}(M - \omega P + \Sigma)u^{\mathrm{ec}}_{\omega} &= f \in \Big(\mathsf{L}^2_1(\mathbb{R}^3) \cap \mathbf{H}(\operatorname{div} 0; \Omega_{\mathrm{ext}}) \cap \mathring{\mathsf{B}}(\Omega_{\mathrm{ext}})^{\perp_{\mathsf{L}^2(\Omega_{\mathrm{ext}})}}\Big) \times \mathsf{L}^2(\mathbb{R}^3) \\ \operatorname{div} \varepsilon E^{\mathrm{ec}}_{\omega}|_{\Omega_{\mathrm{ext}}} &= 0 \\ \varepsilon E^{\mathrm{ec}}_{\omega}|_{\Omega_{\mathrm{ext}}} &\perp_{\mathsf{L}^2(\Omega_{\mathrm{ext}})} \mathring{\mathsf{B}}(\Omega_{\mathrm{ext}}) \\ u^{\mathrm{ec}}_{\omega} &\in \mathbf{H}_{-1}(\mathrm{rot}; \mathbb{R}^3) \times \mathbf{H}(\mathrm{rot}; \mathbb{R}^3)\end{aligned}$$

is

$$\mathcal{L}^{\mathrm{ec}}_{\sigma,\omega} : \Big(\mathsf{L}^2_1(\mathbb{R}^3) \cap \mathbf{H}(\operatorname{div} 0; \Omega_{\mathrm{ext}}) \cap \mathring{\mathsf{B}}(\Omega_{\mathrm{ext}})^{\perp_{\mathsf{L}^2(\Omega_{\mathrm{ext}})}}\Big) \times \mathsf{L}^2(\mathbb{R}^3) \to \mathbf{H}_{-1}(\mathrm{rot}; \mathbb{R}^3) \times \mathbf{H}(\mathrm{rot}; \mathbb{R}^3)$$

looking just on the weights

$$\mathcal{L}^{\mathrm{ec}}_{\sigma,\omega} : \mathsf{L}^2_1(\mathbb{R}^3) \times \mathsf{L}^2(\mathbb{R}^3) \to \mathsf{L}^2_{-1}(\mathbb{R}^3) \times \mathsf{L}^2(\mathbb{R}^3)$$

of course: only f sich that $\mathcal{L}^{\mathrm{ec}}_{\sigma,0} f = \mathcal{L}_{\sigma,0} f$ is well def.

LOW FREQUENCY ASYMPTOTICS FOR THE EDDY-CURRENT APPROXIMATION

then roughly:

$$\begin{aligned} & (M-\omega P+\Sigma)u_\omega^{ec} = f \\ \Leftrightarrow \quad & (M+\Sigma)u_\omega^{ec} = f+\omega P u_\omega^{ec} \\ \Leftrightarrow \quad & u_\omega^{ec} = \mathcal{L}_{\sigma,0} f+\omega \mathcal{L}_{\sigma,0} P u_\omega^{ec} \\ \Leftrightarrow \quad & (\mathrm{Id}-\omega \mathcal{L}_{\sigma,0} P)u_\omega^{ec} = \mathcal{L}_{\sigma,0} f \end{aligned}$$

now, unlike in the full time-harmonic Maxwell case, $\mathcal{L}_{\sigma,0}P$ is a nice bounded operator mapping some Hilbert space into itself. For this observe that

$$(U_\omega^{ec}, V_\omega^{ec}) = \mathcal{L}_{\sigma,0} P u_\omega^{ec} = \mathcal{L}_{\sigma,0} P(E_\omega^{ec}, H_\omega^{ec}) = \mathcal{L}_{\sigma,0}(0, H_\omega^{ec})$$

solves

$$\begin{aligned} \sigma U_\omega^{ec} - \operatorname{rot} V_\omega^{ec} &= 0 \\ \operatorname{rot} U_\omega^{ec} &= H_\omega^{ec}, \end{aligned}$$

which is a very weakly coupled system.
looking again just on the weights: $P u_\omega^{ec} = (0, H_\omega^{ec}) \in \mathsf{L}^2_{-1}(\mathbb{R}^3) \times \mathsf{L}^2(\mathbb{R}^3)$

$$H_\omega^{ec} \in \mathsf{L}^2(\mathbb{R}^3) \quad \Rightarrow \quad U_\omega^{ec} \in \mathsf{L}^2_{-1}(\mathbb{R}^3) \quad \text{(no towers)} \quad \Rightarrow \quad V_\omega^{ec} \in \mathsf{L}^2(\mathbb{R}^3) \quad \text{(no towers)},$$

as σU_ω^{ec} has got compact support and hence belongs to any weighted L^2-space

LOW FREQUENCY ASYMPTOTICS FOR THE EDDY-CURRENT APPROXIMATION

looking again just on the weights:

$$\mathcal{L}_{\sigma,0}P : \mathsf{L}^2_{-1}(\mathbb{R}^3) \times \mathsf{L}^2(\mathbb{R}^3) \to \mathsf{L}^2_{-1}(\mathbb{R}^3) \times \mathsf{L}^2(\mathbb{R}^3)$$

more precisely

$$\mathcal{L}_{\sigma,0}P : \mathsf{L}^2_{-1}(\mathbb{R}^3) \times \mathsf{L}^2(\mathbb{R}^3) \to \mathsf{L}^2_{-1}(\mathbb{R}^3) \times \mathsf{L}^2(\mathbb{R}^3)$$

Then

$$(\mathrm{Id} - \omega \mathcal{L}_{\sigma,0}P)u^{\mathrm{ec}}_{\omega} = \mathcal{L}_{\sigma,0}f$$

shows that the asymptotic is simply given by Neumann's series (as in the bounded domain case)

$$\mathcal{L}^{\mathrm{ec}}_{\sigma,\omega}f = u^{\mathrm{ec}}_{\omega} = \sum_{j=1}^{\infty}(\omega\mathcal{L}_{\sigma,0}P)^j\mathcal{L}_{\sigma,0}f = \sum_{j=1}^{\infty}\omega^j(\mathcal{L}_{\sigma,0}P)^j\mathcal{L}_{\sigma,0}f,$$

provided that $|\omega|$ is sufficient small. The series converges in $\mathsf{L}^2_{-1}(\mathbb{R}^3) \times \mathsf{L}^2(\mathbb{R}^3)$, even in $\mathbf{H}_{-1}(\mathrm{rot};\mathbb{R}^3) \times \mathbf{H}(\mathrm{rot};\mathbb{R}^3)$

LOW FREQUENCY ASYMPTOTICS FOR THE DIFFERENCE OF THE FULL TIME-HARMONIC MAXWELL PROBLEM AND THE EDDY-CURRENT APPROXIMATION

now compare (again in $\mathbb{R}^N$ and with differential forms) the asymptotic series

$$\mathcal{L}_{\sigma,\omega} - \sum_{j=0}^{J-1} \omega^j \mathcal{L}_{\sigma,0}^{j+1} - \omega^{N-1} \sum_{j=0}^{J-N} \omega^j \Gamma_j = O\big(|\omega|^J\big)$$

in the norm of bounded linear operators from $\mathsf{L}_s^{2,q,q+1}(\mathbb{R}^N)$ to $\mathsf{L}_t^{2,q,q+1}(\mathbb{R}^N)$ (s large, t small) and the Neumann series

$$\mathcal{L}_{\sigma,\omega}^{\text{ec}} = \sum_{j=1}^{\infty} \omega^j (\mathcal{L}_{\sigma,0} P)^j \mathcal{L}_{\sigma,0}$$

converging in $\mathsf{L}_{-1}^{2,q,q+1}(\mathbb{R}^N) \times \mathsf{L}^{2,q,q+1}(\mathbb{R}^N)$.

LOW FREQUENCY ASYMPTOTICS FOR THE DIFFERENCE OF THE FULL TIME-HARMONIC MAXWELL PROBLEM AND THE EDDY-CURRENT APPROXIMATION

Theorem (low frequency asymptotics)

For all small enough $\mathbb{C}_+ \setminus \{0\} \ni \omega \to 0$ *the following asymptotics hold:*

(i) *If* $f \in \mathrm{Reg}_1^{q,0}(\mathbb{R}^N)$ *then for all* $t < (1-N)/2$

$$\|(\mathcal{L}_{\sigma,\omega} - \mathcal{L}^{ec}_{\sigma,\omega})f\|_{\mathrm{L}^{2,q,q+1}_t(\mathbb{R}^N)} \xrightarrow{\omega \to 0} 0. \qquad \text{(approx. of order 0)}$$

(ii) *If* $f \in \mathrm{Reg}_s^{q,1}(\mathbb{R}^N)$ *with* $s \in (3/2, N/2+1) \setminus \mathbb{I}$ *then for* $t := s - (N+3)/2$

$$\|(\mathcal{L}_{\sigma,\omega} - \mathcal{L}^{ec}_{\sigma,\omega})f\|_{\mathrm{L}^{2,q,q+1}_t(\mathbb{R}^N)} = O\big(|\omega|\big)\,\|f\|_{\mathrm{L}^{2,q,q+1}_s(\Omega)}. \qquad \text{(approx. of order 1)}$$

note in $\mathbb{R}^3$:

$$\mathrm{Reg}_s^{q,0}(\mathbb{R}^3) = \Big(\mathrm{L}^2_s(\mathbb{R}^3) \cap \mathbf{H}_s(\mathrm{div}\,0;\Omega_{\mathrm{ext}}) \cap \mathring{\mathrm{B}}(\Omega_{\mathrm{ext}})^{\perp_{\mathrm{L}^2(\Omega_{\mathrm{ext}})}}\Big) \times \mathbf{H}_s(\mathrm{div}\,0;\mathbb{R}^3),$$

$$\mathrm{Reg}_s^{q,j}(\mathbb{R}^3) = \{f \in \mathrm{Reg}_s^{q,j-1}(\mathbb{R}^3) : \mathcal{L}^j_{\sigma,0} f \in \mathrm{L}^{2,q,q+1}_{s-j}(\mathbb{R}^N)\}$$

LOW FREQUENCY ASYMPTOTICS FOR THE DIFFERENCE OF THE FULL TIME-HARMONIC MAXWELL PROBLEM AND THE EDDY-CURRENT APPROXIMATION

Theorem (low frequency asymptotics, continued)

For all small enough $\mathbb{C}_+ \setminus \{0\} \ni \omega \to 0$ *the following asymptotics hold:*

(iii) *If* $f \in \mathrm{Reg}_s^{q,2}(\mathbb{R}^N)$ *with* $s \in (5/2, N/2+2) \setminus \mathbb{I}$ *then for* $t := s - (N+5)/2$

$$\|(\mathcal{L}_{\sigma,\omega} - \mathcal{L}^{ec}_{\sigma,\omega})f\|_{\mathsf{L}^{2,q,q+1}_t(\mathbb{R}^N)} = O\big(|\omega|^2\big)\,\|f\|_{\mathsf{L}^{2,q,q+1}_s(\Omega)}, \qquad \textit{(approx. of order 2)}$$

if and only if $\delta F = 0$ *in* $\mathbb{R}^N$ *and* $G = 0$.
note: in $\mathbb{R}^3$ *this is* $\operatorname{div} F = 0$ *in* $\mathbb{R}^3$ *and* $G = 0$. *(very simply condition, hidden jump condition for* F*)*

(iv) *If* $f \neq 0$*, the approximation can never be better than* $O\big(|\omega|^2\big)$*, even for* supp f *compact.*

(v) *If* supp $\hat{\Lambda}$ *compact, then* $\mathrm{Reg}_s^{q,1}(\mathbb{R}^N)$ *and* $\mathrm{Reg}_s^{q,2}(\mathbb{R}^N)$ *can be replaced by* $\mathrm{Reg}_s^{q,0}(\mathbb{R}^N)$*. correction operators* Γ_j *change asymptotics just from oder* $O\big(|\omega|^N\big)$ *on. Hence no change in asymtotics.*

note: condition $\operatorname{div} F = 0$ in $\mathbb{R}^3$ is much more complicated in Ammari, Buffa, Nedelec, namely

$$\operatorname{div} F = 0 \text{ in } \Omega_{\text{ext}}, \quad \operatorname{div} F = 0 \text{ in } \Omega_{\text{int}}$$

$$+ \text{ jump cond. on } n \cdot F, \quad + \text{ complicated cohomology cond. on } F \text{ and } n \cdot F$$

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