The Closed Range Property of the De Rham Complex in Unbounded Domains

(joint work with **Marcus Waurick**[@TUBAFG])

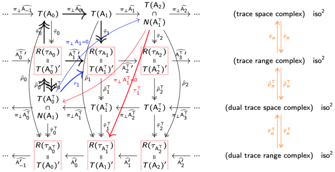
Dirk Pauly

TECHNISCHE UNIVERSITÄT DRESDEN

Rainer Turns 80 in Freiberg @ Dresden (July 27-28, 2026)
A conference on honouring the scientific work of Rainer Picard

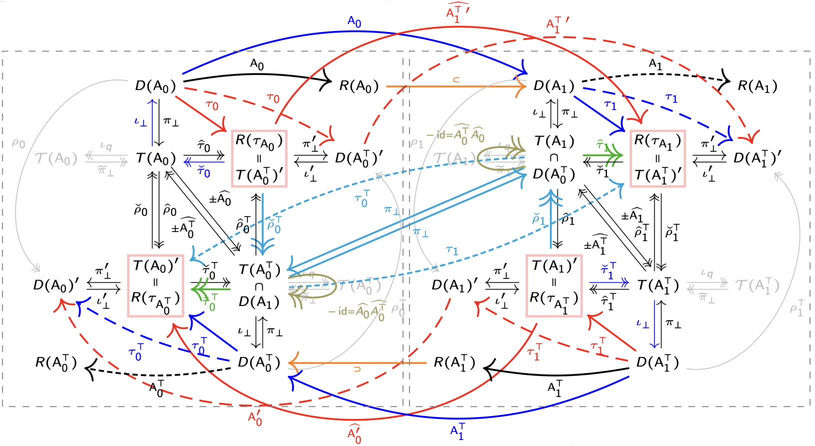
hosts:	Sascha Trostorff	(CAU Kiel)
	Marcus Waurick	(TUBAF)

July 27 @ TUDD

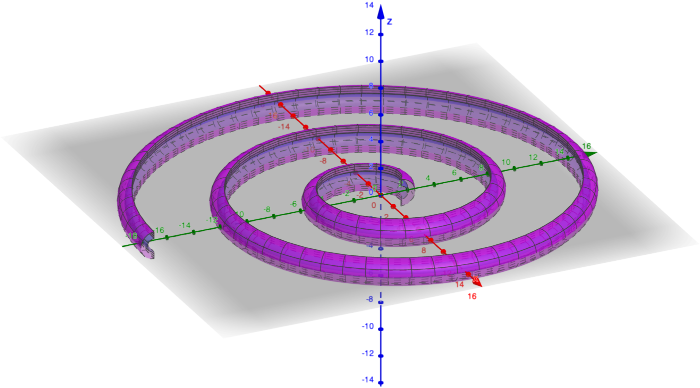


I will not talk about traces...

...but I love this pic!



Does curl have closed range in the infinitely growing half snail shell?



my educated guess one year ago: probably NOT!

now: YES! – paper in the birthday book **Rainer Picard 80**
DP and M. Waurick: *Gaffney's inequality and the Closed Range Property of the de Rham Complex in Unbounded Domains*

ESI Wien
May 5, 2026

Pi80 DD
July 27, 2026

AnaPDE Hannover
September 24, 2026

The Closed Range Property of the De Rham Complex in Unbounded Domains

Hilbert complex / De Rham complex

$$H_0 = L^2 \underset{A_0^* = -\mathrm{div}}{\overset{A_0 = \mathring{\nabla}}{\rightleftarrows}} H_1 = L^2 \underset{A_1^* = \mathrm{rot}}{\overset{A_1 = \mathring{\mathrm{rot}}}{\rightleftarrows}} H_2 = L^2 \underset{A_2^* = -\nabla}{\overset{A_2 = \mathring{\mathrm{div}}}{\rightleftarrows}} H_3 = L^2$$

H_n Hilbert spaces, A_n, A_n^* lddc

fundamental property: $R(A_n)$ closed

closed range thm $\Rightarrow$ $R(A_n)$ cl $\iff$ $R(A_n^*)$ cl

here: $R(\mathring{\nabla})$ cl $\iff$ $R(\mathrm{div})$ cl

$R(\mathring{\mathrm{rot}})$ cl $\iff$ $R(\mathrm{rot})$ cl

$R(\mathring{\mathrm{div}})$ cl $\iff$ $R(\nabla)$ cl

applications

- solution theory / bd inverses
- spectral gap at 0
- low frequency asymptotics (Neumann series)
- exponential stability of the Cauchy problem

reduction step

$A : D(A) \subset H_0 \to H_1$ $\qquad$ $A^* : D(A^*) \subset H_1 \to H_0$

$\mathcal{A} := A|_{N(A)^\perp}$ $\qquad$ $\mathcal{A}^* := A^*|_{N(A^*)^\perp}$ (indeed!)

$D(\mathcal{A}) = D(A) \cap N(A)^\perp$

$N(A)^\perp = \overline{R(A^*)}$ $\qquad$ (projection theo/ Helmholtz) $\qquad$ $N(A^*)^\perp = \overline{R(A)}$

$\Rightarrow$ reduced ops

$\mathcal{A} : D(\mathcal{A}) \subset N(A)^\perp = \overline{R(A^*)} \to \overline{R(A)} = \tilde{H}_1$ $\qquad$ lddc

$\mathcal{A}^* : D(\mathcal{A}^*) \subset N(A^*)^\perp = \overline{R(A)} \to \overline{R(A^*)} = \tilde{H}_0$

$H_0 = N(A) \oplus \overline{R(A^*)}$ $\qquad$ $H_1 = N(A^*) \oplus \overline{R(A)}$

$\Rightarrow$ $D(A) = N(A) \oplus D(\mathcal{A})$ $\qquad$ $D(A^*) = N(A^*) \oplus D(\mathcal{A}^*)$

$\Rightarrow$ $R(A) = R(\mathcal{A})$ (reg. for free) $\qquad$ $R(A^*) = R(\mathcal{A}^*)$

$\Rightarrow$ $\mathcal{A}^{-1} : R(A) \to D(\mathcal{A})$, $\quad (\mathcal{A}^*)^{-1} : R(A^*) \to D(\mathcal{A}^*)$

exist, might be unbd.

fundamental lemma 1

$|\mathcal{A}^{-1}| = |(\mathcal{A}^*)^{-1}|$

(i) $\mathcal{A}^{-1} : R(A) \to D(\mathcal{A})$ bd

(i') $(\mathcal{A}^*)^{-1} : R(A^*) \to D(\mathcal{A}^*)$ bd

(ii) $\exists\, c_A > 0 \ \forall\, x \in D(\mathcal{A})$ $\quad |x| \le c_A |Ax|$ $\qquad$ (Friedrichs)

(ii') $\exists\, c_{A^*} > 0 \ \forall\, y \in D(\mathcal{A}^*)$ $\quad |y| \le c_{A^*} |A^* y|$

(iii) $R(A) = R(\mathcal{A})$ closed

(iii') $R(A^*) = R(\mathcal{A}^*)$ closed

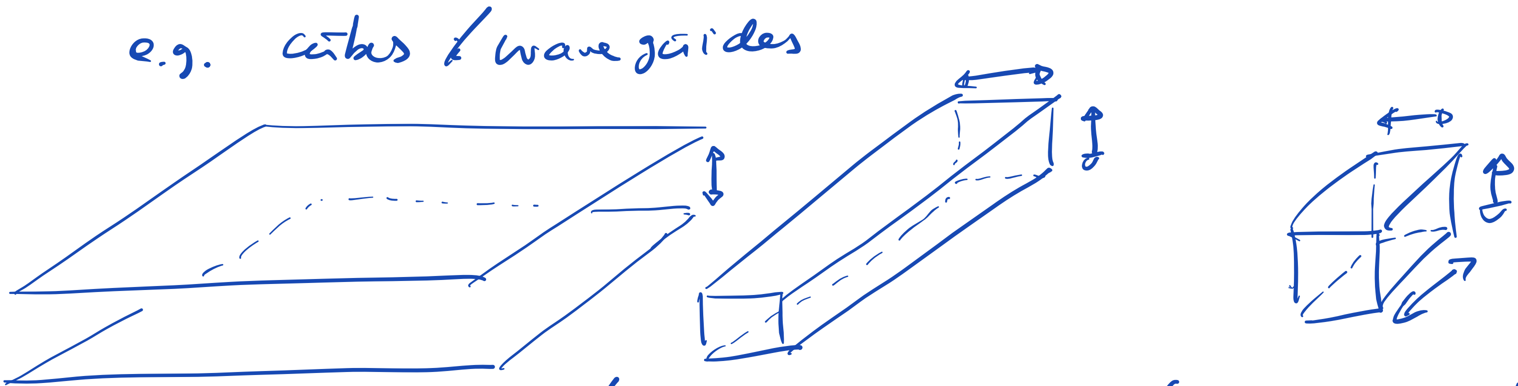
lemma 2 $\quad D(\mathcal{A}) \hookrightarrow H_0 \iff D(\mathcal{A}^*) \hookrightarrow H_1$

$\Rightarrow$ assumptions of lemma 1 hold.

typical situations

- $\Omega \subset \mathbb{R}^N$ bd Lip domain / Riemannian manifold ⇒ cpt emb. realistic ⇒ closed ranges
- $\mathbb{R}^N \setminus \overline{\Omega}$ bd ⇒ cpt emb. fails ⇒ maybe no closed ranges
- our solution: use polynomially weighted Sobolev spaces
- another question: partially bd domains

e.g. cubes / waveguides



bd in 1 direction / bd in 2 directions / bd in 3 directions

well known

- $\mathring{\nabla}: \mathring{H}^1 \subset L^2 \to L^2$, $\Omega \subset \mathbb{R}^3$ bd in 1 dir

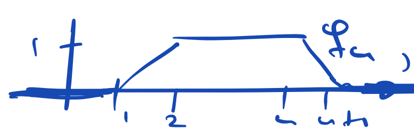
⇒ $R(\mathring{\nabla})$ closed $\quad \left(\text{fund. theo } u(x_3) = \underbrace{u(0)}_{=0} + \int_0^{x_3} \partial_3 u\right)$

⇔ $R(\operatorname{div})$ closed
cl. ran

- $\nabla: H^1 \subset L^2 \to L^2$, $\Omega \subset \mathbb{R}^3$

$R(\nabla)$ closed ⇔ $\Omega \subset \mathbb{R}^3$ bd in 3 dir ⇔ Ω bd

⇔ $R(\mathring{\operatorname{div}})$ closed
cl. ran



$u_n = f_n(|\cdot|)$

⇒ $|u_n|^2_{L^2} \sim n^3$, $|\nabla u_n|^2_{L^2} \sim n^2$

main result $\Omega \text{ (cube)} \subset \mathbb{R}^3$

- $R(\mathring{\nabla})$ cl $\iff$ $R(\operatorname{div})$ cl $\iff$ Ω bd in 1 dir
- $R(\mathring{\operatorname{rot}})$ cl $\iff$ $R(\operatorname{rot})$ cl $\iff$ Ω bd in 2 dir
- $R(\nabla)$ cl $\iff$ $R(\mathring{\operatorname{div}})$ cl $\iff$ Ω bd in 3 dir

moreover $\phi : \Theta \to \Omega$ bi-lipschitz trafo

$(\det \phi' \geq c > 0)$

$$\Rightarrow \quad R(\mathring{\nabla}_\Omega) \text{ cl} \iff R(\mathring{\nabla}_\Theta) \text{ cl}$$

$$R(\mathring{\operatorname{rot}}_\Omega) \text{ cl} \iff R(\mathring{\operatorname{rot}}_\Theta) \text{ cl}$$

$$R(\mathring{\operatorname{div}}_\Omega) \text{ cl} \iff R(\mathring{\operatorname{div}}_\Theta) \text{ cl}$$

n.b. $R(\mathring{\operatorname{rot}}_{\Gamma_t})$ cl $\not\Leftarrow$ Ω bd in 1 dir

mixed bc

$n \cdot E|_{\Gamma_n} = 0$ (normal)

$n \times E \times n|_{\Gamma_t} = 0$ (tangential)

sketch of proof (Ω unbd cube ) ⑤

① show Friedrichs/Poincaré type estimate

$$\forall E \in D(\mathring{\mathrm{rot}}) \cap N(\mathring{\mathrm{rot}})^{\perp} \qquad |E|_{L^2} \le c\, |\mathrm{rot}\, E|_{L^2}$$

② use complex property

$$N(\mathring{\mathrm{rot}})^{\perp} = \overline{R(\mathrm{rot})} \subset N(\mathrm{div}) \subset D(\mathrm{div})$$

$$\Rightarrow D(\mathring{\mathrm{rot}}) \cap N(\mathring{\mathrm{rot}})^{\perp} \subset D(\mathring{\mathrm{rot}}) \cap N(\mathrm{div})$$
$$\subset D(\mathring{\mathrm{rot}}) \cap D(\mathrm{div}) = \mathring{H}(\mathrm{rot}) \cap H(\mathrm{div})$$

③ show/use 1-Gaffney on (unbounded cubes)

$$E \in D(\mathring{\mathrm{rot}}) \cap D(\mathrm{div}) \Rightarrow E \in H^1 \cap D(\mathring{\mathrm{rot}})$$
$$\wedge\ |\nabla E|^2_{L^2} \le |\mathrm{rot}\, E|^2_{L^2} + |\mathrm{div}\, E|^2_{L^2}$$
(even =)

④ $u \in H^1$ with suitable homogeneous Dirichlet bc on each component E_n (zero on one face of the cube)

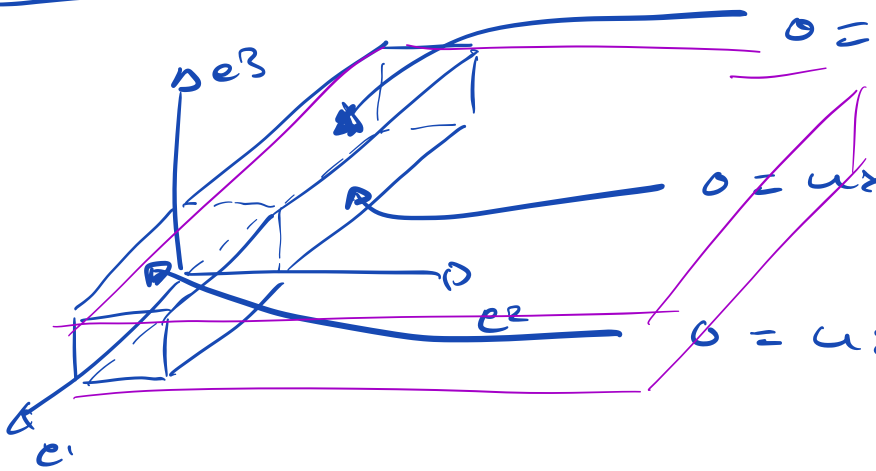
$\Rightarrow$ Friedrichs estimate

$$|u|_{L^2} \le c_F\, |\nabla u|_{L^2}$$

⑤ ③ ∧ ④ $\Rightarrow$

$$|E|^2_{L^2} = \sum_n |E_n|^2_{L^2} \le c_F^2 \sum_n |\nabla E_n|^2_{L^2} = c_F^2\, |\nabla E|^2_{L^2}$$
$$\le c_F^2\, |\mathrm{rot}\, E|^2_{L^2} \quad \text{as} \quad \mathrm{div}\, E = 0$$

observation $\quad E \in H^1 \cap H(\mathrm{rot})$



$$0 = n \cdot E \big|_{x_3 = 1} = E_3$$

$$0 = n \times E \times n \big|_{x_3 = 0} = \begin{bmatrix} E_1 \\ E_2 \\ 0 \end{bmatrix}$$

$$0 = n \times E \times n \big|_{x_2 = 0} \quad \begin{bmatrix} E_1 \\ 0 \\ E_3 \end{bmatrix}$$

$$\Rightarrow \left.\begin{array}{lll} E_1, E_2 = 0 & \text{on} & x_3 = 0 \\ E_1, E_3 = 0 & \text{on} & x_2 = 0 \end{array}\right\} \Rightarrow \checkmark$$

$$\left.\begin{array}{lll} E_1, E_2 = 0 & \text{on} & x_3 = 0 \\ E_3 = 0 & \text{on} & x_3 = 1 \end{array}\right\} \Rightarrow \checkmark$$

additional candies (nice and simple proofs)

- Gaffney estimate / integration by parts
- Lipschitz trafo theo
- applications for waveguides