On some Hilbert complexes, related compact embeddings, . . . and more

Dirk Pauly

Fakultät für Mathematik

UNIVERSITÄT DUISBURG ESSEN

Open-Minded

;-)

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Department of Mathematics, UiO

Hosts: Nils Henrik Risebro & Ragnar Winther

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classical de Rham complex in 3D (∇-rot-div-complex)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot\cup \Gamma_n}$

(electro-magnetics, Maxwell's equations)

$$\{0\} \;\underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\,\mathrm{div}}{\overset{\mathring{\nabla}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\mathrm{rot}}{\overset{\mathring{\mathrm{rot}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\nabla}{\overset{\mathring{\mathrm{div}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}}\; \mathbb{R}$$

mixed boundary conditions and inhomogeneous and anisotropic media

$$\{0\} \text{ or } \mathbb{R} \;\underset{\pi}{\overset{\iota}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\,\mathrm{div}_{\Gamma_n}\varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2_{\varepsilon} \;\underset{\varepsilon^{-1}\,\mathrm{rot}_{\Gamma_n}}{\overset{\mu^{-1}\,\mathrm{rot}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2_{\mu} \;\underset{-\nabla_{\Gamma_n}}{\overset{\mathrm{div}_{\Gamma_t}\,\mu}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota}{\overset{\pi}{\rightleftarrows}}\; \mathbb{R} \text{ or } \{0\}$$

for this talk: $\varepsilon = \mu = 1$ ($=$ id) and no mixed boundary conditions
for all appearing complexes

de Rham complex in ND or on Riemannian manifolds (d-complex)

$\Omega \subset \mathbb{R}^N$ bd w. Lip. dom. or Ω Riemannian manifold with cpt cl. and Lip. boundary Γ

(generalized Maxwell equations | the mother of all complexes |)

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^{2,0} \underset{-\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \mathsf{L}^{2,1} \underset{-\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \cdots \boxed{\mathsf{L}^{2,q} \underset{-\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \mathsf{L}^{2,q+1}} \cdots \mathsf{L}^{2,N-1} \underset{-\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \mathsf{L}^{2,N} \underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}} \mathbb{R}$$

elasticity complex in 3D ($\operatorname{sym}\nabla$-$\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}$-$\operatorname{Div}_{\mathbb{S}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{Div}_{\mathbb{S}}}{\overset{\operatorname{sym}\mathring{\nabla}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}}{\overset{\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^{\top}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{-\operatorname{sym}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathsf{RM}}}{\overset{\pi_{\mathsf{RM}}}{\rightleftarrows}} \mathsf{RM}$$

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\operatorname{Rot}_{\mathbb{S}}$-$\operatorname{Div}_{\mathbb{T}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \;\underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\nabla\nabla}}{\rightleftarrows}}\; \mathsf{L}^2_{\mathbb{S}} \;\underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}}\; \mathsf{L}^2_{\mathbb{T}} \;\underset{-\operatorname{dev}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota_{\mathsf{RT}}}{\overset{\pi_{\mathsf{RT}}}{\rightleftarrows}}\; \mathsf{RT}$$

general complex

$$A_0 : D(A_0) \subset H_0 \to H_1, \qquad A_1 : D(A_1) \subset H_1 \to H_2$$
$$A_0^* : D(A_0^*) \subset H_1 \to H_0, \qquad A_1^* : D(A_1^*) \subset H_2 \to H_1 \qquad \text{(lddc)}$$

general complex property $A_1 A_0 = 0$,
i.e., $R(A_0) \subset N(A_1)$ and/or eq $R(A_1^*) \subset N(A_0^*)$

$$\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} H_0 \underset{A_0^*}{\overset{A_0}{\rightleftarrows}} H_1 \underset{A_1^*}{\overset{A_1}{\rightleftarrows}} H_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$$

general observations

$$Ax = f$$

general theory

- solution theory
- Friedrichs/Poincaré estimates and constants
- Helmholtz/Hodge/Weyl decompositions
- compact embeddings
- continuous and compact inverse operators
- closed ranges
- variational formulations
- functional a posteriori error estimates
- generalized div-curl-lemma
- ...

idea: solve problem with general and simple linear functional analysis

$\Rightarrow$ functional analysis toolbox (fa-toolbox) ...

general observations

$$\mathrm{A}x = f$$

let's say $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ linear and H_0, H_1 Hilbert spaces

question: How to solve?

general observations

$$\mathrm{A}x = f$$

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ linear

solution theory in the sense of Hadamard

- existence $\Leftrightarrow$ $f \in R(\mathrm{A})$
- uniqueness $\Leftrightarrow$ A inj $\Leftrightarrow$ $N(\mathrm{A}) = \{0\}$ $\Leftrightarrow$ A^{-1} exists
- cont dep on f $\Leftrightarrow$ A^{-1} cont

$\Rightarrow$ $x = \mathrm{A}^{-1} f \in D(\mathrm{A})$ and cont estimate (Friedrichs/Poincaré type estimate)

$$|x|_{\mathrm{H}_0} = |\mathrm{A}^{-1} f|_{\mathrm{H}_0} \leq c_{\mathrm{A}} |f|_{\mathrm{H}_1} = c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$$

$\Rightarrow$ best constant $c_{\mathrm{A}} = |\mathrm{A}^{-1}|_{R(\mathrm{A}),\mathrm{H}_0}$ $|\mathrm{A}^{-1}|_{R(\mathrm{A}),D(\mathrm{A})} = (c_{\mathrm{A}}^2 + 1)^{1/2}$

general observations

$A : D(A) \subset H_0 \to H_1$

$A^* : D(A^*) \subset H_1 \to H_0$ Hilbert space adjoint

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$H_1 = \overline{R(A)} \oplus N(A^*) \qquad H_0 = N(A) \oplus \overline{R(A^*)}$$

$$Ax = f$$

solution theory in the sense of Hadamard

- existence $\Leftrightarrow$ $f \in R(A) = N(A^*)^\perp$
- uniqueness $\Leftrightarrow$ A inj $\Leftrightarrow$ $N(A) = \{0\}$ $\Leftrightarrow$ A^{-1} exists
- cont dep on f $\Leftrightarrow$ A^{-1} cont $\Leftrightarrow$ $R(A)$ cl (cl range theo)

fund range cond: $\boxed{R(A) = \overline{R(A)} \text{ closed}}$ (must hold $\rightsquigarrow$ right setting!)

kernel cond: $\boxed{N(A) = \{0\}}$ (fails in gen $\rightsquigarrow$ proj onto $N(A)^\perp = \overline{R(A^*)}$)

general observations

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$\mathsf{H}_1 = \overline{R(\mathsf{A})} \oplus N(\mathsf{A}^*) \qquad \mathsf{H}_0 = N(\mathsf{A}) \oplus \overline{R(\mathsf{A}^*)}$$

remarkable observations

- time-dependent problems are simple
  in gen $\mathsf{A} : D(\mathsf{A}) \subset \mathsf{H} \to \mathsf{H}$, $\mathsf{A} = \partial_t + \mathsf{T}$ (gen T skew-sa, or alt lsast Re $\mathsf{T} \geq 0$)

  $$N(\mathsf{A}) = \{0\} \qquad N(\mathsf{A}^*) = \{0\} \qquad R(\mathsf{A}) \text{ (cl) } = N(\mathsf{A}^*)^\perp = \mathsf{H}$$

- time-harmonic problems are more complicated
  in gen $\mathsf{A} : D(\mathsf{A}) \subset \mathsf{H} \to \mathsf{H}$, $\mathsf{A} = -\omega + \mathsf{T}$

  $$N(\mathsf{A}),\ N(\mathsf{A}^*) \text{ (fin dim)} \qquad R(\mathsf{A}) \text{ (cl, fin co-dim) } = N(\mathsf{A}^*)^\perp$$

  (Fredholm alternative)

- stat problems are most complicated
  in gen $\mathsf{A} : D(\mathsf{A}) \subset \mathsf{H}_0 \to \mathsf{H}_1$

  $$\dim N(\mathsf{A}) = \dim N(\mathsf{A}^*) = \infty \text{ (possibly)} \qquad R(\mathsf{A}) \text{ (cl, infin co-dim) } = N(\mathsf{A}^*)^\perp$$

fa-toolbox for linear (first order) problems/systems

$$Ax = f$$

general theory

- solution theory
- Friedrichs/Poincaré estimates and constants
- Helmholtz/Hodge/Weyl decompositions
- compact embeddings
- continuous and compact inverse operators
- closed ranges
- variational formulations
- functional a posteriori error estimates
- generalized div-curl-lemma
- ...

idea: solve problem with general and simple linear functional analysis
($\Rightarrow$ fa-toolbox) ...

literature: probably very well known for ages, but hard to find ...

Friedrichs, Weyl, Hörmander, Fredholm, von Neumann, Riesz, Banach, ... ?

Why not rediscover and extend/modify for our purposes?

1st fundamental observations

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ lddc, $\quad \mathrm{A}^* : D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0$ Hilbert space adjoint

$(\mathrm{A}, \mathrm{A}^*)$ dual pair as $(\mathrm{A}^*)^* = \overline{\mathrm{A}} = \mathrm{A}$

A, A^* may not be inj

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$\mathrm{H}_1 = N(\mathrm{A}^*) \oplus \overline{R(\mathrm{A})} \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

reduced operators restr to $N(\mathrm{A})^\perp$ and $N(\mathrm{A}^*)^\perp$

$$\mathcal{A} := \mathrm{A}|_{N(\mathrm{A})^\perp} = \mathrm{A}|_{\overline{R(\mathrm{A}^*)}} \qquad \mathcal{A}^* := \mathrm{A}^*|_{N(\mathrm{A}^*)^\perp} = \mathrm{A}^*|_{\overline{R(\mathrm{A})}}$$

$\mathcal{A}$, $\mathcal{A}^*$ inj $\quad \Rightarrow \quad \mathcal{A}^{-1}$, $(\mathcal{A}^*)^{-1}$ ex

1st fundamental observations

$\mathrm{A}: D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1, \quad \mathrm{A}^*: D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0$ lddc $\qquad (\mathrm{A}, \mathrm{A}^*)$ dual pair

$$\mathrm{H}_1 = N(\mathrm{A}^*) \oplus \overline{R(\mathrm{A})} \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

more precisely

$$\mathcal{A} := \mathrm{A}|_{\overline{R(\mathrm{A}^*)}} : D(\mathcal{A}) \subset \overline{R(\mathrm{A}^*)} \to \overline{R(\mathrm{A})}, \qquad D(\mathcal{A}) := D(\mathrm{A}) \cap N(\mathrm{A})^\perp = D(\mathrm{A}) \cap \overline{R(\mathrm{A}^*)}$$
$$\mathcal{A}^* := \mathrm{A}^*|_{\overline{R(\mathrm{A})}} : D(\mathcal{A}^*) \subset \overline{R(\mathrm{A})} \to \overline{R(\mathrm{A}^*)}, \quad D(\mathcal{A}^*) := D(\mathrm{A}^*) \cap N(\mathrm{A}^*)^\perp = D(\mathrm{A}^*) \cap \overline{R(\mathrm{A})}$$

$(\mathcal{A}, \mathcal{A}^*)$ dual pair and $\mathcal{A}$, $\mathcal{A}^*$ inj $\Rightarrow$

inverse ops exist (and bij)

$$\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A}) \qquad (\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$$

refined decompositions

$$D(\mathrm{A}) = N(\mathrm{A}) \oplus D(\mathcal{A}) \qquad D(\mathrm{A}^*) = N(\mathrm{A}^*) \oplus D(\mathcal{A}^*)$$

$\Rightarrow$

$$R(\mathrm{A}) = R(\mathcal{A}) \qquad R(\mathrm{A}^*) = R(\mathcal{A}^*)$$

1st fundamental observations

closed range theorem & closed graph theorem $\Rightarrow$

Lemma (Friedrichs-Poincaré type est/cl range/cont inv)

The following assertions are equivalent:

- **(i)** $\exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathrm{H}_0} \le c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$
- **(i*)** $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathrm{H}_1} \le c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$
- **(ii)** $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_1.
- **(ii*)** $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_0.
- **(iii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is continuous and bijective.*
- **(iii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is continuous and bijective.*

In case that one of the latter assertions is true, e.g., (ii), $R(\mathrm{A})$ *is closed, we have*

$$\mathrm{H}_0 = N(\mathrm{A}) \oplus R(\mathrm{A}^*) \qquad \mathrm{H}_1 = N(\mathrm{A}^*) \oplus R(\mathrm{A})$$
$$D(\mathrm{A}) = N(\mathrm{A}) \oplus D(\mathcal{A}) \qquad D(\mathrm{A}^*) = N(\mathrm{A}^*) \oplus D(\mathcal{A}^*)$$
$$D(\mathcal{A}) = D(\mathrm{A}) \cap R(\mathrm{A}^*) \qquad D(\mathcal{A}^*) = D(\mathrm{A}^*) \cap R(\mathrm{A})$$

and $\quad \mathcal{A} : D(\mathcal{A}) \subset R(\mathrm{A}^*) \to R(\mathrm{A}), \quad \mathcal{A}^* : D(\mathcal{A}^*) \subset R(\mathrm{A}) \to R(\mathrm{A}^*).$

1st fundamental observations

recall

$$\textbf{(i)}\quad \exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathrm{H}_0} \le c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$$

$$\textbf{(i}^*\textbf{)}\quad \exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathrm{H}_1} \le c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$$

'best' consts in **(i)** and **(i*)** equal norms of the inv ops and Rayleigh quotients

$$c_{\mathrm{A}} = |\mathcal{A}^{-1}|_{R(\mathrm{A}),R(\mathrm{A}^*)} \qquad c_{\mathrm{A}^*} = |(\mathcal{A}^*)^{-1}|_{R(\mathrm{A}^*),R(\mathrm{A})}$$

$$\frac{1}{c_{\mathrm{A}}} = \inf_{0 \ne x \in D(\mathcal{A})} \frac{|\mathrm{A}x|_{\mathrm{H}_1}}{|x|_{\mathrm{H}_0}} \qquad \frac{1}{c_{\mathrm{A}^*}} = \inf_{0 \ne y \in D(\mathcal{A}^*)} \frac{|\mathrm{A}^* y|_{\mathrm{H}_0}}{|y|_{\mathrm{H}_1}}$$

Lemma (Friedrichs-Poincaré type const)

$c_{\mathrm{A}} = c_{\mathrm{A}^*}$

1st fundamental observations

Lemma (cpt emb/cpt inv)

The following assertions are equivalent:

- **(i)** $D(\mathcal{A}) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_0$ *is compact.*
- **(i*)** $D(\mathcal{A}^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1$ *is compact.*
- **(ii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to R(\mathrm{A}^*)$ *is compact.*
- **(ii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to R(\mathrm{A})$ *is compact.*

Lemma (Friedrichs-Poincaré type est/cl range/cont inv)

$\Downarrow$ $\boxed{D(\mathcal{A}) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_0 \textit{ compact}}$

- **(i)** $\exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \quad |x|_{\mathrm{H}_0} \le c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$
- **(i*)** $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \quad |y|_{\mathrm{H}_1} \le c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$
- **(ii)** $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_1.
- **(ii*)** $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_0.
- **(iii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is continuous and bijective.*
- **(iii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is continuous and bijective.*

(i)-(iii*) *equi & the resp Helm deco hold &* $|\mathcal{A}^{-1}| = c_{\mathrm{A}} = c_{\mathrm{A}^*} = |(\mathcal{A}^*)^{-1}|$

2nd fundamental observations

So far no complex...

$A_0 : D(A_0) \subset H_0 \to H_1, \quad A_1 : D(A_1) \subset H_1 \to H_2$ (lddc)
$A_0^* : D(A_0^*) \subset H_1 \to H_0, \quad A_1^* : D(A_1^*) \subset H_2 \to H_1$ (lddc)

general complex ($A_1 A_0 = 0$, i.e., $R(A_0) \subset N(A_1)$ and $R(A_1^*) \subset N(A_0^*)$)

$$\cdots \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad H_0 \quad \underset{A_0^*}{\overset{A_0}{\rightleftarrows}} \quad H_1 \quad \underset{A_1^*}{\overset{A_1}{\rightleftarrows}} \quad H_2 \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad \cdots$$

recall Helmholtz deco

$$\begin{aligned} H_1 &= \overline{R(A_0)} \oplus N(A_0^*) \\ &\qquad \cap \qquad\quad \cup \\ &= N(A_1) \oplus \overline{R(A_1^*)} \end{aligned} \qquad \Rightarrow \quad \text{(e.g.)} \quad N(A_1) = \overline{R(A_0)} \oplus \big(\underbrace{N(A_1) \cap N(A_0^*)}_{=:K_1} \big)$$

$\Rightarrow$ refined Helmholtz deco

$$H_1 = \overline{R(A_0)} \oplus K_1 \oplus \overline{R(A_1^*)}$$

2nd fundamental observations

recall

$$D(\mathrm{A}_1) = D(\mathcal{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \qquad R(\mathrm{A}_1) = R(\mathcal{A}_1) \qquad R(\mathrm{A}_1^*) = R(\mathcal{A}_1^*)$$

$$D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \cap \overline{R(\mathrm{A}_0)} \qquad R(\mathrm{A}_0^*) = R(\mathcal{A}_0^*) \qquad R(\mathrm{A}_0) = R(\mathcal{A}_0)$$

cohomology group $K_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)$

Lemma (Helmholtz deco I)

$$\mathrm{H}_1 = \overline{R(\mathrm{A}_0)} \oplus N(\mathrm{A}_0^*) \qquad \mathrm{H}_1 = \overline{R(\mathrm{A}_1^*)} \oplus N(\mathrm{A}_1)$$

$$D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus N(\mathrm{A}_0^*) \qquad D(\mathrm{A}_1) = D(\mathcal{A}_1) \oplus N(\mathrm{A}_1)$$

$$N(\mathrm{A}_1) = D(\mathcal{A}_0^*) \oplus K_1 \qquad N(\mathrm{A}_0^*) = D(\mathcal{A}_1) \oplus K_1$$

$$D(\mathrm{A}_1) = \overline{R(\mathrm{A}_0)} \oplus \big(D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)\big) \qquad D(\mathrm{A}_0^*) = \overline{R(\mathrm{A}_1^*)} \oplus \big(D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1)\big)$$

Lemma (Helmholtz deco II)

$$\mathrm{H}_1 = \overline{R(\mathrm{A}_0)} \oplus K_1 \oplus \overline{R(\mathrm{A}_1^*)}$$

$$D(\mathrm{A}_1) = \overline{R(\mathrm{A}_0)} \oplus K_1 \oplus D(\mathcal{A}_1)$$

$$D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus K_1 \oplus \overline{R(\mathrm{A}_1^*)}$$

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus K_1 \oplus D(\mathcal{A}_1)$$

2nd fundamental observations

$$K_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \qquad D(\mathrm{A}_1) = D(\mathcal{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \qquad D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \cap \overline{R(\mathrm{A}_0)}$$

Lemma (cpt emb II)

The following assertions are equivalent:

(i) $D(\mathcal{A}_0) \hookrightarrow\!\!\!\!\to \mathrm{H}_0$, $D(\mathcal{A}_1) \hookrightarrow\!\!\!\!\to \mathrm{H}_1$, *and* $K_1 \hookrightarrow\!\!\!\!\to \mathrm{H}_1$ *are compact.*

(ii) $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\to \mathrm{H}_1$ *is compact.*

In this case $K_1 < \infty$.

Theorem (fa-toolbox I)

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\to \mathrm{H}_1 \text{ compact}}$

(i) *all emb cpt, i.e.,* $D(\mathcal{A}_0) \hookrightarrow\!\!\!\!\to \mathrm{H}_0$, $D(\mathcal{A}_1) \hookrightarrow\!\!\!\!\to \mathrm{H}_1$, $D(\mathcal{A}_0^*) \hookrightarrow\!\!\!\!\to \mathrm{H}_1$, $D(\mathcal{A}_1^*) \hookrightarrow\!\!\!\!\to \mathrm{H}_2$ *cpt*

(ii) *cohomology group* K_1 *finite dim*

(iii) *all ranges closed, i.e.,* $R(\mathrm{A}_0)$, $R(\mathrm{A}_0^*)$, $R(\mathrm{A}_1)$, $R(\mathrm{A}_1^*)$ *cl*

(iv) *all Friedrichs-Poincaré type est hold*

(v) *all Hodge-Helmholtz-Weyl type deco I & II hold with closed ranges*

2nd fundamental observations

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

Theorem (fa-toolbox I (Friedrichs-Poincaré type est))

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1 \ \textit{compact}}$ $\Rightarrow$ $\exists$ $|\mathcal{A}_i^{-1}| = c_{\mathrm{A}_i} = c_{\mathrm{A}_i^*} = |(\mathcal{A}_i^*)^{-1}| \in (0,\infty)$

- **(i)** $\forall\, x \in D(\mathcal{A}_0)$ $\quad |x|_{\mathrm{H}_0} \leq c_{\mathrm{A}_0} |\mathrm{A}_0 x|_{\mathrm{H}_1}$
- **(i*)** $\forall\, y \in D(\mathcal{A}_0^*)$ $\quad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}_0} |\mathrm{A}_0^* y|_{\mathrm{H}_0}$
- **(ii)** $\forall\, y \in D(\mathcal{A}_1)$ $\quad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |\mathrm{A}_1 y|_{\mathrm{H}_2}$
- **(ii*)** $\forall\, z \in D(\mathcal{A}_1^*)$ $\quad |z|_{\mathrm{H}_2} \leq c_{\mathrm{A}_1} |\mathrm{A}_1^* z|_{\mathrm{H}_1}$
- **(iii)** $\forall\, y \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ $\quad |(1-\pi_{K_1})y|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |\mathrm{A}_1 y|_{\mathrm{H}_2} + c_{\mathrm{A}_0} |\mathrm{A}_0^* y|_{\mathrm{H}_0}$

note $\pi_{K_1} y \in K_1$ and $(1-\pi_{K_1})y \in K_1^{\perp}$

Remark

enough $R(\mathrm{A}_0)$ *and* $R(\mathrm{A}_1)$ *cl*

2nd fundamental observations

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

Theorem (fa-toolbox I (Helmholtz deco))

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1 \text{ compact}}$

$$\begin{aligned}
\mathrm{H}_1 &= R(\mathrm{A}_0) \oplus N(\mathrm{A}_0^*) \\
D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus N(\mathrm{A}_0^*) \\
N(\mathrm{A}_1) &= D(\mathcal{A}_0^*) \oplus K_1 \\
D(\mathrm{A}_1) &= R(\mathrm{A}_0) \oplus \big(D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)\big)
\end{aligned}$$

$$\begin{aligned}
\mathrm{H}_1 &= R(\mathrm{A}_1^*) \oplus N(\mathrm{A}_1) \\
D(\mathrm{A}_1) &= D(\mathcal{A}_1) \oplus N(\mathrm{A}_1) \\
N(\mathrm{A}_0^*) &= D(\mathcal{A}_1) \oplus K_1 \\
D(\mathrm{A}_0^*) &= R(\mathrm{A}_1^*) \oplus \big(D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1)\big)
\end{aligned}$$

$$\begin{aligned}
\mathrm{H}_1 &= R(\mathrm{A}_0) \oplus K_1 \oplus R(\mathrm{A}_1^*) \\
D(\mathrm{A}_1) &= R(\mathrm{A}_0) \oplus K_1 \oplus D(\mathcal{A}_1) \\
D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus K_1 \oplus R(\mathrm{A}_1^*) \\
D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus K_1 \oplus D(\mathcal{A}_1)
\end{aligned}$$

Remark

enough $R(\mathrm{A}_0)$ *and* $R(\mathrm{A}_1)$ *cl*

(stat) first order system - solution theory

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

$\boxed{\mathrm{A}_1 x = f}$ $\qquad \dim N(\mathrm{A}_1) = \infty$

find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ such that the fos

$$\begin{aligned} \mathrm{A}_1 x &= f \\ \mathrm{A}_0^* x &= g \\ \pi_{K_1} x &= k \end{aligned} \qquad \text{think of} \qquad \begin{aligned} (\mathring{\mathrm{rot}} E &= F) \\ (-\operatorname{div} E &= g) \\ (\pi_{\mathrm{D}} E &= K) \end{aligned}$$

kernel = cohomology group = $K_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)$

trivially necessary $\quad f \in R(\mathrm{A}_1) \quad g \in R(\mathrm{A}_0^*) \quad k \in K_1$

apply fa-toolbox

(stat) first order system - solution theory

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ st fos $\quad \mathrm{A}_1 x = f \qquad \mathrm{A}_0^* x = g \qquad \pi_{K_1} x = k$

Theorem (fa-toolbox II (solution theory))

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1 \textit{ compact}}$

fos is uniq sol $\Leftrightarrow$ $f \in R(\mathrm{A}_1) \qquad g \in R(\mathrm{A}_0^*) \qquad k \in K_1$

$$x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus K_1 = D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$$

$$\boxed{x_f := \mathcal{A}_1^{-1} f} \in D(\mathcal{A}_1)$$

$$\boxed{x_g := (\mathcal{A}_0^*)^{-1} g} \in D(\mathcal{A}_0^*)$$

dep cont on data $|x|_{\mathrm{H}_1} \le |x_f|_{\mathrm{H}_1} + |x_g|_{\mathrm{H}_1} + |k|_{\mathrm{H}_1} \le c_{\mathrm{A}_1} |f|_{\mathrm{H}_2} + c_{\mathrm{A}_0} |g|_{\mathrm{H}_0} + |k|_{\mathrm{H}_1}$

moreover

$$\pi_{R(\mathrm{A}_1^*)} x = x_f \qquad \pi_{R(\mathrm{A}_0)} x = x_g \qquad \pi_{K_1} x = k \qquad |x|_{\mathrm{H}_1}^2 = |x_f|_{\mathrm{H}_1}^2 + |x_g|_{\mathrm{H}_1}^2 + |k|_{\mathrm{H}_1}^2$$

Remark

enough $R(\mathrm{A}_0)$ *and* $R(\mathrm{A}_1)$ *cl*

(stat) first order system - variational formulations

$$x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus K_1 = D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$$
$$x_f := \mathcal{A}_1^{-1} f \in D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap R(\mathrm{A}_1^*) = D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \cap K_1^\perp$$
$$x_g := (\mathcal{A}_0^*)^{-1} g \in D(\mathcal{A}_0^*) = D(\mathrm{A}_0^*) \cap R(\mathrm{A}_0) = D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1) \cap K_1^\perp$$

$$\begin{aligned} \mathrm{A}_1 x &= f \\ \mathrm{A}_0^* x &= g \\ \pi_{K_1} x &= k \end{aligned} \qquad \begin{aligned} \mathrm{A}_1 x_f &= f \\ \mathrm{A}_0^* x_f &= 0 \\ \pi_{K_1} x_f &= 0 \end{aligned} \qquad \begin{aligned} \mathrm{A}_1 x_g &= 0 \\ \mathrm{A}_0^* x_g &= g \\ \pi_{K_1} x_g &= 0 \end{aligned} \qquad \begin{aligned} \mathrm{A}_1 k &= 0 \\ \mathrm{A}_0^* k &= 0 \\ \pi_{K_1} k &= k \end{aligned}$$

- option I: find x_f and x_g separately $\Rightarrow$ $x = x_f + x_g + k$
- option II: find x directly

(stat) first order system - variational formulations I

finding

$$x_f := \mathcal{A}_1^{-1} f \in D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap \underbrace{R(\mathrm{A}_1^*)}_{=R(\mathcal{A}_1^*)} = D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \cap K_1^{\perp}$$

$$\begin{aligned} \mathrm{A}_1 x_f &= f \\ \mathrm{A}_0^* x_f &= 0 \\ \pi_{K_1} x_f &= 0 \end{aligned}$$

at least two options

- option Ia: multiply $\mathrm{A}_1 x_f = f$ by $\mathrm{A}_1 \xi$ $\Rightarrow$

$$\forall\, \xi \in D(\mathcal{A}_1) \qquad \langle \mathrm{A}_1 x_f, \mathrm{A}_1 \xi \rangle_{\mathrm{H}_2} = \langle f, \mathrm{A}_1 \xi \rangle_{\mathrm{H}_2}$$

  weak form of $\boxed{\mathrm{A}_1^* \mathrm{A}_1 x_f = \mathrm{A}_1^* f}$

- option Ib: repr $x_f = \mathrm{A}_1^* y_f$ with potential $y_f = (\mathcal{A}_1^*)^{-1} x_f \in D(\mathcal{A}_1^*)$
  and mult by x_f by $\mathrm{A}_1^* \phi$ $\Rightarrow$

$$\forall\, \phi \in D(\mathcal{A}_1^*) \qquad \langle \mathrm{A}_1^* y_f, \mathrm{A}_1^* \phi \rangle_{\mathrm{H}_1} = \langle x_f, \mathrm{A}_1^* \phi \rangle_{\mathrm{H}_1} = \langle \mathrm{A}_1 x_f, \phi \rangle_{\mathrm{H}_2} = \langle f, \phi \rangle_{\mathrm{H}_2}$$

  weak form of $\boxed{\mathrm{A}_1 x_f = f}$ and $\boxed{\mathrm{A}_1 \mathrm{A}_1^* y_f = f}$

analogously for x_g

(stat) first order system - a posteriori error estimates

problem: find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ st $\mathrm{A}_1 x = f$ $\mathrm{A}_0^* x = g$ $\pi_{K_1} x = k$

'very' non-conforming 'approximation' of x: $\tilde{x} \in \mathrm{H}_1$

def., dcmp. err. $e = x - \tilde{x} = \pi_{R(\mathrm{A}_0)} e + \pi_{K_1} e + \pi_{R(\mathrm{A}_1^*)} e \in \mathrm{H}_1 = R(\mathrm{A}_0) \oplus K_1 \oplus R(\mathrm{A}_1^*)$

Theorem (sharp upper bounds)

Let $\tilde{x} \in \mathrm{H}_1$ and $e = x - \tilde{x}$. Then

$$|e|_{\mathrm{H}_1}^2 = |\pi_{R(\mathrm{A}_0)} e|_{\mathrm{H}_1}^2 + |\pi_{K_1} e|_{\mathrm{H}_1}^2 + |\pi_{R(\mathrm{A}_1^*)} e|_{\mathrm{H}_1}^2$$

$$|\pi_{R(\mathrm{A}_0)} e|_{\mathrm{H}_1} = \min_{\phi \in D(\mathrm{A}_0^*)} \big(c_{\mathrm{A}_0} |\mathrm{A}_0^* \phi - g|_{\mathrm{H}_0} + |\phi - \tilde{x}|_{\mathrm{H}_1} \big)$$

reg $(\mathrm{A}_0 \mathrm{A}_0^* + 1)$-prbl in $D(\mathrm{A}_0^*)$

$$|\pi_{R(\mathrm{A}_1^*)} e|_{\mathrm{H}_1} = \min_{\varphi \in D(\mathrm{A}_1)} \big(c_{\mathrm{A}_1} |\mathrm{A}_1 \varphi - f|_{\mathrm{H}_2} + |\varphi - \tilde{x}|_{\mathrm{H}_1} \big)$$

reg $(\mathrm{A}_1^* \mathrm{A}_1 + 1)$-prbl in $D(\mathrm{A}_1)$

$$|\pi_{K_1} e|_{\mathrm{H}_1} = |\pi_{K_1} \tilde{x} - k|_{\mathrm{H}_1} = \min_{\substack{\xi \in D(\mathrm{A}_0) \\ \zeta \in D(\mathrm{A}_1^*)}} |\mathrm{A}_0 \xi + \mathrm{A}_1^* \zeta + \tilde{x} - k|_{\mathrm{H}_1}$$

cpld $(\mathrm{A}_0^* \mathrm{A}_0)$-$(\mathrm{A}_1 \mathrm{A}_1^*)$-sys in $D(\mathcal{A}_0)$-$D(\mathcal{A}_1^*)$

Remark

Even $\pi_{K_1} e = k - \pi_{K_1} \tilde{x}$ and the minima are attained at

$$\hat{\phi} = \pi_{R(\mathrm{A}_0)} e + \tilde{x}, \qquad \hat{\varphi} = \pi_{R(\mathrm{A}_1^*)} e + \tilde{x}, \qquad \mathrm{A}_0 \hat{\xi} + \mathrm{A}_1^* \hat{\zeta} = (\pi_{K_1} - 1) \tilde{x}.$$

A_0^*-A_1-lemma (generalized global div-curl-lemma)

Lemma (A_0^*-A_1-lemma)

Let $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathrm{H}_1$ be compact, and

(i) (x_n) *bounded in* $D(\mathrm{A}_1)$,

(ii) (y_n) *bounded in* $D(\mathrm{A}_0^*)$.

$\Rightarrow \exists\ x \in D(\mathrm{A}_1),\ y \in D(\mathrm{A}_0^*)$ *and subsequences st*

$x_n \rightharpoonup x$ *in* $D(\mathrm{A}_1)$ *and* $y_n \rightharpoonup y$ *in* $D(\mathrm{A}_0^*)$ *as well as*

$$\langle x_n, y_n \rangle_{\mathrm{H}_1} \to \langle x, y \rangle_{\mathrm{H}_1}.$$

classical de Rham complex in 3D (∇-rot-div-complex)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot\cup \Gamma_n}$

(electro-magneto dynamics, Maxwell's equations)

$$\{0\} \;\underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\operatorname{div}}{\overset{\mathring{\nabla}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\operatorname{rot}}{\overset{\mathring{\operatorname{rot}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\nabla}{\overset{\mathring{\operatorname{div}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}}\; \mathbb{R}$$

mixed boundary conditions and inhomogeneous and anisotropic media

$$\{0\} \text{ or } \mathbb{R} \;\underset{\pi}{\overset{\iota}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\operatorname{div}_{\Gamma_n}\varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2_{\varepsilon} \;\underset{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota}{\overset{\pi}{\rightleftarrows}}\; \mathbb{R} \text{ or } \{0\}$$

classical de Rham complex in 3D (∇-rot-div-complex)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot\cup \Gamma_n}$

(electro-magneto dynamics, Maxwell's equations with mixed boundary conditions)

$$\{0\} \text{ or } \mathbb{R} \;\underset{\pi}{\overset{\iota}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\operatorname{div}_{\Gamma_n}\varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2_\varepsilon \;\underset{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota}{\overset{\pi}{\rightleftarrows}}\; \mathbb{R} \text{ or } \{0\}$$

related fos

$$\begin{array}{ll|ll|ll|ll} \nabla_{\Gamma_t} u = A & \text{in } \Omega & \operatorname{rot}_{\Gamma_t} E = J & \text{in } \Omega & \operatorname{div}_{\Gamma_t} H = k & \text{in } \Omega & \pi v = b & \text{in } \Omega \\ \pi u = a & \text{in } \Omega & -\operatorname{div}_{\Gamma_n} \varepsilon E = j & \text{in } \Omega & \varepsilon^{-1}\operatorname{rot}_{\Gamma_n} H = K & \text{in } \Omega & -\nabla_{\Gamma_n} v = B & \text{in } \Omega \end{array}$$

related sos

$$\begin{array}{ll|ll|ll} -\operatorname{div}_{\Gamma_n} \varepsilon \nabla_{\Gamma_t} u = j & \text{in } \Omega & \varepsilon^{-1}\operatorname{rot}_{\Gamma_n}\operatorname{rot}_{\Gamma_t} E = K & \text{in } \Omega & -\nabla_{\Gamma_n}\operatorname{div}_{\Gamma_t} H = B & \text{in } \Omega \\ \pi u = a & \text{in } \Omega & -\operatorname{div}_{\Gamma_n} \varepsilon E = j & \text{in } \Omega & \varepsilon^{-1}\operatorname{rot}_{\Gamma_n} H = K & \text{in } \Omega \end{array}$$

corresponding compact embeddings:

$$D(\nabla_{\Gamma_t}) \cap D(\pi) = D(\nabla_{\Gamma_t}) = \mathsf{H}^1_{\Gamma_t} \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

$$D(\operatorname{rot}_{\Gamma_t}) \cap D(-\operatorname{div}_{\Gamma_n}\varepsilon) = \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n} \hookrightarrow \mathsf{L}^2_\varepsilon \qquad \text{(Weck's selection theorem, '74)}$$

$$D(\operatorname{div}_{\Gamma_t}) \cap D(\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}) = \mathsf{D}_{\Gamma_t} \cap \mathsf{R}_{\Gamma_n} \hookrightarrow \mathsf{L}^2 \qquad \text{(Weck's selection theorem, '74)}$$

$$D(\nabla_{\Gamma_n}) \cap D(\pi) = D(\nabla_{\Gamma_n}) = \mathsf{H}^1_{\Gamma_n} \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

Weck's selection theorem for weak Lip. dom. and mixed bc: Bauer/Py/Schomburg ('16)

Weck's selection theorem (Weck '74, (Habil. '72) stimulated by Rolf Leis)
(Weber '80, Picard '84, Costabel '90, Witsch '93, Jochmann '97, Kuhn '99, Picard/Weck/Witsch '01, Py '96, '03, '06, '07, '08)

classical de Rham complex in 3D (∇-rot-div-complex)

$$\begin{aligned} \operatorname{rot} E &= F && \text{in } \Omega \\ -\operatorname{div} \varepsilon E &= g && \text{in } \Omega \\ \nu \times E &= 0 && \text{at } \Gamma_t \\ \nu \cdot \varepsilon E &= 0 && \text{at } \Gamma_n \end{aligned}$$

non-trivial kernel $\mathcal{H}_{\mathrm{D},\varepsilon} = \{H \in \mathrm{L}^2 : \operatorname{rot} H = 0, \operatorname{div} \varepsilon H = 0, \nu \times H|_{\Gamma_t} = 0, \nu \cdot \varepsilon H|_{\Gamma_n} = 0\}$
additional condition on Dirichlet/Neumann fields for uniqueness

$$\pi_{\mathrm{D}} E = K \in \mathcal{H}_{\mathrm{D},\varepsilon}$$

$$\{0\} \text{ or } \mathbb{R} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathrm{L}^2 \underset{-\operatorname{div}_{\Gamma_n} \varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}} \mathrm{L}^2_\varepsilon \underset{\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}} \mathrm{L}^2 \underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}} \mathrm{L}^2 \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathbb{R} \text{ or } \{0\}$$

$$\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_{-1} \underset{\mathrm{A}_{-1}^*}{\overset{\mathrm{A}_{-1}}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\mathrm{A}_2^*}{\overset{\mathrm{A}_2}{\rightleftarrows}} \mathrm{H}_3 \underset{\mathrm{A}_3^*}{\overset{\mathrm{A}_3}{\rightleftarrows}} \mathrm{H}_4 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$$

find $E \in \mathrm{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1} \mathrm{D}_{\Gamma_n}(\Omega)$ st (fos) find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ st

$$\begin{aligned} \operatorname{rot}_{\Gamma_t} E &= F & \qquad & & \mathrm{A}_1 x &= f \\ -\operatorname{div}_{\Gamma_n} \varepsilon E &= g & & \text{translation} & \mathrm{A}_0^* x &= g \\ \pi_{\mathrm{D/N}} E &= K & & & \pi_{K_1} x &= k \end{aligned}$$

classical de Rham complex in 3D (∇-rot-div-complex)

$c_{A_0} = c_{fp}$ (Friedrichs/Poincaré constant) and $c_{A_1} = c_m$ (Maxwell constant)

Lemma/Theorem $\Downarrow$ $D(A_1) \cap D(A_0^*) \hookrightarrow\!\!\!\rightarrow L^2_\varepsilon(\Omega)$ compact

(i) all Friedrichs-Poincaré type est hold

$$\begin{aligned}
&\forall\, \varphi \in D(\mathcal{A}_0) && |\varphi|_{H_0} \le c_{A_0} |A_0\varphi|_{H_1} && \Leftrightarrow && \forall\, \varphi \in H^1_{\Gamma_t} && |\varphi|_{L^2} \le c_{fp} |\nabla\varphi|_{L^2_\varepsilon} \\
&\forall\, \phi \in D(\mathcal{A}_0^*) && |\phi|_{H_1} \le c_{A_0} |A_0^*\phi|_{H_0} && \Leftrightarrow && \forall\, \Phi \in \varepsilon^{-1} D_{\Gamma_n} \cap \nabla H^1_{\Gamma_t} && |\Phi|_{L^2_\varepsilon} \le c_{fp} |\operatorname{div} \varepsilon\Phi|_{L^2} \\
&\forall\, \varphi \in D(\mathcal{A}_1) && |\varphi|_{H_1} \le c_{A_1} |A_1\varphi|_{H_2} && \Leftrightarrow && \forall\, \Phi \in R_{\Gamma_t} \cap \varepsilon^{-1} \operatorname{rot} R_{\Gamma_n} && |\Phi|_{L^2_\varepsilon} \le c_m |\operatorname{rot}\Phi|_{L^2} \\
&\forall\, \psi \in D(\mathcal{A}_1^*) && |\psi|_{H_2} \le c_{A_1} |A_1^*\psi|_{H_1} && \Leftrightarrow && \forall\, \Psi \in R_{\Gamma_n} \cap \operatorname{rot} R_{\Gamma_t} && |\Psi|_{L^2} \le c_m |\operatorname{rot}\Psi|_{L^2_\varepsilon}
\end{aligned}$$

(ii) all ranges $R(A_0) = \nabla H^1_{\Gamma_t}$, $R(A_1) = \operatorname{rot} R_{\Gamma_t}$, $R(A_0^*) = \operatorname{div} D_{\Gamma_n}$ are cl in L^2

(iii) the inverse ops $(\widetilde{\nabla}_{\Gamma_t})^{-1}$, $(\widetilde{\operatorname{div}_{\Gamma_n}\varepsilon})^{-1}$, $(\widetilde{\operatorname{rot}}_{\Gamma_t})^{-1}$, $(\widetilde{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}})^{-1}$ are cont, even cpt

(iv) all Helmholtz decomposition hold, e.g.,

$$H_1 = R(A_0) \oplus K_1 \oplus R(A_1^*) \quad \Leftrightarrow \quad L^2_\varepsilon = \nabla H^1_{\Gamma_t} \oplus_{L^2_\varepsilon} \mathcal{H}_{D,\varepsilon} \oplus_{L^2_\varepsilon} \varepsilon^{-1} \operatorname{rot} R_{\Gamma_n}$$

(v) solution theory

(vi) variational formulations

(vii) functional a posteriori error estimates

(viii) div-curl-lemma

(ix) ...

classical de Rham complex in 3D (∇-rot-div-complex)

Theorem (sharp upper bounds)

Let $\boxed{\tilde{E} \in \mathsf{L}^2_\varepsilon}$ *(very non-conforming approximation of E!) and* $\boxed{e := E - \tilde{E}}$. *Then*

$$|e|^2_{\mathsf{L}^2_\varepsilon} = |\pi_{R(\nabla_{\Gamma_t})} e|^2_{\mathsf{L}^2_\varepsilon} + |\pi_{R(\varepsilon^{-1}\operatorname{rot}_{\Gamma_n})} e|^2_{\mathsf{L}^2_\varepsilon} + |\pi_{\mathcal{H}_{\mathrm{D},\varepsilon}} e|^2_{\mathsf{L}^2_\varepsilon}$$

$$= \min_{\Phi \in \varepsilon^{-1}\mathsf{D}_{\Gamma_n}} \Big(c_{\mathrm{fp}} |\operatorname{div} \varepsilon\Phi + g|_{\mathsf{L}^2} + |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon} \Big)^2$$

reg $(-\nabla_{\Gamma_t} \operatorname{div}_{\Gamma_n} + 1)$*-prbl in* D_{Γ_n}

$$+ \min_{\Phi \in \mathsf{R}_{\Gamma_t}} \Big(c_{\mathrm{m}} |\operatorname{rot} \Phi - F|_{\mathsf{L}^2} + |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon} \Big)^2$$

reg $(\operatorname{rot}_{\Gamma_n} \operatorname{rot}_{\Gamma_t} + 1)$*-prbl in* R_{Γ_t}

$$+ \min_{\phi \in \mathsf{H}^1_{\Gamma_t}, \Psi \in \mathsf{R}_{\Gamma_n}} |\nabla\phi + \varepsilon^{-1} \operatorname{rot} \Psi + \tilde{E} - K|^2_{\mathsf{L}^2_\varepsilon}$$

cpld $(-\operatorname{div}_{\Gamma_n} \nabla_{\Gamma_t})$-$(\operatorname{rot}_{\Gamma_t} \operatorname{rot}_{\Gamma_n})$*-sys in* $\mathsf{H}^1_{\Gamma_t}$-R_{Γ_n}

Remark

- $(\operatorname{rot}_{\Gamma_t} \operatorname{rot}_{\Gamma_n})$*-prbl needs saddle point formulation*
- Ω *top trv* $\Rightarrow \pi_{\mathrm{D}} = 0$ *and* $\mathsf{R}_{\Gamma_t,0} = \nabla \mathsf{H}^1_{\Gamma_t}$ *and* $\mathsf{D}_{\Gamma_n,0} = \operatorname{rot} \mathsf{R}_{\Gamma_n}$
- $\boxed{\Omega \text{ convex and } \varepsilon = \mu = 1 \text{ and } \Gamma_t = \Gamma \text{ or } \Gamma_n = \Gamma \Rightarrow c_{\mathrm{f}} \le c_{\mathrm{m}} \le c_{\mathrm{p}} \le \frac{\operatorname{diam}_\Omega}{\pi}}$

div-curl-lemma

Lemma (div-curl-lemma (global version))

Assumptions:

- **(i)** (E_n) *bounded in* $\mathsf{L}^2(\Omega)$
- **(i')** (H_n) *bounded in* $\mathsf{L}^2(\Omega)$
- **(ii)** $(\operatorname{rot} E_n)$ *bounded in* $\mathsf{L}^2(\Omega)$
- **(ii')** $(\operatorname{div} \varepsilon H_n)$ *bounded in* $\mathsf{L}^2(\Omega)$
- **(iii)** $\nu \times E_n = 0$ *on* Γ_t, *i.e.,* $E_n \in \mathsf{R}_{\Gamma_t}(\Omega)$
- **(iii')** $\nu \cdot \varepsilon H_n = 0$ *on* Γ_n, *i.e.,* $H_n \in \varepsilon^{-1}\mathsf{D}_{\Gamma_n}(\Omega)$

$\Rightarrow \exists\ E, H$ *and subsequences st*

$E_n \rightharpoonup E$, $\operatorname{rot} E_n \rightharpoonup \operatorname{rot} E$ *and* $H_n \rightharpoonup H$, $\operatorname{div} H_n \rightharpoonup \operatorname{div} H$ *in* $\mathsf{L}^2(\Omega)$ *and*

$$\langle E_n, H_n \rangle_{\mathsf{L}^2_\varepsilon(\Omega)} \to \langle E, H \rangle_{\mathsf{L}^2_\varepsilon(\Omega)}$$

de Rham complex in ND or on Riemannian manifolds (d-complex)

$\Omega \subset \mathbb{R}^N$ bd w. Lip. dom. or Ω Riemannian manifold with cpt cl. and Lip. boundary Γ

(generalized Maxwell equations)

$$\{0\} \quad \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \quad \mathsf{L}^{2,0} \quad \underset{-\,\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \quad \mathsf{L}^{2,1} \quad \underset{-\,\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \quad \cdots \quad \mathsf{L}^{2,q} \quad \underset{-\,\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \quad \mathsf{L}^{2,q+1} \ldots \mathsf{L}^{2,N-1} \quad \underset{-\,\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \quad \mathsf{L}^{2,N} \quad \underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}} \quad \mathbb{R}$$

de Rham complex in ND or on Riemannian manifolds (d-complex)

$\Omega \subset \mathbb{R}^N$ bd w. Lip. dom. or Ω Riemannian manifold with cpt cl. and Lip. boundary Γ

(generalized Maxwell equations)

$$\{0\} \text{ or } \mathbb{R} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathsf{L}^{2,0} \underset{-\delta^1_{\Gamma_n}}{\overset{\mathrm{d}^0_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^{2,1} \underset{-\delta^2_{\Gamma_n}}{\overset{\mathrm{d}^1_{\Gamma_t}}{\rightleftarrows}} \cdots \mathsf{L}^{2,q} \underset{-\delta^{q+1}_{\Gamma_n}}{\overset{\mathrm{d}^q_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^{2,q+1} \ldots \mathsf{L}^{2,N-1} \underset{-\delta^N_{\Gamma_n}}{\overset{\mathrm{d}^{N-1}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^{2,N} \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathbb{R} \text{ or } \{0\}$$

related fos

$$\mathrm{d}^q_{\Gamma_t} E = F \quad \text{in } \Omega$$
$$-\delta^q_{\Gamma_n} E = G \quad \text{in } \Omega$$

related sos

$$-\delta^{q+1}_{\Gamma_n} \mathrm{d}^q_{\Gamma_t} E = F \quad \text{in } \Omega$$
$$-\delta^q_{\Gamma_n} E = G \quad \text{in } \Omega$$

includes: EMS rot / div, Laplacian, rot rot, and more...
corresponding compact embeddings:

$$D(\mathrm{d}^q_{\Gamma_t}) \cap D(\delta^q_{\Gamma_n}) \hookrightarrow \mathsf{L}^{2,q} \qquad \text{(Weck's selection theorems, '74)}$$

Weck's selection theorem for Lip. manifolds and mixed bc: Bauer/Py/Schomburg ('17)

elasticity complex in 3D (sym ∇-Rot $\mathrm{Rot}_{\mathbb{S}}^{\top}$-$\mathrm{Div}_{\mathbb{S}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \;\underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\,\mathrm{Div}_{\mathbb{S}}}{\overset{\mathring{\mathrm{sym}}\,\nabla}{\rightleftarrows}}\; \mathsf{L}^2_{\mathbb{S}} \;\underset{\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top}}{\overset{\mathring{\mathrm{Rot}}\,\mathrm{Rot}_{\mathbb{S}}^{\top}}{\rightleftarrows}}\; \mathsf{L}^2_{\mathbb{S}} \;\underset{-\,\mathrm{sym}\,\nabla}{\overset{\mathring{\mathrm{Div}}_{\mathbb{S}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota_{\mathsf{RM}}}{\overset{\pi_{\mathsf{RM}}}{\rightleftarrows}}\; \mathsf{RM}$$

elasticity complex in 3D (sym ∇-Rot $\mathrm{Rot}_{\mathbb{S}}^{\top}$-$\mathrm{Div}_{\mathbb{S}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\mathrm{Div}_{\mathbb{S}}}{\overset{\mathring{\mathrm{sym}}\,\nabla}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top}}{\overset{\mathring{\mathrm{Rot}}\,\mathrm{Rot}_{\mathbb{S}}^{\top}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{-\mathrm{sym}\,\nabla}{\overset{\mathring{\mathrm{Div}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathsf{RM}}}{\overset{\pi_{\mathsf{RM}}}{\rightleftarrows}} \mathsf{RM}$$

related fos ($\mathring{\mathrm{Rot}}\,\mathrm{Rot}_{\mathbb{S}}^{\top}$, $\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top}$ first order operators!)

$$\begin{array}{llll} \mathring{\mathrm{sym}}\,\nabla v = M \ \text{in } \Omega & \mathring{\mathrm{Rot}}\,\mathrm{Rot}_{\mathbb{S}}^{\top} M = F \ \text{in } \Omega & \mathring{\mathrm{Div}}_{\mathbb{S}} N = g \ \text{in } \Omega & \pi v = r \ \text{in } \Omega \\ \pi v = 0 \ \text{in } \Omega & -\mathrm{Div}_{\mathbb{S}} M = f \ \text{in } \Omega & \mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top} N = G \ \text{in } \Omega & -\mathrm{sym}\,\nabla v = M \ \text{in } \Omega \end{array}$$

related sos ($\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top}\,\mathring{\mathrm{Rot}}\,\mathrm{Rot}_{\mathbb{S}}^{\top}$ second order operator!)

$$\begin{array}{lll} -\mathrm{Div}_{\mathbb{S}}\,\mathring{\mathrm{sym}}\,\nabla v = f \ \text{in } \Omega & \mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top}\,\mathring{\mathrm{Rot}}\,\mathrm{Rot}_{\mathbb{S}}^{\top} M = G \ \text{in } \Omega & -\mathrm{sym}\,\nabla\,\mathring{\mathrm{Div}}_{\mathbb{S}} N = M \ \text{in } \Omega \\ \pi v = 0 \ \text{in } \Omega & -\mathrm{Div}_{\mathbb{S}} M = f \ \text{in } \Omega & \mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top} N = G \ \text{in } \Omega \end{array}$$

corresponding compact embeddings:

$$D(\mathring{\mathrm{sym}}\,\nabla) \cap D(\pi) = D(\mathring{\nabla}) = \mathring{\mathsf{H}}^1 \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem and Korn ineq.)}$$

$$D(\mathring{\mathrm{Rot}}\,\mathrm{Rot}_{\mathbb{S}}^{\top}) \cap D(\mathrm{Div}_{\mathbb{S}}) \hookrightarrow \mathsf{L}^2_{\mathbb{S}} \qquad \text{(new selection theorem)}$$

$$D(\mathring{\mathrm{Div}}_{\mathbb{S}}) \cap D(\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top}) \hookrightarrow \mathsf{L}^2_{\mathbb{S}} \qquad \text{(new selection theorem)}$$

$$D(\pi) \cap D(\mathrm{sym}\,\nabla) = D(\nabla) = \mathsf{H}^1 \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem and Korn ineq.)}$$

two new selection theorems for strong Lip. dom.: Py/Schomburg/Zulehner ('18)

elasticity complex in 3D (sym ∇-Rot $\mathrm{Rot}_{\mathbb{S}}^{\top}$-$\mathrm{Div}_{\mathbb{S}}$-complex)

Lemma/Theorem $\Downarrow$ $D(\mathrm{A}_1)\cap D(\mathrm{A}_0^*)\hookrightarrow\!\!\!\rightarrow \mathrm{H}_1,\quad D(\mathrm{A}_2)\cap D(\mathrm{A}_1^*)\hookrightarrow\!\!\!\rightarrow \mathrm{H}_2\quad$ cpt

(i) all Friedrichs-Poincaré type est hold

est for $\mathcal{A}_0 \quad\Leftrightarrow\quad \forall\,\varphi\in D(\mathrm{sym}\,\mathring{\nabla})\cap R(\mathrm{Div}_{\mathbb{S}})=\mathring{\mathrm{H}}^1 \qquad |\varphi|_{\mathrm{L}^2}\le c_0|\mathrm{sym}\,\nabla\varphi|_{\mathrm{L}^2}$

est for $\mathcal{A}_0^* \quad\Leftrightarrow\quad \forall\,\Phi\in D(\mathrm{Div}_{\mathbb{S}})\cap R(\mathrm{sym}\,\mathring{\nabla}) \qquad |\Phi|_{\mathrm{L}^2}\le c_0|\mathrm{Div}\,\Phi|_{\mathrm{L}^2}$

est for $\mathcal{A}_1 \quad\Leftrightarrow\quad \forall\,\Phi\in D(\mathrm{Rot}\,\mathring{\mathrm{Rot}}_{\mathbb{S}}^{\top})\cap R(\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top}) \qquad |\Phi|_{\mathrm{L}^2}\le c_1|\mathrm{Rot}\,\mathrm{Rot}^{\top}\Phi|_{\mathrm{L}^2}$

est for $\mathcal{A}_1^* \quad\Leftrightarrow\quad \forall\,\Phi\in D(\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top})\cap R(\mathrm{Rot}\,\mathring{\mathrm{Rot}}_{\mathbb{S}}^{\top}) \qquad |\Phi|_{\mathrm{L}^2}\le c_1|\mathrm{Rot}\,\mathrm{Rot}^{\top}\Phi|_{\mathrm{L}^2}$

est for $\mathcal{A}_2 \quad\Leftrightarrow\quad \forall\,\Phi\in D(\mathring{\mathrm{Div}}_{\mathbb{S}})\cap R(\mathrm{sym}\,\nabla) \qquad |\Phi|_{\mathrm{L}^2}\le c_2|\mathrm{Div}\,\Phi|_{\mathrm{L}^2}$

est for $\mathcal{A}_2^* \quad\Leftrightarrow\quad \forall\,\varphi\in D(\mathrm{sym}\,\nabla)\cap R(\mathring{\mathrm{Div}}_{\mathbb{S}})=\mathrm{H}^1\cap\mathrm{RM}^{\perp} \qquad |\varphi|_{\mathrm{L}^2}\le c_2|\mathrm{sym}\,\nabla\varphi|_{\mathrm{L}^2}$

(ii) all ranges $R(\mathrm{A}_n)=R(\mathcal{A}_n),\quad R(\mathrm{A}_n^*)=R(\mathcal{A}_n^*)$ are cl in L^2

(iii) all inverse ops $\mathcal{A}_n^{-1}$, $(\mathcal{A}_n^*)^{-1}$ are cont, even cpt

(iv) all Helmholtz decomposition hold, e.g.,

$$\mathrm{H}_1=R(\mathrm{A}_0)\oplus K_1\oplus R(\mathrm{A}_1^*)\quad\Leftrightarrow\quad \mathrm{L}^2=R(\mathrm{sym}\,\mathring{\nabla})\oplus_{\mathrm{L}^2}\mathcal{H}_{D,\mathbb{S}}\oplus_{\mathrm{L}^2}R(\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top})$$

(v) solution theories

(vi) variational formulations

(vii) functional a posteriori error estimates

(viii) div-curl-lemmas

(ix) ...

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\mathrm{Rot}_{\mathbb{S}}$-$\mathrm{Div}_{\mathbb{T}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\nabla\nabla}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}} \underset{-\operatorname{dev}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathsf{RT}}}{\overset{\pi_{\mathsf{RT}}}{\rightleftarrows}} \mathsf{RT}$$

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\mathrm{Rot}_{\mathbb{S}}$-$\mathrm{Div}_{\mathbb{T}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\nabla}\nabla}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}} \underset{-\operatorname{dev}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathsf{RT}}}{\overset{\pi_{\mathsf{RT}}}{\rightleftarrows}} \mathsf{RT}$$

related fos ($\mathring{\nabla}\nabla$, $\operatorname{div}\operatorname{Div}_{\mathbb{S}}$ first order operators!)

$$\begin{array}{rlcrlcrlcrl} \mathring{\nabla}\nabla u = M & \text{in } \Omega & | & \mathring{\operatorname{Rot}}_{\mathbb{S}} M = F & \text{in } \Omega & | & \mathring{\operatorname{Div}}_{\mathbb{T}} N = g & \text{in } \Omega & | & \pi v = r & \text{in } \Omega \\ \pi u = 0 & \text{in } \Omega & | & \operatorname{div}\operatorname{Div}_{\mathbb{S}} M = f & \text{in } \Omega & | & \operatorname{sym}\operatorname{Rot}_{\mathbb{T}} N = G & \text{in } \Omega & | & -\operatorname{dev}\nabla v = T & \text{in } \Omega \end{array}$$

related sos ($\operatorname{div}\operatorname{Div}_{\mathbb{S}} \mathring{\nabla}\nabla = \mathring{\Delta}^2$ second order operator!)

$$\begin{array}{rlcrlcrl} \operatorname{div}\operatorname{Div}_{\mathbb{S}} \mathring{\nabla}\nabla u = \mathring{\Delta}^2 u = f & \text{in } \Omega & | & \operatorname{sym}\operatorname{Rot}_{\mathbb{T}} \mathring{\operatorname{Rot}}_{\mathbb{S}} M = G & \text{in } \Omega & | & -\operatorname{dev}\nabla\mathring{\operatorname{Div}}_{\mathbb{T}} N = T & \text{in } \Omega \\ \pi u = 0 & \text{in } \Omega & | & \operatorname{div}\operatorname{Div}_{\mathbb{S}} M = f & \text{in } \Omega & | & \operatorname{sym}\operatorname{Rot}_{\mathbb{T}} N = G & \text{in } \Omega \end{array}$$

corresponding compact embeddings:

$$\begin{aligned} D(\mathring{\nabla}\nabla) \cap D(\pi) = D(\mathring{\nabla}\nabla) = \mathring{\mathsf{H}}^2 &\hookrightarrow \mathsf{L}^2 && \text{(Rellich's selection theorem)} \\ D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \cap D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) &\hookrightarrow \mathsf{L}^2_{\mathbb{S}} && \text{(new selection theorem)} \\ D(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) &\hookrightarrow \mathsf{L}^2_{\mathbb{T}} && \text{(new selection theorem)} \\ D(\pi) \cap D(\operatorname{dev}\nabla) = D(\operatorname{dev}\nabla) = D(\nabla) = \mathsf{H}^1 &\hookrightarrow \mathsf{L}^2 && \text{(Rellich's selection theorem and Korn type ineq.)} \end{aligned}$$

two new selection theorems for strong Lip. dom. and Korn Type ineq.: Py/Zulehner ('16)

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\operatorname{Rot}_{\mathbb{S}}$-$\operatorname{Div}_{\mathbb{T}}$-complex)

Lemma/Theorem $\Downarrow$ $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1, \quad D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_2 \quad \text{cpt}$

(i) all Friedrichs-Poincaré type est hold

est for $\mathcal{A}_0 \quad \Leftrightarrow \quad \forall\, \varphi \in D(\mathring{\nabla}\nabla) \cap R(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) = \mathring{\mathrm{H}}^2 \qquad |\varphi|_{\mathrm{L}^2} \leq c_0 |\nabla\nabla\varphi|_{\mathrm{L}^2}$

est for $\mathcal{A}_0^* \quad \Leftrightarrow \quad \forall\, \Phi \in D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \cap R(\mathring{\nabla}\nabla) \qquad |\Phi|_{\mathrm{L}^2} \leq c_0 |\operatorname{div}\operatorname{Div}\Phi|_{\mathrm{L}^2}$

est for $\mathcal{A}_1 \quad \Leftrightarrow \quad \forall\, \Phi \in D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \cap R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \qquad |\Phi|_{\mathrm{L}^2} \leq c_1 |\operatorname{Rot}\Phi|_{\mathrm{L}^2}$

est for $\mathcal{A}_1^* \quad \Leftrightarrow \quad \forall\, \Phi \in D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \cap R(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \qquad |\Phi|_{\mathrm{L}^2} \leq c_1 |\operatorname{sym}\operatorname{Rot}\Phi|_{\mathrm{L}^2}$

est for $\mathcal{A}_2 \quad \Leftrightarrow \quad \forall\, \Phi \in D(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap R(\operatorname{dev}\nabla) \qquad |\Phi|_{\mathrm{L}^2} \leq c_2 |\operatorname{Div}\Phi|_{\mathrm{L}^2}$

est for $\mathcal{A}_2^* \quad \Leftrightarrow \quad \forall\, \varphi \in D(\operatorname{dev}\nabla) \cap R(\mathring{\operatorname{Div}}_{\mathbb{T}}) = \mathrm{H}^1 \cap \mathrm{RT}^{\perp} \qquad |\varphi|_{\mathrm{L}^2} \leq c_2 |\operatorname{dev}\nabla\varphi|_{\mathrm{L}^2}$

(ii) all ranges $R(\mathrm{A}_n) = R(\mathcal{A}_n), \quad R(\mathrm{A}_n^*) = R(\mathcal{A}_n^*)$ are cl in L^2

(iii) all inverse ops $\mathcal{A}_n^{-1}$, $(\mathcal{A}_n^*)^{-1}$ are cont, even cpt

(iv) all Helmholtz decomposition hold, e.g.,

$$\mathrm{H}_1 = R(\mathrm{A}_0) \oplus K_1 \oplus R(\mathrm{A}_1^*) \quad \Leftrightarrow \quad \mathrm{L}^2_{\mathbb{S}} = R(\mathring{\nabla}\nabla) \oplus_{\mathrm{L}^2_{\mathbb{S}}} \mathcal{H}_{\mathrm{D},\mathbb{S}} \oplus_{\mathrm{L}^2_{\mathbb{S}}} R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}),$$

$$\mathrm{H}_2 = R(\mathrm{A}_1) \oplus K_2 \oplus R(\mathrm{A}_2^*) \quad \Leftrightarrow \quad \mathrm{L}^2_{\mathbb{T}} = R(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \oplus_{\mathrm{L}^2_{\mathbb{T}}} \mathcal{H}_{\mathrm{N},\mathbb{T}} \oplus_{\mathrm{L}^2_{\mathbb{T}}} R(\operatorname{dev}\nabla)$$

(v)-(ix) solution theories, variational formulations, functional a posteriori error estimates, div-curl-lemmas, . . .

literature (fa-toolbox, complexes, a posteriori error estimates, ...)

results of this talk:

- Py: *Solution Theory and Functional A Posteriori Error Estimates for General First Order Systems with Applications to Electro-Magneto-Statics*, (NFAO) Numerical Functional Analysis and Optimization, 2018

(paper contains main results of this talk)

literature (complexes, Friedrichs type constants, Maxwell constants)

results of this talk:

- Py: *On Constants in Maxwell Inequalities for Bounded and Convex Domains*, Zapiski POMI/ (JMS)Journal of Mathematical Sciences (Springer New York), 2015
- Py: *On Maxwell's and Poincare's Constants*, (DCDS) Discrete and Continuous Dynamical Systems - Series S, 2015
- Py: *On the Maxwell Constants in 3D*, (M2AS) Mathematical Methods in the Applied Sciences, 2017
- Py: *On the Maxwell and Friedrichs/Poincaré Constants in ND*, (MZ) Mathematische Zeitschrift, 2018

- Py: *. . . some (so far) unpublished results*

literature (complexes, Friedrichs type constants, compact embeddings)

- Weck, N.: *Maxwell's boundary value problems on Riemannian manifolds with nonsmooth boundaries*,
  (JMA2) Journal of Mathematical Analysis and Applications, 1974 (1972)
- Picard, R.: *An elementary proof for a compact imbedding result in generalized electromagnetic theory*,
  (MZ) Mathematische Zeitschrift, 1984
- Witsch, K.-J.: *A remark on a compactness result in electromagnetic theory*,
  (M2AS) Mathematical Methods in the Applied Sciences, 1993

results of this talk:

- Bauer, S., Py, Schomburg, M.: *The Maxwell Compactness Property in Bounded Weak Lipschitz Domains with Mixed Boundary Conditions*,
  (SIMA) SIAM Journal on Mathematical Analysis, 2016
- Py, Zulehner, W.: *The divDiv-Complex and Applications to Biharmonic Equations*,
  (AA) Applicable Analysis, 2018
- Py, Zulehner, W.: *The Elasticity Complex*,
  submitted, 2019

literature (div-curl-lemma)

original papers (local div-curl-lemma):

- Murat, F.: *Compacité par compensation*,
  Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 1978
- Tartar, L.: *Compensated compactness and applications to partial differential equations*,
  Nonlinear analysis and mechanics, Heriot-Watt symposium, 1979

literature (div-curl-lemma)

recent papers (global div-curl-lemma, H^1-detour):

- Gloria, A., Neukamm, S., Otto, F.: *Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics*, (IM) Invent. Math., 2015
- Kozono, H., Yanagisawa, T.: *Global compensated compactness theorem for general differential operators of first order*, (ARMA) Arch. Ration. Mech. Anal., 2013
- Schweizer, B.: *On Friedrichs inequality, Helmholtz decomposition, vector potentials, and the div-curl lemma*, accepted preprint, 2018

recent papers (global div-curl-lemma, general results/this talk):

- Waurick, M.: *A Functional Analytic Perspective to the div-curl Lemma*, (JOP) J. Operator Theory, 2018
- Py: *A Global div-curl-Lemma for Mixed Boundary Conditions in Weak Lipschitz Domains and a Corresponding Generalized* A_0^*-A_1*-Lemma in Hilbert Spaces*, (ANA) Analysis (Munich), 2018

literature (full time-dependent Maxwell equations)

- Py, Picard, R.: *A Note on the Justification of the Eddy Current Model in Electrodynamics*,
  (M2AS) Mathematical Methods in the Applied Sciences, 2017
- Py, Picard, R., Trostorff, S., Waurick, M.: *On a Class of Degenerate Abstract Parabolic Problems and Applications to Some Eddy Current Models*,
  submitted, 2019

literature (Maxwell's equations and more...)

upcoming books:

- Langer, U., Py, Repin, S. (Eds): *Maxwell's equations. Analysis and numerics*, Radon Series on Applied Mathematics, De Gruyter, 2019
- Py: *Maxwell's Equations: Hilbert Space Methods for the Theory of Electromagnetism*, Radon Series on Applied Mathematics, De Gruyter, 2020

(last book: contains all results of this talk and more...)

...the world is full of complexes ...;)

⇒ relaxing at (and you're all invited!)

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