Non-Standard Partial Integration: Implications to Maxwell and Korn Inequalities or How one cannot apply the Closed Graph Theorem

Universität Konstanz

Oberseminar Partielle Differentialgleichungen

Gastgeber: Reinhard Racke

Dirk Pauly

Universität Duisburg-Essen

Universität Duisburg Essen

Open-Minded :-)

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Overview

NON-STANDARD INTEGRATION BY PARTS

Proposition (Grisvard's book and older...)

Let $\Omega \subset \mathbb{R}^N$ be piecewise C^2. Then for all $v \in \overset{\circ}{\mathrm{C}}{}^{\infty}(\overline{\Omega})$

$$|\operatorname{div} v|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{rot} v|^2_{\mathrm{L}^2(\Omega)} - |\nabla v|^2_{\mathrm{L}^2(\Omega)} = \sum_{\ell=1}^{L} \int_{\Gamma_\ell} \underbrace{\left(\operatorname{div} \nu \, |v_{\mathrm{n}}|^2 + ((\nabla \nu)\, v_{\mathrm{t}}) \cdot v_{\mathrm{t}}\right)}_{\text{curvature, sign!}} + \sum_{\ell=1}^{L} \int_{\Gamma_\ell} \underbrace{\left(v_{\mathrm{n}} \operatorname{div}_\Gamma v_{\mathrm{t}} - v_{\mathrm{t}} \cdot \nabla_\Gamma v_{\mathrm{n}}\right)}_{\text{boundary conditions, no sign!}}$$

and for all $v \in \overset{\circ}{\mathrm{C}}{}^{\infty}_{\mathrm{t,n}}(\Omega)$

$$|\operatorname{div} v|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{rot} v|^2_{\mathrm{L}^2(\Omega)} - |\nabla v|^2_{\mathrm{L}^2(\Omega)} = \sum_{\ell=1}^{L} \int_{\Gamma_\ell} \left(\operatorname{div} \nu \, |v_{\mathrm{n}}|^2 + ((\nabla \nu)\, v_{\mathrm{t}}) \cdot v_{\mathrm{t}}\right)$$

$$\begin{cases} \leq 0 & \text{, if } \Omega \text{ is piecewise } \mathrm{C}^2\text{-concave,} \\ = 0 & \text{, if } \Omega \text{ is a polyhedron,} \\ \geq 0 & \text{, if } \Omega \text{ is piecewise } \mathrm{C}^2\text{-convex.} \end{cases}$$

TWO MAXWELL INEQUALITIES

$\Omega \subset \mathbb{R}^3$ bounded, weak Lipschitz (even weaker possible)

$$\Rightarrow \qquad \mathring{\mathrm{R}}(\Omega) \cap \operatorname{rot} \mathrm{R}(\Omega) \hookrightarrow \mathrm{L}^2(\Omega) \quad \Leftrightarrow \quad \mathrm{R}(\Omega) \cap \operatorname{rot} \mathring{\mathrm{R}}(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$$

$\Rightarrow$ Maxwell estimates:

$$\exists\, \mathring{c}_{\mathrm{m}} > 0 \qquad \forall\, E \in \mathring{\mathrm{R}}(\Omega) \cap \operatorname{rot} \mathrm{R}(\Omega) \qquad |E|_{\mathrm{L}^2(\Omega)} \leq \mathring{c}_{\mathrm{m}} |\operatorname{rot} E|_{\mathrm{L}^2(\Omega)}$$

$$\exists\, c_{\mathrm{m}} > 0 \qquad \forall\, H \in \mathrm{R}(\Omega) \cap \operatorname{rot} \mathring{\mathrm{R}}(\Omega) \qquad |H|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{m}} |\operatorname{rot} H|_{\mathrm{L}^2(\Omega)}$$

note: best constants

$$\frac{1}{\mathring{c}_{\mathrm{m}}} = \inf_{0 \neq E \in \mathring{\mathrm{R}}(\Omega) \cap \operatorname{rot} \mathrm{R}(\Omega)} \frac{|\operatorname{rot} E|_{\mathrm{L}^2(\Omega)}}{|E|_{\mathrm{L}^2(\Omega)}}, \quad \frac{1}{c_{\mathrm{m}}} = \inf_{0 \neq H \in \mathrm{R}(\Omega) \cap \operatorname{rot} \mathring{\mathrm{R}}(\Omega)} \frac{|\operatorname{rot} H|_{\mathrm{L}^2(\Omega)}}{|H|_{\mathrm{L}^2(\Omega)}}$$

Theorem

(i) $\mathring{c}_{\mathrm{m}} = c_{\mathrm{m}}$ **(ii)** Ω *convex* $\Rightarrow c_{\mathrm{m}} \leq c_{\mathrm{p}}$

Poincaré estimate: $\exists\, c_{\mathrm{p}} > 0 \qquad \forall\, u \in \mathrm{H}^1(\Omega) \cap \mathbb{R}^{\perp} \qquad |u|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{p}} |\nabla u|_{\mathrm{L}^2(\Omega)}$

best constant: $\displaystyle \frac{1}{c_{\mathrm{p}}} = \inf_{0 \neq u \in \mathrm{H}^1(\Omega) \cap \mathbb{R}^{\perp}} \frac{|\nabla u|_{\mathrm{L}^2(\Omega)}}{|u|_{\mathrm{L}^2(\Omega)}}$

PROOF OF MAXWELL INEQUALITIES

<u>step one:</u> two lin., cl., dens. def. op. and their reduced op.

$$A : D(A) \subset X \to Y, \qquad \mathcal{A} : D(\mathcal{A}) := D(A) \cap R(A^*) \subset R(A^*) \to R(A),$$
$$A^* : D(A^*) \subset Y \to X, \qquad \mathcal{A}^* : D(\mathcal{A}^*) := D(A^*) \cap R(A) \subset R(A) \to R(A^*)$$

crucial assumption: $D(\mathcal{A}) \hookrightarrow X \; \big(\Leftrightarrow \quad D(\mathcal{A}^*) \hookrightarrow Y\big)$

$\Downarrow$

gen. Poincaré estimates:

$$\exists\, c_A > 0 \qquad \forall\, x \in D(\mathcal{A}) \qquad |x| \leq c_A |Ax|$$
$$\exists\, c_{A^*} > 0 \qquad \forall\, y \in D(\mathcal{A}^*) \qquad |y| \leq c_{A^*} |A^* y|$$

note: best constants

$$\frac{1}{c_A} = \inf_{0 \neq x \in D(\mathcal{A})} \frac{|Ax|}{|x|}, \quad \frac{1}{c_{A^*}} = \inf_{0 \neq y \in D(\mathcal{A}^*)} \frac{|A^* y|}{|y|}$$

Theorem

$c_A = c_{A^*}$

PROOF OF MAXWELL INEQUALITIES

step two: two lin., cl., den. def. op. and their reduced op.

$$A : D(A) \subset X \to Y, \qquad \mathcal{A} : D(\mathcal{A}) := D(A) \cap R(A^*) \subset R(A^*) \to R(A),$$
$$A^* : D(A^*) \subset Y \to X, \qquad \mathcal{A}^* : D(\mathcal{A}^*) := D(A^*) \cap R(A) \subset R(A) \to R(A^*)$$

choose

$$A := \mathring{\mathrm{rot}} : \mathring{\mathrm{R}}(\Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega), \qquad \mathring{\mathcal{rot}} : \mathring{\mathrm{R}}(\Omega) \cap \mathrm{rot}\,\mathrm{R}(\Omega) \subset \mathrm{rot}\,\mathrm{R}(\Omega) \to \mathrm{rot}\,\mathring{\mathrm{R}}(\Omega),$$
$$\mathrm{rot} = \mathring{\mathrm{rot}}^* : \mathrm{R}(\Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega), \quad \mathcal{rot} = \mathring{\mathcal{rot}}^* : \mathrm{R}(\Omega) \cap \mathrm{rot}\,\mathring{\mathrm{R}}(\Omega) \subset \mathrm{rot}\,\mathring{\mathrm{R}}(\Omega) \to \mathrm{rot}\,\mathrm{R}(\Omega)$$

crucial assumption: $\mathring{\mathrm{R}}(\Omega) \cap \mathrm{rot}\,\mathrm{R}(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$ $\big(\Leftrightarrow \ \mathrm{R}(\Omega) \cap \mathrm{rot}\,\mathring{\mathrm{R}}(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)\big)$

$\Downarrow$

gen. Poincaré estimates (Maxwell estimates):

$$\exists\, \mathring{c}_{\mathrm{m}} > 0 \qquad \forall\, E \in \mathring{\mathrm{R}}(\Omega) \cap \mathrm{rot}\,\mathrm{R}(\Omega) \qquad |E|_{\mathrm{L}^2(\Omega)} \leq \mathring{c}_{\mathrm{m}} |\,\mathrm{rot}\,E|_{\mathrm{L}^2(\Omega)}$$
$$\exists\, c_{\mathrm{m}} > 0 \qquad \forall\, H \in \mathrm{R}(\Omega) \cap \mathrm{rot}\,\mathring{\mathrm{R}}(\Omega) \qquad |H|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{m}} |\,\mathrm{rot}\,H|_{\mathrm{L}^2(\Omega)}$$

Theorem

$\mathring{c}_{\mathrm{m}} = c_{\mathrm{m}}$

PROOF OF MAXWELL INEQUALITIES

step three:

Proposition (integration by parts (Grisvard's book and older...))

Let $\Omega \subset \mathbb{R}^3$ be piecewise C^2. Then for all $E \in \overset{\circ}{\mathsf{C}}{}^\infty(\overline{\Omega})$

$$|\operatorname{div} E|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)} - |\nabla E|^2_{\mathsf{L}^2(\Omega)}$$

$$= \sum_{\ell=1}^{L} \int_{\Gamma_\ell} \underbrace{\left(\operatorname{div}\nu\, |E_{\mathsf{n}}|^2 + ((\nabla\nu)\, E_{\mathsf{t}}) \cdot E_{\mathsf{t}}\right)}_{\substack{\text{curvature, sign!} \\ \ldots\, \geq 0, \text{ if } \Omega \text{ convex.}}} + \sum_{\ell=1}^{L} \int_{\Gamma_\ell} \underbrace{\left(E_{\mathsf{n}} \operatorname{div}_\Gamma E_{\mathsf{t}} - E_{\mathsf{t}} \cdot \nabla_\Gamma E_{\mathsf{n}}\right)}_{\text{boundary conditions, no sign!}}.$$

approx. convex Ω from inside by convex and smooth $(\Omega_k)_k \quad \Rightarrow$

Corollary (Gaffney's inequality)

Let $\Omega \subset \mathbb{R}^3$ be convex and $E \in \overset{\circ}{\mathsf{R}}(\Omega) \cap \mathsf{D}(\Omega)$ or $E \in \mathsf{R}(\Omega) \cap \overset{\circ}{\mathsf{D}}(\Omega)$.
Then $E \in \mathsf{H}^1(\Omega)$ and

$$|\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathsf{L}^2(\Omega)} - |\nabla E|^2_{\mathsf{L}^2(\Omega)} \geq 0.$$

PROOF OF MAXWELL INEQUALITIES

<u>step four:</u>

$$\text{(Poincaré)} \qquad \exists\, c_{\mathrm{p}} > 0 \quad \forall\, u \in \mathrm{H}^1(\Omega) \cap \mathbb{R}^{\perp} \qquad |u|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{p}} |\nabla u|_{\mathrm{L}^2(\Omega)}$$

Let Ω be convex and $E \in \mathrm{R}(\Omega) \cap \overset{\circ}{\mathrm{D}}_0(\Omega)$. Note $\overset{\circ}{\mathrm{D}}_0(\Omega) = \operatorname{rot} \overset{\circ}{\mathrm{R}}(\Omega)$.

Cor. (Gaffney) $\Rightarrow$ $E \in \mathrm{H}^1(\Omega)$ and $E = \operatorname{rot} H$ with $H \in \overset{\circ}{\mathrm{R}}(\Omega)$.

$\Rightarrow$ $E \in \mathrm{H}^1(\Omega) \cap (\mathbb{R}^3)^{\perp} \cap \overset{\circ}{\mathrm{D}}_0(\Omega)$, since $\langle E, a \rangle_{\mathrm{L}^2(\Omega)} = \langle \operatorname{rot} H, a \rangle_{\mathrm{L}^2(\Omega)} = 0$ for $a \in \mathbb{R}^3$

$\Downarrow$

$$|E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{p}} |\nabla E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{p}} |\operatorname{rot} E|_{\mathrm{L}^2(\Omega)}$$

$\Downarrow$

$$c_{\mathrm{m}} \leq c_{\mathrm{p}}$$

□

Theorem

Ω convex $\Rightarrow$ $\overset{\circ}{c}_{\mathrm{p}} \leq \overset{\circ}{c}_{\mathrm{m}} = c_{\mathrm{m}} \leq c_{\mathrm{p}}$

Here:

$$\text{(Poincaré/Friedrichs)} \qquad \exists\, \overset{\circ}{c}_{\mathrm{p}} > 0 \quad \forall\, u \in \overset{\circ}{\mathrm{H}}{}^1(\Omega) \qquad |u|_{\mathrm{L}^2(\Omega)} \leq \overset{\circ}{c}_{\mathrm{p}} |\nabla u|_{\mathrm{L}^2(\Omega)}$$

MATRICES

Let $A \in \mathbb{R}^{N\times N}$.

$$\begin{matrix}\text{sym}\\ \text{skw}\end{matrix} A := \frac{1}{2}(A \pm A^\top), \qquad \text{id}_A := \frac{\text{tr}\, A}{N}\,\text{id}, \qquad \text{tr}\, A := A \cdot \text{id}, \qquad \text{dev}\, A := A - \text{id}_A$$

(pointwise orthogonality) $\Rightarrow$

$$|A|^2 = |\,\text{dev}\, A|^2 + \frac{1}{N}|\,\text{tr}\, A|^2, \quad |A|^2 = |\,\text{sym}\, A|^2 + |\,\text{skw}\, A|^2, \quad |\,\text{sym}\, A|^2 = |\,\text{dev}\,\text{sym}\, A|^2 + \frac{1}{N}|\,\text{tr}\, A|^2$$

$$\Rightarrow \quad |\,\text{dev}\, A|, N^{-1/2}|\,\text{tr}\, A|, |\,\text{sym}\, A|, |\,\text{skw}\, A| \leq |A|$$

$\Omega \subset \mathbb{R}^N$ and $A := \nabla v := J_v^\top$ for $v \in \mathsf{H}^1(\Omega)$ $\Rightarrow$ (pointwise)

$$|\,\text{skw}\,\nabla v|^2 = \frac{1}{2}|\,\text{rot}\, v|^2, \quad \text{tr}\,\nabla v = \text{div}\, v,$$

$$|\nabla v|^2 = |\,\text{dev}\,\text{sym}\,\nabla v|^2 + \frac{1}{N}|\,\text{div}\, v|^2 + \frac{1}{2}|\,\text{rot}\, v|^2 \tag{1}$$

Moreover

$$|\nabla v|^2 = |\,\text{rot}\, v|^2 + \langle \nabla v, (\nabla v)^\top \rangle$$

since $\quad 2|\,\text{skw}\,\nabla v|^2 = \frac{1}{2}|\nabla v - (\nabla v)^\top|^2 = |\nabla v|^2 - \langle \nabla v, (\nabla v)^\top \rangle.$

KORN'S FIRST INEQUALITY: STANDARD BOUNDARY CONDITIONS

Lemma (Korn's first inequality: $\overset{\circ}{\mathrm{H}}{}^1$-version)

Let Ω be an open subset of $\mathbb{R}^N$ with $2 \leq N \in \mathbb{N}$. Then for all $v \in \overset{\circ}{\mathrm{H}}{}^1(\Omega)$

$$|\nabla v|^2_{\mathrm{L}^2(\Omega)} = 2|\operatorname{dev}\operatorname{sym}\nabla v|^2_{\mathrm{L}^2(\Omega)} + \frac{2-N}{N}|\operatorname{div} v|^2_{\mathrm{L}^2(\Omega)} \leq 2|\operatorname{dev}\operatorname{sym}\nabla v|^2_{\mathrm{L}^2(\Omega)}$$

and equality holds if and only if $\operatorname{div} v = 0$ *or* $N = 2$.

Proof.

note: $-\Delta = \operatorname{rot}^* \operatorname{rot} - \nabla \operatorname{div}$ (vector Laplacian)

$$\Rightarrow \quad \forall\, v \in \overset{\circ}{\mathrm{C}}{}^\infty(\Omega) \quad |\nabla v|^2_{\mathrm{L}^2(\Omega)} = |\operatorname{rot} v|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} v|^2_{\mathrm{L}^2(\Omega)} \quad \text{(Gaffney's equality)} \quad (2)$$

(2) extends to all $v \in \overset{\circ}{\mathrm{H}}{}^1(\Omega)$ by continuity. Then

$$|\nabla v|^2_{\mathrm{L}^2(\Omega)} = |\operatorname{dev}\operatorname{sym}\nabla v|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2}|\nabla v|^2_{\mathrm{L}^2(\Omega)} + \frac{2-N}{2N}|\operatorname{div} v|^2_{\mathrm{L}^2(\Omega)}$$

follows by (1), i.e., $|\nabla v|^2 = |\operatorname{dev}\operatorname{sym}\nabla v|^2 + \frac{1}{N}|\operatorname{div} v|^2 + \frac{1}{2}|\operatorname{rot} v|^2$, and (2). □

KORN'S FIRST INEQUALITY: TANGENTIAL/NORMAL BOUNDARY CONDITIONS

main result:

Theorem (Korn's first inequality: tangential/normal version)

Let $\Omega \subset \mathbb{R}^N$ be piecewise C^2-concave and $v \in \overset{\circ}{\mathsf{H}}{}^1_{\mathsf{t},\mathsf{n}}(\Omega)$. Then Korn's first inequality

$$|\nabla v|_{\mathsf{L}^2(\Omega)} \leq \sqrt{2}|\operatorname{dev}\operatorname{sym}\nabla v|_{\mathsf{L}^2(\Omega)}$$

holds. If Ω is a polyhedron, even

$$|\nabla v|^2_{\mathsf{L}^2(\Omega)} = 2|\operatorname{dev}\operatorname{sym}\nabla v|^2_{\mathsf{L}^2(\Omega)} + \frac{2-N}{N}|\operatorname{div} v|^2_{\mathsf{L}^2(\Omega)} \leq 2|\operatorname{dev}\operatorname{sym}\nabla v|^2_{\mathsf{L}^2(\Omega)}$$

is true and equality holds if and only if $\operatorname{div} v = 0$ *or* $N = 2$.

KORN'S FIRST INEQUALITY: TANGENTIAL/NORMAL BOUNDARY CONDITIONS

tools:

Proposition (integration by parts (Grisvard's book and older...))

Let $\Omega \subset \mathbb{R}^N$ *be piecewise* C^2. *Then*

$$|\operatorname{div} v|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{rot} v|^2_{\mathsf{L}^2(\Omega)} - |\nabla v|^2_{\mathsf{L}^2(\Omega)} = \sum_{\ell=1}^{L} \int_{\Gamma_\ell} \underbrace{\big(\operatorname{div}\nu\,|v_{\mathsf{n}}|^2 + ((\nabla\nu)\,v_{\mathsf{t}})\cdot v_{\mathsf{t}}\big)}_{\text{curvature, sign!}} + \sum_{\ell=1}^{L} \int_{\Gamma_\ell} \underbrace{(v_{\mathsf{n}} \operatorname{div}_\Gamma v_{\mathsf{t}} - v_{\mathsf{t}} \cdot \nabla_\Gamma v_{\mathsf{n}})}_{\text{boundary conditions, no sign!}},$$

$$|\operatorname{div} v|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{rot} v|^2_{\mathsf{L}^2(\Omega)} - |\nabla v|^2_{\mathsf{L}^2(\Omega)} = \sum_{\ell=1}^{L} \int_{\Gamma_\ell} \big(\operatorname{div}\nu\,|v_{\mathsf{n}}|^2 + ((\nabla\nu)\,v_{\mathsf{t}})\cdot v_{\mathsf{t}}\big).$$

holds for all $v \in \mathring{\mathsf{C}}^\infty(\overline{\Omega})$ *resp.* $v \in \mathring{\mathsf{C}}^\infty_{\mathsf{t,n}}(\Omega)$.

Corollary (Gaffney's inequalities)

Let $\Omega \subset \mathbb{R}^N$ *be piecewise* C^2 *and* $v \in \mathring{\mathsf{H}}^1_{\mathsf{t,n}}(\Omega)$. *Then*

$$|\operatorname{rot} v|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{div} v|^2_{\mathsf{L}^2(\Omega)} - |\nabla v|^2_{\mathsf{L}^2(\Omega)} \begin{cases} \leq 0 & \text{, if } \Omega \text{ is piecewise } \mathsf{C}^2\text{-concave,} \\ = 0 & \text{, if } \Omega \text{ is a polyhedron,} \\ \geq 0 & \text{, if } \Omega \text{ is piecewise } \mathsf{C}^2\text{-convex.} \end{cases}$$

KORN'S FIRST INEQUALITY: TANGENTIAL/NORMAL BOUNDARY CONDITIONS

Proof.

(1), i.e., $|\nabla v|^2 = |\operatorname{dev}\operatorname{sym}\nabla v|^2 + \frac{1}{N}|\operatorname{div} v|^2 + \frac{1}{2}|\operatorname{rot} v|^2$, and the corollary $\Rightarrow$

$$|\nabla v|^2_{\mathsf{L}^2(\Omega)} \leq |\operatorname{dev}\operatorname{sym}\nabla v|^2_{\mathsf{L}^2(\Omega)} + \frac{1}{2}|\nabla v|^2_{\mathsf{L}^2(\Omega)} + \frac{2-N}{2N}|\operatorname{div} v|^2_{\mathsf{L}^2(\Omega)}$$

$\Rightarrow$ first estimate

Ω polyhedron $\Rightarrow$ equality holds □

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  On a variant of Korn's inequality arising in statistical mechanics. A tribute to J.L. Lions.
- Desvillettes, L. and Villani, C.: Invent. Math., (2005)
  On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation

CITATIONS

- Desvillettes, L. and Villani, C.: ESAIM Control Optim. Calc. Var., (2002)
  On a variant of Korn's inequality arising in statistical mechanics.
  A tribute to J.L. Lions.
    - page 607
    - page 608
    - page 609
    - Proposition 5
    - (end of) Theorem 3 (continued)
    - page 609 (closed graph theorem)

- Desvillettes, L. and Villani, C.: Invent. Math., (2005)
  On the trend to global equilibrium for spatially inhomogeneous kinetic systems:
  the Boltzmann equation
    - page 306

HOW ONE CANNOT APPLY THE CLOSED GRAPH THEOREM!

generally: compact embedding or regularity + closed graph theorem

$\Rightarrow$ Poincaré type estimate

(hard analysis to do!)

surprisingly: $\exists$ people closed graph / open mapping / bounded inverse theorem

$\Rightarrow$ Poincaré type estimate

(example on next slide)

!!! THIS IS WRONG !!!

How one cannot apply the closed graph theorem!

4. Our primary goal was to obtain fully explicit lower bounds for $K(\Omega)$ in terms of simple geometrical information about Ω; to achieve this completely with our method, we would have to give quantitative estimates on C_H. Unfortunately, we have been unable to find explicit estimates about C_H in the literature, although it seems unlikely that nobody has been interested in this problem. Of course, when $N = 3$ and Ω is simply connected, estimate (10) is equivalent to

$$\|\nabla u\|^2_{L^2(\Omega)} \le C_H(\Omega)\big(\|\nabla \cdot u\|^2_{L^2(\Omega)} + \|\nabla \wedge u\|^2_{L^2(\Omega)}\big), \qquad (13)$$

up to possible replacement of C_H by $C_H + 1$. This is an estimate which is well-known to many people, but for which it seems very difficult to find an accurate reference. Inequality (10) can be seen as a consequence of the closed graph theorem; for instance, in the case of a simply connected domain, one just needs to note that (i) $\|\nabla^{\mathrm{a}} u\|^2_{L^2} + \|\nabla \cdot u\|^2_{L^2}$ is bounded by $\|\nabla u\|^2_{L^2}$, (ii) the identities $\nabla \cdot u = 0$, $\nabla^{\mathrm{a}} u = 0$, $u \cdot n = 0$ (on the boundary), together imply $u = 0$; so in fact the norms appearing on the left and on the right-hand side of (10) have to be equivalent. The proof of point (ii) is as follows: from Poincaré's lemma in a simply connected domain, there exists a real-valued function ψ such that $\nabla\psi = u$; then ψ is a harmonic function with homogeneous Neumann boundary condition, so it has to be a constant, and $u = 0$.

Of course this argument gives no insight on how to estimate the constants. As pointed out to us independently by Druet and by Serre, one can choose $C_H(\Omega) = 1$ if Ω is convex, but the general case seems to be much harder. Anyway this is a separate issue which has nothing to do with axisymmetry; all the relevant information about axisymmetry lies in our estimates on $G(\Omega)^{-1}$.

- *$C_H = C_H(\Omega)$ is a constant related to the homology of Ω and the Hodge decomposition, defined by the inequality*

$$\|\nabla^{\mathrm{sym}} v\|^2_{L^2(\Omega)/V_0(\Omega)} \le C_H \left(\|\nabla \cdot v\|^2_{L^2(\Omega)} + \|\nabla^{\mathrm{a}} v\|^2_{L^2(\Omega)}\right), \qquad (10)$$

or (almost) equivalently by inequality (13) below. Here $\nabla \cdot v$ stands for the divergence of the vector field v, $\nabla \cdot v = \sum_i \partial v_i/\partial x_i$, and $V_0(\Omega)$ is the space of all vector fields $v_0 \in H^1(\Omega; \mathbb{R}^N)$ such that

$$\nabla \cdot v_0 = 0, \qquad \nabla^{\mathrm{a}} v_0 = 0.$$

We recall that V_0 is a finite-dimensional vector space whose dimension depends only on the topology of Ω;

(copies from original paper...)