Hilbert Complexes and PDEs

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Hilbert Complexes and PDEs

Some Hilbert Complexes

PDEs: de Rham complex 3D

grad-complex

$$\{0\} \underset{\pi_0}{\overset{\iota_0}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{div}}{\overset{\mathring{\operatorname{grad}}}{\rightleftarrows}} \vec{\mathsf{L}}^2 \underset{\operatorname{rot}}{\overset{\mathring{\operatorname{rot}}}{\rightleftarrows}} \vec{\mathsf{L}}^2 \underset{-\operatorname{grad}}{\overset{\mathring{\operatorname{div}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}} \mathbb{R}$$

PDEs

$$\partial_t^n - \underbrace{\operatorname{div}\mathring{\operatorname{grad}}}_{=\Delta_{\mathrm{D}}}, \qquad \partial_t^n - \underbrace{\mathring{\operatorname{div}}\operatorname{grad}}_{=\Delta_{\mathrm{N}}}, \qquad \partial_t^n + \underbrace{\operatorname{rot}\mathring{\operatorname{rot}}}_{=\vec{\square}_{\mathrm{t}}}, \qquad \partial_t^n + \underbrace{\operatorname{rot}\mathring{\operatorname{rot}} - \mathring{\operatorname{grad}}\operatorname{div}}_{=-\vec{\Delta}_{\mathrm{t}}}$$

elliptic ($n = 0$) / parabolic ($n = 1$) / hyperbolic ($n = 2$)

or skew-selfadjoint FOSs

$$\partial_t^m - \underbrace{\begin{bmatrix} 0 & \operatorname{div} \\ \mathring{\operatorname{grad}} & 0 \end{bmatrix}}_{=\mathrm{Maxwell}_{\mathrm{D,acoustic}}}, \quad \partial_t^m - \underbrace{\begin{bmatrix} 0 & -\operatorname{rot} \\ \mathring{\operatorname{rot}} & 0 \end{bmatrix}}_{=\mathrm{Maxwell}_{\mathrm{t,electromagnetic}}}, \quad \partial_t^m - \underbrace{\begin{bmatrix} 0 & \operatorname{div} & 0 & 0 \\ \mathring{\operatorname{grad}} & 0 & -\operatorname{rot} & 0 \\ 0 & \mathring{\operatorname{rot}} & 0 & \operatorname{grad} \\ 0 & 0 & \mathring{\operatorname{div}} & 0 \end{bmatrix}}_{=\mathrm{Picard's\ extended\ Maxwell}_{\mathrm{t,electromagnetic}}}$$

PDEs: de Rham complex 3D

Picard's extended Maxwell $\Rightarrow$ factorisation of the Laplacian

$$\begin{bmatrix} 0 & \operatorname{div} & 0 & 0 \\ \mathring{\operatorname{grad}} & 0 & -\operatorname{rot} & 0 \\ 0 & \mathring{\operatorname{rot}} & 0 & \operatorname{grad} \\ 0 & 0 & \mathring{\operatorname{div}} & 0 \end{bmatrix}^2 = \begin{bmatrix} \operatorname{div}\mathring{\operatorname{grad}} & 0 & 0 & 0 \\ 0 & \mathring{\operatorname{grad}}\operatorname{div} - \operatorname{rot}\mathring{\operatorname{rot}} & 0 & 0 \\ 0 & 0 & -\mathring{\operatorname{rot}}\operatorname{rot} + \operatorname{grad}\mathring{\operatorname{div}} & 0 \\ 0 & 0 & 0 & \mathring{\operatorname{div}}\operatorname{grad} \end{bmatrix}$$

$$= \begin{bmatrix} \Delta_{\mathsf{D}} & 0 & 0 & 0 \\ 0 & \vec{\Delta}_{\mathsf{t}} & 0 & 0 \\ 0 & 0 & \vec{\Delta}_{\mathsf{n}} & 0 \\ 0 & 0 & 0 & \Delta_{\mathsf{N}} \end{bmatrix}$$

solving Picard's extended Maxwell $\Rightarrow$ Helmholtz/Weyl decompositions

$$\begin{bmatrix} 0 & \operatorname{div} & 0 & 0 \\ \mathring{\operatorname{grad}} & 0 & -\operatorname{rot} & 0 \\ 0 & \mathring{\operatorname{rot}} & 0 & \operatorname{grad} \\ 0 & 0 & \mathring{\operatorname{div}} & 0 \end{bmatrix} \Rightarrow \begin{aligned} \mathsf{L}^2 &= R(\operatorname{div}) \\ \vec{\mathsf{L}}^2 &= R(\mathring{\operatorname{grad}}) \oplus R(\operatorname{rot}) \oplus \text{1st cohomology group} \\ \vec{\mathsf{L}}^2 &= R(\operatorname{grad}) \oplus R(\mathring{\operatorname{rot}}) \oplus \text{2nd cohomology group} \\ \mathsf{L}^2 &= R(\mathring{\operatorname{div}}) \oplus \mathbb{R} \end{aligned}$$

PDEs: de Rham complex ND / manifolds

d-complex (mother of all complexes)

$$\{0\} \underset{\pi_0}{\overset{\iota_0}{\rightleftarrows}} \cdots \quad \mathsf{L}^{2,q-1} \underset{-\,\delta_q}{\overset{\mathring{\mathsf{d}}_{q-1}}{\rightleftarrows}} \mathsf{L}^{2,q} \underset{-\,\delta_{q+1}}{\overset{\mathring{\mathsf{d}}_q}{\rightleftarrows}} \mathsf{L}^{2,q+1} \quad \cdots \underset{\iota_{*\mathbb{R}}}{\overset{\pi_{*\mathbb{R}}}{\rightleftarrows}} *\,\mathbb{R}$$

PDEs

$$\partial_t^n - \delta\,\mathring{\mathsf{d}}, \qquad \partial_t^n - \mathring{\delta}\,\mathsf{d}, \qquad \partial_t^n - \delta\,\mathring{\mathsf{d}}, \qquad \partial_t^n \underbrace{-\,\delta\,\mathring{\mathsf{d}} - \mathring{\mathsf{d}}\,\delta}_{=-\mathring{\Delta}_{q,\mathsf{t}}}$$

elliptic ($n = 0$) / parabolic ($n = 1$) / hyperbolic ($n = 2$)

or skew-selfadjoint FOSs $\partial_t^m - \underbrace{\begin{bmatrix} 0 & \delta \\ \mathring{\mathsf{d}} & 0 \end{bmatrix}}_{=\mathrm{Maxwell}_{\mathsf{t}}}$, $\partial_t^m - \underbrace{\begin{bmatrix} 0 & \delta & 0 & 0 \\ \mathring{\mathsf{d}} & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \delta \\ 0 & 0 & \mathring{\mathsf{d}} & 0 \end{bmatrix}}_{=\mathrm{Picard's\ extended\ Maxwell}_{\mathsf{t}}}$

PDEs: de Rham complex ND / manifolds

Picard's extended Maxwell $\Rightarrow$ factorisation of the Hodge-Laplacian

$$\begin{bmatrix} 0 & \delta & 0 & 0 \\ \mathring{\mathrm{d}} & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \delta \\ 0 & 0 & \mathring{\mathrm{d}} & 0 \end{bmatrix}^2 = \begin{bmatrix} \delta\,\mathring{\mathrm{d}} & 0 & 0 & 0 & 0 \\ 0 & \mathring{\mathrm{d}}\,\delta + \delta\,\mathring{\mathrm{d}} & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \mathring{\mathrm{d}}\,\delta + \delta\,\mathring{\mathrm{d}} & 0 \\ 0 & 0 & 0 & 0 & \mathring{\mathrm{d}}\,\delta \end{bmatrix} = \begin{bmatrix} \mathring{\Delta}_{0,\mathrm{t}} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathring{\Delta}_{N,\mathrm{t}} \end{bmatrix}$$

solving Picard's extended Maxwell $\Rightarrow$ Hodge/Helmholtz/Weyl decompositions

$$\begin{bmatrix} 0 & \delta & 0 & 0 \\ \mathring{\mathrm{d}} & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \delta \\ 0 & 0 & \mathring{\mathrm{d}} & 0 \end{bmatrix} \Rightarrow \begin{aligned} \mathrm{L}^{2,0} &= R(\delta) \\ &\vdots \\ \mathrm{L}^{2,q} &= R(\mathring{\mathrm{d}}) \oplus R(\delta) \oplus \text{(q-1)th cohomology group} \\ &\vdots \\ \mathrm{L}^{2,N} &= R(\mathring{\mathrm{d}}) \oplus *\mathbb{R} \end{aligned}$$

PDEs: elasticity complex 3D

symGrad-complex

$$\{0\} \underset{\pi_0}{\overset{\iota_0}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\operatorname{symGrad}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{RotRot}^\top_{\mathbb{S}}}{\overset{\mathring{\operatorname{RotRot}}^\top_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{-\operatorname{symGrad}}{\overset{\mathring{\operatorname{Div}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathbb{RM}}}{\overset{\pi_{\mathbb{RM}}}{\rightleftarrows}} \mathbb{RM}$$

PDEs

$$\partial_t^n - \underbrace{\operatorname{Div}_{\mathbb{S}} \mathring{\operatorname{symGrad}}}_{=\vec{\Delta}_{\mathsf{D}}}, \qquad \partial_t^n - \underbrace{\mathring{\operatorname{Div}}_{\mathbb{S}} \operatorname{symGrad}}_{=\vec{\Delta}_{\mathsf{N}}}, \qquad \partial_t^n + \operatorname{RotRot}^\top_{\mathbb{S}} \mathring{\operatorname{RotRot}}^\top_{\mathbb{S}},$$

$$\partial_t^n + \underbrace{\operatorname{RotRot}^\top_{\mathbb{S}} \mathring{\operatorname{RotRot}}^\top_{\mathbb{S}}}_{\text{4th order}} - \underbrace{\mathring{\operatorname{symGrad}} \operatorname{Div}_{\mathbb{S}}}_{\text{2nd order}}$$

(apparently mixed order type, but NOT compact perturbation!)

elliptic $(n = 0)$ / parabolic $(n = 1)$ / hyperbolic $(n = 2)$

PDEs: elasticity complex 3D

extended elasticity $\Rightarrow$ factorisation of the gen Laplacian

$$\begin{bmatrix} 0 & \operatorname{Div}_{\mathbb{S}} & 0 & 0 \\ \operatorname{sym}\mathring{\operatorname{Grad}} & 0 & -\operatorname{RotRot}_{\mathbb{S}}^{\top} & 0 \\ 0 & \mathring{\operatorname{RotRot}}_{\mathbb{S}}^{\top} & 0 & \operatorname{symGrad} \\ 0 & 0 & \mathring{\operatorname{Div}}_{\mathbb{S}} & 0 \end{bmatrix}^2 = \begin{bmatrix} \vec{\Delta}_{\mathrm{D}} & 0 & 0 & 0 \\ 0 & \mathring{\Delta}_{\mathbb{S},t} & 0 & 0 \\ 0 & 0 & \mathring{\Delta}_{\mathbb{S},n} & 0 \\ 0 & 0 & 0 & \vec{\Delta}_{\mathrm{N}} \end{bmatrix}$$

$$= \begin{bmatrix} \operatorname{Div}_{\mathbb{S}} \operatorname{sym}\mathring{\operatorname{Grad}} & 0 & 0 & 0 \\ 0 & \operatorname{sym}\mathring{\operatorname{Grad}} \operatorname{Div}_{\mathbb{S}} - \operatorname{RotRot}_{\mathbb{S}}^{\top} \mathring{\operatorname{RotRot}}_{\mathbb{S}}^{\top} & 0 & 0 \\ 0 & 0 & -\mathring{\operatorname{RotRot}}_{\mathbb{S}}^{\top} \operatorname{RotRot}_{\mathbb{S}}^{\top} + \operatorname{symGrad} \mathring{\operatorname{Div}}_{\mathbb{S}} & 0 \\ 0 & 0 & 0 & \mathring{\operatorname{Div}}_{\mathbb{S}} \operatorname{symGrad} \end{bmatrix}$$

solving extended elasticity $\Rightarrow$ gen Hodge/Helmholtz/Weyl decompositions

$$\begin{bmatrix} 0 & \operatorname{Div}_{\mathbb{S}} & 0 & 0 \\ \operatorname{sym}\mathring{\operatorname{Grad}} & 0 & -\operatorname{RotRot}_{\mathbb{S}}^{\top} & 0 \\ 0 & \mathring{\operatorname{RotRot}}_{\mathbb{S}}^{\top} & 0 & \operatorname{symGrad} \\ 0 & 0 & \mathring{\operatorname{Div}}_{\mathbb{S}} & 0 \end{bmatrix} \Rightarrow \begin{aligned} \vec{\mathrm{L}}^2 &= R(\operatorname{Div}_{\mathbb{S}}) \\ \mathrm{L}^2_{\mathbb{S}} &= R(\operatorname{sym}\mathring{\operatorname{Grad}}) \oplus R(\operatorname{RotRot}_{\mathbb{S}}^{\top}) \oplus \text{1st cohomology group} \\ \mathrm{L}^2_{\mathbb{S}} &= R(\operatorname{symGrad}) \oplus R(\mathring{\operatorname{RotRot}}_{\mathbb{S}}^{\top}) \oplus \text{2nd cohomology group} \\ \vec{\mathrm{L}}^2 &= R(\mathring{\operatorname{Div}}_{\mathbb{S}}) \oplus \mathbb{RM} \end{aligned}$$

PDEs: 1st and 2nd biharmonic / general relativity complexes 3D

Gradgrad-complex

$$\{0\} \underset{\pi_0}{\overset{\iota_0}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{divDiv}_{\mathbb{S}}}{\overset{\mathring{\operatorname{Gradgrad}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{symRot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}} \underset{-\operatorname{devGrad}}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathbb{RT}}}{\overset{\pi_{\mathbb{RT}}}{\rightleftarrows}} \mathbb{RT}$$

devGrad-complex

$$\{0\} \underset{\pi_0}{\overset{\iota_0}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{Div}_{\mathbb{T}}}{\overset{\mathring{\operatorname{devGrad}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}} \underset{\operatorname{Rot}_{\mathbb{S}}}{\overset{\mathring{\operatorname{symRot}}_{\mathbb{T}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{Gradgrad}}{\overset{\mathring{\operatorname{divDiv}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathbb{P}_1}}{\overset{\pi_{\mathbb{P}_1}}{\rightleftarrows}} \mathbb{P}_1$$

PDEs

$$\partial_t^n + \underbrace{\operatorname{divDiv}_{\mathbb{S}} \mathring{\operatorname{Gradgrad}}}_{=\Delta_{\mathsf{D}}^2}, \qquad \partial_t^n - \underbrace{\mathring{\operatorname{Div}}_{\mathbb{T}} \operatorname{devGrad}}_{=\vec{\Delta}_{\mathsf{N}}}, \qquad \partial_t^n + \operatorname{Rot}_{\mathbb{S}} \mathring{\operatorname{symRot}}_{\mathbb{T}},$$

$$\partial_t^n + \underbrace{\operatorname{symRot}_{\mathbb{T}} \mathring{\operatorname{Rot}}_{\mathbb{S}}}_{\text{2nd order}} + \underbrace{\mathring{\operatorname{Gradgrad}} \operatorname{divDiv}_{\mathbb{S}}}_{\text{4th order}}$$

(apparently mixed order type, but NOT compact perturbation!)

elliptic ($n = 0$) / parabolic ($n = 1$) / hyperbolic ($n = 2$)

BGG

whole zoo more . . .

BGG resolution

I.N. Bernstein, I.M. Gelfand, S.I. Gelfand ('70s)

M. Eastwood

D. Arnold, R. Falk, R. Winther, . . .

also important for FEM, FEEC, DEC, . . .
(construction and analysis)

General Complex / $\rightsquigarrow$ FA-ToolBox

densely defined and closed (unbounded) linear operators

$$\begin{aligned} \mathrm{A}_0 &: D(\mathrm{A}_0) \subset \mathrm{H}_0 \to \mathrm{H}_1, & \mathrm{A}_1 &: D(\mathrm{A}_1) \subset \mathrm{H}_1 \to \mathrm{H}_2 \\ \mathrm{A}_0^* &: D(\mathrm{A}_0^*) \subset \mathrm{H}_1 \to \mathrm{H}_0, & \mathrm{A}_1^* &: D(\mathrm{A}_1^*) \subset \mathrm{H}_2 \to \mathrm{H}_1 \end{aligned}$$

general complex property $\boxed{\mathrm{A}_1\mathrm{A}_0 = 0}$, i.e., $R(\mathrm{A}_0) \subset N(\mathrm{A}_1)$ $\quad (\Leftrightarrow \; R(\mathrm{A}_1^*) \subset N(\mathrm{A}_0^*))$

$\boxed{\text{Hilbert complex}}$

$$N(\mathrm{A}_0) \underset{\pi_{N(\mathrm{A}_0)}}{\overset{\iota_{N(\mathrm{A}_0)}}{\rightleftarrows}} \cdots \; \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \; \cdots \underset{\iota_{N(\mathrm{A}_n^*)}}{\overset{\pi_{N(\mathrm{A}_n^*)}}{\rightleftarrows}} N(\mathrm{A}_n^*)$$

General Complex $\rightsquigarrow$ FA-ToolBox

Hilbert complex

$$N(\mathrm{A}_0) \underset{\pi_{N(\mathrm{A}_0)}}{\overset{\iota_{N(\mathrm{A}_0)}}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \cdots \mathrm{H}_{n-1} \underset{\mathrm{A}_{n-1}^*}{\overset{\mathrm{A}_{n-1}}{\rightleftarrows}} \mathrm{H}_n \underset{\mathrm{A}_n^*}{\overset{\mathrm{A}_n}{\rightleftarrows}} \mathrm{H}_{n+1} \cdots \mathrm{H}_N \underset{\mathrm{A}_N^*}{\overset{\mathrm{A}_N}{\rightleftarrows}} \mathrm{H}_{N+1} \underset{\iota_{N(\mathrm{A}_N^*)}}{\overset{\pi_{N(\mathrm{A}_N^*)}}{\rightleftarrows}} N(\mathrm{A}_N^*)$$

some equations

$$\partial_t^n + \mathrm{A}_0^*\mathrm{A}_0, \qquad \partial_t^n + \mathrm{A}_1\mathrm{A}_1^*, \qquad \partial_t^n + \mathrm{A}_1^*\mathrm{A}_1 + \mathrm{A}_0\mathrm{A}_0^*, \qquad \partial_t^m - \begin{bmatrix} 0 & -\mathrm{A}_1^* \\ \mathrm{A}_1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} \partial_t^m & \mathrm{A}_1^* \\ -\mathrm{A}_1 & \partial_t^\ell \end{bmatrix}$$

elliptic ($n = 0$) / parabolic ($n = 1$) / hyperbolic ($n = 2$) $\qquad m, \ell \in \{0, 1\}$

ell ($m = \ell = 0$) / para ($m = 1, \ell = 0$ or $m = 0, \ell = 1$) / hyper ($m = \ell = 1$)

$$\partial_t^m - \underbrace{\begin{bmatrix} 0 & -\mathrm{A}_0^* & 0 & 0 \\ \mathrm{A}_0 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -\mathrm{A}_N^* \\ 0 & 0 & \mathrm{A}_N & 0 \end{bmatrix}}_{=\text{extended complex operator}}$$

General Complex $\rightsquigarrow$ FA-ToolBox

extended complex operator $\Rightarrow$ factorisation of gen Hodge-Laplacian

$$\begin{bmatrix} 0 & -A_0^* & 0 & 0 \\ A_0 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -A_N^* \\ 0 & 0 & A_N & 0 \end{bmatrix}^2 = -\begin{bmatrix} A_0^*A_0 & 0 & 0 & 0 & 0 \\ 0 & A_0A_0^* \oplus A_1^*A_1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & A_{N-1}A_{N-1}^* \oplus A_N^*A_N & 0 \\ 0 & 0 & 0 & 0 & A_NA_N^* \end{bmatrix}$$

$$= -\begin{bmatrix} \Delta_{A,0} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \Delta_{A,N+1} \end{bmatrix}$$

solving extended complex operator $\Rightarrow$ gen Hodge/Helmholtz/Weyl decompositions

$$\begin{bmatrix} 0 & -A_0^* & 0 & 0 \\ A_0 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -A_N^* \\ 0 & 0 & A_N & 0 \end{bmatrix} \Rightarrow \begin{aligned} H_0 &= R(A_0^*) \oplus N(A_0) \\ &\vdots \\ H_n &= R(A_{n-1}) \oplus R(A_n^*) \oplus \big(N(A_n) \cap N(A_{n-1}^*)\big) \\ &\vdots \\ H_{N+1} &= R(A_N) \oplus N(A_N^*) \end{aligned}$$

General Complex $\rightsquigarrow$ FA-ToolBox

extended complex operator

$$\begin{bmatrix} 0 & -A_0^* & 0 & 0 \\ A_0 & 0 & -A_1^* & 0 \\ 0 & A_1 & 0 & -A_2^* \\ 0 & 0 & A_2 & 0 \end{bmatrix}$$

another nice operator is

$$\begin{bmatrix} A_2 & 0 \\ A_1^* & A_0 \end{bmatrix}$$

solving the gen div-curl system (g st $y = 0$)

and giving the gen Hodge/Helmholtz/Weyl decomposition (eg $f = 0$)

$$\begin{aligned} A_2 x &= f \in R(A_2) && (\perp \text{ kernel of } A_2^*) \\ A_1^* x + A_0 y &= g \in R(A_1^*) \oplus R(A_0) && (\perp \text{ cohomology group}) \end{aligned}$$

think of

$$\begin{bmatrix} A_2 & 0 \\ A_1^* & A_0 \end{bmatrix} = \begin{bmatrix} \mathring{\operatorname{div}} & 0 \\ \operatorname{rot} & \mathring{\operatorname{grad}} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathring{\operatorname{div}}\, x & = f \\ \operatorname{rot} x + \mathring{\operatorname{grad}}\, y & = g \end{bmatrix}$$

$\mathring{\operatorname{grad}}\, y$ sometimes Lagrange parameter

Hilbert Complexes and PDEs

OVERVIEW

(I) general theory

FA-ToolBox (Hilbert complexes, tailor-made and simple functional analysis)

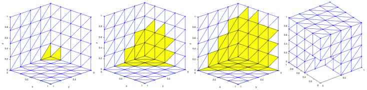
$$Ax = f$$

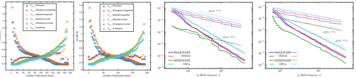
- solution theories
- closed ranges and continuous inverse operators
- Friedrichs/Poincaré estimates
- inf sup lemmas
- results about constants/eigenvalues
- Helmholtz/Hodge/Weyl decompositions
- compact embeddings
- compact inverse operators
- (compact) trace Hilbert complexes
- variational formulations
- regular potentials and regular decompositions
- generalized div-curl-lemmas
- index theorems
- dimensions and bases of cohomology groups
- functional a posteriori error estimates
- optimal control problems
- finding new FEs
- ...

(II) applications to pdes (complexes)

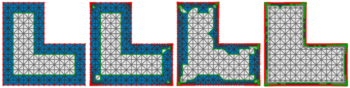
- $\cdots \; \mathsf{L}^2 \; \underset{-\,\mathrm{div}}{\overset{\mathring{\mathrm{grad}}}{\rightleftarrows}} \; \mathsf{L}^2 \; \underset{\mathrm{rot}}{\overset{\mathring{\mathrm{rot}}}{\rightleftarrows}} \; \mathsf{L}^2 \; \underset{-\,\mathrm{grad}}{\overset{\mathring{\mathrm{div}}}{\rightleftarrows}} \; \mathsf{L}^2 \; \cdots$ (de Rham)
- $\cdots \; \mathsf{L}^2 \; \underset{-\,\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \; \mathsf{L}^2 \; \underset{-\,\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \; \mathsf{L}^2 \; \cdots$ (de Rham)
- $\cdots \; \mathsf{L}^2 \; \underset{-\,\mathrm{Div}_{\mathbb{S}}}{\overset{\mathrm{sym}\,\mathring{\mathrm{Grad}}}{\rightleftarrows}} \; \mathsf{L}^2_{\mathbb{S}} \; \underset{\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^\top}{\overset{\mathrm{Rot}\,\mathring{\mathrm{Rot}}_{\mathbb{S}}^\top}{\rightleftarrows}} \; \mathsf{L}^2_{\mathbb{S}} \; \underset{-\,\mathrm{sym}\,\mathrm{Grad}}{\overset{\mathring{\mathrm{Div}}_{\mathbb{S}}}{\rightleftarrows}} \; \mathsf{L}^2 \; \cdots$ (elasticity)
- $\cdots \; \mathsf{L}^2 \; \underset{\mathrm{div}\,\mathrm{Div}_{\mathbb{S}}}{\overset{\mathrm{Grad}\,\mathring{\mathrm{grad}}}{\rightleftarrows}} \; \mathsf{L}^2_{\mathbb{S}} \; \underset{\mathrm{sym}\,\mathrm{Rot}_{\mathbb{T}}}{\overset{\mathring{\mathrm{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \; \mathsf{L}^2_{\mathbb{T}} \; \underset{-\,\mathrm{dev}\,\mathrm{Grad}}{\overset{\mathring{\mathrm{Div}}_{\mathbb{T}}}{\rightleftarrows}} \; \mathsf{L}^2 \; \cdots$ (biharmonic/general relativity)
- $\cdots$ (. . . much more complexes)
- $\cdots \; \mathsf{H}_0 \; \underset{\mathsf{A}_0^*}{\overset{\mathsf{A}_0}{\rightleftarrows}} \; \mathsf{H}_1 \; \underset{\mathsf{A}_1^*}{\overset{\mathsf{A}_1}{\rightleftarrows}} \; \mathsf{H}_2 \; \cdots$ (general)

(III) numerical applications to pdes

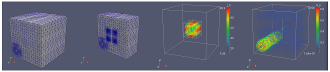


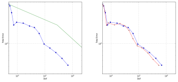


- **Friedrichs/Poincaré/Maxwell constants**
  monotonicity?
  analytical results and FEM computations
  (with colleagues from Praha, Wien)



- **functional a posteriori error estimates**
  for BEM
  computations with BEM and FEM
  (with colleagues from Darmstadt, München, Wien, St. Petersburg, Bosch GmbH)





- **functional a posteriori error estimates**
  electro-magneto static optimal control problems
  computations with FEM
  (with colleagues from Darmstadt, Essen)

- **DEC-FA-ToolBox**
  DEC: Discrete Exterior Calculus as discrete version of FA-ToolBox
  (with colleagues from Jyväskylä)
  $\rightsquigarrow$ https://sites.google.com/jyu.fi/gfd/method/online-time-integrator

Solving PDEs with Hilbert Complexes

Introduction and Motivation

Solving PDEs with Hilbert Complexes

FA-ToolBox

general observations

$$Ax = f$$

general observations

$$\boxed{\mathrm{A}x = f}$$

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ (lin, dd, cl, H_0, H_1 Hilbert spaces)

$$\boxed{? \qquad x = \mathrm{A}^{-1} f \qquad ?}$$

general observations

$$\boxed{Ax = f}$$

$A : D(A) \subset H_0 \to H_1$ (lin, dd, cl)

solution theory in the sense of Hadamard

- existence $\Leftrightarrow$ $f \in R(A)$
- uniqueness $\Leftrightarrow$ A inj $\Leftrightarrow$ $N(A) = \{0\}$ $\Leftrightarrow$ A^{-1} exists
- cont dep on f $\Leftrightarrow$ A^{-1} cont

aim: $x = A^{-1}f \in D(A)$ and cont estimate (Friedrichs/Poincaré type estimate)

$$|x|_{H_0} = |A^{-1}f|_{H_0} \leq c_A |f|_{H_1} = c_A |Ax|_{H_1}$$

note: best constant $c_A = |A^{-1}|_{R(A),H_0} = \frac{1}{\lambda_0}$,

λ_0^2 smallest eigenvalue of A^*A (and AA^*)

general observations

$A : D(A) \subset H_0 \to H_1$

$A^* : D(A^*) \subset H_1 \to H_0$ Hilbert space adjoint

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$H_1 = \overline{R(A)} \oplus N(A^*), \qquad H_0 = N(A) \oplus \overline{R(A^*)}$$

$$\boxed{Ax = f}$$

solution theory in the sense of Hadamard

- existence $\Leftrightarrow$ $f \in R(A) = N(A^*)^\perp$ (Fredholm alt, if $R(A)$ cl)
- uniqueness $\Leftrightarrow$ A inj $\Leftrightarrow$ $N(A) = \{0\}$ $\Leftrightarrow$ A^{-1} exists
- cont dep on f $\Leftrightarrow$ A^{-1} cont $\Leftrightarrow$ $R(A)$ cl (cl graph theo)

fund range cond: $\boxed{R(A) = \overline{R(A)} \text{ closed}}$ (must hold $\rightsquigarrow$ right setting!)

kernel cond: $\boxed{N(A) = \{0\}}$ (fails in gen $\rightsquigarrow$ proj onto $N(A)^\perp = \overline{R(A^*)} = R(A^*)$)

general observations

observations (from this perspective)

- time-dependent problems are simple

  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H} \to \mathrm{H}$, $\boxed{\mathrm{A} = \partial_t + \mathrm{T}}$ (gen T skw-sa, or at least Re $\mathrm{T} \geq 0$)

  $$N(\mathrm{A}) = \{0\} \qquad N(\mathrm{A}^*) = \{0\} \qquad R(\mathrm{A}) \text{ (cl) } = N(\mathrm{A}^*)^\perp = \mathrm{H}$$

- time-harmonic problems are more complicated

  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H} \to \mathrm{H}$, $\boxed{\mathrm{A} = -\omega + \mathrm{T}}$

  $$N(\mathrm{A}),\ N(\mathrm{A}^*) \text{ (fin dim)} \qquad R(\mathrm{A}) \text{ (cl, fin co-dim) } = N(\mathrm{A}^*)^\perp$$

  (Fredholm alternative)

- static problems are most complicated

  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$, $\boxed{\mathrm{A} = 0 + \mathrm{T}}$

  $$\dim N(\mathrm{A}) = \dim N(\mathrm{A}^*) = \infty \text{ (possible/standard)} \qquad R(\mathrm{A}) \text{ (cl, infin co-dim) } = N(\mathrm{A}^*)^\perp$$

compactness $\Rightarrow$ cl (closed range)

general key observations I

$$A : D(A) \subset H_0 \to H_1 \qquad \text{(lddc, } H_0, H_1 \text{ Hilbert spaces)}$$
$$A^* : D(A^*) \subset H_1 \to H_0 \qquad \text{(Hilbert space adjoint, } (A, A^*) \text{ dual pair)}$$

A, A^* may not be inj $\Rightarrow$ Helmholtz/Hodge/Weyl decos (proj theorem)

$$H_1 = N(A^*) \oplus \overline{R(A)} \qquad H_0 = N(A) \oplus \overline{R(A^*)}$$

$$\mathcal{A} := A|_{N(A)^\perp} = A|_{\overline{R(A^*)}} : D(A) \cap N(A)^\perp \subset \overline{R(A^*)} \to \overline{R(A)} \qquad \text{(reduced operators)}$$
$$\mathcal{A}^* := A^*|_{N(A^*)^\perp} = A|_{\overline{R(A)}} : D(A^*) \cap N(A^*)^\perp \subset \overline{R(A)} \to \overline{R(A^*)}$$

$\mathcal{A}$, $\mathcal{A}^*$ inj $\Rightarrow$ $\mathcal{A}^{-1} : R(A) \to D(\mathcal{A})$, $(\mathcal{A}^*)^{-1} : R(A^*) \to D(\mathcal{A}^*)$ ex, ? bounded ?

general key observations I

FA-ToolBox

first simple lemmas

general key observations I

Lemma (FA-ToolBox Lemma 1)

The following assertions are equivalent:

- $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *bd*
- $\forall\, x \in D(\mathcal{A}) \quad |x|_{\mathrm{H}_0} \leq c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$
- $R(\mathrm{A}) = R(\mathcal{A})$ *cl*
- $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *bd*
- $\forall\, y \in D(\mathcal{A}^*) \quad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}} |\mathrm{A}^* y|_{\mathrm{H}_0}$
- $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *cl*

note: best const $c_{\mathrm{A}} = |\mathrm{A}^{-1}|_{R(\mathrm{A}),\mathrm{H}_0} = |(\mathrm{A}^*)^{-1}|_{R(\mathrm{A}^*),\mathrm{H}_1} = \frac{1}{\lambda_0}$,

λ_0^2 smallest pos ev of $\mathrm{A}^*\mathrm{A}$ and $\mathrm{A}\mathrm{A}^*$

Lemma (FA-ToolBox Lemma 2)

The following assertions are equivalent:

- $D(\mathcal{A}) \hookrightarrow \mathrm{H}_0$ *cpt*
- $\mathcal{A}^{-1} : R(\mathrm{A}) \to \mathrm{H}_0$ *cpt*
- $D(\mathcal{A}^*) \hookrightarrow \mathrm{H}_1$ *cpt*
- $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to \mathrm{H}_1$ *cpt*

Lemma (FA-ToolBox Lemma 3)

$D(\mathcal{A}) \hookrightarrow \mathrm{H}_0$ *cpt* $\quad \Rightarrow \quad$ *assertions of Lemma 1 hold.*

general key observations II

FA-ToolBox

so far no complex

time-dependent problems: OK
time-harmonic problems: OK
static problems: NOT OK

general key observations II

$$\begin{aligned} &\mathrm{A}_0 : D(\mathrm{A}_0) \subset \mathrm{H}_0 \to \mathrm{H}_1, && \mathrm{A}_0^* : D(\mathrm{A}_0^*) \subset \mathrm{H}_1 \to \mathrm{H}_0 && \text{(lddc)} \\ &\mathrm{A}_1 : D(\mathrm{A}_1) \subset \mathrm{H}_1 \to \mathrm{H}_2, && \mathrm{A}_1^* : D(\mathrm{A}_1^*) \subset \mathrm{H}_2 \to \mathrm{H}_1 \end{aligned}$$

gen cmplx $\boxed{\mathrm{A}_1\mathrm{A}_0 \subset 0}$ $\quad (\Leftrightarrow\ R(\mathrm{A}_0) \subset N(\mathrm{A}_1) \ \Leftrightarrow\ R(\mathrm{A}_1^*) \subset N(\mathrm{A}_0^*) \ \Leftrightarrow\ \mathrm{A}_0^*\mathrm{A}_1^* \subset 0)$

$$\boxed{\cdots \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad \mathrm{H}_0 \quad \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \quad \mathrm{H}_1 \quad \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \quad \mathrm{H}_2 \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad \cdots}$$

recall Helmholtz deco

$$\begin{aligned} \mathrm{H}_1 &= \overline{R(\mathrm{A}_0)} \oplus N(\mathrm{A}_0^*) \\ & \quad\ \ \bigcap \qquad\quad \bigcup \\ &= N(\mathrm{A}_1) \oplus \overline{R(\mathrm{A}_1^*)} \end{aligned} \quad \Rightarrow \text{ (e.g.)} \quad N(\mathrm{A}_1) = \overline{R(\mathrm{A}_0)} \oplus \underbrace{\big(N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \big)}_{=:N_{0,1} \text{ cohom gr}}$$

$\Rightarrow$ refined Helmholtz deco $\boxed{\mathrm{H}_1 = \overline{R(\mathrm{A}_0)} \oplus N_{0,1} \oplus \overline{R(\mathrm{A}_1^*)}}$

general key observations II

refined Helmholtz deco

$$H_1 = \overbrace{R(A_0)}^{=N(A_0^*)^\perp} \oplus N_{0,1} \oplus \overbrace{R(A_1^*)}^{=N(A_1)^\perp}$$

$$D(A_1) \cap D(A_0^*) = D(\mathcal{A}_0^*) \oplus N_{0,1} \oplus D(\mathcal{A}_1)$$

Lemma (FA-ToolBox Lemma 4)

$D(A_1) \cap D(A_0^*) \hookrightarrow H_1$ *compact* $\quad\Rightarrow\quad$ *Lemma 1 holds*

Lemma (FA-ToolBox Lemma 5)

The following assertions are equivalent:

- $D(A_1) \cap D(A_0^*) \hookrightarrow H_1$ *compact*
- $D(\mathcal{A}_0) \hookrightarrow H_0$ *and* $D(\mathcal{A}_1) \hookrightarrow H_1$ *and* $N_{0,1} \hookrightarrow H_1$ *compact*

In this case $\dim N_{0,1} < \infty$

Remark (FA-ToolBox Remark 1)

cohomology group: $N(A_1) \cap N(A_0^*) = N(A_1) \cap R(A_0)^\perp \cong N(A_1)/R(A_0)$

general key observations II

Remark (FA-ToolBox Remark 2)

$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathrm{H}_1$ compact $\quad\Rightarrow\quad$ everything holds!

Remark

Question: How to prove $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathrm{H}_1 \text{ compact}}$?

Answer: $\boxed{\text{FA-ToolBox}}$ + $\boxed{\text{regular decompositions and regular potentials}}$

Hilbert Complexes and PDEs

Some Selected Results

Assumption (on the domain where the PDEs are posed)

Let $\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain with bounded weak Lipschitz interface (mixed boundary conditions).

FA-ToolBox: Basics

Theorem (compact complexes)

All the latter Hilbert complexes are compact, i.e., for all n

$$D(\mathrm{A}_n) \cap D(\mathrm{A}_{n-1}^*) \hookrightarrow \mathrm{H}_n \text{ compact.}$$

best and gen assumption

Corollary (closed complexes)

All the latter Hilbert complexes are closed, i.e., for all n $R(\mathrm{A}_n)$, $R(\mathrm{A}_n^*)$ *closed.*

Corollary (Helmholtz/Hodge/Weyl decompositions)

For all n, e.g., $\mathrm{H}_n = R(\mathrm{A}_{n-1}) \oplus N_{n-1,n} \oplus R(\mathrm{A}_n^*)$

Corollary (Friedrichs/Poincaré estimates)

All the latter operators admit Friedrichs/Poincaré type estimates, i.e., for all n

$$\forall\, x \in D(\mathcal{A}_n) \quad |x|_{\mathrm{H}_n} \leq c_{\mathrm{A}_n} |\mathrm{A}_n x|_{\mathrm{H}_{n+1}}.$$

FA-ToolBox: Applications I

application: solution theories

e.g.: FOS (gen electro-magneto statics / saddle point problems)

$$\mathrm{A}_1 x = f$$
$$\mathrm{A}_0^* x = g$$
$$\pi_{N_{0,1}} x = k$$

Theorem (static solution theory)

FOS uniquely solvable $\Leftrightarrow$ $f \in R(\mathrm{A}_1)$, $g \in R(\mathrm{A}_0^*)$, $k \in N_{0,1}$

Moreover

$$x = x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus N_{0,1} = D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$$

with

$$\boxed{x_f := \mathcal{A}_1^{-1} f} \in D(\mathcal{A}_1), \qquad \boxed{x_g := (\mathcal{A}_0^*)^{-1} g} \in D(\mathcal{A}_0^*)$$

and dep cont on data $|x|_{\mathrm{H}_1}^2 = |x_f|_{\mathrm{H}_1}^2 + |x_g|_{\mathrm{H}_1}^2 + |k|_{\mathrm{H}_1}^2 \leq c_{\mathrm{A}_1}^2 |f|_{\mathrm{H}_2}^2 + c_{\mathrm{A}_0}^2 |g|_{\mathrm{H}_0}^2 + |k|_{\mathrm{H}_1}^2$

note: $x = \mathcal{A}_1^{-1} f + (\mathcal{A}_0^*)^{-1} g + k$

FA-ToolBox: Applications I

application: solution theories

same for other systems such as

$$\begin{aligned} A_1^* A_1 x &= h \\ A_0^* x &= g \\ \pi_{N_{0,1}} x &= k \\ x &= \mathcal{A}_1^{-1} (\mathcal{A}_1^*)^{-1} h + (\mathcal{A}_0^*)^{-1} g + k \end{aligned}$$

$$\begin{aligned} A_1 x &= f \\ A_0 A_0^* x &= j \\ \pi_{N_{0,1}} x &= k \\ x &= \mathcal{A}_1^{-1} f + (\mathcal{A}_0^*)^{-1} \mathcal{A}_0^{-1} j + k \end{aligned}$$

$$\begin{aligned} A_1^* A_1 x &= h \\ A_0 A_0^* x &= j \\ \pi_{N_{0,1}} x &= k \\ x &= \mathcal{A}_1^{-1} (\mathcal{A}_1^*)^{-1} h + (\mathcal{A}_0^*)^{-1} \mathcal{A}_0^{-1} j + k \end{aligned}$$

$$\begin{aligned} (A_1^* A_1 + A_0 A_0^*) x &= l \\ \pi_{N_{0,1}} x &= k \\ x &= \mathcal{A}_1^{-1} (\mathcal{A}_1^*)^{-1} \pi_{R(A_1^*)} l + k \\ &\quad + (\mathcal{A}_0^*)^{-1} \mathcal{A}_0^{-1} \pi_{R(A_0)} l \end{aligned}$$

FA-ToolBox: Applications I

application: a posteriori error estimates for FOS/SOS

problem: find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ st $\mathrm{A}_1 x = f \quad \mathrm{A}_0^* x = g \quad \pi_{N_1} x = k$

'very' non-conforming 'approximation' of x: $\tilde{x} \in \mathrm{H}_1$

def., dcmp. err. $e = x - \tilde{x} = \pi_{R(\mathrm{A}_0)} e + \pi_{N_1} e + \pi_{R(\mathrm{A}_1^*)} e \in \mathrm{H}_1 = R(\mathrm{A}_0) \oplus N_1 \oplus R(\mathrm{A}_1^*)$

Theorem (sharp upper bounds)

Let $\tilde{x} \in \mathrm{H}_1$ *and* $e = x - \tilde{x}$. *Then*

$$|e|_{\mathrm{H}_1}^2 = |\pi_{R(\mathrm{A}_0)} e|_{\mathrm{H}_1}^2 + |\pi_{N_1} e|_{\mathrm{H}_1}^2 + |\pi_{R(\mathrm{A}_1^*)} e|_{\mathrm{H}_1}^2$$

$$|\pi_{R(\mathrm{A}_0)} e|_{\mathrm{H}_1} = \min_{\phi \in D(\mathrm{A}_0^*)} \big(c_{\mathrm{A}_0} |\mathrm{A}_0^* \phi - g|_{\mathrm{H}_0} + |\phi - \tilde{x}|_{\mathrm{H}_1} \big)$$

reg $(\mathrm{A}_0 \mathrm{A}_0^* + 1)$*-prbl in* $D(\mathrm{A}_0^*)$

$$|\pi_{R(\mathrm{A}_1^*)} e|_{\mathrm{H}_1} = \min_{\varphi \in D(\mathrm{A}_1)} \big(c_{\mathrm{A}_1} |\mathrm{A}_1 \varphi - f|_{\mathrm{H}_2} + |\varphi - \tilde{x}|_{\mathrm{H}_1} \big)$$

reg $(\mathrm{A}_1^* \mathrm{A}_1 + 1)$*-prbl in* $D(\mathrm{A}_1)$

$$|\pi_{N_1} e|_{\mathrm{H}_1} = |\pi_{N_1} \tilde{x} - k|_{\mathrm{H}_1} = \min_{\substack{\xi \in D(\mathrm{A}_0) \\ \zeta \in D(\mathrm{A}_1^*)}} |\mathrm{A}_0 \xi + \mathrm{A}_1^* \zeta + \tilde{x} - k|_{\mathrm{H}_1}$$

cpld $(\mathrm{A}_0^* \mathrm{A}_0)$-$(\mathrm{A}_1 \mathrm{A}_1^*)$*-sys in* $D(\mathcal{A}_0)$-$D(\mathcal{A}_1^*)$

Remark

also share lower bounds

FA-ToolBox: Applications I

application: div-curl lemmas

Theorem (div-curl lemma / A_0^*-A_1 lemma)

(x_n) bounded in $D(A_1)$ and (y_n) bounded in $D(A_0^*)$

$\Rightarrow \exists\, x \in D(A_1)$, $y \in D(A_0^*)$ and subseq st $x_n \rightharpoonup x$ in $D(A_1)$ and $y_n \rightharpoonup y$ in $D(A_0^*)$ and

$$\langle x_n, y_n \rangle_{H_1} \to \langle x, y \rangle_{H_1}$$

FA-ToolBox: Applications I

application: div-curl lemmas

Theorem (gen div-curl lemma / gen A_0^*-A_1 lemma)

enough: $R(A_0)$, $R(A_1$ *closed and* $\dim N_{0,1} < \infty$. *Then:* $(x_n), (y_n) \subset H_1$ *bounded and*

(i) $(\widetilde{A}_1 x_n)$ *is relatively compact in* $D(A_1^*)'$

(ii) $(\widetilde{A}_0^* y_n)$ *is relatively compact in* $D(A_0)'$

$\Rightarrow \exists\, x, y \in H_1$ *and subseq st* $x_n \rightharpoonup x$ *in* H_1 *and* $y_n \rightharpoonup y$ *in* H_1 *and*

$$\langle x_n, y_n \rangle_{H_1} \to \langle x, y \rangle_{H_1}.$$

proof uses key observation by Marcus Waurick

Lemma

Let $R(A)$ *be closed. For* $(x_n) \subset H_0$ *the following statements are equivalent:*

(i) $(\widetilde{A} x_n)$ *is relatively compact in* $D(A^*)'$.

(ii) $(\pi_{R(A^*)} x_n)$ *is relatively compact in* $R(A^*)$ *resp.* H_1.

If $x_n \rightharpoonup x$ *in* H_1, *then either of cond. (i) or (ii) implies* $\pi_{R(A^*)} x_n \to \pi_{R(A^*)} x$ *in* H_1.

FA-ToolBox: Continued

Corollary (compact resolvents)

All the latter corresponding inverses (of the reduced operators) are compact, i.e., for all n

$$\mathcal{A}_n^{-1},\ (\mathcal{A}_n^*)^{-1} \text{ compact.}$$

Corollary (spectra)

All the latter operators have discrete point spectra with finite eigenspaces, i.e., for all n

$$\sigma\Big(\begin{bmatrix} 0 & \mathrm{A}_n^* \\ \mathrm{A}_n & 0 \end{bmatrix}\Big) \setminus \{0\} = \pm\sqrt{\sigma(\mathrm{A}_n^*\mathrm{A}_n)} \setminus \{0\} = \pm\sqrt{\sigma(\mathrm{A}_n\mathrm{A}_n^*)} \setminus \{0\}$$

and $\sigma(\mathrm{A}_n^*\mathrm{A}_n) \setminus \{0\} = \{0 < \lambda_1^2 < \lambda_2^2 < \cdots < \lambda_\ell^2 \to \infty\}$ *and* λ_ℓ^2 *finite multiplicity.*

FA-ToolBox: Continued

Corollary (spectral theorems)

All the latter operators admit a spectral representation, i.e., for all n there exist orthonormal bases (ξ_n) *and* (ζ_n) *with, e.g.,*

$$x = \sum_\ell x_\ell \xi_\ell, \qquad \mathrm{A}_n x = \sum_\ell \lambda_\ell x_\ell \zeta_\ell, \qquad \mathrm{A}_n^* \mathrm{A}_n x = \sum_\ell \lambda_\ell^2 x_\ell \xi_\ell,$$

$$y = \sum_\ell y_\ell \zeta_\ell, \qquad \mathrm{A}_n^* y = \sum_\ell \lambda_\ell y_\ell \xi_\ell, \qquad \mathrm{A}_n \mathrm{A}_n^* y = \sum_\ell \lambda_\ell^2 y_\ell \zeta_\ell.$$

Corollary (Friedrichs/Poincaré estimates)

Friedrichs/Poincaré estimates type for higher eigenspaces.

FA-ToolBox: Continued

key ingredients

Lemma (bounded regular decompositions)

All the latter Hilbert complexes admit bounded regular decompositions, i.e., for all n

$$D(\mathrm{A}_n) = \mathrm{H}_n^+ + \mathrm{A}_{n-1}\mathrm{H}_{n-1}^+.$$

Corollary (bounded regular potentials)

All the latter Hilbert complexes admit bounded regular potentials, i.e., for all n

$$R(\mathrm{A}_n) = \mathrm{A}_n\mathrm{H}_n^+.$$

FA-ToolBox: Applications II

application: characterisation of duals

Theorem (characterisation of dual spaces)

Define regular dual spaces $H_n^- := (H_n^+)'$. *Then:*

regular decomposition	*dual space*
$D(A_1) = H_1^+ + A_0 H_0^+$	$D(A_1)' = \{\Phi \in H_1^- : A_0' \Phi \in H_0^-\}$
$D(A_0^*) = H_1^+ + A_1^* H_2^+$	$D(A_1^*)' = \{\Phi \in H_1^- : (A_1^*)' \Phi \in H_2^-\}$

think of

regular decomposition	characterisation of dual space
$D(\mathring{\mathrm{rot}}) = \mathring{H}^1 + \mathring{\mathrm{grad}}\, \mathring{H}^1$	$D(\mathring{\mathrm{rot}})' = \{\Phi \in H^{-1} : \mathrm{div}\, \Phi \in H^{-1}\}$
$D(\mathrm{div}) = H^1 + \mathrm{rot}\, H^1$	$D(\mathrm{div})' = \{\Phi \in \mathring{H}^{-1} : \mathrm{rot}\, \Phi \in \mathring{H}^{-1}\}$

with $\mathring{H}^1 := D(\mathring{\mathrm{grad}})$ and $H^1 := D(\mathrm{grad})$ and $H^{-1} := (\mathring{H}^1)'$ and $\mathring{H}^{-1} := (H^1)'$
(everything works with mixed boundary conditions $H^{-1}_{\gamma_n} := (H^1_{\gamma_t})'$)

FA-ToolBox: Applications II

application: cohomology groups

Theorem (Dirichlet-Neumann fields / cohomology groups)

The dimensions of the cohomology groups (generalised Dirichlet-Neumann fields) are independent of metric *and* Sobolev order *and admit a* duality*. In particular, there exists a finite C^∞-smooth per-bases (bases after projection).*

number of Dirichlet-Neumann fields

$$\mathcal{H}_{\epsilon,\gamma_t,\gamma_n} := N(\mathrm{rot}_{\gamma_t}) \cap N(\mathrm{div}_{\gamma_n}\,\epsilon) = N(\mathrm{rot}_{\gamma_t}) \cap R(\mathrm{grad}_{\gamma_t})^{\perp_{L^2_\epsilon}} \cong N(\mathrm{rot}_{\gamma_t})/R(\mathrm{grad}_{\gamma_t})$$

is independent of metric and Sobolev order, i.e.,

Lemma (Dirichlet-Neumann fields / cohomology groups)

For all k (Sobolev order)

$$\dim N(\mathrm{rot}^k_{\gamma_t})/R(\mathrm{grad}^k_{\gamma_t}) = \dim N(\mathrm{div}^k_{\gamma_n})/R(\mathrm{rot}^k_{\gamma_n}) = \dim \mathcal{H}_{\epsilon,\gamma_t,\gamma_n} < \infty.$$

FA-ToolBox: Applications II

application: cohomology groups

More precisely:

Lemma (Dirichlet-Neumann fields / cohomology groups)

There exist smooth pre-bases of Dirichlet-Neumann fields

$$\mathcal{B}_{\mathrm{rot},\gamma_t} \subset N(\mathrm{rot}^{\infty}_{\gamma_t}) = \mathsf{C}^{\infty}_{\gamma_t}(\overline{\Omega}) \cap N(\mathrm{rot}), \qquad \textit{(finite set)}$$

$$\mathcal{B}_{\mathrm{div},\gamma_n} \subset N(\mathrm{div}^{\infty}_{\gamma_n}) = \mathsf{C}^{\infty}_{\gamma_n}(\overline{\Omega}) \cap N(\mathrm{div}), \qquad \textit{(finite set)}$$

such that

$$\mathcal{H}_{\epsilon,\gamma_t,\gamma_n} = \mathrm{lin}\, \pi_{N(\mathrm{div}_{\gamma_n}\, \epsilon)} \mathcal{B}_{\mathrm{rot},\gamma_t} = \mathrm{lin}\, \pi_{N(\mathrm{rot}_{\gamma_t})} \mathcal{B}_{\mathrm{div},\gamma_n}. \qquad \textit{(bases)}$$

Corollary (Dirichlet-Neumann fields / cohomology groups)

For all Sobolev order k

$$N(\mathrm{rot}^{k}_{\gamma_t})/R(\mathrm{grad}^{k}_{\gamma_t}) \cong \mathrm{lin}\, \mathcal{B}_{\mathrm{rot},\gamma_t} \cong \mathcal{H}_{\epsilon,\gamma_t,\gamma_n} \cong \mathrm{lin}\, \mathcal{B}_{\mathrm{div},\gamma_n} \cong N(\mathrm{div}^{k}_{\gamma_n})/R(\mathrm{rot}^{k}_{\gamma_n})$$

FA-ToolBox: Applications II

application: biharmonic split

biharmonic equation $\Leftrightarrow$ to 3 elliptic 2nd order problems

$$\Delta_{\mathrm{D}}^2 u = f \quad \Leftrightarrow \quad \operatorname{divDiv}_{\mathbb{S}} \mathring{\operatorname{Gradgrad}} u = f$$

$$\Leftrightarrow \quad \begin{aligned} p &= \Delta_{\mathrm{D}}^{-1} f, \\ E &= (\mathring{\operatorname{Rot}}_{\mathbb{S}} \operatorname{symRot}_{\mathbb{T}})^{-1}_{\operatorname{Div}_{\mathbb{T}}=0} \operatorname{spn} \operatorname{grad} p, \\ u &= \Delta_{\mathrm{D}}^{-1}(3p + \operatorname{tr} \operatorname{symRot}_{\mathbb{T}} E) \end{aligned}$$

FEs for $\operatorname{symRot}_{\mathbb{T}}$ needed!

FA-ToolBox: some literature

some related literature

- Py: Discrete and Continuous Dynamical Systems 2015
On Maxwell's and Poincare's Constants

- Py: Mathematical Methods in the Applied Sciences 2017
On the Maxwell Constants in 3D

- Py and Irwin Yousept: Mathematical Modelling and Numerical Analysis 2017
A Posteriori Error Analysis for the Optimal Control of Magneto-Static Fields

- Py and Immanuel Anjam: Computational Methods in Applied Mathematics 2019
An Elementary Method of Deriving A Posteriori Error Equalities and Estimates for Linear Partial Differential Equations

- Py: Mathematische Zeitschrift 2019
On the Maxwell and Friedrichs/Poincaré Constants in ND

- Py: Analysis 2019
A Global div-curl-Lemma for Mixed Boundary Conditions in Weak Lipschitz Domains and a Corresponding Generalized A_0^*-A_1*-Lemma in Hilbert Spaces*

- Py: Numerical Functional Analysis and Optimization 2020
Solution Theory, Variational Formulations, and Functional a Posteriori Error Estimates for General First Order Systems with Applications to Electro-Magneto-Statics and More

- Py and Jan Valdman: Computers and Mathematics with Applications 2020
Poincar-Friedrichs Type Constants for Operators Involving grad, curl, and div: Theory and Numerical Experiments

- Py and Walter Zulehner: Applicable Analysis 2020
The divDiv-Complex and Applications to Biharmonic Equations

FA-ToolBox: some literature

some recent related literature

- Stefan Kurz and Py and Dirk Praetorius and Sergey Repin and Daniel Sebastian: Numerische Mathematik 2021
Functional A Posteriori Error Estimates for Boundary Element Methods

- Py and Rainer Picard and Sascha Trostorff and Marcus Waurick: Journal of Functional Analysis 2021
On a Class of Degenerate Abstract Parabolic Problems and Applications to Some Eddy Current Models

- Py and Walter Zulehner: Applicable Analysis 2022
The Elasticity Complex: Compact Embeddings and Regular Decompositions

- Py and Michael Schomburg: Mathematical Methods in the Applied Sciences 2022
Hilbert Complexes with Mixed Boundary Conditions - Part 1: De Rham Complex

- Py and Michael Schomburg: Mathematical Methods in the Applied Sciences 2022
Hilbert Complexes with Mixed Boundary Conditions - Part 2: Elasticity Complex

- Py and Michael Schomburg: Mathematical Methods in the Applied Sciences 2023
Hilbert Complexes with Mixed Boundary Conditions - Part 3: Biharmonic Complex

- Py and Marcus Waurick: Mathematische Zeitschrift 2022
*The Index of some Mixed Order Dirac-Type Operators
and Generalised Dirichlet-Neumann Tensor Fields*

- Py and Nathanel Skrepek: Annali dell' Universita di Ferrara 2022
A Compactness Result for the div-curl System with Inhomogeneous Mixed Boundary Conditions for Bounded Lipschitz Domains and Some Applications

- Ralf Hiptmair and Py and Erick Schulz: Journal of Functional Analysis 2023
Traces for Hilbert Complexes

FA-ToolBox: This is Finnish but not the end.

Thank you