On Some Hilbert Complexes and Applications

(or: Solving PDEs with Hilbert Complexes)

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Solving PDEs with Hilbert Complexes

Some Hilbert Complexes

classical de Rham complex in 3D (∇-rot-div-complex)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot\cup \Gamma_n}$

(electro-magnetics, Maxwell's equations)

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{div}}{\overset{\mathring{\nabla}}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{rot}}{\overset{\mathring{\operatorname{rot}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\nabla}{\overset{\mathring{\operatorname{div}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}} \mathbb{R}$$

mixed boundary conditions and inhomogeneous and anisotropic media

$$\{0\} \text{ or } \mathbb{R} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{div}_{\Gamma_n} \varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_\varepsilon \underset{\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}}{\overset{\mu^{-1} \operatorname{rot}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_\mu \underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t} \mu}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathbb{R} \text{ or } \{0\}$$

this talk:

- $\varepsilon = \mu = 1$ (= id)
- no mixed boundary conditions, i.e., $\Gamma_t = \Gamma$ and $\Gamma_n = \emptyset$

for all appearing complexes (Maxwell, elasticity, biharmonic, general relativity, ...)

de Rham complex in ND or on Riemannian manifolds (d-complex)

$\Omega \subset \mathbb{R}^N$ bd w. Lip. dom. or Ω Riemannian manifold with cpt cl. and Lip. boundary Γ

(generalized Maxwell equations **the mother of all complexes**)

$$\{0\} \overset{\iota_{\{0\}}}{\underset{\pi_{\{0\}}}{\rightleftarrows}} \mathsf{L}^{2,0} \overset{\mathring{\mathrm{d}}}{\underset{-\delta}{\rightleftarrows}} \mathsf{L}^{2,1} \overset{\mathring{\mathrm{d}}}{\underset{-\delta}{\rightleftarrows}} \cdots \boxed{\mathsf{L}^{2,q} \overset{\mathring{\mathrm{d}}}{\underset{-\delta}{\rightleftarrows}} \mathsf{L}^{2,q+1}} \cdots \mathsf{L}^{2,N-1} \overset{\mathring{\mathrm{d}}}{\underset{-\delta}{\rightleftarrows}} \mathsf{L}^{2,N} \overset{\pi_{\mathbb{R}}}{\underset{\iota_{\mathbb{R}}}{\rightleftarrows}} \mathbb{R}$$

elasticity complex in 3D (sym ∇-Rot $\mathrm{Rot}_{\mathbb{S}}^{\top}$-$\mathrm{Div}_{\mathbb{S}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \;\underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\operatorname{Div}_{\mathbb{S}}}{\overset{\operatorname{sym}\mathring{\nabla}}{\rightleftarrows}}\; \mathsf{L}^2_{\mathbb{S}} \;\underset{\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}}{\overset{\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^{\top}}{\rightleftarrows}}\; \mathsf{L}^2_{\mathbb{S}} \;\underset{-\operatorname{sym}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{S}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota_{\mathsf{RM}}}{\overset{\pi_{\mathsf{RM}}}{\rightleftarrows}}\; \mathsf{RM}$$

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\operatorname{Rot}_{\mathbb{S}}$-$\operatorname{Div}_{\mathbb{T}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \overset{\iota_{\{0\}}}{\underset{\pi_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \overset{\mathring{\nabla\nabla}}{\underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}} \overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\underset{-\operatorname{dev}\nabla}{\rightleftarrows}} \mathsf{L}^2 \overset{\pi_{\mathsf{RT}}}{\underset{\iota_{\mathsf{RT}}}{\rightleftarrows}} \mathsf{RT}$$

general complex

$$\begin{aligned} \mathrm{A}_0 : D(\mathrm{A}_0) \subset \mathrm{H}_0 \to \mathrm{H}_1, \qquad & \mathrm{A}_1 : D(\mathrm{A}_1) \subset \mathrm{H}_1 \to \mathrm{H}_2 \\ \mathrm{A}_0^* : D(\mathrm{A}_0^*) \subset \mathrm{H}_1 \to \mathrm{H}_0, \qquad & \mathrm{A}_1^* : D(\mathrm{A}_1^*) \subset \mathrm{H}_2 \to \mathrm{H}_1 \end{aligned} \qquad \text{(lddc)}$$

general complex property $\boxed{\mathrm{A}_1\mathrm{A}_0 = 0}$,

i.e., $R(\mathrm{A}_0) \subset N(\mathrm{A}_1)$ and/or eq $R(\mathrm{A}_1^*) \subset N(\mathrm{A}_0^*)$

$$\cdots \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad \mathrm{H}_0 \quad \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \quad \mathrm{H}_1 \quad \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \quad \mathrm{H}_2 \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad \cdots$$

Solving PDEs with Hilbert Complexes

FA-ToolBox

general observations

$$Ax = f$$

general observations

$$Ax = f$$

general theory

- solution theory
- closed ranges
- Friedrichs/Poincaré estimates and constants
- Helmholtz/Hodge/Weyl decompositions
- compact embeddings
- continuous and compact inverse operators
- regular potentials and regular decompositions
- variational formulations
- generalized div-curl-lemma
- functional a posteriori error estimates
- ...

idea: solve problem with general and simple linear functional analysis

$\Rightarrow$ functional analysis toolbox (FA-ToolBox) ...

general observations

$$\mathrm{A}x = f$$

let's say $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ linear and H_0, H_1 Hilbert spaces

question: How to solve?

general observations

$$\mathrm{A}x = f$$

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ linear

solution theory in the sense of Hadamard

- existence $\Leftrightarrow$ $f \in R(\mathrm{A})$
- uniqueness $\Leftrightarrow$ A inj $\Leftrightarrow$ $N(\mathrm{A}) = \{0\}$ $\Leftrightarrow$ A^{-1} exists
- cont dep on f $\Leftrightarrow$ A^{-1} cont

$\Rightarrow$ $x = \mathrm{A}^{-1}f \in D(\mathrm{A})$ and cont estimate (Friedrichs/Poincaré type estimate)

$$|x|_{\mathrm{H}_0} = |\mathrm{A}^{-1}f|_{\mathrm{H}_0} \leq c_{\mathrm{A}}|f|_{\mathrm{H}_1} = c_{\mathrm{A}}|\mathrm{A}x|_{\mathrm{H}_1}$$

$\Rightarrow$ best constant $c_{\mathrm{A}} = |\mathrm{A}^{-1}|_{R(\mathrm{A}),\mathrm{H}_0}$ $|\mathrm{A}^{-1}|_{R(\mathrm{A}),D(\mathrm{A})} = (c_{\mathrm{A}}^2 + 1)^{1/2}$

general observations

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$

$\mathrm{A}^* : D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0$ Hilbert space adjoint

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$\mathrm{H}_1 = \overline{R(\mathrm{A})} \oplus N(\mathrm{A}^*) \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

$$\mathrm{A}x = f$$

solution theory in the sense of Hadamard

- existence $\Leftrightarrow$ $f \in R(\mathrm{A}) = N(\mathrm{A}^*)^\perp$ (Fredholm alternative)
- uniqueness $\Leftrightarrow$ A inj $\Leftrightarrow$ $N(\mathrm{A}) = \{0\}$ $\Leftrightarrow$ A^{-1} exists
- cont dep on f $\Leftrightarrow$ A^{-1} cont $\Leftrightarrow$ $R(\mathrm{A})$ cl (cl range theo)

fund range cond: $\boxed{R(\mathrm{A}) = \overline{R(\mathrm{A})} \text{ closed}}$ (must hold $\rightsquigarrow$ right setting!)

kernel cond: $\boxed{N(\mathrm{A}) = \{0\}}$ $\big($fails in gen $\rightsquigarrow$ proj onto $N(\mathrm{A})^\perp = \overline{R(\mathrm{A}^*)} = R(\mathrm{A}^*)\big)$

general observations

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$\mathrm{H}_1 = \overline{R(\mathrm{A})} \oplus N(\mathrm{A}^*) \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

observations

- time-dependent problems are simple
  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H} \to \mathrm{H}$, $\boxed{\mathrm{A} = \partial_t \Lambda + \mathrm{T}}$ (gen T skw-sa, or at least Re T ≥ 0)
  $$N(\mathrm{A}) = \{0\} \qquad N(\mathrm{A}^*) = \{0\} \qquad R(\mathrm{A}) \text{ (cl) } = N(\mathrm{A}^*)^{\perp} = \mathrm{H}$$
- time-harmonic problems are more complicated
  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H} \to \mathrm{H}$, $\boxed{\mathrm{A} = -\omega\Lambda + \mathrm{T}}$
  $$N(\mathrm{A}),\ N(\mathrm{A}^*) \text{ (fin dim)} \qquad R(\mathrm{A}) \text{ (cl, fin co-dim) } = N(\mathrm{A}^*)^{\perp}$$
  (Fredholm alternative)
- stat problems are most complicated
  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$, $\boxed{\mathrm{A} = 0 + \mathrm{T}}$
  $$\dim N(\mathrm{A}) = \dim N(\mathrm{A}^*) = \infty \text{ (possibly)} \qquad R(\mathrm{A}) \text{ (cl, infin co-dim) } = N(\mathrm{A}^*)^{\perp}$$

FA-ToolBox for linear (first order) problems/systems

$$Ax = f$$

general theory

- solution theory
- closed ranges
- Friedrichs/Poincaré estimates and constants
- Helmholtz/Hodge/Weyl decompositions
- compact embeddings
- continuous and compact inverse operators
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- variational formulations
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- ...

idea: solve problem with general and simple linear functional analysis
($\Rightarrow$ FA-ToolBox) ...

literature: many parts probably very well known for ages, but hard to find ...

Friedrichs, Weyl, Hörmander, Fredholm, von Neumann, Riesz, Banach, ... ?

Why not rediscover and extend/modify for our purposes?

1st fundamental observations

$\mathrm{A}: D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ lddc, $\quad \mathrm{A}^*: D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0$ Hilbert space adjoint

$(\mathrm{A}, \mathrm{A}^*)$ dual pair as $(\mathrm{A}^*)^* = \overline{\mathrm{A}} = \mathrm{A}$

A, A^* may not be inj

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$\mathrm{H}_1 = N(\mathrm{A}^*) \oplus \overline{R(\mathrm{A})} \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

reduced operators restr to $N(\mathrm{A})^\perp$ and $N(\mathrm{A}^*)^\perp$

$$\boxed{\mathcal{A} := \mathrm{A}|_{N(\mathrm{A})^\perp} = \mathrm{A}|_{\overline{R(\mathrm{A}^*)}}} \qquad \boxed{\mathcal{A}^* := \mathrm{A}^*|_{N(\mathrm{A}^*)^\perp} = \mathrm{A}^*|_{\overline{R(\mathrm{A})}}}$$

$\mathcal{A}$, $\mathcal{A}^*$ inj $\quad \Rightarrow \quad \mathcal{A}^{-1}$, $(\mathcal{A}^*)^{-1}$ ex

1st fundamental observations

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1, \quad \mathrm{A}^* : D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0$ lddc $\qquad (\mathrm{A}, \mathrm{A}^*)$ dual pair

$$\mathrm{H}_1 = N(\mathrm{A}^*) \oplus \overline{R(\mathrm{A})} \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

more precisely

$$\mathcal{A} := \mathrm{A}|_{\overline{R(\mathrm{A}^*)}} : D(\mathcal{A}) \subset \overline{R(\mathrm{A}^*)} \to \overline{R(\mathrm{A})}, \qquad D(\mathcal{A}) := D(\mathrm{A}) \cap N(\mathrm{A})^\perp = D(\mathrm{A}) \cap \overline{R(\mathrm{A}^*)}$$
$$\mathcal{A}^* := \mathrm{A}^*|_{\overline{R(\mathrm{A})}} : D(\mathcal{A}^*) \subset \overline{R(\mathrm{A})} \to \overline{R(\mathrm{A}^*)}, \quad D(\mathcal{A}^*) := D(\mathrm{A}^*) \cap N(\mathrm{A}^*)^\perp = D(\mathrm{A}^*) \cap \overline{R(\mathrm{A})}$$

$(\mathcal{A}, \mathcal{A}^*)$ dual pair $\quad$ and $\quad \mathcal{A}, \mathcal{A}^*$ inj $\quad \Rightarrow$

inverse ops exist (and bij)

$$\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A}) \qquad (\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$$

refined decompositions

$$D(\mathrm{A}) = N(\mathrm{A}) \oplus D(\mathcal{A}) \qquad D(\mathrm{A}^*) = N(\mathrm{A}^*) \oplus D(\mathcal{A}^*)$$

$\Rightarrow$

$$R(\mathrm{A}) = R(\mathcal{A}) \qquad R(\mathrm{A}^*) = R(\mathcal{A}^*)$$

1st fundamental observations

closed range theorem & closed graph theorem $\Rightarrow$

Lemma (Friedrichs-Poincaré type est/cl range/cont inv)

The following assertions are equivalent:

- **(i)** $\exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathrm{H}_0} \le c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$
- **(i*)** $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathrm{H}_1} \le c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$
- **(ii)** $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_1.
- **(ii*)** $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_0.
- **(iii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is continuous and bijective.*
- **(iii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is continuous and bijective.*

Note: trivial equivalence of (i) or (i*) to inf-sup condition

In case that one of the latter assertions is true, e.g., (ii), $R(\mathrm{A})$ is closed, we have

$$\mathrm{H}_0 = N(\mathrm{A}) \oplus R(\mathrm{A}^*) \qquad \mathrm{H}_1 = N(\mathrm{A}^*) \oplus R(\mathrm{A})$$
$$D(\mathrm{A}) = N(\mathrm{A}) \oplus D(\mathcal{A}) \qquad D(\mathrm{A}^*) = N(\mathrm{A}^*) \oplus D(\mathcal{A}^*)$$
$$D(\mathcal{A}) = D(\mathrm{A}) \cap R(\mathrm{A}^*) \qquad D(\mathcal{A}^*) = D(\mathrm{A}^*) \cap R(\mathrm{A})$$

and $\quad \mathcal{A} : D(\mathcal{A}) \subset R(\mathrm{A}^*) \to R(\mathrm{A}), \quad \mathcal{A}^* : D(\mathcal{A}^*) \subset R(\mathrm{A}) \to R(\mathrm{A}^*).$

1st fundamental observations

recall

(i) $\exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathrm{H}_0} \leq c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$

(i*) $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$

'best' consts in **(i)** and **(i*)** equal norms of the inv ops and Rayleigh quotients

$$c_{\mathrm{A}} = |\mathcal{A}^{-1}|_{R(\mathrm{A}),R(\mathrm{A}^*)} \qquad c_{\mathrm{A}^*} = |(\mathcal{A}^*)^{-1}|_{R(\mathrm{A}^*),R(\mathrm{A})}$$

$$\frac{1}{c_{\mathrm{A}}} = \inf_{0 \neq x \in D(\mathcal{A})} \frac{|\mathrm{A}x|_{\mathrm{H}_1}}{|x|_{\mathrm{H}_0}} \qquad \frac{1}{c_{\mathrm{A}^*}} = \inf_{0 \neq y \in D(\mathcal{A}^*)} \frac{|\mathrm{A}^* y|_{\mathrm{H}_0}}{|y|_{\mathrm{H}_1}}$$

Lemma (Friedrichs-Poincaré type const)

$c_{\mathrm{A}} = c_{\mathrm{A}^*}$

1st fundamental observations

Lemma (cpt emb/cpt inv)

The following assertions are equivalent:

- **(i)** $D(\mathcal{A}) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_0$ *is compact.*
- **(i*)** $D(\mathcal{A}^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$ *is compact.*
- **(ii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to R(\mathrm{A}^*)$ *is compact.*
- **(ii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to R(\mathrm{A})$ *is compact.*

Lemma (Friedrichs-Poincaré type est/cl range/cont inv)

$\Downarrow$ $\boxed{D(\mathcal{A}) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_0 \textit{ compact}}$

- **(i)** $\exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathrm{H}_0} \leq c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$
- **(i*)** $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$
- **(ii)** $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_1.
- **(ii*)** $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_0.
- **(iii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is continuous and bijective.*
- **(iii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is continuous and bijective.*

(i)-(iii*) equi & the resp Helm deco hold & $|\mathcal{A}^{-1}| = c_{\mathrm{A}} = c_{\mathrm{A}^*} = |(\mathcal{A}^*)^{-1}|$

2nd fundamental observations

So far no complex...

$$A_0 : D(A_0) \subset H_0 \to H_1, \quad A_1 : D(A_1) \subset H_1 \to H_2 \text{ (lddc)}$$
$$A_0^* : D(A_0^*) \subset H_1 \to H_0, \quad A_1^* : D(A_1^*) \subset H_2 \to H_1 \text{ (lddc)}$$

general complex ($A_1 A_0 = 0$, i.e., $R(A_0) \subset N(A_1)$ and $R(A_1^*) \subset N(A_0^*)$)

$$\boxed{\cdots \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad H_0 \quad \underset{A_0^*}{\overset{A_0}{\rightleftarrows}} \quad H_1 \quad \underset{A_1^*}{\overset{A_1}{\rightleftarrows}} \quad H_2 \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad \cdots}$$

recall Helmholtz deco

$$\begin{aligned} H_1 &= \overline{R(A_0)} \oplus N(A_0^*) \\ & \qquad \bigcap \qquad\quad \bigcup \\ &= N(A_1) \oplus \overline{R(A_1^*)} \end{aligned} \qquad \Rightarrow \text{ (e.g.) } N(A_1) = \overline{R(A_0)} \oplus \Big(\underbrace{N(A_1) \cap N(A_0^*)}_{=:N_1} \Big)$$

$\Rightarrow$ refined Helmholtz deco

$$H_1 = \overline{R(A_0)} \oplus N_1 \oplus \overline{R(A_1^*)}$$

2nd fundamental observations

recall

$$D(\mathrm{A}_1) = D(\mathcal{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \qquad R(\mathrm{A}_1) = R(\mathcal{A}_1) \qquad R(\mathrm{A}_1^*) = R(\mathcal{A}_1^*)$$
$$D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \cap \overline{R(\mathrm{A}_0)} \qquad R(\mathrm{A}_0^*) = R(\mathcal{A}_0^*) \qquad R(\mathrm{A}_0) = R(\mathcal{A}_0)$$

cohomology group $N_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)$

Lemma (Helmholtz deco I)

$$\mathrm{H}_1 = \overline{R(\mathrm{A}_0)} \oplus N(\mathrm{A}_0^*) \qquad \mathrm{H}_1 = \overline{R(\mathrm{A}_1^*)} \oplus N(\mathrm{A}_1)$$
$$D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus N(\mathrm{A}_0^*) \qquad D(\mathrm{A}_1) = D(\mathcal{A}_1) \oplus N(\mathrm{A}_1)$$
$$N(\mathrm{A}_1) = D(\mathcal{A}_0^*) \oplus N_1 \qquad N(\mathrm{A}_0^*) = D(\mathcal{A}_1) \oplus N_1$$
$$D(\mathrm{A}_1) = \overline{R(\mathrm{A}_0)} \oplus \big(D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)\big) \qquad D(\mathrm{A}_0^*) = \overline{R(\mathrm{A}_1^*)} \oplus \big(D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1)\big)$$

Lemma (Helmholtz deco II)

$$\mathrm{H}_1 = \overline{R(\mathrm{A}_0)} \oplus N_1 \oplus \overline{R(\mathrm{A}_1^*)}$$
$$D(\mathrm{A}_1) = \overline{R(\mathrm{A}_0)} \oplus N_1 \oplus D(\mathcal{A}_1)$$
$$D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus N_1 \oplus \overline{R(\mathrm{A}_1^*)}$$
$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus N_1 \oplus D(\mathcal{A}_1)$$

2nd fundamental observations

$$N_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \qquad D(\mathrm{A}_1) = D(\mathcal{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \qquad D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \cap \overline{R(\mathrm{A}_0)}$$

Lemma (cpt emb II)

The following assertions are equivalent:

(i) $D(\mathcal{A}_0) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_0$, $D(\mathcal{A}_1) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$, *and* $N_1 \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$ *are compact.*

(ii) $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$ *is compact.*

In this case $N_1 < \infty$.

Theorem (FA-ToolBox I)

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1 \text{ compact}}$

(i) *all emb cpt, i.e.,* $D(\mathcal{A}_0) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_0$, $D(\mathcal{A}_1) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$, $D(\mathcal{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$, $D(\mathcal{A}_1^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_2$ *cpt*

(ii) *cohomology group* N_1 *finite dim*

(iii) *all ranges closed, i.e.,* $R(\mathrm{A}_0)$, $R(\mathrm{A}_0^*)$, $R(\mathrm{A}_1)$, $R(\mathrm{A}_1^*)$ *cl*

(iv) *all Friedrichs-Poincaré type est hold*

(v) *all Hodge-Helmholtz-Weyl type deco I & II hold with closed ranges*

2nd fundamental observations

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

Theorem (FA-ToolBox I (Friedrichs-Poincaré type est))

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1 \ \textit{compact}} \quad \Rightarrow \quad \exists \quad |\mathcal{A}_i^{-1}| = c_{\mathrm{A}_i} = c_{\mathrm{A}_i^*} = |(\mathcal{A}_i^*)^{-1}| \in (0, \infty)$

- **(i)** $\forall\, x \in D(\mathcal{A}_0)$ $\quad |x|_{\mathrm{H}_0} \leq c_{\mathrm{A}_0} |\mathrm{A}_0 x|_{\mathrm{H}_1}$
- **(i*)** $\forall\, y \in D(\mathcal{A}_0^*)$ $\quad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}_0} |\mathrm{A}_0^* y|_{\mathrm{H}_0}$
- **(ii)** $\forall\, y \in D(\mathcal{A}_1)$ $\quad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |\mathrm{A}_1 y|_{\mathrm{H}_2}$
- **(ii*)** $\forall\, z \in D(\mathcal{A}_1^*)$ $\quad |z|_{\mathrm{H}_2} \leq c_{\mathrm{A}_1} |\mathrm{A}_1^* z|_{\mathrm{H}_1}$
- **(iii)** $\forall\, y \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ $\quad |(1 - \pi_{N_1}) y|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |\mathrm{A}_1 y|_{\mathrm{H}_2} + c_{\mathrm{A}_0} |\mathrm{A}_0^* y|_{\mathrm{H}_0}$

note $\pi_{N_1} y \in N_1$ and $(1 - \pi_{N_1}) y \in N_1^\perp$

Remark

enough $R(\mathrm{A}_0)$ *and* $R(\mathrm{A}_1)$ *cl*

2nd fundamental observations

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

Theorem (FA-ToolBox I (Helmholtz deco))

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1 \text{ compact}}$

$$\begin{aligned} \mathrm{H}_1 &= R(\mathrm{A}_0) \oplus N(\mathrm{A}_0^*) & \mathrm{H}_1 &= R(\mathrm{A}_1^*) \oplus N(\mathrm{A}_1) \\ D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus N(\mathrm{A}_0^*) & D(\mathrm{A}_1) &= D(\mathcal{A}_1) \oplus N(\mathrm{A}_1) \\ N(\mathrm{A}_1) &= D(\mathcal{A}_0^*) \oplus N_1 & N(\mathrm{A}_0^*) &= D(\mathcal{A}_1) \oplus N_1 \\ D(\mathrm{A}_1) &= R(\mathrm{A}_0) \oplus \big(D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)\big) & D(\mathrm{A}_0^*) &= R(\mathrm{A}_1^*) \oplus \big(D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1)\big) \end{aligned}$$

$$\begin{aligned} \mathrm{H}_1 &= R(\mathrm{A}_0) \oplus N_1 \oplus R(\mathrm{A}_1^*) \\ D(\mathrm{A}_1) &= R(\mathrm{A}_0) \oplus N_1 \oplus D(\mathcal{A}_1) \\ D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus N_1 \oplus R(\mathrm{A}_1^*) \\ D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus N_1 \oplus D(\mathcal{A}_1) \end{aligned}$$

Remark

enough $R(\mathrm{A}_0)$ and $R(\mathrm{A}_1)$ cl

Solving PDEs with Hilbert Complexes

(Static) First Order Systems

(stat) first order system - solution theory

complex $\quad \cdots \overset{\cdots}{\underset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \overset{\mathrm{A}_0}{\underset{\mathrm{A}_0^*}{\rightleftarrows}} \mathrm{H}_1 \overset{\mathrm{A}_1}{\underset{\mathrm{A}_1^*}{\rightleftarrows}} \mathrm{H}_2 \overset{\cdots}{\underset{\cdots}{\rightleftarrows}} \cdots$

$\boxed{\mathrm{A}_1 x = f}$ $\qquad \dim N(\mathrm{A}_1) = \infty$

find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ such that the fos

$$\begin{aligned} \mathrm{A}_1 x &= f \\ \mathrm{A}_0^* x &= g \\ \pi_{N_1} x &= k \end{aligned} \qquad \text{think of} \qquad \begin{aligned} (\mathring{\mathrm{rot}} E &= F) \\ (-\operatorname{div} E &= g) \\ (\pi_{\mathrm{D}} E &= K) \end{aligned}$$

kernel = cohomology group = $N_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)$

trivially necessary $\quad f \in R(\mathrm{A}_1) \quad g \in R(\mathrm{A}_0^*) \quad k \in N_1$

apply FA-ToolBox

(stat) first order system - solution theory

complex $\cdots \overset{\cdots}{\underset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \overset{\mathrm{A}_0}{\underset{\mathrm{A}_0^*}{\rightleftarrows}} \mathrm{H}_1 \overset{\mathrm{A}_1}{\underset{\mathrm{A}_1^*}{\rightleftarrows}} \mathrm{H}_2 \overset{\cdots}{\underset{\cdots}{\rightleftarrows}} \cdots$

find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ st fos $\quad \mathrm{A}_1 x = f \quad \mathrm{A}_0^* x = g \quad \pi_{N_1} x = k$

Theorem (FA-ToolBox II (solution theory))

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1 \text{ compact}}$

fos is uniq sol $\Leftrightarrow$ $f \in R(\mathrm{A}_1) \quad g \in R(\mathrm{A}_0^*) \quad k \in N_1$

$$x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus N_1 = D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$$

$$\boxed{x_f := \mathcal{A}_1^{-1} f} \in D(\mathcal{A}_1)$$

$$\boxed{x_g := (\mathcal{A}_0^*)^{-1} g} \in D(\mathcal{A}_0^*)$$

dep cont on data $|x|_{\mathrm{H}_1} \le |x_f|_{\mathrm{H}_1} + |x_g|_{\mathrm{H}_1} + |k|_{\mathrm{H}_1} \le c_{\mathrm{A}_1} |f|_{\mathrm{H}_2} + c_{\mathrm{A}_0} |g|_{\mathrm{H}_0} + |k|_{\mathrm{H}_1}$

moreover

$$\pi_{R(\mathrm{A}_1^*)} x = x_f \qquad \pi_{R(\mathrm{A}_0)} x = x_g \qquad \pi_{N_1} x = k \qquad |x|_{\mathrm{H}_1}^2 = |x_f|_{\mathrm{H}_1}^2 + |x_g|_{\mathrm{H}_1}^2 + |k|_{\mathrm{H}_1}^2$$

Remark

enough $R(\mathrm{A}_0)$ *and* $R(\mathrm{A}_1)$ *cl*

(stat) first order system - a posteriori error estimates

problem: find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ st $\mathrm{A}_1 x = f$ $\mathrm{A}_0^* x = g$ $\pi_{N_1} x = k$

'very' non-conforming 'approximation' of x: $\tilde{x} \in \mathrm{H}_1$

def., dcmp. err. $e = x - \tilde{x} = \pi_{R(\mathrm{A}_0)} e + \pi_{N_1} e + \pi_{R(\mathrm{A}_1^*)} e \in \mathrm{H}_1 = R(\mathrm{A}_0) \oplus N_1 \oplus R(\mathrm{A}_1^*)$

Theorem (sharp upper bounds)

Let $\tilde{x} \in \mathrm{H}_1$ and $e = x - \tilde{x}$. Then

$$|e|_{\mathrm{H}_1}^2 = |\pi_{R(\mathrm{A}_0)} e|_{\mathrm{H}_1}^2 + |\pi_{N_1} e|_{\mathrm{H}_1}^2 + |\pi_{R(\mathrm{A}_1^*)} e|_{\mathrm{H}_1}^2$$

$$|\pi_{R(\mathrm{A}_0)} e|_{\mathrm{H}_1} = \min_{\phi \in D(\mathrm{A}_0^*)} \big(c_{\mathrm{A}_0} |\mathrm{A}_0^* \phi - g|_{\mathrm{H}_0} + |\phi - \tilde{x}|_{\mathrm{H}_1}\big)$$

reg $(\mathrm{A}_0 \mathrm{A}_0^* + 1)$-prbl in $D(\mathrm{A}_0^*)$

$$|\pi_{R(\mathrm{A}_1^*)} e|_{\mathrm{H}_1} = \min_{\varphi \in D(\mathrm{A}_1)} \big(c_{\mathrm{A}_1} |\mathrm{A}_1 \varphi - f|_{\mathrm{H}_2} + |\varphi - \tilde{x}|_{\mathrm{H}_1}\big)$$

reg $(\mathrm{A}_1^* \mathrm{A}_1 + 1)$-prbl in $D(\mathrm{A}_1)$

$$|\pi_{N_1} e|_{\mathrm{H}_1} = |\pi_{N_1} \tilde{x} - k|_{\mathrm{H}_1} = \min_{\substack{\xi \in D(\mathrm{A}_0) \\ \zeta \in D(\mathrm{A}_1^*)}} |\mathrm{A}_0 \xi + \mathrm{A}_1^* \zeta + \tilde{x} - k|_{\mathrm{H}_1}$$

cpld $(\mathrm{A}_0^* \mathrm{A}_0)$-$(\mathrm{A}_1 \mathrm{A}_1^*)$-sys in $D(\mathcal{A}_0)$-$D(\mathcal{A}_1^*)$

Remark

Even $\pi_{N_1} e = k - \pi_{N_1} \tilde{x}$ and the minima are attained at

$$\hat{\phi} = \pi_{R(\mathrm{A}_0)} e + \tilde{x}, \qquad \hat{\varphi} = \pi_{R(\mathrm{A}_1^*)} e + \tilde{x}, \qquad \mathrm{A}_0 \hat{\xi} + \mathrm{A}_1^* \hat{\zeta} = (\pi_{N_1} - 1)\tilde{x}.$$

A_0^*-A_1-lemma (generalized global div-curl-lemma)

Lemma (A_0^*-A_1-lemma)

Let $D(A_1) \cap D(A_0^*) \hookrightarrow H_1$ *be compact, and*

(i) (x_n) *bounded in* $D(A_1)$,

(ii) (y_n) *bounded in* $D(A_0^*)$.

$\Rightarrow \exists\ x \in D(A_1)$, $y \in D(A_0^*)$ *and subsequences st*

$x_n \rightharpoonup x$ *in* $D(A_1)$ *and* $y_n \rightharpoonup y$ *in* $D(A_0^*)$ *as well as*

$$\langle x_n, y_n \rangle_{H_1} \to \langle x, y \rangle_{H_1}.$$

A_0^*-A_1-lemma (generalized global div-curl-lemma)

Lemma (generalized A_0^*-A_1-lemma)

Let $R(A_0)$ and $R(A_1$ be closed, and let N_1 be finite dimensional. Moreover, let $(x_n), (y_n) \subset H_1$ be bounded such that

(i) $\widetilde{A}_1(x_n)$ *is relatively compact in* $D(A_1^*)'$,

(ii) $\widetilde{A}_0^*(y_n)$ *is relatively compact in* $D(A_0)'$.

$\Rightarrow \exists\, x, y \in H_1$ *and subsequences st* $x_n \rightharpoonup x$ *in* H_1 *and* $y_n \rightharpoonup y$ *in* H_1 *as well as*

$$\langle x_n, y_n \rangle_{H_1} \to \langle x, y \rangle_{H_1}.$$

proof uses key observation

Lemma

Let $R(A)$ be closed. For $(x_n) \subset H_0$ the following statements are equivalent:

(i) $\widetilde{A}x_n$ *is relatively compact in* $D(A^*)'$.

(ii) $\pi_{R(A^*)}x_n$ *is relatively compact in* $R(A^*)$ *resp.* H_1.

If $x_n \rightharpoonup x$ in H_1, then either of cond. (i) or (ii) implies $\pi_{R(A^*)}x_n \to \pi_{R(A^*)}x$ in H_1.

nice results and joint work with Marcus Waurick

Solving PDEs with Hilbert Complexes

Applications: FOS & SOS (First and Second Order Systems)

classical de Rham complex in 3D (∇-rot-div-complex)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot{\cup} \Gamma_n}$

(electro-magneto dynamics, Maxwell's equations)

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{div}}{\overset{\mathring{\nabla}}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{rot}}{\overset{\mathring{\operatorname{rot}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\nabla}{\overset{\mathring{\operatorname{div}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}} \mathbb{R}$$

mixed boundary conditions and inhomogeneous and anisotropic media

$$\{0\} \text{ or } \mathbb{R} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{div}_{\Gamma_n} \varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_\varepsilon \underset{\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathbb{R} \text{ or } \{0\}$$

classical de Rham complex in 3D (∇-rot-div-complex)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot\cup \Gamma_n}$

(electro-magneto dynamics, Maxwell's equations with mixed boundary conditions)

$$\{0\} \text{ or } \mathbb{R} \;\underset{\pi}{\overset{\iota}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\operatorname{div}_{\Gamma_n}\varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2_\varepsilon \;\underset{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota}{\overset{\pi}{\rightleftarrows}}\; \mathbb{R} \text{ or } \{0\}$$

related fos

$$\begin{array}{llllllll} \nabla_{\Gamma_t} u = A & \text{in } \Omega & | \quad \operatorname{rot}_{\Gamma_t} E = J & \text{in } \Omega & | \quad \operatorname{div}_{\Gamma_t} H = k & \text{in } \Omega & | \quad \pi v = b & \text{in } \Omega \\ \pi u = a & \text{in } \Omega & | \quad -\operatorname{div}_{\Gamma_n} \varepsilon E = j & \text{in } \Omega & | \quad \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} H = K & \text{in } \Omega & | \quad -\nabla_{\Gamma_n} v = B & \text{in } \Omega \end{array}$$

related sos

$$\begin{array}{llllll} -\operatorname{div}_{\Gamma_n} \varepsilon \nabla_{\Gamma_t} u = j & \text{in } \Omega & | \quad \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} \operatorname{rot}_{\Gamma_t} E = K & \text{in } \Omega & | \quad -\nabla_{\Gamma_n} \operatorname{div}_{\Gamma_t} H = B & \text{in } \Omega \\ \pi u = a & \text{in } \Omega & | \quad -\operatorname{div}_{\Gamma_n} \varepsilon E = j & \text{in } \Omega & | \quad \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} H = K & \text{in } \Omega \end{array}$$

corresponding compact embeddings:

$$D(\nabla_{\Gamma_t}) \cap D(\pi) = D(\nabla_{\Gamma_t}) = \mathsf{H}^1_{\Gamma_t} \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

$$D(\operatorname{rot}_{\Gamma_t}) \cap D(-\operatorname{div}_{\Gamma_n} \varepsilon) = \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n} \hookrightarrow \mathsf{L}^2_\varepsilon \qquad \text{(Weck's selection theorem, '74)}$$

$$D(\operatorname{div}_{\Gamma_t}) \cap D(\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}) = \mathsf{D}_{\Gamma_t} \cap \mathsf{R}_{\Gamma_n} \hookrightarrow \mathsf{L}^2 \qquad \text{(Weck's selection theorem, '74)}$$

$$D(\nabla_{\Gamma_n}) \cap D(\pi) = D(\nabla_{\Gamma_n}) = \mathsf{H}^1_{\Gamma_n} \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

Weck's selection theorem for weak Lip. dom. and mixed bc: Bauer/Py/Schomburg ('16)

Weck's selection theorem (Weck '74, (Habil. '72) stimulated by Rolf Leis)
(Weber '80, Picard '84, Costabel '90, Witsch '93, Jochmann '97, Kuhn '99, Picard/Weck/Witsch '01, Py '96, '03, '06, '07, '08)

classical de Rham complex in 3D (∇-rot-div-complex)

$$
\begin{aligned}
\operatorname{rot} E &= F && \text{in } \Omega\\
-\operatorname{div} \varepsilon E &= g && \text{in } \Omega\\
\nu \times E &= 0 && \text{at } \Gamma_t\\
\nu \cdot \varepsilon E &= 0 && \text{at } \Gamma_n
\end{aligned}
$$

non-trivial kernel $\mathcal{H}_{\mathrm{D},\varepsilon} = \{H \in \mathrm{L}^2 : \operatorname{rot} H = 0,\ \operatorname{div} \varepsilon H = 0,\ \nu \times H|_{\Gamma_t} = 0,\ \nu \cdot \varepsilon H|_{\Gamma_n} = 0\}$
additional condition on Dirichlet/Neumann fields for uniqueness

$$\pi_{\mathrm{D}} E = K \in \mathcal{H}_{\mathrm{D},\varepsilon}$$

$$\{0\} \text{ or } \mathbb{R} \;\underset{\pi}{\overset{\iota}{\rightleftarrows}}\; \mathrm{L}^2 \;\underset{-\operatorname{div}_{\Gamma_n} \varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}}\; \mathrm{L}^2_\varepsilon \;\underset{\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}}\; \mathrm{L}^2 \;\underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}}\; \mathrm{L}^2 \;\underset{\iota}{\overset{\pi}{\rightleftarrows}}\; \mathbb{R} \text{ or } \{0\}$$

$$\cdots \;\underset{\cdots}{\overset{\cdots}{\rightleftarrows}}\; \mathrm{H}_{-1} \;\underset{\mathrm{A}_{-1}^*}{\overset{\mathrm{A}_{-1}}{\rightleftarrows}}\; \mathrm{H}_0 \;\underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}}\; \mathrm{H}_1 \;\underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}}\; \mathrm{H}_2 \;\underset{\mathrm{A}_2^*}{\overset{\mathrm{A}_2}{\rightleftarrows}}\; \mathrm{H}_3 \;\underset{\mathrm{A}_3^*}{\overset{\mathrm{A}_3}{\rightleftarrows}}\; \mathrm{H}_4 \;\underset{\cdots}{\overset{\cdots}{\rightleftarrows}}\; \cdots$$

find $E \in \mathrm{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\mathrm{D}_{\Gamma_n}(\Omega)$ st (fos) find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ st

$$
\begin{aligned}
\operatorname{rot}_{\Gamma_t} E &= F & \qquad & & \mathrm{A}_1 x &= f\\
-\operatorname{div}_{\Gamma_n} \varepsilon E &= g & & \text{translation} & \mathrm{A}_0^* x &= g\\
\pi_{\mathrm{D/N}} E &= K & & & \pi_{N_1} x &= k
\end{aligned}
$$

classical de Rham complex in 3D (∇-rot-div-complex)

$c_{A_0} = c_{fp}$ (Friedrichs/Poincaré constant) and $c_{A_1} = c_m$ (Maxwell constant)

Lemma/Theorem $\Downarrow$ $D(A_1) \cap D(A_0^*) \hookrightarrow\!\!\!\!\rightarrow L^2_\varepsilon(\Omega)$ compact

(i) all Friedrichs-Poincaré type est hold

$$\forall\, \varphi \in D(\mathcal{A}_0) \quad |\varphi|_{H_0} \le c_{A_0} |A_0 \varphi|_{H_1} \quad \Leftrightarrow \quad \forall\, \varphi \in H^1_{\Gamma_t} \quad |\varphi|_{L^2} \le c_{fp} |\nabla \varphi|_{L^2_\varepsilon}$$

$$\forall\, \phi \in D(\mathcal{A}_0^*) \quad |\phi|_{H_1} \le c_{A_0} |A_0^* \phi|_{H_0} \quad \Leftrightarrow \quad \forall\, \Phi \in \varepsilon^{-1} D_{\Gamma_n} \cap \nabla H^1_{\Gamma_t} \quad |\Phi|_{L^2_\varepsilon} \le c_{fp} |\operatorname{div} \varepsilon \Phi|_{L^2}$$

$$\forall\, \phi \in D(\mathcal{A}_1) \quad |\phi|_{H_1} \le c_{A_1} |A_1 \phi|_{H_2} \quad \Leftrightarrow \quad \forall\, \Phi \in R_{\Gamma_t} \cap \varepsilon^{-1} \operatorname{rot} R_{\Gamma_n} \quad |\Phi|_{L^2_\varepsilon} \le c_m |\operatorname{rot} \Phi|_{L^2}$$

$$\forall\, \psi \in D(\mathcal{A}_1^*) \quad |\psi|_{H_2} \le c_{A_1} |A_1^* \psi|_{H_1} \quad \Leftrightarrow \quad \forall\, \Psi \in R_{\Gamma_n} \cap \operatorname{rot} R_{\Gamma_t} \quad |\Psi|_{L^2} \le c_m |\operatorname{rot} \Psi|_{L^2_\varepsilon}$$

(ii) all ranges $R(A_0) = \nabla H^1_{\Gamma_t}$, $R(A_1) = \operatorname{rot} R_{\Gamma_t}$, $R(A_0^*) = \operatorname{div} D_{\Gamma_n}$ are cl in L^2

(iii) the inverse ops $(\widetilde{\nabla}_{\Gamma_t})^{-1}$, $(\widetilde{\operatorname{div}_{\Gamma_n} \varepsilon})^{-1}$, $(\widetilde{\operatorname{rot}}_{\Gamma_t})^{-1}$, $(\widetilde{\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}})^{-1}$ are cont, even cpt

(iv) all Helmholtz decomposition hold, e.g.,

$$H_1 = R(A_0) \oplus N_1 \oplus R(A_1^*) \quad \Leftrightarrow \quad L^2_\varepsilon = \nabla H^1_{\Gamma_t} \oplus_{L^2_\varepsilon} \mathcal{H}_{D,\varepsilon} \oplus_{L^2_\varepsilon} \varepsilon^{-1} \operatorname{rot} R_{\Gamma_n}$$

(v) solution theory

(vi) variational formulations

(vii) functional a posteriori error estimates

(viii) div-curl-lemma

(ix) ...

classical de Rham complex in 3D (∇-rot-div-complex)

find $E \in \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n}$ s.t. / think of $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$

$$\operatorname{rot}_{\Gamma_t} E = F \qquad \mathrm{A}_1 x = f$$
$$\operatorname{div}_{\Gamma_n} \varepsilon E = g \qquad / \quad \text{think of} \quad \mathrm{A}_0^* x = g$$
$$\pi_{\mathcal{H}_{\mathrm{D},\varepsilon}} E = K \qquad \pi_{K_1} x = k$$

sol is simply $x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus K_1 = D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$

with $\boxed{x_f := \mathcal{A}_1^{-1} f} \in D(\mathcal{A}_1)$ and $\boxed{x_g := (\mathcal{A}_0^*)^{-1} g} \in D(\mathcal{A}_0^*)$

i.e., $E = E_F + E_g + K,$ where

$$\boxed{E_F := (\widetilde{\operatorname{rot}}_{\Gamma_t})^{-1} F} \in D(\widetilde{\operatorname{rot}}_{\Gamma_t}) = \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1} \operatorname{rot} \mathsf{R}_{\Gamma_n} = \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n,0} \cap \mathcal{H}_{\mathrm{D},\varepsilon}^{\perp},$$
$$\boxed{E_g := (\widetilde{\operatorname{div}}_{\Gamma_n}\varepsilon)^{-1} g} \in D(\widetilde{\operatorname{div}}_{\Gamma_n}\varepsilon) = \varepsilon^{-1}\mathsf{D}_{\Gamma_n} \cap \nabla \mathsf{H}_{\Gamma_t}^1 = \varepsilon^{-1}\mathsf{D}_{\Gamma_n} \cap \mathsf{R}_{\Gamma_t,0} \cap \mathcal{H}_{\mathrm{D},\varepsilon}^{\perp}$$

classical de Rham complex in 3D (∇-rot-div-complex)

Theorem (sharp upper bounds)

Let $\boxed{\tilde{E} \in \mathsf{L}^2_\varepsilon}$ *(very non-conforming approximation of E!) and* $\boxed{e := E - \tilde{E}}$. *Then*

$$|e|^2_{\mathsf{L}^2_\varepsilon} = |\pi_{R(\nabla_{\Gamma_t})} e|^2_{\mathsf{L}^2_\varepsilon} + |\pi_{R(\varepsilon^{-1}\operatorname{rot}_{\Gamma_n})} e|^2_{\mathsf{L}^2_\varepsilon} + |\pi_{\mathcal{H}_{\mathrm{D},\varepsilon}} e|^2_{\mathsf{L}^2_\varepsilon}$$

$$= \min_{\Phi \in \varepsilon^{-1}\mathsf{D}_{\Gamma_n}} \big(c_{\mathrm{fp}} |\operatorname{div} \varepsilon\Phi + g|_{\mathsf{L}^2} + |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon} \big)^2$$

$\boxed{\textit{reg } (-\nabla_{\Gamma_t} \operatorname{div}_{\Gamma_n} + 1)\textit{-prbl in } \mathsf{D}_{\Gamma_n}}$

$$+ \min_{\Phi \in \mathsf{R}_{\Gamma_t}} \big(c_{\mathrm{m}} |\operatorname{rot} \Phi - F|_{\mathsf{L}^2} + |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon} \big)^2$$

$\boxed{\textit{reg } (\operatorname{rot}_{\Gamma_n} \operatorname{rot}_{\Gamma_t} + 1)\textit{-prbl in } \mathsf{R}_{\Gamma_t}}$

$$+ \min_{\phi \in \mathsf{H}^1_{\Gamma_t}, \Psi \in \mathsf{R}_{\Gamma_n}} |\nabla\phi + \varepsilon^{-1}\operatorname{rot}\Psi + \tilde{E} - K|^2_{\mathsf{L}^2_\varepsilon}$$

$\boxed{\textit{cpld } (-\operatorname{div}_{\Gamma_n} \nabla_{\Gamma_t})\text{-}(\operatorname{rot}_{\Gamma_t} \operatorname{rot}_{\Gamma_n})\textit{-sys in } \mathsf{H}^1_{\Gamma_t}\text{-}\mathsf{R}_{\Gamma_n}}$

Remark

- $(\operatorname{rot}_{\Gamma_t} \operatorname{rot}_{\Gamma_n})$*-prbl needs saddle point formulation*
- Ω *top trv* $\Rightarrow \pi_{\mathrm{D}} = 0$ *and* $\mathsf{R}_{\Gamma_t,0} = \nabla \mathsf{H}^1_{\Gamma_t}$ *and* $\mathsf{D}_{\Gamma_n,0} = \operatorname{rot} \mathsf{R}_{\Gamma_n}$
- $\boxed{\Omega \textit{ convex and } \varepsilon = \mu = 1 \textit{ and } \Gamma_t = \Gamma \textit{ or } \Gamma_n = \Gamma \Rightarrow c_{\mathrm{f}} \le c_{\mathrm{m}} \le c_{\mathrm{p}} \le \frac{\operatorname{diam}_\Omega}{\pi}}$

div-curl-lemma

Lemma (div-curl-lemma (global version))

Assumptions:

- **(i)** (E_n) *bounded in* $\mathrm{L}^2(\Omega)$
- **(i')** (H_n) *bounded in* $\mathrm{L}^2(\Omega)$
- **(ii)** $(\operatorname{rot} E_n)$ *bounded in* $\mathrm{L}^2(\Omega)$
- **(ii')** $(\operatorname{div} \varepsilon H_n)$ *bounded in* $\mathrm{L}^2(\Omega)$
- **(iii)** $\nu \times E_n = 0$ *on* Γ_t, *i.e.*, $E_n \in \mathrm{R}_{\Gamma_t}(\Omega)$
- **(iii')** $\nu \cdot \varepsilon H_n = 0$ *on* Γ_n, *i.e.*, $H_n \in \varepsilon^{-1}\mathrm{D}_{\Gamma_n}(\Omega)$

$\Rightarrow \exists\ E, H$ *and subsequences st*

$E_n \rightharpoonup E$, $\operatorname{rot} E_n \rightharpoonup \operatorname{rot} E$ *and* $H_n \rightharpoonup H$, $\operatorname{div} H_n \rightharpoonup \operatorname{div} H$ *in* $\mathrm{L}^2(\Omega)$ *and*

$$\langle E_n, H_n \rangle_{\mathrm{L}^2_\varepsilon(\Omega)} \to \langle E, H \rangle_{\mathrm{L}^2_\varepsilon(\Omega)}$$

$\Rightarrow$ classical local version

de Rham complex in ND or on Riemannian manifolds (d-complex)

$\Omega \subset \mathbb{R}^N$ bd w. Lip. dom. or Ω Riemannian manifold with cpt cl. and Lip. boundary Γ

(generalized Maxwell equations)

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^{2,0} \underset{-\,\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \mathsf{L}^{2,1} \underset{-\,\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \cdots \mathsf{L}^{2,q} \underset{-\,\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \mathsf{L}^{2,q+1} \ldots \mathsf{L}^{2,N-1} \underset{-\,\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \mathsf{L}^{2,N} \underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}} \mathbb{R}$$

de Rham complex in ND or on Riemannian manifolds (d-complex)

$\Omega \subset \mathbb{R}^N$ bd w. Lip. dom. or Ω Riemannian manifold with cpt cl. and Lip. boundary Γ

(generalized Maxwell equations)

$$\{0\} \text{ or } \mathbb{R} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathrm{L}^{2,0} \underset{-\delta^1_{\Gamma_n}}{\overset{\mathrm{d}^0_{\Gamma_t}}{\rightleftarrows}} \mathrm{L}^{2,1} \underset{-\delta^2_{\Gamma_n}}{\overset{\mathrm{d}^1_{\Gamma_t}}{\rightleftarrows}} \cdots \mathrm{L}^{2,q} \underset{-\delta^{q+1}_{\Gamma_n}}{\overset{\mathrm{d}^q_{\Gamma_t}}{\rightleftarrows}} \mathrm{L}^{2,q+1} \ldots \mathrm{L}^{2,N-1} \underset{-\delta^N_{\Gamma_n}}{\overset{\mathrm{d}^{N-1}_{\Gamma_t}}{\rightleftarrows}} \mathrm{L}^{2,N} \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathbb{R} \text{ or } \{0\}$$

related fos

$$\begin{aligned} \mathrm{d}^q_{\Gamma_t} E &= F && \text{in } \Omega \\ -\delta^q_{\Gamma_n} E &= G && \text{in } \Omega \end{aligned}$$

related sos

$$\begin{aligned} -\delta^{q+1}_{\Gamma_n} \mathrm{d}^q_{\Gamma_t} E &= F && \text{in } \Omega \\ -\delta^q_{\Gamma_n} E &= G && \text{in } \Omega \end{aligned}$$

includes: EMS rot / div, Laplacian, rot rot, and more...
corresponding compact embeddings:

$$D(\mathrm{d}^q_{\Gamma_t}) \cap D(\delta^q_{\Gamma_n}) \hookrightarrow \mathrm{L}^{2,q} \qquad \text{(Weck's selection theorems, '74)}$$

Weck's selection theorem for Lip. manifolds and mixed bc: Bauer/Py/Schomburg ('17)

elasticity complex in 3D (sym ∇-Rot $\operatorname{Rot}_{\mathbb{S}}^{\top}$-$\operatorname{Div}_{\mathbb{S}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \quad \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{-\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\operatorname{sym}} \nabla}{\rightleftarrows}} \quad \mathsf{L}^2_{\mathbb{S}} \quad \underset{\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}}{\overset{\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^{\top}}{\rightleftarrows}} \quad \mathsf{L}^2_{\mathbb{S}} \quad \underset{-\operatorname{sym}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{S}}}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{\iota_{\mathsf{RM}}}{\overset{\pi_{\mathsf{RM}}}{\rightleftarrows}} \quad \mathsf{RM}$$

elasticity complex in 3D (sym ∇-Rot $\text{Rot}_{\mathbb{S}}^{\top}$-$\text{Div}_{\mathbb{S}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\text{Div}_{\mathbb{S}}}{\overset{\mathring{\text{sym}}\,\nabla}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\text{Rot}\,\text{Rot}_{\mathbb{S}}^{\top}}{\overset{\mathring{\text{Rot}}\,\mathring{\text{Rot}}_{\mathbb{S}}^{\top}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{-\text{sym}\,\nabla}{\overset{\mathring{\text{Div}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\text{RM}}}{\overset{\pi_{\text{RM}}}{\rightleftarrows}} \text{RM}$$

related fos ($\mathring{\text{Rot}}\,\mathring{\text{Rot}}_{\mathbb{S}}^{\top}$, $\text{Rot}\,\text{Rot}_{\mathbb{S}}^{\top}$ first order operators!)

$\mathring{\text{sym}}\,\nabla v = M$ in Ω	$\mathring{\text{Rot}}\,\mathring{\text{Rot}}_{\mathbb{S}}^{\top} M = F$ in Ω	$\mathring{\text{Div}}_{\mathbb{S}} N = g$ in Ω	$\pi v = r$ in Ω
$\pi v = 0$ in Ω	$-\text{Div}_{\mathbb{S}} M = f$ in Ω	$\text{Rot}\,\text{Rot}_{\mathbb{S}}^{\top} N = G$ in Ω	$-\text{sym}\,\nabla v = M$ in Ω

related sos ($\text{Rot}\,\text{Rot}_{\mathbb{S}}^{\top}\,\mathring{\text{Rot}}\,\mathring{\text{Rot}}_{\mathbb{S}}^{\top}$ second order operator!)

$-\text{Div}_{\mathbb{S}}\,\mathring{\text{sym}}\,\nabla v = f$ in Ω	$\text{Rot}\,\text{Rot}_{\mathbb{S}}^{\top}\,\mathring{\text{Rot}}\,\mathring{\text{Rot}}_{\mathbb{S}}^{\top} M = G$ in Ω	$-\text{sym}\,\nabla\,\mathring{\text{Div}}_{\mathbb{S}} N = M$ in Ω
$\pi v = 0$ in Ω	$-\text{Div}_{\mathbb{S}} M = f$ in Ω	$\text{Rot}\,\text{Rot}_{\mathbb{S}}^{\top} N = G$ in Ω

corresponding compact embeddings:

$$D(\mathring{\text{sym}}\,\nabla) \cap D(\pi) = D(\mathring{\nabla}) = \mathring{\mathsf{H}}^1 \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem and Korn ineq.)}$$

$$D(\mathring{\text{Rot}}\,\mathring{\text{Rot}}_{\mathbb{S}}^{\top}) \cap D(\text{Div}_{\mathbb{S}}) \hookrightarrow \mathsf{L}^2_{\mathbb{S}} \qquad \text{(new selection theorem)}$$

$$D(\mathring{\text{Div}}_{\mathbb{S}}) \cap D(\text{Rot}\,\text{Rot}_{\mathbb{S}}^{\top}) \hookrightarrow \mathsf{L}^2_{\mathbb{S}} \qquad \text{(new selection theorem)}$$

$$D(\pi) \cap D(\text{sym}\,\nabla) = D(\nabla) = \mathsf{H}^1 \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem and Korn ineq.)}$$

two new selection theorems for strong Lip. dom.: Py/Schomburg/Zulehner ('18)

elasticity complex in 3D (sym ∇-Rot $\mathrm{Rot}_{\mathbb{S}}^{\top}$-$\mathrm{Div}_{\mathbb{S}}$-complex)

Lemma/Theorem $\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1, \quad D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_2 \quad \text{cpt}}$

(i) all Friedrichs-Poincaré type est hold

est for $\mathcal{A}_0$ $\Leftrightarrow$ $\forall\, \varphi \in D(\mathrm{sym}\,\mathring{\nabla}) \cap R(\mathrm{Div}_{\mathbb{S}}) = \mathring{\mathrm{H}}^1$ $\quad |\varphi|_{\mathrm{L}^2} \le c_0 |\mathrm{sym}\,\nabla\varphi|_{\mathrm{L}^2}$

est for $\mathcal{A}_0^*$ $\Leftrightarrow$ $\forall\, \Phi \in D(\mathrm{Div}_{\mathbb{S}}) \cap R(\mathrm{sym}\,\mathring{\nabla})$ $\quad |\Phi|_{\mathrm{L}^2} \le c_0 |\mathrm{Div}\,\Phi|_{\mathrm{L}^2}$

est for $\mathcal{A}_1$ $\Leftrightarrow$ $\forall\, \Phi \in D(\mathrm{Rot}\,\mathring{\mathrm{Rot}}_{\mathbb{S}}^{\top}) \cap R(\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top})$ $\quad |\Phi|_{\mathrm{L}^2} \le c_1 |\mathrm{Rot}\,\mathrm{Rot}^{\top}\,\Phi|_{\mathrm{L}^2}$

est for $\mathcal{A}_1^*$ $\Leftrightarrow$ $\forall\, \Phi \in D(\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top}) \cap R(\mathrm{Rot}\,\mathring{\mathrm{Rot}}_{\mathbb{S}}^{\top})$ $\quad |\Phi|_{\mathrm{L}^2} \le c_1 |\mathrm{Rot}\,\mathrm{Rot}^{\top}\,\Phi|_{\mathrm{L}^2}$

est for $\mathcal{A}_2$ $\Leftrightarrow$ $\forall\, \Phi \in D(\mathring{\mathrm{Div}}_{\mathbb{S}}) \cap R(\mathrm{sym}\,\nabla)$ $\quad |\Phi|_{\mathrm{L}^2} \le c_2 |\mathrm{Div}\,\Phi|_{\mathrm{L}^2}$

est for $\mathcal{A}_2^*$ $\Leftrightarrow$ $\forall\, \varphi \in D(\mathrm{sym}\,\nabla) \cap R(\mathring{\mathrm{Div}}_{\mathbb{S}}) = \mathrm{H}^1 \cap \mathrm{RM}^{\perp}$ $\quad |\varphi|_{\mathrm{L}^2} \le c_2 |\mathrm{sym}\,\nabla\varphi|_{\mathrm{L}^2}$

(ii) all ranges $R(\mathrm{A}_n) = R(\mathcal{A}_n)$, $\quad R(\mathrm{A}_n^*) = R(\mathcal{A}_n^*)$ are cl in L^2

(iii) all inverse ops $\mathcal{A}_n^{-1}$, $(\mathcal{A}_n^*)^{-1}$ are cont, even cpt

(iv) all Helmholtz decomposition hold, e.g.,

$$\mathrm{H}_1 = R(\mathrm{A}_0) \oplus N_1 \oplus R(\mathrm{A}_1^*) \quad \Leftrightarrow \quad \mathrm{L}^2 = R(\mathrm{sym}\,\mathring{\nabla}) \oplus_{\mathrm{L}^2} \mathcal{H}_{\mathrm{D},\mathbb{S}} \oplus_{\mathrm{L}^2} R(\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top})$$

(v) solution theories

(vi) variational formulations

(vii) functional a posteriori error estimates

(viii) div-curl-lemmas

(ix) ...

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\mathrm{Rot}_{\mathbb{S}}$-$\mathrm{Div}_{\mathbb{T}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\nabla}\nabla}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}} \underset{-\operatorname{dev}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathsf{RT}}}{\overset{\pi_{\mathsf{RT}}}{\rightleftarrows}} \mathsf{RT}$$

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\mathrm{Rot}_{\mathbb{S}}$-$\mathrm{Div}_{\mathbb{T}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\nabla}\nabla}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}} \underset{-\operatorname{dev}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathsf{RT}}}{\overset{\pi_{\mathsf{RT}}}{\rightleftarrows}} \mathsf{RT}$$

related fos ($\mathring{\nabla}\nabla$, $\operatorname{div}\operatorname{Div}_{\mathbb{S}}$ first order operators!)

$$\begin{array}{rl|rl|rl|rl} \mathring{\nabla}\nabla u = M & \text{in } \Omega & \mathring{\operatorname{Rot}}_{\mathbb{S}} M = F & \text{in } \Omega & \mathring{\operatorname{Div}}_{\mathbb{T}} N = g & \text{in } \Omega & \pi v = r & \text{in } \Omega \\ \pi u = 0 & \text{in } \Omega & \operatorname{div}\operatorname{Div}_{\mathbb{S}} M = f & \text{in } \Omega & \operatorname{sym}\operatorname{Rot}_{\mathbb{T}} N = G & \text{in } \Omega & -\operatorname{dev}\nabla v = T & \text{in } \Omega \end{array}$$

related sos ($\operatorname{div}\operatorname{Div}_{\mathbb{S}}\mathring{\nabla}\nabla = \mathring{\Delta}^2$ second order operator!)

$$\begin{array}{rl|rl|rl} \operatorname{div}\operatorname{Div}_{\mathbb{S}}\mathring{\nabla}\nabla u = \mathring{\Delta}^2 u = f & \text{in } \Omega & \operatorname{sym}\operatorname{Rot}_{\mathbb{T}}\mathring{\operatorname{Rot}}_{\mathbb{S}} M = G & \text{in } \Omega & -\operatorname{dev}\nabla\mathring{\operatorname{Div}}_{\mathbb{T}} N = T & \text{in } \Omega \\ \pi u = 0 & \text{in } \Omega & \operatorname{div}\operatorname{Div}_{\mathbb{S}} M = f & \text{in } \Omega & \operatorname{sym}\operatorname{Rot}_{\mathbb{T}} N = G & \text{in } \Omega \end{array}$$

corresponding compact embeddings:

$$\begin{aligned} D(\mathring{\nabla}\nabla) \cap D(\pi) = D(\mathring{\nabla}\nabla) = \mathring{\mathsf{H}}^2 &\hookrightarrow \mathsf{L}^2 && \text{(Rellich's selection theorem)} \\ D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \cap D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) &\hookrightarrow \mathsf{L}^2_{\mathbb{S}} && \text{(new selection theorem)} \\ D(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) &\hookrightarrow \mathsf{L}^2_{\mathbb{T}} && \text{(new selection theorem)} \\ D(\pi) \cap D(\operatorname{dev}\nabla) = D(\operatorname{dev}\nabla) = D(\nabla) = \mathsf{H}^1 &\hookrightarrow \mathsf{L}^2 && \text{(Rellich's selection theorem and Korn type ineq.)} \end{aligned}$$

two new selection theorems for strong Lip. dom. and Korn Type ineq.: Py/Zulehner ('16)

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\mathrm{Rot}_{\mathbb{S}}$-$\mathrm{Div}_{\mathbb{T}}$-complex)

Lemma/Theorem

$\Downarrow$ $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1, \quad D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_2 \quad \text{cpt}$

(i) all Friedrichs-Poincaré type est hold

est for $\mathcal{A}_0$	$\Leftrightarrow$	$\forall\, \varphi \in D(\mathring{\nabla}\nabla) \cap R(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) = \mathring{\mathrm{H}}^2$	$\lvert\varphi\rvert_{\mathrm{L}^2} \le c_0 \lvert\nabla\nabla\varphi\rvert_{\mathrm{L}^2}$
est for $\mathcal{A}_0^*$	$\Leftrightarrow$	$\forall\, \Phi \in D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \cap R(\mathring{\nabla}\nabla)$	$\lvert\Phi\rvert_{\mathrm{L}^2} \le c_0 \lvert\operatorname{div}\operatorname{Div}\Phi\rvert_{\mathrm{L}^2}$
est for $\mathcal{A}_1$	$\Leftrightarrow$	$\forall\, \Phi \in D(\mathring{\mathrm{Rot}}_{\mathbb{S}}) \cap R(\operatorname{sym}\mathrm{Rot}_{\mathbb{T}})$	$\lvert\Phi\rvert_{\mathrm{L}^2} \le c_1 \lvert\mathrm{Rot}\,\Phi\rvert_{\mathrm{L}^2}$
est for $\mathcal{A}_1^*$	$\Leftrightarrow$	$\forall\, \Phi \in D(\operatorname{sym}\mathrm{Rot}_{\mathbb{T}}) \cap R(\mathring{\mathrm{Rot}}_{\mathbb{S}})$	$\lvert\Phi\rvert_{\mathrm{L}^2} \le c_1 \lvert\operatorname{sym}\mathrm{Rot}\,\Phi\rvert_{\mathrm{L}^2}$
est for $\mathcal{A}_2$	$\Leftrightarrow$	$\forall\, \Phi \in D(\mathring{\mathrm{Div}}_{\mathbb{T}}) \cap R(\operatorname{dev}\nabla)$	$\lvert\Phi\rvert_{\mathrm{L}^2} \le c_2 \lvert\mathrm{Div}\,\Phi\rvert_{\mathrm{L}^2}$
est for $\mathcal{A}_2^*$	$\Leftrightarrow$	$\forall\, \varphi \in D(\operatorname{dev}\nabla) \cap R(\mathring{\mathrm{Div}}_{\mathbb{T}}) = \mathrm{H}^1 \cap \mathrm{RT}^{\perp}$	$\lvert\varphi\rvert_{\mathrm{L}^2} \le c_2 \lvert\operatorname{dev}\nabla\varphi\rvert_{\mathrm{L}^2}$

(ii) all ranges $R(\mathrm{A}_n) = R(\mathcal{A}_n)$, $R(\mathrm{A}_n^*) = R(\mathcal{A}_n^*)$ are cl in L^2

(iii) all inverse ops $\mathcal{A}_n^{-1}$, $(\mathcal{A}_n^*)^{-1}$ are cont, even cpt

(iv) all Helmholtz decomposition hold, e.g.,

$$\mathrm{H}_1 = R(\mathrm{A}_0) \oplus N_1 \oplus R(\mathrm{A}_1^*) \quad \Leftrightarrow \quad \mathrm{L}^2_{\mathbb{S}} = R(\mathring{\nabla}\nabla) \oplus_{\mathrm{L}^2_{\mathbb{S}}} \mathcal{H}_{\mathrm{D},\mathbb{S}} \oplus_{\mathrm{L}^2_{\mathbb{S}}} R(\operatorname{sym}\mathrm{Rot}_{\mathbb{T}}),$$

$$\mathrm{H}_2 = R(\mathrm{A}_1) \oplus N_2 \oplus R(\mathrm{A}_2^*) \quad \Leftrightarrow \quad \mathrm{L}^2_{\mathbb{T}} = R(\mathring{\mathrm{Rot}}_{\mathbb{S}}) \oplus_{\mathrm{L}^2_{\mathbb{T}}} \mathcal{H}_{\mathrm{N},\mathbb{T}} \oplus_{\mathrm{L}^2_{\mathbb{T}}} R(\operatorname{dev}\nabla)$$

(v)-(ix) solution theories, variational formulations, functional a posteriori error estimates, div-curl-lemmas, ...

Solving PDEs with Hilbert Complexes

APPENDIX: Literature

literature (FA-ToolBox, complexes, a posteriori error estimates, ...)

some results of this talk:

- Py: *Solution Theory, Variational Formulations, and Functional a Posteriori Error Estimates for General First Order Systems with Applications to Electro-Magneto-Statics and More*,
  (NFAO) Numerical Functional Analysis and Optimization, 2019

literature (complexes, Friedrichs type constants, Maxwell constants)

results of this talk:

- Py: *On Constants in Maxwell Inequalities for Bounded and Convex Domains*, Zapiski POMI/ (JMS)Journal of Mathematical Sciences (Springer New York), 2015
- Py: *On Maxwell's and Poincare's Constants*, (DCDS) Discrete and Continuous Dynamical Systems - Series S, 2015
- Py: *On the Maxwell Constants in 3D*, (M2AS) Mathematical Methods in the Applied Sciences, 2017
- Py: *On the Maxwell and Friedrichs/Poincaré Constants in ND*, (MZ) Mathematische Zeitschrift, 2019

- Py: . . . *some (so far) unpublished results*

literature (complexes, Friedrichs type constants, compact embeddings)

- Weck, N.: *Maxwell's boundary value problems on Riemannian manifolds with nonsmooth boundaries*,
  (JMA2) Journal of Mathematical Analysis and Applications, 1974 (1972)
- Picard, R.: *An elementary proof for a compact imbedding result in generalized electromagnetic theory*,
  (MZ) Mathematische Zeitschrift, 1984
- Witsch, K.-J.: *A remark on a compactness result in electromagnetic theory*,
  (M2AS) Mathematical Methods in the Applied Sciences, 1993

results of this talk:

- Bauer, S., Py, Schomburg, M.: *The Maxwell Compactness Property in Bounded Weak Lipschitz Domains with Mixed Boundary Conditions*,
  (SIMA) SIAM Journal on Mathematical Analysis, 2016
- Py, Zulehner, W.: *The divDiv-Complex and Applications to Biharmonic Equations*,
  (AA) Applicable Analysis, 2019
- Py, Zulehner, W.: *The Elasticity Complex*,
  submitted, 2019

literature (div-curl-lemma)

original papers (local div-curl-lemma):

- Murat, F.: *Compacité par compensation*,
  Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 1978
- Tartar, L.: *Compensated compactness and applications to partial differential equations*,
  Nonlinear analysis and mechanics, Heriot-Watt symposium, 1979

literature (div-curl-lemma)

recent papers (global div-curl-lemma, H^1-detour):

- Gloria, A., Neukamm, S., Otto, F.: *Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics*, (IM) Invent. Math., 2015
- Kozono, H., Yanagisawa, T.: *Global compensated compactness theorem for general differential operators of first order*, (ARMA) Arch. Ration. Mech. Anal., 2013
- Schweizer, B.: *On Friedrichs inequality, Helmholtz decomposition, vector potentials, and the div-curl lemma*, accepted preprint, 2018

recent papers (global div-curl-lemma, general results/this talk):

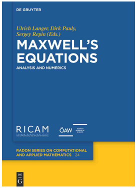
- Waurick, M.: *A Functional Analytic Perspective to the div-curl Lemma*, (JOP) J. Operator Theory, 2018
- Py: *A Global div-curl-Lemma for Mixed Boundary Conditions in Weak Lipschitz Domains and a Corresponding Generalized* A_0^*-A_1*-Lemma in Hilbert Spaces*, (ANA) Analysis (Munich), 2019

literature (full time-dependent Maxwell equations)

- Py, Picard, R.: *A Note on the Justification of the Eddy Current Model in Electrodynamics*,
  (M2AS) Mathematical Methods in the Applied Sciences, 2017
- Py, Picard, R., Trostorff, S., Waurick, M.: *On a Class of Degenerate Abstract Parabolic Problems and Applications to Some Eddy Current Models*,
  submitted, 2019

literature (Maxwell's equations and more...)

books:



- Langer, U., Py, Repin, S. (Eds): *Maxwell's equations. Analysis and numerics*, Radon Series on Applied Mathematics, De Gruyter, July 2019
- Py: *Maxwell's Equations: Hilbert Space Methods for the Theory of Electromagnetism*, Radon Series on Applied Mathematics, De Gruyter, 2020

(last book: contains all results of this talk and more...)

Solving PDEs with Hilbert Complexes

APPENDIX: Solution Theory FOSs and Variational Formulations

(stat) first order system - solution theory

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ st fos $\quad \mathrm{A}_1 x = f \quad \mathrm{A}_0^* x = g \quad \pi_{N_1} x = k$

Theorem (FA-ToolBox II (solution theory))

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\!\rightarrow \mathrm{H}_1 \text{ compact}}$

fos is uniq sol $\Leftrightarrow$ $f \in R(\mathrm{A}_1) \quad g \in R(\mathrm{A}_0^*) \quad k \in N_1$

$$x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus N_1 = D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$$

$$\boxed{x_f := \mathcal{A}_1^{-1} f} \in D(\mathcal{A}_1)$$

$$\boxed{x_g := (\mathcal{A}_0^*)^{-1} g} \in D(\mathcal{A}_0^*)$$

dep cont on data $|x|_{\mathrm{H}_1} \leq |x_f|_{\mathrm{H}_1} + |x_g|_{\mathrm{H}_1} + |k|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |f|_{\mathrm{H}_2} + c_{\mathrm{A}_0} |g|_{\mathrm{H}_0} + |k|_{\mathrm{H}_1}$

moreover

$\pi_{R(\mathrm{A}_1^*)} x = x_f \quad \pi_{R(\mathrm{A}_0)} x = x_g \quad \pi_{N_1} x = k \quad |x|_{\mathrm{H}_1}^2 = |x_f|_{\mathrm{H}_1}^2 + |x_g|_{\mathrm{H}_1}^2 + |k|_{\mathrm{H}_1}^2$

Remark

enough $R(\mathrm{A}_0)$ *and* $R(\mathrm{A}_1)$ *cl*

(stat) first order system - variational formulations

$$x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus N_1 = D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$$

$$x_f := \mathcal{A}_1^{-1} f \in D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap R(\mathrm{A}_1^*) = D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \cap N_1^\perp$$

$$x_g := (\mathcal{A}_0^*)^{-1} g \in D(\mathcal{A}_0^*) = D(\mathrm{A}_0^*) \cap R(\mathrm{A}_0) = D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1) \cap N_1^\perp$$

$$\begin{aligned} \mathrm{A}_1 x &= f & \mathrm{A}_1 x_f &= f & \mathrm{A}_1 x_g &= 0 & \mathrm{A}_1 k &= 0 \\ \mathrm{A}_0^* x &= g & \mathrm{A}_0^* x_f &= 0 & \mathrm{A}_0^* x_g &= g & \mathrm{A}_0^* k &= 0 \\ \pi_{N_1} x &= k & \pi_{N_1} x_f &= 0 & \pi_{N_1} x_g &= 0 & \pi_{N_1} k &= k \end{aligned}$$

- option I: find x_f and x_g separately $\Rightarrow$ $x = x_f + x_g + k$
- option II: find x directly

(stat) first order system - variational formulations I

finding

$$x_f := \mathcal{A}_1^{-1} f \in D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap \underbrace{R(\mathrm{A}_1^*)}_{=R(\mathcal{A}_1^*)} = D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \cap N_1^\perp$$

$$\begin{aligned} \mathrm{A}_1 x_f &= f \\ \mathrm{A}_0^* x_f &= 0 \\ \pi_{N_1} x_f &= 0 \end{aligned}$$

at least two options

- option Ia: multiply $\mathrm{A}_1 x_f = f$ by $\mathrm{A}_1 \xi$ $\Rightarrow$

$$\boxed{\forall\, \xi \in D(\mathcal{A}_1) \qquad \langle \mathrm{A}_1 x_f, \mathrm{A}_1 \xi \rangle_{\mathrm{H}_2} = \langle f, \mathrm{A}_1 \xi \rangle_{\mathrm{H}_2}}$$

  weak form of $\boxed{\mathrm{A}_1^* \mathrm{A}_1 x_f = \mathrm{A}_1^* f}$

- option Ib: repr $x_f = \mathrm{A}_1^* y_f$ with potential $y_f = (\mathcal{A}_1^*)^{-1} x_f \in D(\mathcal{A}_1^*)$
  and mult by x_f by $\mathrm{A}_1^* \phi$ $\Rightarrow$

$$\boxed{\forall\, \phi \in D(\mathcal{A}_1^*) \qquad \langle \mathrm{A}_1^* y_f, \mathrm{A}_1^* \phi \rangle_{\mathrm{H}_1} = \langle x_f, \mathrm{A}_1^* \phi \rangle_{\mathrm{H}_1} = \langle \mathrm{A}_1 x_f, \phi \rangle_{\mathrm{H}_2} = \langle f, \phi \rangle_{\mathrm{H}_2}}$$

  weak form of $\boxed{\mathrm{A}_1 x_f = f}$ and $\boxed{\mathrm{A}_1 \mathrm{A}_1^* y_f = f}$

analogously for x_g

Solving PDEs with Hilbert Complexes

APPENDIX: Crucial Property is a Suitable Compact Embedding

key tools to prove compact embeddings

crucial tool: compact embeddings

- localisation to top triv domains by partition of unity
  fine for first order operators $\surd$
  tech prob for second order operators
  sol: regular decompositions in H^{-1}, ...
- Helmholtz decompositions
- regular potentials
  here is the hard analysis: weak/strong Lipschitz domains, mixed bc, ...
- boiling down to Rellich's selection theorem

Solving PDEs with Hilbert Complexes

APPENDIX: Regular Potentials and Regular Decompositions

joint work with

Ralf Hiptmair, Clemens Pechstein, Michael Schomburg, Walter Zulehner

generalises and improves work/results of Costabel and McIntosh

regular potentials

Theorem (regular potentials)

Let (Ω, Γ_t) be a bounded strong Lipschitz pair and $k \geq 0$. Then there exists a continuous linear operator

$$\mathcal{S}^q_{\mathrm{d},k} : \mathring{\mathsf{H}}^{k,q}_{\Gamma_t}(\Omega) \cap \mathring{\mathsf{D}}^q_{\Gamma_t,0}(\Omega) \cap \mathcal{H}^q(\Omega)^\perp \longrightarrow \mathring{\mathsf{H}}^{k+1,q-1}_{\Gamma_t}(\Omega),$$

such that $\mathrm{d}\,\mathcal{S}^q_{\mathrm{d},k} = \mathrm{id}\,|_{\mathring{\mathsf{H}}^{k,q}_{\Gamma_t}(\Omega)\cap\mathring{\mathsf{D}}^q_{\Gamma_t,0}(\Omega)\cap\mathcal{H}^q(\Omega)^\perp}$. *In particular,*

$$\begin{aligned}\mathring{\mathsf{H}}^{k,q}_{\Gamma_t}(\Omega) \cap \mathring{\mathsf{D}}^q_{\Gamma_t,0}(\Omega) \cap \mathcal{H}^q(\Omega)^\perp &= \mathring{\mathsf{H}}^{k,q}_{\Gamma_t}(\Omega) \cap \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_t}(\Omega)\\ &= \mathrm{d}\,\mathring{\mathsf{H}}^{k+1,q-1}_{\Gamma_t}(\Omega)\\ &= \mathrm{d}\,\mathring{\mathsf{D}}^{k,q-1}_{\Gamma_t}(\Omega)\end{aligned}$$

and the regular $\mathring{\mathsf{H}}^{k+1,q-1}_{\Gamma_t}(\Omega)$*-potential depends continuously on the data. Especially, these spaces are closed subspaces of* $\mathring{\mathsf{H}}^{k,q}(\Omega)$ *and* $\mathcal{S}^q_{\mathrm{d},k}$ *is a right inverse to* d.

regular docompositions

Theorem (regular decompositions)

Let (Ω, Γ_t) be a bounded strong Lipschitz pair and $k \geq 0$. Then the regular decompositions

$$\mathring{\mathsf{D}}^{k,q}_{\Gamma_t}(\Omega) = \mathring{\mathsf{H}}^{k+1,q}_{\Gamma_t}(\Omega) + \mathrm{d}\,\mathring{\mathsf{H}}^{k+1,q-1}_{\Gamma_t}(\Omega)$$

$$\bigcap \qquad\qquad \bigcup$$

$$= \mathcal{S}^{q+1}_{\mathrm{d},k}\,\mathrm{d}\,\mathring{\mathsf{D}}^{k,q}_{\Gamma_t}(\Omega) \dot{+} \big(\mathring{\mathsf{H}}^{k,q}_{\Gamma_t}(\Omega) \cap \mathring{\mathsf{D}}^{q}_{\Gamma_t,0}(\Omega)\big)$$

hold with linear and continuous regular decomposition resp. potential operators, which can be defined explicitly by the orthonormal Helmholtz projectors and the operators $\mathcal{S}^{q}_{\mathrm{d},k}$.

dual regular potentials and decompositions

Dual regular potentials and decompositions involving

$$\mathring{\mathsf{H}}^{-k,q}_{\Gamma_n}(\Omega) = \mathring{\mathsf{H}}^{k,q}_{\Gamma_t}(\Omega)'$$

can be proved by Banach space duality. E.g.:

- $\mathring{\mathsf{D}}^{k,q}_{\Gamma_t}(\Omega)' = \mathring{\Delta}^{-k-1,q}_{\Gamma_n}(\Omega) := \{E' \in \mathring{\mathsf{H}}^{-k-1,q}_{\Gamma_n}(\Omega) \,:\, \delta\, E' \in \mathring{\mathsf{H}}^{-k-1,q-1}_{\Gamma_n}(\Omega)\}$
- dual ranges are closed
- dual Friedrichs/Poincaré typ estimates, inf-sup condition, i.e.,

$$\delta^{-1} : \delta\, \mathring{\mathsf{H}}^{-k,q}_{\Gamma_n}(\Omega) \longrightarrow \big(\mathrm{d}\, \mathring{\mathsf{H}}^{k+1,q-1}_{\Gamma_t}(\Omega)\big)' \qquad \text{cont}$$

$$\Updownarrow$$

$$\forall\, H' \in \big(\mathrm{d}\, \mathring{\mathsf{H}}^{k+1,q-1}_{\Gamma_t}(\Omega)\big)' \qquad \frac{1}{c}|H'|_{\mathring{\mathsf{H}}^{-k,q}_{\Gamma_n}(\Omega)} \leq |\,\delta\, H'|_{\mathring{\mathsf{H}}^{-k-1,q-1}_{\Gamma_n}(\Omega)} \leq c|H'|_{\mathring{\mathsf{H}}^{-k,q}_{\Gamma_n}(\Omega)}$$

$$\Updownarrow$$

$$0 < \inf_{0 \neq H \in \mathrm{d}\, \mathring{\mathsf{H}}^{k+1,q-1}_{\Gamma_t}(\Omega)} \sup_{0 \neq E \in \mathring{\mathsf{H}}^{k+1,q-1}_{\Gamma_t}(\Omega)} \frac{\langle H, \mathrm{d}\, E\rangle_{\mathring{\mathsf{H}}^{k,q}_{\Gamma_t}(\Omega)}}{|H|_{\mathring{\mathsf{H}}^{k,q}_{\Gamma_t}(\Omega)}|E|_{\mathring{\mathsf{H}}^{k+1,q-1}_{\Gamma_t}(\Omega)}}$$

dual regular potentials and decompositions

Ω top triv $\quad\Rightarrow$

$$\mathrm{d}\,\mathring{\mathsf{H}}^{k+1,q-1}_{\Gamma_t}(\Omega) = \mathring{\mathsf{H}}^{k,q}_{\Gamma_t}(\Omega) \cap \mathring{\mathsf{D}}^{q}_{\Gamma_t,0}(\Omega)$$

$$\delta\,\mathring{\mathsf{H}}^{-k,q}_{\Gamma_n}(\Omega) = \mathring{\Delta}^{-k-1,q-1}_{\Gamma_n,0}(\Omega) = \{H' \in \mathring{\mathsf{H}}^{-k-1,q-1}_{\Gamma_n}(\Omega) \,:\, \delta H' = 0\}$$

Solving PDEs with Hilbert Complexes

APPENDIX: Friedrichs/Poincaré/Maxwell constants (numerics)

joint work with

Jan Valdman

Friedrichs/Poincaré/Maxwell constants

assumption: $\varepsilon = \mu = 1$ and $\Gamma_t = \Gamma$, i.e., $c_{\mathrm{fp}} = c_{\mathrm{f}}$ or $\Gamma_n = \Gamma$, i.e., $c_{\mathrm{fp}} = c_{\mathrm{p}}$

Lemma (Maxwell-Poincaré constants)

Ω convex and bounded $\Rightarrow$ $c_{\mathrm{m}} \leq c_{\mathrm{p}} \leq \dfrac{\mathrm{diam}_\Omega}{\pi}$

Mild Conjecture (Maxwell-Poincaré constants)

Ω convex and bounded $\Rightarrow$ $c_{\mathrm{f}} \leq c_{\mathrm{m}} \leq c_{\mathrm{p}} \leq \frac{\mathrm{diam}_\Omega}{\pi}$

Theorem (FA-ToolBox / Friedrichs-Poincaré type estimates and constants)

$$\forall\, \varphi \in D(\mathcal{A}_0) \quad |\varphi|_{\mathrm{H}_0} \leq c_{\mathrm{A}_0} |\mathrm{A}_0 \varphi|_{\mathrm{H}_1} \quad \Leftrightarrow \quad \forall\, \varphi \in \mathrm{H}^1_\Gamma \quad |\varphi|_{\mathrm{L}^2} \leq c_{\mathrm{f}} |\nabla \varphi|_{\mathrm{L}^2}$$

$$\forall\, \phi \in D(\mathcal{A}_0^*) \quad |\phi|_{\mathrm{H}_1} \leq c_{\mathrm{A}_0} |\mathrm{A}_0^* \phi|_{\mathrm{H}_0} \quad \Leftrightarrow \quad \forall\, \Phi \in \mathrm{D} \cap \nabla \mathrm{H}^1_\Gamma \quad |\Phi|_{\mathrm{L}^2} \leq c_{\mathrm{f}} |\operatorname{div} \Phi|_{\mathrm{L}^2}$$

$$\forall\, \phi \in D(\mathcal{A}_1) \quad |\phi|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |\mathrm{A}_1 \phi|_{\mathrm{H}_2} \quad \Leftrightarrow \quad \forall\, \Phi \in \mathrm{R}_\Gamma \cap \operatorname{rot} \mathrm{R} \quad |\Phi|_{\mathrm{L}^2} \leq c_{\mathrm{m}} |\operatorname{rot} \Phi|_{\mathrm{L}^2}$$

$$\forall\, \psi \in D(\mathcal{A}_1^*) \quad |\psi|_{\mathrm{H}_2} \leq c_{\mathrm{A}_1} |\mathrm{A}_1^* \psi|_{\mathrm{H}_1} \quad \Leftrightarrow \quad \forall\, \Psi \in \mathrm{R} \cap \operatorname{rot} \mathrm{R}_\Gamma \quad |\Psi|_{\mathrm{L}^2} \leq c_{\mathrm{m}} |\operatorname{rot} \Psi|_{\mathrm{L}^2}$$

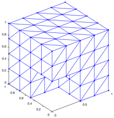
$$\forall\, \psi \in D(\mathcal{A}_2) \quad |\psi|_{\mathrm{H}_2} \leq c_{\mathrm{A}_2} |\mathrm{A}_2 \psi|_{\mathrm{H}_3} \quad \Leftrightarrow \quad \forall\, \Psi \in \mathrm{D}_\Gamma \cap \nabla \mathrm{H}^1 \quad |\Psi|_{\mathrm{L}^2} \leq c_{\mathrm{p}} |\operatorname{div} \Psi|_{\mathrm{L}^2}$$

$$\forall\, \xi \in D(\mathcal{A}_2^*) \quad |\xi|_{\mathrm{H}_3} \leq c_{\mathrm{A}_2} |\mathrm{A}_2^* \xi|_{\mathrm{H}_2} \quad \Leftrightarrow \quad \forall\, \zeta \in \mathrm{H}^1 \cap \mathbb{R}^\perp \quad |\zeta|_{\mathrm{L}^2} \leq c_{\mathrm{p}} |\nabla \zeta|_{\mathrm{L}^2}$$

Friedrichs/Poincaré/Maxwell constants

surprise numerical tests show even for non-convex domains and mixed bc

e.g., Fichera corner domain

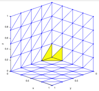
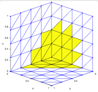
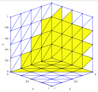
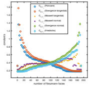
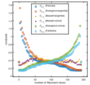


Conjecture (Maxwell-Poincaré constants)

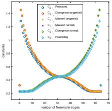
$$c_{\mathrm{f}} \leq \min\{c_{\mathrm{fp}}, c_{\mathrm{pf}}\} \leq c_{\mathrm{m}} \leq \max\{c_{\mathrm{fp}}, c_{\mathrm{pf}}\} \leq \sup_{\Gamma_t \neq \emptyset}\{c_{\mathrm{fp}}\} < \infty$$

Theorem (FA-ToolBox / Friedrichs-Poincaré type estimates and constants)

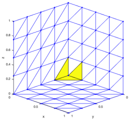
$\forall\, \varphi \in D(\mathcal{A}_0)$	$\lvert\varphi\rvert_{H_0} \leq c_{A_0} \lvert A_0 \varphi\rvert_{H_1}$	$\Leftrightarrow$	$\forall\, \varphi \in H^1_{\Gamma_t}$	$\lvert\varphi\rvert_{L^2} \leq c_{\mathrm{fp}} \lvert\nabla\varphi\rvert_{L^2}$
$\forall\, \phi \in D(\mathcal{A}_0^*)$	$\lvert\phi\rvert_{H_1} \leq c_{A_0} \lvert A_0^* \phi\rvert_{H_0}$	$\Leftrightarrow$	$\forall\, \Phi \in D_{\Gamma_n} \cap \nabla H^1_{\Gamma_t}$	$\lvert\Phi\rvert_{L^2} \leq c_{\mathrm{fp}} \lvert \operatorname{div} \Phi\rvert_{L^2}$
$\forall\, \phi \in D(\mathcal{A}_1)$	$\lvert\phi\rvert_{H_1} \leq c_{A_1} \lvert A_1 \phi\rvert_{H_2}$	$\Leftrightarrow$	$\forall\, \Phi \in R_{\Gamma_t} \cap \operatorname{rot} R_{\Gamma_n}$	$\lvert\Phi\rvert_{L^2} \leq c_{\mathrm{m}} \lvert \operatorname{rot} \Phi\rvert_{L^2}$
$\forall\, \psi \in D(\mathcal{A}_1^*)$	$\lvert\psi\rvert_{H_2} \leq c_{A_1} \lvert A_1^* \psi\rvert_{H_1}$	$\Leftrightarrow$	$\forall\, \Psi \in R_{\Gamma_n} \cap \operatorname{rot} R_{\Gamma_t}$	$\lvert\Psi\rvert_{L^2} \leq c_{\mathrm{m}} \lvert \operatorname{rot} \Psi\rvert_{L^2}$
$\forall\, \psi \in D(\mathcal{A}_2)$	$\lvert\psi\rvert_{H_2} \leq c_{A_2} \lvert A_2 \psi\rvert_{H_3}$	$\Leftrightarrow$	$\forall\, \Psi \in D_{\Gamma_t} \cap \nabla H^1_{\Gamma_n}$	$\lvert\Psi\rvert_{L^2} \leq c_{\mathrm{pf}} \lvert \operatorname{div} \Psi\rvert_{L^2}$
$\forall\, \xi \in D(\mathcal{A}_2^*)$	$\lvert\xi\rvert_{H_3} \leq c_{A_2} \lvert A_2^* \xi\rvert_{H_2}$	$\Leftrightarrow$	$\forall\, \zeta \in H^1_{\Gamma_n}$	$\lvert\zeta\rvert_{L^2} \leq c_{\mathrm{pf}} \lvert\nabla\zeta\rvert_{L^2}$

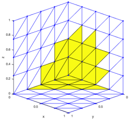






Friedrichs/Poincaré/Maxwell constants

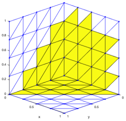


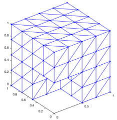
2D unit square



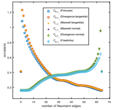


3D unit cube





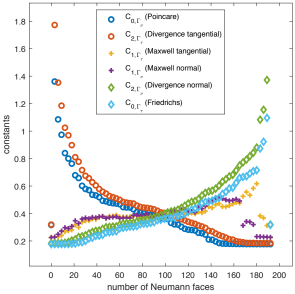
3D Fichera corner



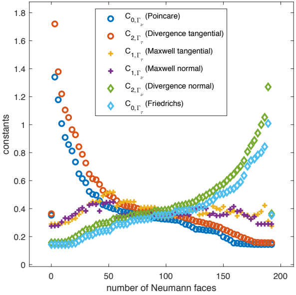
2D L-shape

Conjecture (Maxwell-Poincaré constants)

$$c_{\mathrm{f}} \le \min\{c_{\mathrm{fp}}, c_{\mathrm{pf}}\} \le c_{\mathrm{m}} \le \max\{c_{\mathrm{fp}}, c_{\mathrm{pf}}\} \le \sup_{\Gamma_t \neq \emptyset}\{c_{\mathrm{fp}}\} < \infty$$



3D unit cube



3D Fichera corner domain

Solving PDEs with Hilbert Complexes

APPENDIX: A Posteriori Error Estimates for BEM (Boundary Element Method)

joint work with

Stefan Kurz, Dirk Praetorius, Sergey Repin, Daniel Sebastian

functional a posteriori error estimates for BEM

problem: num approx with BEM

$$\Delta u = 0 \quad \text{in } \Omega, \qquad\qquad u|_\Gamma = g \quad \text{on } \Gamma.$$

functional a posteriori error estimates: num approx with FEM

$$\max_{\substack{E \in \mathrm{L}^2(\Omega)\\ \operatorname{div} E = 0}} \Big(2\langle n \cdot E, g - \widetilde{u}|_\Gamma \rangle_{\mathrm{H}^{-1/2}(\Gamma)} - |E|^2_{\mathrm{L}^2(\Omega)} \Big) = |\nabla (u - \widetilde{u})|^2_{\mathrm{L}^2(\Omega)} = \min_{\substack{v \in \mathrm{H}^1(\Omega)\\ v|_\Gamma = g - \widetilde{u}|_\Gamma}} |\nabla v|^2_{\mathrm{L}^2(\Omega)}$$

natural energy norm ($\mathrm{H}^1(\Omega)$-volume norm)

idea: compute upper and lower bounds in a thin boundary layer using FEM

functional a posteriori error estimates for BEM

$$\max_{\substack{E\in\mathrm{L}^2(\Omega)\\ \operatorname{div} E=0}} \Big(2\langle n\cdot E, g-\widetilde{u}|_\Gamma\rangle_{\mathrm{H}^{-1/2}(\Gamma)} - |E|^2_{\mathrm{L}^2(\Omega)}\Big) = |\nabla(u-\widetilde{u})|^2_{\mathrm{L}^2(\Omega)} = \min_{\substack{v\in\mathrm{H}^1(\Omega)\\ v|_\Gamma = g-\widetilde{u}|_\Gamma}} |\nabla v|^2_{\mathrm{L}^2(\Omega)}$$

minimiser of upper bound $\overline{v} = u - \widetilde{u}$: standard Dirichlet-Laplacian

$$\Delta v = 0 \quad \text{in } \Omega, \qquad v|_\Gamma = g - \widetilde{u}|_\Gamma \quad \text{on } \Gamma$$

exact solution is $v = \overline{v}$ $\Rightarrow$ standard FEM on boundary layer for v

maximiser of lower bound $\underline{E} = \nabla\overline{v} = \nabla(u-\widetilde{u})$: Neumann-type-Laplacian

$$\Delta v = 0 \quad \text{in } \Omega, \qquad n\cdot\nabla v|_\Gamma = \langle g - \widetilde{u}|_\Gamma, n\cdot\nabla\widehat{(\,\cdot\,)}|_\Gamma\rangle \quad \text{in } \mathrm{H}^{-1/2}(\Gamma)$$

(here $\widehat{(\,\cdot\,)}$ harmonic extension and $n\cdot\nabla\widehat{(\,\cdot\,)}|_\Gamma$ Dirichlet2Neumann operator)

exact solution is $v = \overline{v}$ and $\nabla v = \underline{E}$ $\Rightarrow$ non-standard FEM on bd layer for E

$\Rightarrow$ saddle point formulation (mixed/dual Laplacian)

Find $(E, v) \in \mathrm{H}(\operatorname{div},\Omega)\times\mathrm{L}^2(\Omega)$ s.t. for all $(\Phi,\varphi)\in\mathrm{H}(\operatorname{div},\Omega)\times\mathrm{L}^2(\Omega)$

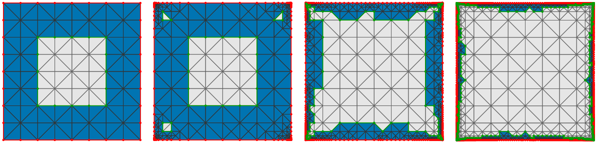
$$\langle E,\Phi\rangle_{\mathrm{L}^2(\Omega)} + \langle \operatorname{div}\Phi, v\rangle_{\mathrm{L}^2(\Omega)} = \langle n\cdot\Phi, g-\widetilde{u}|_\Gamma\rangle_{\mathrm{H}^{-1/2}(\Gamma)},$$

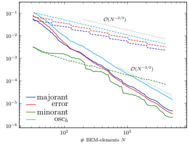
$$\langle \operatorname{div} E,\varphi\rangle_{\mathrm{L}^2(\Omega)} = 0$$

unique sol $(E, v) = (\underline{E}, \overline{v})$

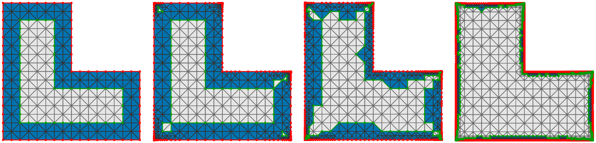
functional a posteriori error estimates for BEM - some pics

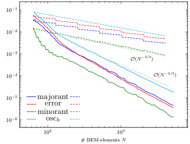
Ω: unit square, $u(x) = cosh(x_1)cos(x_2)$, known smooth solution u



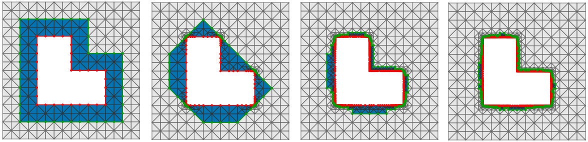


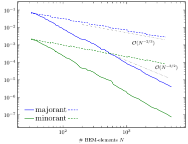
Ω: L-shaped domain, $u(x) = u(r, \varphi) = r^{2/3}\cos(2/3\varphi)$, known non-smooth solution u





Ω: L-shaped exterior domain, g (bd data) given by double-layer potential operator, unknown exact solution u





oscillatory error upper bound exact error lower bound convergence rates

adaptive mesh-ref with Dörfler marking (solid lines) vs. unif mesh-ref (dashed lines)

Solving PDEs with Hilbert Complexes

APPENDIX: Electro-Magneto-Static Optimal Control
with A Posteriori Error Estimates

joint work with

Irwin Yousept

Analysis

- $\emptyset \neq \omega \subset \Omega$: beschränkte Lipschitz Gebiete mit Rändern γ und Γ.
- $\varepsilon, \mu : \Omega \to \mathbb{R}^{3\times 3}$: L^∞-Materialeigenschaften des Mediums

<u>Aufgabenstellung:</u> (klassische Formulierungen)

<u>gegeben:</u> $\widehat{H} \in \mathsf{L}^2(\Omega)$ erwünschtes magnetisches Feld, $\widehat{j} \in \mathsf{L}^2(\omega)$ erwünschte Kontrolle, $\widehat{J} \in \mathsf{L}^2(\Omega)$ externe Stromdichte

<u>gesucht:</u> $\bar{j}$ optimale Stromdichte in ω, Lösung des Optimierungsproblems

$$F(\bar{j}) = \min_{j \in \mathcal{J}} F(j), \qquad F(j) := |H(j) - \widehat{H}|^2_{\mathsf{L}^2_\mu(\Omega)} + \kappa |j - \widehat{j}|^2_{\mathsf{L}^2_\varepsilon(\omega)} \tag{1}$$

unter der Nebenbedingung des magnetostatischen Problems (MSP) für $H = H(j)$

$$\begin{aligned} \operatorname{rot} H &= \varepsilon\pi(\zeta j + J) && \text{in } \Omega,\\ \operatorname{div} \mu H &= 0 && \text{in } \Omega,\\ n \cdot \mu H &= 0 && \text{auf } \Gamma,\\ \mu H &\perp_{\mathsf{L}^2(\Omega)} \mathcal{H}_N(\Omega). \end{aligned}$$

Hier: Neumann Felder $\mathcal{H}_N(\Omega) = \{E \in \mathsf{L}^2(\Omega) : \operatorname{rot} E = 0, \operatorname{div} \mu H = 0, n \cdot \mu H|_\Gamma = 0\}$, ζ Nullerweiterung, π Orthonormalprojektor auf Wertebereich von ε^{-1}-Rotationen, $\mathcal{J}$ nicht-trivialer und abg Teilraum vom $\mathsf{L}^2_\varepsilon(\omega)$, z.B. $\mathcal{J} = \varepsilon^{-1}\mathring{\mathsf{D}}_0(\omega) \subset_\zeta \varepsilon^{-1}\mathring{\mathsf{D}}_0(\Omega)$

Analysis

Finde $\bar{j}$

$$F(\bar{j}) = \min_{j \in \mathcal{J}} F(j), \qquad F(j) = |H(j) - \widehat{H}|^2_{\mathsf{L}^2_\mu(\Omega)} + \kappa\, |j - \widehat{j}|^2_{\mathsf{L}^2_\varepsilon(\omega)}$$

NB (MSP) für $H = H(j)$

$$\begin{aligned} \operatorname{rot} H &= \varepsilon\pi(\zeta j + J) && \text{in } \Omega, \\ \operatorname{div} \mu H &= 0 && \text{in } \Omega, \\ n \cdot \mu H &= 0 && \text{auf } \Gamma, \\ \mu H &\perp_{\mathsf{L}^2(\Omega)} \mathcal{H}_N(\Omega). \end{aligned}$$

Idee: FA-ToolBox ... PDGl (MSP für H)
$\leadsto$ Operator A (AH)
$\leadsto$ reduzierter Operator $\mathcal{A} = \mathrm{A}|_{N(\mathrm{A})^\perp}$ ($\mathcal{A}H$) mit allen NB
(div, RB, Kern)

Analysis

$H \rightsquigarrow x$
alle rechten Seiten $\rightsquigarrow f$
MSP $\rightsquigarrow$ A bzw. $\mathcal{A}$

$$\boxed{\mathrm{A}\,x = f}$$

Dann: Finde $\bar{j}$

$$F(\bar{j}) = \min_{j \in \mathcal{J}} F(j), \qquad F(j) = |H(j) - \widehat{H}|^2_{\mathsf{L}^2_\mu(\Omega)} + \kappa\,|j - \widehat{j}|^2_{\mathsf{L}^2_\varepsilon(\omega)}$$

NB (MSP) für $H = H(j)$

$$\rightsquigarrow \quad H = H(j) = \mathcal{A}^{-1}\pi(\zeta j + J) \qquad \text{(affin linear)}$$

$\Rightarrow$ Finde $\bar{j}$

$$\boxed{F(\bar{j}) = \min_{j \in \mathcal{J}} F(j), \qquad F(j) = |\mathcal{A}^{-1}\pi(\zeta j + J) - \widehat{H}|^2_{\mathsf{L}^2_\mu(\Omega)} + \kappa\,|j - \widehat{j}|^2_{\mathsf{L}^2_\varepsilon(\omega)}}$$

Analysis

$\rightsquigarrow$ Tafel

Analysis

Finde $\bar{j}$

$$F(\bar{j}) = \min_{j \in \mathcal{J}} F(j), \qquad F(j) = |\mathcal{A}^{-1}\pi(\zeta j + J) - \widehat{H}|^2_{\mathsf{L}^2_\mu(\Omega)} + \kappa\,|j - \widehat{j}|^2_{\mathsf{L}^2_\varepsilon(\omega)}$$

... ein wenig FA ... $\Rightarrow$
Die eindeutige Lösung $\bar{j}$ is gegeben durch
die eindeutige Lösung $(\bar{j}, \bar{E}, \bar{H}) \in \mathcal{J} \times D(\mathcal{A}) \times D(\mathcal{A}^*)$ des Systems

$$\begin{aligned} \bar{j} &= \widehat{j} - \frac{1}{\kappa}\pi_\omega \zeta^* \bar{E}, && \text{(Opt. Kontr. durch adj. Zustand } \bar{E}\text{)} \\ \bar{E} &= \mathcal{A}^{-1}(\bar{H} - \widehat{H}), && \text{(Adjungierte Zustandsgleichung)} \\ \bar{H} &= (\mathcal{A}^*)^{-1}\pi(\zeta \bar{j} + J), && \text{(Zustandsgleichung)} \end{aligned}$$

Numerik $\Rightarrow$ variationelle Formulierung: $\bar{j} = \widehat{j} - \frac{1}{\kappa}\pi_\omega \zeta^* \bar{E}$ und $\bar{E} \in D(\mathcal{A})$ löst

$$\forall\, \Phi \in D(\mathcal{A}) \qquad \langle \mathrm{A}\bar{E}, \mathrm{A}\Phi \rangle_{\mathsf{L}^2_\mu(\Omega)} + \frac{1}{\kappa}\langle \pi_\omega \zeta^* \bar{E}, \pi_\omega \zeta^* \Phi \rangle_{\mathsf{L}^2_\varepsilon(\omega)} = \langle \zeta \widehat{j} + J, \Phi \rangle_{\mathsf{L}^2_\varepsilon(\Omega)} - \langle \widehat{H}, \mathrm{A}\Phi \rangle_{\mathsf{L}^2_\mu(\Omega)}$$

Analysis und ein wenig Numerik

variationelle Formulierung: $\bar{j} = \widehat{j} - \frac{1}{\kappa}\pi_\omega\zeta^*\bar{E}$ und $\bar{E} \in D(\mathcal{A})$ löst

$$\forall\, \Phi \in D(\mathcal{A}) \qquad \langle \mathrm{A}\bar{E}, \mathrm{A}\Phi\rangle_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{\kappa}\langle \pi_\omega\zeta^*\bar{E}, \pi_\omega\zeta^*\Phi\rangle_{\mathrm{L}^2_\varepsilon(\omega)} = \langle \zeta\widehat{j} + J, \Phi\rangle_{\mathrm{L}^2_\varepsilon(\Omega)} - \langle \widehat{H}, \mathrm{A}\Phi\rangle_{\mathrm{L}^2_\mu(\Omega)}$$

$$\uparrow$$

degeneriert $\Rightarrow$ Sattelpunktformulierung

$$N(\mathrm{A}) = H_0(\mathrm{rot},\Omega) \supset \nabla H^1_0(\Omega)$$

$\Rightarrow$ Finde

$(\bar{E}, \bar{u}, \bar{v}) \in H_0(\mathrm{rot},\Omega) \times H^1_0(\Omega) \times H^1(\omega)$, so dass für alle
$(\Phi, \varphi, \phi) \in H_0(\mathrm{rot},\Omega) \times H^1_0(\Omega) \times H^1(\omega)$

$$\begin{aligned} \langle \mu^{-1}\,\mathrm{rot}\,\bar{E}, \mathrm{rot}\,\Phi\rangle_{\mathrm{L}^2(\Omega)} &+ \frac{1}{\kappa}\langle \varepsilon\zeta^*\bar{E}, \zeta^*\Phi\rangle_{\mathrm{L}^2(\omega)} \\ &+ \langle \varepsilon\Phi, \nabla\bar{u}\rangle_{\mathrm{L}^2(\Omega)} + \frac{1}{\kappa}\langle \varepsilon\zeta^*\Phi, \nabla\bar{v}\rangle_{\mathrm{L}^2(\omega)} = \langle \varepsilon(\zeta\widehat{j} + J), \Phi\rangle_{\mathrm{L}^2(\Omega)} - \langle \widehat{H}, \mathrm{rot}\,\Phi\rangle_{\mathrm{L}^2(\Omega)}, \\ \langle \varepsilon\bar{E}, \nabla\varphi\rangle_{\mathrm{L}^2(\Omega)} &= 0, \\ \langle \varepsilon\zeta^*\bar{E}, \nabla\phi\rangle_{\mathrm{L}^2(\omega)} + \langle \varepsilon\nabla\bar{v}, \nabla\phi\rangle_{\mathrm{L}^2(\omega)} &= 0 \end{aligned}$$

Analysis und ein wenig Numerik

... nun funktionale a posteriori Fehlerschätzer im Stile von Repin ...

ein wenig Numerik

Gebiete

$$\Omega := (-0.5, 1)^3, \qquad \Xi := (-0.5, 0) \times (-0.5, 0) \times (-0.5, 1),$$
$$\omega := (0, 0.5)^3, \qquad \Theta := (0, 0.5) \times (0, 0.5) \times (-0.5, 1)$$

$\varepsilon := 1$, $\kappa := 1$, stückweise konstante magnetische Permeabilität $\mu := \begin{cases} 10 & \text{in } \Xi, \\ 1 & \text{sonst.} \end{cases}$

Konstruktion analytischer Lösungen:

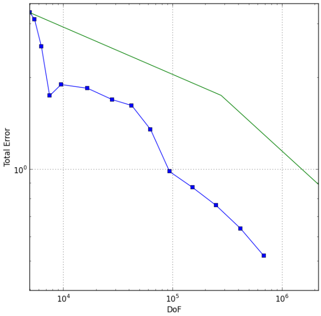
$$E(x) := \frac{\mu^2(x)}{8\pi^2} \sin^2(2\pi x_1) \sin^2(2\pi x_2) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad x \in \Omega$$

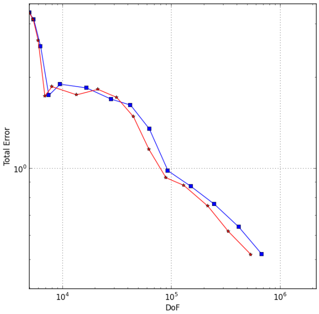
$$\bar{j}(x) := 100 \begin{bmatrix} \sin(2\pi x_1) \cos(2\pi x_2) \\ -\cos(2\pi x_1) \sin(2\pi x_2) \\ 0 \end{bmatrix}, \qquad x \in \omega$$

$$\bar{E} := \chi_{\Omega \setminus \Theta} E, \qquad \bar{H} := \mu^{-1} \operatorname{rot} E, \qquad \widehat{H} := \chi_\Theta \bar{H}, \qquad \widehat{j} := \bar{j}, \qquad J := \operatorname{rot} \bar{H} - \chi_\Omega \bar{j}$$

ein wenig Numerik

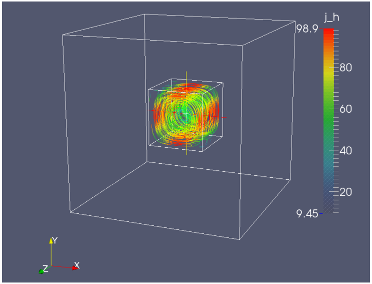
Gesamtfehler bei uniformer Gitterverfeinerung (grüne Linie)
Gesamtfehler bei adaptiver Gitterverf basierend auf dem exakten Fehler (rote Linie)
GF bei adaptiver Gitterverf bas auf unserem a posteriori Fehlerschätzer (blaue Linie)

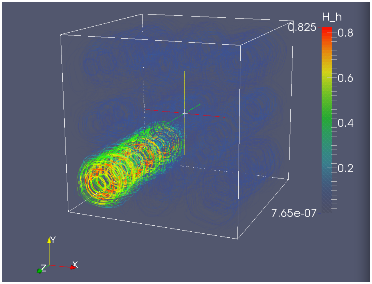




ein wenig Numerik

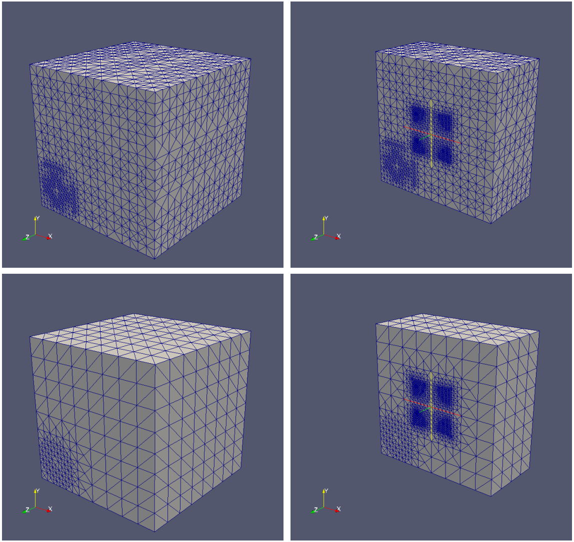
berechnete optimale Kontrolle (linkes Bildchen)
berechnetes optimales Magnetfeld (rechtes Bildchen)
(alles auf dem feinsten adaptiven Gitter)





ein wenig Numerik

adaptives Gitter resul aus unserem Fehlerschätzer (oben) und exaktem Fehler (unten)



COMMERCIALS

. . . the world is full of complexes . . . ;-)

⇒ relaxing at (and you're all invited!)

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13th Workshop on **A**nalysis and **A**dvanced **N**umerical **M**ethods for **P**artial **D**ifferential **E**quations (not only) for **J**unior **S**cientists

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September 28 - October 2 2020
Pitsidia, Crete, Greece

INVITED KEY NOTE SPEAKERS:

Joachim Schöberl (Wien)
Carsten Trunk (Ilmenau)
Ragnar Winther (Oslo)

ORGANIZERS: Johannes Kraus, Dirk Pauly, Sergey Repin, Marcus Waurick