# Static Maxwell Type Problems: Functional A Posteriori Error Estimates and Estimates for the Maxwell Constant in 3D

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# Part 1

## Functional A Posteriori Error Estimates for Static Maxwell Type Problems

# Introduction: (simple) Electro Static Maxwell Problem

- simple: $\varepsilon, \mu = \mathrm{id}$, only hom. bc, only tang. (electric) bc, no mixed bc
- $\Omega \subset \mathbb{R}^3$ bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$
- $E : \Omega \to \mathbb{R}^3$ electric vector field
- $F : \Omega \to \mathbb{R}^3$, $G : \Omega \to \mathbb{R}$ given right hand side data
- $\tau$ tangential trace, i.e., $\tau E = n \times E|_\Gamma : \Gamma \to \mathbb{R}^3$ (for smooth $E$)
- $\perp$ orthogonality w.r.t. $\mathrm{L}^2(\Omega)$-scalar product $\langle E, H\rangle_{\mathrm{L}^2(\Omega)} := \int_\Omega E \cdot H$
- $E \in \mathcal{H}_D(\Omega)$ Dirichlet field $\Leftrightarrow$ $E \in \mathrm{L}^2(\Omega)$ & $\operatorname{rot} E = 0$, $\operatorname{div} E = 0$ and $\tau E = 0$

$$
\begin{aligned}
\operatorname{rot} E &= F && \text{in } \Omega \\
\operatorname{div} E &= G && \text{in } \Omega \\
\tau E &= 0 && \text{on } \Gamma \\
E &\perp \mathcal{H}_D(\Omega)
\end{aligned}
$$

- GOAL: non-conforming estimates for error $e := E - \tilde{E}$, $\tilde{E}$ approximation of $E$

  JUST: $\tilde{E} \in \mathrm{L}^2(\Omega)$

# Introduction: Dirichlet Laplace Problem

- $u \in \mathrm{H}^1(\Omega)$ with
$$\Delta u = \operatorname{div} \nabla u = G, \qquad u|_\Gamma = 0$$
- set $E := \nabla u$
- note: $\operatorname{rot} E = 0$, $n \times E|_\Gamma = \nabla_\Gamma u|_\Gamma = 0$ (since $\mathrm{d}\,\iota^* = \iota^*\,\mathrm{d}$) and $E \perp \mathcal{H}_D(\Omega)$

$$\Rightarrow \quad \begin{aligned} \operatorname{rot} E &= 0 && \text{in } \Omega \\ \operatorname{div} E &= G && \text{in } \Omega \\ \tau E &= 0 && \text{on } \Gamma \\ E &\perp \mathcal{H}_D(\Omega) \end{aligned}$$

- $\Rightarrow$ non-conforming estimates for error $e := \nabla u - \tilde{E}$

  JUST: $\tilde{E} \in \mathrm{L}^2(\Omega)$ approximation of $E = \nabla u$
- $\Rightarrow$ non-conforming estimates of energy norm

# Introduction: rot rot-Problem

- $U \in \mathrm{R}(\Omega)$ with
$$\operatorname{rot}\operatorname{rot} U = F, \qquad n \times \operatorname{rot} U|_\Gamma = 0$$
- set $E := \operatorname{rot} U$
- note: $\operatorname{div} E = 0$, $n \times E|_\Gamma = 0$ and $E \perp \mathcal{H}_D(\Omega)$

$$\Rightarrow \quad \begin{aligned} \operatorname{rot} E &= F && \text{in } \Omega \\ \operatorname{div} E &= 0 && \text{in } \Omega \\ \tau E &= 0 && \text{on } \Gamma \\ E &\perp \mathcal{H}_D(\Omega) \end{aligned}$$

- $\Rightarrow$ non-conforming estimates for error $e := \operatorname{rot} U - \tilde{E}$

  JUST: $\tilde{E} \in \mathrm{L}^2(\Omega)$ approximation of $E = \operatorname{rot} U$
- $\Rightarrow$ non-conforming estimates of energy norm

# Solution Theory

$$\begin{aligned} \operatorname{rot} E &= F && \text{in } \Omega \\ \operatorname{div} E &= G && \text{in } \Omega \\ \tau E &= 0 && \text{on } \Gamma \\ E &\perp \mathcal{H}_D(\Omega) \end{aligned}$$

$$\overset{\text{split}}{\Rightarrow} \quad \begin{aligned} \operatorname{rot} E_{\text{rot}} &= F, & \operatorname{rot} E_{\nabla} &= 0 && \text{in } \Omega \\ \operatorname{div} E_{\text{rot}} &= 0, & \operatorname{div} E_{\nabla} &= G && \text{in } \Omega \\ \tau E_{\text{rot}} &= 0, & \tau E_{\nabla} &= 0 && \text{on } \Gamma \\ E_{\text{rot}} &\perp \mathcal{H}_D(\Omega), & E_{\nabla} &\perp \mathcal{H}_D(\Omega) \end{aligned}$$

(2 electro static Maxwell problems)

introducing scalar and vector potentials $u$ and $U$ solving

$$\begin{aligned} \Delta U = \operatorname{rot}\operatorname{rot} U &= F, & \Delta u = \operatorname{div} \nabla u &= G && \text{in } \Omega \\ \operatorname{div} U &= 0 & & && \text{in } \Omega \\ \tau \operatorname{rot} U &= 0, & u &= 0 && \text{on } \Gamma \end{aligned}$$

variational formulations for $u$ and $U$ (right Hilbert spaces)
$\Rightarrow E_{\text{rot}} := \operatorname{rot} U$ and $E_{\nabla} := \nabla u$ as well as $E := E_{\text{rot}} + E_{\nabla}$

# Method for Error Estimates

$$\begin{aligned} \operatorname{rot} E &= F && \text{in } \Omega \\ \operatorname{div} E &= G && \text{in } \Omega \qquad \text{(electro static Maxwell problem)} \\ \tau E &= 0 && \text{on } \Gamma \\ E &\perp \mathcal{H}_D(\Omega) \end{aligned}$$

- method: funct. a post. error est. for linear second order elliptic problems
  pioneering work of Sergey Repin starting 1990's
  later extended to 'all' linear and non-linear second order elliptic problems
  (Laplace, elastic, parabolic, hyperbolic, even order problems, ...)
- Maxwell system is first order! What to do?
- solution: Helmholtz decomposition $\Rightarrow$ scalar and vector potentials
  $\Rightarrow$ second order methods for the potentials

# Sobolev Spaces

spaces

$$\begin{aligned}
\mathrm{R}(\Omega) &:= \{E \in \mathrm{L}^2(\Omega) \, : \, \operatorname{rot} E \in \mathrm{L}^2(\Omega)\} \\
\mathrm{R}_0(\Omega) &:= \{E \in \mathrm{R}(\Omega) \, : \, \operatorname{rot} E = 0\} \\
\mathring{\mathrm{R}}(\Omega) &:= \{E \in \mathrm{R}(\Omega) \, : \, \tau E = 0\} = \overline{\mathring{\mathrm{C}}^\infty(\Omega)}^{\mathrm{R}(\Omega)} \qquad \text{(Gauß' theorem)} \\
\mathring{\mathrm{R}}_0(\Omega) &:= \mathring{\mathrm{R}}(\Omega) \cap \mathrm{R}_0(\Omega)
\end{aligned}$$

analogously

$$\mathrm{D}(\Omega), \quad \mathrm{D}_0(\Omega), \quad \mathring{\mathrm{D}}(\Omega), \quad \mathring{\mathrm{D}}_0(\Omega)$$

and

$$\begin{aligned}
\mathcal{H}_D(\Omega) &:= \mathring{\mathrm{R}}_0(\Omega) \cap \mathrm{D}_0(\Omega) \quad \text{(finite dimensionl since } \mathring{\mathrm{R}}(\Omega) \cap \mathrm{D}(\Omega) \hookrightarrow \mathrm{L}^2(\Omega) \text{ compact)} \\
&= \{E \in \mathrm{L}^2(\Omega) \, : \, \operatorname{rot} E = 0, \ \operatorname{div} E = 0, \ \tau E = 0\}
\end{aligned}$$

# Results: Upper Bounds for Non-Conforming Approximations

$\tilde{E} \in \mathsf{L}^2(\Omega)$ approximation of $E \Rightarrow$

**Theorem ('11 DP, S.I. Repin)**

*For all* $\tilde{E} \in \mathsf{L}^2(\Omega)$ *and all* $D \in \mathcal{H}_D(\Omega)$

$$|E - \tilde{E} - D|^2_{\mathsf{L}^2(\Omega)} \leq \inf_{X \in \mathring{\mathsf{R}}(\Omega)} M^2_{\mathrm{rot}}(\tilde{E}; X) + \inf_{Y \in \mathsf{D}(\Omega)} M^2_{\mathrm{div}}(\tilde{E}; Y) + |\pi(\tilde{E} - D)|^2_{\mathbb{R}^d}$$

*holds. Here,* $\pi : \mathcal{H}_D(\Omega) \to \mathbb{R}^d$ *(e.g. isomorphic) and*

$$M_{\mathrm{rot}}(\tilde{E}; X) := c_{\mathrm{m}} |F - \operatorname{rot} X|_{\mathsf{L}^2(\Omega)} + |\tilde{E} - X|_{\mathsf{L}^2(\Omega)},$$

$$M_{\mathrm{div}}(\tilde{E}; Y) := c_{\mathrm{p},\circ} |G - \operatorname{div} Y|_{\mathsf{L}^2(\Omega)} + |\tilde{E} - Y|_{\mathsf{L}^2(\Omega)}.$$

only natural and 'well known' continuity constants involved:

| | | | |
|---|---|---|---|
| $c_{\mathrm{p},\circ}$ | Poincaré constant | $\forall\, u \in \mathring{\mathsf{H}}^1(\Omega)$ | $\lvert u\rvert_{\mathsf{L}^2(\Omega)} \leq c_{\mathrm{p},\circ} \lvert\nabla u\rvert_{\mathsf{L}^2(\Omega)}$ |
| $c_{\mathrm{m}}$ | Maxwell constant | $\forall\, E \in \mathbb{H}$ | $\lvert E\rvert_{\mathsf{L}^2(\Omega)} \leq c_{\mathrm{m}} \lvert\operatorname{rot} E\rvert_{\mathsf{L}^2(\Omega)}$ |

here $\mathbb{H} := \mathsf{R}(\Omega) \cap \operatorname{rot} \mathring{\mathsf{R}}(\Omega) = \mathsf{R}(\Omega) \cap \mathring{\mathsf{D}}_0(\Omega) \cap \mathcal{H}_N(\Omega)^\perp$

# Proof: Tools

1. Rellich's selection theorems, i.e., $\overset{\circ}{\mathrm{H}}{}^1(\Omega), \mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$ compact
   $\Rightarrow$ Poincaré estimates, i.e.,
   $$\forall\, u \in \overset{\circ}{\mathrm{H}}{}^1(\Omega) \qquad |u|_{\mathrm{L}^2(\Omega)} \le c_{\mathrm{p},\circ} |\nabla u|_{\mathrm{L}^2(\Omega)}$$
   $$\forall\, u \in \mathrm{H}^1(\Omega) \cap \mathbb{R}^\perp \qquad |u|_{\mathrm{L}^2(\Omega)} \le c_{\mathrm{p}} |\nabla u|_{\mathrm{L}^2(\Omega)}$$
2. Maxwell selection theorems, i.e.,
   $$\overset{\circ}{\mathrm{R}}(\Omega) \cap \mathrm{D}(\Omega), \mathrm{R}(\Omega) \cap \overset{\circ}{\mathrm{D}}(\Omega) \hookrightarrow \mathrm{L}^2(\Omega) \text{ compact}$$
   $\Rightarrow$ Maxwell estimates, i.e.,
   $$\forall\, E \in \tilde{\mathbb{H}} = \overset{\circ}{\mathrm{R}}(\Omega) \cap \mathrm{D}_0(\Omega) \cap \mathcal{H}_D(\Omega)^\perp \qquad |E|_{\mathrm{L}^2(\Omega)} \le c_{\mathrm{m}} |\operatorname{rot} E|_{\mathrm{L}^2(\Omega)}$$
   $$\forall\, E \in \mathbb{H} = \mathrm{R}(\Omega) \cap \overset{\circ}{\mathrm{D}}_0(\Omega) \cap \mathcal{H}_N(\Omega)^\perp \qquad |E|_{\mathrm{L}^2(\Omega)} \le c_{\mathrm{m}} |\operatorname{rot} E|_{\mathrm{L}^2(\Omega)}$$
3. Maxwell selection theorems $\Rightarrow \dim \mathcal{H}_D(\Omega), \dim \mathcal{H}_N(\Omega) < \infty$ (Betti numbers)
4. Helmholtz decompositions (all 6 images are closed in $\mathrm{L}^2(\Omega)$)

$$\mathrm{L}^2(\Omega) = \nabla \overset{\circ}{\mathrm{H}}{}^1(\Omega) \oplus \overbrace{\mathcal{H}_D(\Omega) \oplus \operatorname{rot} \mathrm{R}(\Omega)}^{=\mathrm{D}_0(\Omega)}, \qquad \operatorname{rot} \mathrm{R}(\Omega) = \operatorname{rot} \mathbb{H}$$
$$\mathrm{L}^2(\Omega) = \nabla \mathrm{H}^1(\Omega) \oplus \underbrace{\mathcal{H}_N(\Omega) \oplus \operatorname{rot} \overset{\circ}{\mathrm{R}}(\Omega)}_{=\overset{\circ}{\mathrm{D}}_0(\Omega)}, \qquad \operatorname{rot} \overset{\circ}{\mathrm{R}}(\Omega) = \operatorname{rot} \tilde{\mathbb{H}}$$

# Proof

$e = E - \tilde{E} \in \mathsf{L}^2(\Omega)$

- Helmholtz decomposition of error $\Rightarrow$

$$e = e_\nabla + e_\mathcal{H} + e_{\text{rot}} \in \nabla \overset{\circ}{\mathsf{H}}{}^1(\Omega) \oplus \mathcal{H}_D(\Omega) \oplus \operatorname{rot} \mathbb{H}$$

- $e_\nabla = \nabla u$ with scalar potential $u \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)$
- $e_{\text{rot}} = \operatorname{rot} U$ with vector potential $U \in \mathbb{H}$
- $|e|^2_{\mathsf{L}^2(\Omega)} = |e_\nabla|^2_{\mathsf{L}^2(\Omega)} + |e_\mathcal{H}|^2_{\mathsf{L}^2(\Omega)} + |e_{\text{rot}}|^2_{\mathsf{L}^2(\Omega)}$

# Proof...

1. $e_\nabla = \nabla u,\ u \in \overset{\circ}{\mathrm{H}}{}^1(\Omega)$: $\quad \forall\, \varphi \in \overset{\circ}{\mathrm{H}}{}^1(\Omega) \quad \forall\, Y \in \mathrm{D}(\Omega)$

$$\langle e_\nabla, \nabla\varphi\rangle_{\mathrm{L}^2(\Omega)} = \langle e, \nabla\varphi\rangle_{\mathrm{L}^2(\Omega)} = \langle E, \nabla\varphi\rangle_{\mathrm{L}^2(\Omega)} - \langle \tilde{E}, \nabla\varphi\rangle_{\mathrm{L}^2(\Omega)}$$

$$= \langle \operatorname{div} Y - G, \varphi\rangle_{\mathrm{L}^2(\Omega)} + \langle Y - \tilde{E}, \nabla\varphi\rangle_{\mathrm{L}^2(\Omega)}$$

$$\leq |\operatorname{div} Y - G|_{\mathrm{L}^2(\Omega)} \underbrace{|\varphi|_{\mathrm{L}^2(\Omega)}}_{\leq c_{\mathrm{p},\circ}|\nabla\varphi|_{\mathrm{L}^2(\Omega)}} + |Y - \tilde{E}|_{\mathrm{L}^2(\Omega)}|\nabla\varphi|_{\mathrm{L}^2(\Omega)}$$

$$\varphi := u \quad \Rightarrow \quad |e_\nabla|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{p},\circ}|\operatorname{div} Y - G|_{\mathrm{L}^2(\Omega)} + |Y - \tilde{E}|_{\mathrm{L}^2(\Omega)}$$

2. $e_{\mathrm{rot}} = \operatorname{rot} U,\ U \in \mathbb{H}$: $\quad \forall\, \Phi \in \mathbb{H} \quad \forall\, X \in \overset{\circ}{\mathrm{R}}(\Omega)$

$$\langle e_{\mathrm{rot}}, \operatorname{rot}\Phi\rangle_{\mathrm{L}^2(\Omega)} = \langle e, \operatorname{rot}\Phi\rangle_{\mathrm{L}^2(\Omega)} = \langle E, \operatorname{rot}\Phi\rangle_{\mathrm{L}^2(\Omega)} - \langle \tilde{E}, \operatorname{rot}\Phi\rangle_{\mathrm{L}^2(\Omega)}$$

$$= \langle F - \operatorname{rot} X, \Phi\rangle_{\mathrm{L}^2(\Omega)} + \langle X - \tilde{E}, \operatorname{rot}\Phi\rangle_{\mathrm{L}^2(\Omega)}$$

$$\leq |F - \operatorname{rot} X|_{\mathrm{L}^2(\Omega)} \underbrace{|\Phi|_{\mathrm{L}^2(\Omega)}}_{\leq c_{\mathrm{m}}|\operatorname{rot}\Phi|_{\mathrm{L}^2(\Omega)}} + |X - \tilde{E}|_{\mathrm{L}^2(\Omega)}|\operatorname{rot}\Phi|_{\mathrm{L}^2(\Omega)}$$

$$\Phi := U \quad \Rightarrow \quad |e_{\mathrm{rot}}|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{m}}|F - \operatorname{rot} X|_{\mathrm{L}^2(\Omega)} + |X - \tilde{E}|_{\mathrm{L}^2(\Omega)}$$

3. $e_{\mathcal{H}}$: simple algebraic manipulation $\square$
$\Rightarrow |e_{\mathcal{H}}|_{\mathrm{L}^2(\Omega)} \leq |\pi(\tilde{E} - D)|_{\mathbb{R}^d}$

## Maxwell Constants

# Part 2

# The Maxwell Constants in 3D

# Estimates for the Maxwell Constants

open problem: estimates for the Maxwell constants $c_{\mathtt{m}}$ in 3D or nD?

$$\forall\, E \in \overset{\circ}{\mathrm{R}}(\Omega) \cap \mathrm{D}(\Omega) \cap \mathcal{H}_D(\Omega)^\perp \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m,t}} \big( |\operatorname{rot} E|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)} \big)^{1/2}$$

$$\forall\, E \in \mathrm{R}(\Omega) \cap \overset{\circ}{\mathrm{D}}(\Omega) \cap \mathcal{H}_N(\Omega)^\perp \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m,n}} \big( |\operatorname{rot} E|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)} \big)^{1/2}$$

question:

$$\boxed{? \leq c_{\mathtt{m,t}}, c_{\mathtt{m,n}} \leq ?}$$

in 2D well known

$$c_{\mathtt{p},\circ} \leq c_{\mathtt{m,t}}, c_{\mathtt{m,n}} \leq c_{\mathtt{p}}$$

with Poincaré constants

$$\forall\, u \in \overset{\circ}{\mathrm{H}}{}^1(\Omega) \qquad |u|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{p},\circ} |\nabla u|_{\mathrm{L}^2(\Omega)}$$

$$\forall\, u \in \mathrm{H}^1(\Omega) \cap \mathbb{R}^\perp \qquad |u|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{p}} |\nabla u|_{\mathrm{L}^2(\Omega)}$$

note always

$$c_{\mathtt{p},\circ} = \frac{1}{\sqrt{\lambda_1}} < \frac{1}{\sqrt{\mu_2}} = c_{\mathtt{p}}$$

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# Step 1: Problem Reduction by Helmholtz Decomposition

$$\forall\, E \in \overset{\circ}{\mathrm{R}}(\Omega) \cap \mathrm{D}(\Omega) \cap \mathcal{H}_D(\Omega)^{\perp} \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m},\mathtt{t}} \big( |\operatorname{rot} E|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)} \big)^{1/2}$$

$$\forall\, E \in \mathrm{R}(\Omega) \cap \overset{\circ}{\mathrm{D}}(\Omega) \cap \mathcal{H}_N(\Omega)^{\perp} \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m},\mathtt{n}} \big( |\operatorname{rot} E|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)} \big)^{1/2}$$

Helmholtz decomposition $\Rightarrow$ splits 2 problems into 4 'nicer' problems

$$\forall\, E \in \mathrm{D}(\Omega) \cap \underbrace{\overset{\circ}{\mathrm{R}}_0(\Omega) \cap \mathcal{H}_D(\Omega)^{\perp}}_{=\nabla \overset{\circ}{\mathrm{H}}{}^1(\Omega)} \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m},\mathtt{t},\mathrm{div}} |\operatorname{div} E|_{\mathrm{L}^2(\Omega)}$$

$$\forall\, E \in \overset{\circ}{\mathrm{R}}(\Omega) \cap \underbrace{\mathrm{D}_0(\Omega) \cap \mathcal{H}_D(\Omega)^{\perp}}_{=\operatorname{rot} \mathrm{R}(\Omega)} \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m},\mathtt{t},\mathrm{rot}} |\operatorname{rot} E|_{\mathrm{L}^2(\Omega)}$$

$$\forall\, E \in \overset{\circ}{\mathrm{D}}(\Omega) \cap \underbrace{\mathrm{R}_0(\Omega) \cap \mathcal{H}_N(\Omega)^{\perp}}_{=\nabla \mathrm{H}^1(\Omega)} \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m},\mathtt{n},\mathrm{div}} |\operatorname{div} E|_{\mathrm{L}^2(\Omega)}$$

$$\forall\, E \in \mathrm{R}(\Omega) \cap \underbrace{\overset{\circ}{\mathrm{D}}_0(\Omega) \cap \mathcal{H}_N(\Omega)^{\perp}}_{=\operatorname{rot} \overset{\circ}{\mathrm{R}}(\Omega)} \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m},\mathtt{n},\mathrm{rot}} |\operatorname{rot} E|_{\mathrm{L}^2(\Omega)}$$

## Step 2: First Results

reminder

$$\forall\, E \in \overset{\circ}{\mathrm{R}}(\Omega) \cap \mathrm{D}(\Omega) \cap \mathcal{H}_D(\Omega)^{\perp} \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m},\mathtt{t}} \big( |\operatorname{rot} E|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)} \big)^{1/2}$$

$$\forall\, E \in \mathrm{R}(\Omega) \cap \overset{\circ}{\mathrm{D}}(\Omega) \cap \mathcal{H}_N(\Omega)^{\perp} \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m},\mathtt{n}} \big( |\operatorname{rot} E|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)} \big)^{1/2}$$

$$\forall\, E \in \mathrm{D}(\Omega) \cap \nabla \overset{\circ}{\mathrm{H}}{}^1(\Omega) \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m},\mathtt{t},\mathrm{div}} |\operatorname{div} E|_{\mathrm{L}^2(\Omega)}$$

$$\forall\, E \in \overset{\circ}{\mathrm{R}}(\Omega) \cap \operatorname{rot} \mathrm{R}(\Omega) \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m},\mathtt{t},\mathrm{rot}} |\operatorname{rot} E|_{\mathrm{L}^2(\Omega)}$$

$$\forall\, E \in \overset{\circ}{\mathrm{D}}(\Omega) \cap \nabla \mathrm{H}^1(\Omega) \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m},\mathtt{n},\mathrm{div}} |\operatorname{div} E|_{\mathrm{L}^2(\Omega)}$$

$$\forall\, E \in \mathrm{R}(\Omega) \cap \operatorname{rot} \overset{\circ}{\mathrm{R}}(\Omega) \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m},\mathtt{n},\mathrm{rot}} |\operatorname{rot} E|_{\mathrm{L}^2(\Omega)}$$

trivially: $c_{\mathtt{m},\mathtt{t},\mathrm{rot}}, c_{\mathtt{m},\mathtt{t},\mathrm{div}} \leq c_{\mathtt{m},\mathtt{t}}$ and $c_{\mathtt{m},\mathtt{n},\mathrm{rot}}, c_{\mathtt{m},\mathtt{n},\mathrm{div}} \leq c_{\mathtt{m},\mathtt{n}}$
trivially: $c_{\mathtt{m},\mathtt{t}} \leq \max\{c_{\mathtt{m},\mathtt{t},\mathrm{rot}}, c_{\mathtt{m},\mathtt{t},\mathrm{div}}\}$ and $c_{\mathtt{m},\mathtt{n}} \leq \max\{c_{\mathtt{m},\mathtt{n},\mathrm{rot}}, c_{\mathtt{m},\mathtt{n},\mathrm{div}}\}$ (Helmholtz)
trivially: $c_{\mathtt{m},\mathtt{t}} = \max\{c_{\mathtt{m},\mathtt{t},\mathrm{rot}}, c_{\mathtt{m},\mathtt{t},\mathrm{div}}\}$ and $c_{\mathtt{m},\mathtt{n}} = \max\{c_{\mathtt{m},\mathtt{n},\mathrm{rot}}, c_{\mathtt{m},\mathtt{n},\mathrm{div}}\}$

**Theorem ('13 DP)**

$c_{\mathtt{m},\mathtt{t},\mathrm{div}} = c_{\mathtt{p},\circ} \qquad c_{\mathtt{m},\mathtt{n},\mathrm{div}} = c_{\mathtt{p}} \qquad c_{\mathtt{m},\mathtt{t},\mathrm{rot}} = c_{\mathtt{m},\mathtt{n},\mathrm{rot}}$

remains to estimate $\boxed{c_{\mathtt{m},\mathrm{rot}} := c_{\mathtt{m},\mathtt{t},\mathrm{rot}} = c_{\mathtt{m},\mathtt{n},\mathrm{rot}}}$

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# Step 3: Main Results

$$\forall\, E \in \overset{\circ}{\mathrm{R}}(\Omega) \cap \mathrm{D}(\Omega) \cap \mathcal{H}_D(\Omega)^{\perp} \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m,t}} \big( |\operatorname{rot} E|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)} \big)^{1/2}$$

$$\forall\, E \in \mathrm{R}(\Omega) \cap \overset{\circ}{\mathrm{D}}(\Omega) \cap \mathcal{H}_N(\Omega)^{\perp} \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m,n}} \big( |\operatorname{rot} E|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)} \big)^{1/2}$$

$$\forall\, E \in \mathrm{D}(\Omega) \cap \nabla \overset{\circ}{\mathrm{H}}{}^1(\Omega) \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{p},\circ} |\operatorname{div} E|_{\mathrm{L}^2(\Omega)}$$

$$\forall\, E \in \overset{\circ}{\mathrm{R}}(\Omega) \cap \operatorname{rot} \mathrm{R}(\Omega) \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m,rot}} |\operatorname{rot} E|_{\mathrm{L}^2(\Omega)}$$

$$\forall\, E \in \overset{\circ}{\mathrm{D}}(\Omega) \cap \nabla \mathrm{H}^1(\Omega) \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{p}} |\operatorname{div} E|_{\mathrm{L}^2(\Omega)}$$

$$\forall\, E \in \mathrm{R}(\Omega) \cap \operatorname{rot} \overset{\circ}{\mathrm{R}}(\Omega) \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m,rot}} |\operatorname{rot} E|_{\mathrm{L}^2(\Omega)}$$

trivially: $c_{\mathtt{m,t}} = \max\{c_{\mathtt{m,rot}}, c_{\mathtt{p},\circ}\}$ and $c_{\mathtt{m,n}} = \max\{c_{\mathtt{m,rot}}, c_{\mathtt{p}}\}$
remains to estimate $\boxed{c_{\mathtt{m,rot}}}$

**Theorem ('13 DP)**

*Let $\Omega$ be bounded and convex. Then $c_{\mathtt{m,rot}} \leq c_{\mathtt{p}}$. Moreover, $c_{\mathtt{p},\circ} \leq c_{\mathtt{m,t}} \leq c_{\mathtt{m,n}} = c_{\mathtt{p}}$.*

equivalent formulation for eigenvalues

# Proof of First Theorem

Proof ... by some functional analysis ...

$A : D(A) \subset \mathrm{H}_1 \to \mathrm{H}_2$ lin., dens. def., closed with adjoint $A^* : D(A^*) \subset \mathrm{H}_2 \to \mathrm{H}_1$
assume $D(A) \cap \overline{R(A^*)} \hookrightarrow \mathrm{H}_1$ compact!
define $M := \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$ and note $M^2 := \begin{bmatrix} A^*A & 0 \\ & AA^* \end{bmatrix}$

$\Rightarrow M, M^2, A^*A, AA^*$ self-adjoint with pure point spectra and

$\sigma_{\mathrm{p}}(M) = \pm\sqrt{\sigma_{\mathrm{p}}(A^*A)} = \pm\sqrt{\sigma_{\mathrm{p}}(AA^*)} = \pm\{\kappa_1, \kappa_2, \ldots\}$, $0 \leq \kappa_n \nearrow \infty$

looking at first resp. second eigenvalues $\Rightarrow$

$$\inf_{0 \neq u \in D(A) \cap R(A^*)} \frac{|Au|^2_{\mathrm{H}_2}}{|u|^2_{\mathrm{H}_1}} = \inf_{0 \neq v \in D(A^*) \cap R(A)} \frac{|A^*v|^2_{\mathrm{H}_1}}{|v|^2_{\mathrm{H}_2}}$$

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# Proof of First Theorem...

$A : D(A) \subset \mathrm{H}_1 \to \mathrm{H}_2$ lin., dens. def., cl., adjoint $A^* : D(A^*) \subset \mathrm{H}_2 \to \mathrm{H}_1$
$D(A) \cap R(A^*) \hookrightarrow \mathrm{H}_1$ cpt (note $R(A^*) = N(A)^\perp$ and $R(A^*)$ cl.)

$$\inf_{0 \neq u \in D(A) \cap R(A^*)} \frac{|Au|^2_{\mathrm{H}_2}}{|u|^2_{\mathrm{H}_1}} = \inf_{0 \neq v \in D(A^*) \cap R(A)} \frac{|A^* v|^2_{\mathrm{H}_1}}{|v|^2_{\mathrm{H}_2}}$$

especially

$A := \mathring{\nabla} : \mathring{\mathrm{H}}^1(\Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$, $A^* := -\operatorname{div} : \mathrm{D}(\Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$

$R(A^*) = N(A)^\perp = \{0\}^\perp = \mathrm{L}^2(\Omega)$

$\mathring{\mathrm{H}}^1(\Omega) \cap \mathrm{L}^2(\Omega) = \mathring{\mathrm{H}}^1(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$ cpt by Rellich's selection theorem

$$\frac{1}{c^2_{\mathtt{p},\circ}} = \lambda_1 = \inf_{0 \neq u \in \mathring{\mathrm{H}}^1(\Omega)} \frac{|\nabla u|^2_{\mathrm{L}^2(\Omega)}}{|u|^2_{\mathrm{L}^2(\Omega)}} = \inf_{0 \neq E \in \mathrm{D}(\Omega) \cap \nabla \mathring{\mathrm{H}}^1(\Omega)} \frac{|\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)}}{|E|^2_{\mathrm{L}^2(\Omega)}} = \frac{1}{c^2_{\mathtt{m},\mathtt{t},\operatorname{div}}}$$

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# Proof of First Theorem......

$A : D(A) \subset \mathsf{H}_1 \to \mathsf{H}_2$ lin., dens. def., cl., adjoint $A^* : D(A^*) \subset \mathsf{H}_2 \to \mathsf{H}_1$
$D(A) \cap R(A^*) \hookrightarrow \mathsf{H}_1$ cpt (note $R(A^*) = N(A)^\perp$ and $R(A^*)$ cl.)

$$\inf_{0 \neq u \in D(A) \cap R(A^*)} \frac{|Au|^2_{\mathsf{H}_2}}{|u|^2_{\mathsf{H}_1}} = \inf_{0 \neq v \in D(A^*) \cap R(A)} \frac{|A^* v|^2_{\mathsf{H}_1}}{|v|^2_{\mathsf{H}_2}}$$

especially

$A := \nabla : \mathsf{H}^1(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega)$, $A^* := -\mathring{\operatorname{div}} : \mathring{\mathsf{D}}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega)$

$R(A^*) = N(A)^\perp = \mathbb{R}^\perp$

$\mathsf{H}^1(\Omega) \cap \mathbb{R}^\perp \subset \mathsf{H}^1(\Omega) \hookrightarrow \mathsf{L}^2(\Omega)$ cpt by Rellich's selection theorem

$$\frac{1}{c_{\mathtt{p}}^2} = \mu_2 = \inf_{0 \neq u \in \mathsf{H}^1(\Omega) \cap \mathbb{R}^\perp} \frac{|\nabla u|^2_{\mathsf{L}^2(\Omega)}}{|u|^2_{\mathsf{L}^2(\Omega)}} = \inf_{0 \neq E \in \mathring{\mathsf{D}}(\Omega) \cap \nabla \mathsf{H}^1(\Omega)} \frac{|\operatorname{div} E|^2_{\mathsf{L}^2(\Omega)}}{|E|^2_{\mathsf{L}^2(\Omega)}} = \frac{1}{c_{\mathtt{m,n,div}}^2}$$

# Proof of First Theorem.........

$A : D(A) \subset \mathrm{H}_1 \to \mathrm{H}_2$ lin., dens. def., cl., adjoint $A^* : D(A^*) \subset \mathrm{H}_2 \to \mathrm{H}_1$
$D(A) \cap R(A^*) \hookrightarrow \mathrm{H}_1$ cpt (note $R(A^*) = N(A)^\perp$ and $R(A^*)$ cl.)

$$\inf_{0 \neq u \in D(A) \cap R(A^*)} \frac{|Au|^2_{\mathrm{H}_2}}{|u|^2_{\mathrm{H}_1}} = \inf_{0 \neq v \in D(A^*) \cap R(A)} \frac{|A^* v|^2_{\mathrm{H}_1}}{|v|^2_{\mathrm{H}_2}}$$

especially

$A := \mathring{\mathrm{rot}} : \mathring{\mathrm{R}}(\Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$, $A^* := \mathrm{rot} : \mathrm{R}(\Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$

$R(A^*) = \mathrm{rot}\,\mathrm{R}(\Omega)$

$\mathring{\mathrm{R}}(\Omega) \cap \mathrm{rot}\,\mathrm{R}(\Omega) \subset \mathring{\mathrm{R}}(\Omega) \cap \mathrm{D}(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$ cpt by MCP

$$\frac{1}{c^2_{\mathtt{m},\mathtt{t},\mathrm{rot}}} = \kappa_2 = \inf_{0 \neq E \in \mathring{\mathrm{R}}(\Omega) \cap \mathrm{rot}\,\mathrm{R}(\Omega)} \frac{|\,\mathrm{rot}\,E|^2_{\mathrm{L}^2(\Omega)}}{|E|^2_{\mathrm{L}^2(\Omega)}} = \inf_{0 \neq E \in \mathrm{R}(\Omega) \cap \mathrm{rot}\,\mathring{\mathrm{R}}(\Omega)} \frac{|\,\mathrm{rot}\,E|^2_{\mathrm{L}^2(\Omega)}}{|E|^2_{\mathrm{L}^2(\Omega)}} = \frac{1}{c^2_{\mathtt{m},\mathtt{n},\mathrm{rot}}}$$

□

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# Proof of Second (Main) Theorem

Proof

crucial estimate for convex domains

**Lemma ('98 C. Amrouche, C. Bernardi, M. Dauge, V. Girault)**

*$\Omega \subset \mathbb{R}$ bd and convex. Then $E \in \overset{\circ}{\mathrm{R}}(\Omega) \cap \mathrm{D}(\Omega), \mathrm{R}(\Omega) \cap \overset{\circ}{\mathrm{D}}(\Omega) \subset \mathrm{H}^1(\Omega)$ continuous and*

$$|\nabla E|^2_{\mathrm{L}^2(\Omega)} \leq 1 \cdot \big(|\operatorname{rot} E|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)}\big).$$

related, earlier, partial results by M. Costabel ('91), J. Saranen ('82), P. Grisvard ('72, '85), R. Leis ('68), J. Kadlec ('64)

$\Rightarrow$ pick $E \in \mathrm{R}(\Omega) \cap \operatorname{rot} \overset{\circ}{\mathrm{R}}(\Omega) = \mathrm{R}(\Omega) \cap \overset{\circ}{\mathrm{D}}_0(\Omega) \cap \mathcal{H}_N(\Omega)^\perp = \mathrm{R}(\Omega) \cap \overset{\circ}{\mathrm{D}}_0(\Omega)$ ($\Omega$ convex)

$\Rightarrow \langle E, a\rangle_{\mathrm{L}^2(\Omega)} = \langle \operatorname{rot} H, a\rangle_{\mathrm{L}^2(\Omega)} = 0$ for all $a \in \mathbb{R}^3$ since $H \in \overset{\circ}{\mathrm{R}}(\Omega)$

$\Rightarrow E \in \mathrm{H}^1(\Omega) \cap (\mathbb{R}^3)^\perp$

$$\Rightarrow |E|_{\mathrm{L}^2(\Omega)} \overset{\text{Poincare}}{\leq} c_{\mathrm{p}} |\nabla E|_{\mathrm{L}^2(\Omega)} \overset{\text{Lemma}}{\leq} 1 \cdot c_{\mathrm{p}} |\operatorname{rot} E|_{\mathrm{L}^2(\Omega)}$$

$\Rightarrow c_{\mathrm{m},\mathrm{rot}} \leq c_{\mathrm{p}}$

very simple! □

# Last Slide!

more results:

Blagodarya / Thank You

Functional A Posteriori Error Estimates:

- lower bounds
- usual features of Sergey's estimates:
  sharpness, only natural constants, simple implementation, ...
- $\Omega$ exterior domain, polynomially weighted estimates
- differential forms, $\Omega \subset \mathbb{R}^N$, $\Omega$ Riemannian manifold
- hyperbolic problems, full time-dependent Maxwell system, eddy current, ...
- with $\varepsilon$, $\mu$ and inhomogeneous bc
- diffusion problem, elasticity, ...
- magnetic problem!!!!!
- mixed boundary conditions

Maxwell Constants:

- also with $\varepsilon$, $\mu$
- also $\Omega \subset \mathbb{R}^N$ with differential forms
- also on non-convex polygons (not too pointy)

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