

On some Hilbert complexes,  
related compact embeddings, the div-curl lemma, ... and more,  
... even some numerics ;-)

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*Open-Minded* :-)

Kalvåg Workshop

Analysis and Computation of Coupled Problems

Knutholmen Hotel, Kalvåg, Norway, November 2019

Host: Jan Nordbotten, U Bergen

# Solving PDEs with Hilbert Complexes

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## Solving PDEs with Hilbert Complexes

### Some Hilbert Complexes



# classical de Rham complex in 3D ( $\nabla$ -rot-div-complex)

$\Omega \subset \mathbb{R}^3$  bounded weak Lipschitz domain,  $\partial\Omega = \Gamma = \overline{\Gamma_t \dot{\cup} \Gamma_n}$

(electro-magnetics, Maxwell's equations)

$$\{0\} \begin{array}{c} \xrightarrow{\iota_{\{0\}}} \\ \xleftarrow{\pi_{\{0\}}} \end{array} L^2 \begin{array}{c} \xrightarrow{\nabla} \\ \xleftarrow{-\operatorname{div}} \end{array} L^2 \begin{array}{c} \xrightarrow{\operatorname{rot}} \\ \xleftarrow{\operatorname{rot}} \end{array} L^2 \begin{array}{c} \xrightarrow{\operatorname{div}} \\ \xleftarrow{-\nabla} \end{array} L^2 \begin{array}{c} \xrightarrow{\pi_{\mathbb{R}}} \\ \xleftarrow{\iota_{\mathbb{R}}} \end{array} \mathbb{R}$$

mixed boundary conditions and inhomogeneous and anisotropic media

$$\{0\} \text{ or } \mathbb{R} \begin{array}{c} \xrightarrow{\iota} \\ \xleftarrow{\pi} \end{array} L^2 \begin{array}{c} \xrightarrow{\nabla_{\Gamma_t}} \\ \xleftarrow{-\operatorname{div}_{\Gamma_n} \varepsilon} \end{array} L^2_{\varepsilon} \begin{array}{c} \xrightarrow{\mu^{-1} \operatorname{rot}_{\Gamma_t}} \\ \xleftarrow{\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}} \end{array} L^2_{\mu} \begin{array}{c} \xrightarrow{\operatorname{div}_{\Gamma_t} \mu} \\ \xleftarrow{-\nabla_{\Gamma_n}} \end{array} L^2 \begin{array}{c} \xrightarrow{\pi} \\ \xleftarrow{\iota} \end{array} \mathbb{R} \text{ or } \{0\}$$

this talk:

- $\varepsilon = \mu = 1$  (= id)
- no mixed boundary conditions, i.e.,  $\Gamma_t = \Gamma$  and  $\Gamma_n = \emptyset$

for all appearing complexes (Maxwell, elasticity, biharmonic, general relativity, ...)





# de Rham complex in ND or on Riemannian manifolds (d-complex)

$\Omega \subset \mathbb{R}^N$  bd w. Lip. dom. or  $\Omega$  Riemannian manifold with cpt cl. and Lip. boundary  $\Gamma$   
 (generalized Maxwell equations the mother of all complexes )

$$\{0\} \begin{array}{c} \xrightarrow{\iota_{\{0\}}} \\ \xrightarrow{\pi_{\{0\}}} \end{array} \mathbb{L}^{2,0} \begin{array}{c} \xrightarrow{\mathring{d}} \\ \xrightarrow{-\delta} \end{array} \mathbb{L}^{2,1} \begin{array}{c} \xrightarrow{\mathring{d}} \\ \xrightarrow{-\delta} \end{array} \dots \boxed{\mathbb{L}^{2,q} \begin{array}{c} \xrightarrow{\mathring{d}} \\ \xrightarrow{-\delta} \end{array} \mathbb{L}^{2,q+1}} \dots \mathbb{L}^{2,N-1} \begin{array}{c} \xrightarrow{\mathring{d}} \\ \xrightarrow{-\delta} \end{array} \mathbb{L}^{2,N} \begin{array}{c} \xrightarrow{\pi_{\mathbb{R}}} \\ \xrightarrow{\iota_{\mathbb{R}}} \end{array} \mathbb{R}$$



# elasticity complex in 3D (sym $\nabla$ -Rot Rot $^T$ -Div $_S$ -complex)

$\Omega \subset \mathbb{R}^3$  bounded strong Lipschitz domain

$$\begin{array}{ccccccc}
 \{0\} & \begin{array}{c} \iota_{\{0\}} \\ \rightleftharpoons \\ \pi_{\{0\}} \end{array} & L^2 & \begin{array}{c} \text{sym } \nabla \\ \rightleftharpoons \\ -\text{Div}_S \end{array} & L^2_S & \begin{array}{c} \text{Rot Rot}_S^T \\ \rightleftharpoons \\ \text{Rot Rot}_S^T \end{array} & L^2_S & \begin{array}{c} \text{Div}_S \\ \rightleftharpoons \\ -\text{sym } \nabla \end{array} & L^2 & \begin{array}{c} \pi_{\text{RM}} \\ \rightleftharpoons \\ \iota_{\text{RM}} \end{array} & \text{RM}
 \end{array}$$

biharmonic / general relativity complex in 3D ( $\nabla\nabla$ -Rot<sub>S</sub>-Div<sub>T</sub>-complex)

$\Omega \subset \mathbb{R}^3$  bounded strong Lipschitz domain

$$\begin{array}{ccccccc}
 \{0\} & \begin{array}{c} \hookrightarrow_{\iota_{\{0\}}} \\ \leftarrow_{\pi_{\{0\}}} \end{array} & L^2 & \begin{array}{c} \nabla\nabla \\ \leftarrow_{\text{div Div}_S} \end{array} & L^2_S & \begin{array}{c} \mathring{\text{Rot}}_S \\ \leftarrow_{\text{sym Rot}_T} \end{array} & L^2_T & \begin{array}{c} \mathring{\text{Div}}_T \\ \leftarrow_{-\text{dev } \nabla} \end{array} & L^2 & \begin{array}{c} \pi_{RT} \\ \leftarrow_{\iota_{RT}} \end{array} & RT
 \end{array}$$



# general complex

$$\begin{array}{ll}
 A_0 : D(A_0) \subset H_0 \rightarrow H_1, & A_1 : D(A_1) \subset H_1 \rightarrow H_2 \\
 A_0^* : D(A_0^*) \subset H_1 \rightarrow H_0, & A_1^* : D(A_1^*) \subset H_2 \rightarrow H_1
 \end{array} \quad (\text{lddc})$$

general complex property  $\boxed{A_1 A_0 = 0}$ ,

i.e.,  $R(A_0) \subset N(A_1)$  and/or eq  $R(A_1^*) \subset N(A_0^*)$

$$\dots \begin{array}{c} \dots \\ \rightleftarrows \\ \dots \end{array} H_0 \begin{array}{c} A_0 \\ \rightleftarrows \\ A_0^* \end{array} H_1 \begin{array}{c} A_1 \\ \rightleftarrows \\ A_1^* \end{array} H_2 \begin{array}{c} \dots \\ \rightleftarrows \\ \dots \end{array} \dots$$



## Solving PDEs with Hilbert Complexes

FA-ToolBox



# general observations

$$Ax = f$$



# general observations

$$Ax = f$$

## general theory

- solution theory
- closed ranges
- Friedrichs/Poincaré estimates and constants
- Helmholtz/Hodge/Weyl decompositions
- compact embeddings
- continuous and compact inverse operators
- regular potentials and regular decompositions
- variational formulations
- generalized div-curl-lemma
- functional a posteriori error estimates
- ...

idea: solve problem with general and simple linear functional analysis

⇒ functional analysis toolbox (FA-ToolBox) ...



# general observations

$$Ax = f$$

let's say  $A : D(A) \subset H_0 \rightarrow H_1$  linear and  $H_0, H_1$  Hilbert spaces

question: How to solve?





# general observations

$$Ax = f$$

$A : D(A) \subset H_0 \rightarrow H_1$  linear

solution theory in the sense of Hadamard

- existence  $\Leftrightarrow f \in R(A)$
- uniqueness  $\Leftrightarrow A$  inj  $\Leftrightarrow N(A) = \{0\} \Leftrightarrow A^{-1}$  exists
- cont dep on  $f \Leftrightarrow A^{-1}$  cont

$\Rightarrow x = A^{-1}f \in D(A)$  and cont estimate (Friedrichs/Poincaré type estimate)

$$|x|_{H_0} = |A^{-1}f|_{H_0} \leq c_A |f|_{H_1} = c_A |Ax|_{H_1}$$

$\Rightarrow$  best constant  $c_A = |A^{-1}|_{R(A), H_0} \quad |A^{-1}|_{R(A), D(A)} = (c_A^2 + 1)^{1/2}$



# general observations

$$A : D(A) \subset H_0 \rightarrow H_1$$

$$A^* : D(A^*) \subset H_1 \rightarrow H_0 \text{ Hilbert space adjoint}$$

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$H_1 = \overline{R(A)} \oplus N(A^*) \quad H_0 = N(A) \oplus \overline{R(A^*)}$$

$$Ax = f$$

solution theory in the sense of Hadamard

- existence  $\Leftrightarrow f \in R(A) = N(A^*)^\perp$  (Fredholm alternative)
- uniqueness  $\Leftrightarrow A \text{ inj} \Leftrightarrow N(A) = \{0\} \Leftrightarrow A^{-1} \text{ exists}$
- cont dep on  $f \Leftrightarrow A^{-1} \text{ cont} \Leftrightarrow R(A) \text{ cl} \quad (\text{cl range theo})$

fund range cond:  $R(A) = \overline{R(A)}$  closed (must hold  $\rightsquigarrow$  right setting!)

kernel cond:  $N(A) = \{0\}$  (fails in gen  $\rightsquigarrow$  proj onto  $N(A)^\perp = \overline{R(A^*)} = R(A^*)$ )



# general observations

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$H_1 = \overline{R(A)} \oplus N(A^*) \quad H_0 = N(A) \oplus \overline{R(A^*)}$$

observations

- time-dependent problems are simple

in gen  $A : D(A) \subset H \rightarrow H$ ,  $A = \partial_t \Lambda + T$  (gen  $T$  skw-sa, or at least  $\operatorname{Re} T \geq 0$ )

$$N(A) = \{0\} \quad N(A^*) = \{0\} \quad R(A) \text{ (cl)} = N(A^*)^\perp = H$$

- time-harmonic problems are more complicated

in gen  $A : D(A) \subset H \rightarrow H$ ,  $A = -\omega \Lambda + T$

$$N(A), N(A^*) \text{ (fin dim)} \quad R(A) \text{ (cl, fin co-dim)} = N(A^*)^\perp$$

(Fredholm alternative)

- stat problems are most complicated

in gen  $A : D(A) \subset H_0 \rightarrow H_1$ ,  $A = 0 + T$

$$\dim N(A) = \dim N(A^*) = \infty \text{ (possibly)} \quad R(A) \text{ (cl, infin co-dim)} = N(A^*)^\perp$$



# FA-ToolBox for linear (first order) problems/systems

$$Ax = f$$

general theory

- solution theory
- closed ranges
- Friedrichs/Poincaré estimates and constants
- Helmholtz/Hodge/Weyl decompositions
- compact embeddings
- continuous and compact inverse operators
- regular potentials and regular decompositions
- variational formulations
- generalized div-curl-lemma
- functional a posteriori error estimates
- ...

idea: solve problem with general and simple linear functional analysis  
( $\Rightarrow$  FA-ToolBox) ...

literature: many parts probably very well known for ages, but hard to find ...

Friedrichs, Weyl, Hörmander, Fredholm, von Neumann, Riesz, Banach, ... ?

Why not rediscover and extend/modify for our purposes?



# 1st fundamental observations

$A : D(A) \subset H_0 \rightarrow H_1$  lddc,  $A^* : D(A^*) \subset H_1 \rightarrow H_0$  Hilbert space adjoint

$(A, A^*)$  dual pair as  $(A^*)^* = \overline{A} = A$

$A, A^*$  may not be inj

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$H_1 = N(A^*) \oplus \overline{R(A)} \quad H_0 = N(A) \oplus \overline{R(A^*)}$$

reduced operators restr to  $N(A)^\perp$  and  $N(A^*)^\perp$

$$\mathcal{A} := A|_{N(A)^\perp} = A|_{\overline{R(A^*)}}$$

$$\mathcal{A}^* := A^*|_{N(A^*)^\perp} = A^*|_{\overline{R(A)}}$$

$\mathcal{A}, \mathcal{A}^*$  inj  $\Rightarrow \mathcal{A}^{-1}, (\mathcal{A}^*)^{-1}$  ex



# 1st fundamental observations

$A : D(A) \subset H_0 \rightarrow H_1$ ,  $A^* : D(A^*) \subset H_1 \rightarrow H_0$  lddc  $(A, A^*)$  dual pair

$$H_1 = N(A^*) \oplus \overline{R(A)} \quad H_0 = N(A) \oplus \overline{R(A^*)}$$

more precisely

$$\mathcal{A} := A|_{\overline{R(A^*)}} : D(\mathcal{A}) \subset \overline{R(A^*)} \rightarrow \overline{R(A)}, \quad D(\mathcal{A}) := D(A) \cap N(A)^\perp = D(A) \cap \overline{R(A^*)}$$

$$\mathcal{A}^* := A^*|_{\overline{R(A)}} : D(\mathcal{A}^*) \subset \overline{R(A)} \rightarrow \overline{R(A^*)}, \quad D(\mathcal{A}^*) := D(A^*) \cap N(A^*)^\perp = D(A^*) \cap \overline{R(A)}$$

$(\mathcal{A}, \mathcal{A}^*)$  dual pair and  $\mathcal{A}, \mathcal{A}^*$  inj  $\Rightarrow$

inverse ops exist (and bij)

$$\mathcal{A}^{-1} : R(A) \rightarrow D(\mathcal{A}) \quad (\mathcal{A}^*)^{-1} : R(A^*) \rightarrow D(\mathcal{A}^*)$$

refined decompositions

$$D(A) = N(A) \oplus D(\mathcal{A}) \quad D(A^*) = N(A^*) \oplus D(\mathcal{A}^*)$$

$\Rightarrow$

$$R(A) = R(\mathcal{A}) \quad R(A^*) = R(\mathcal{A}^*)$$



# 1st fundamental observations

closed range theorem & closed graph theorem  $\Rightarrow$

## Lemma (Friedrichs-Poincaré type est/cl range/cont inv)

*The following assertions are equivalent:*

- (i)  $\exists c_A \in (0, \infty) \quad \forall x \in D(\mathcal{A}) \quad |x|_{H_0} \leq c_A |Ax|_{H_1}$
- (i\*)  $\exists c_{A^*} \in (0, \infty) \quad \forall y \in D(\mathcal{A}^*) \quad |y|_{H_1} \leq c_{A^*} |A^*y|_{H_0}$
- (ii)  $R(A) = R(\mathcal{A})$  is closed in  $H_1$ .
- (ii\*)  $R(A^*) = R(\mathcal{A}^*)$  is closed in  $H_0$ .
- (iii)  $\mathcal{A}^{-1} : R(A) \rightarrow D(\mathcal{A})$  is continuous and bijective.
- (iii\*)  $(\mathcal{A}^*)^{-1} : R(A^*) \rightarrow D(\mathcal{A}^*)$  is continuous and bijective.

Note: trivial equivalence of (i) or (i\*) to inf-sup condition

In case that one of the latter assertions is true, e.g., (ii),  $R(A)$  is closed, we have

$$\begin{aligned}
 H_0 &= N(A) \oplus R(A^*) & H_1 &= N(A^*) \oplus R(A) \\
 D(A) &= N(A) \oplus D(\mathcal{A}) & D(A^*) &= N(A^*) \oplus D(\mathcal{A}^*) \\
 D(\mathcal{A}) &= D(A) \cap R(A^*) & D(\mathcal{A}^*) &= D(A^*) \cap R(A)
 \end{aligned}$$

and  $\mathcal{A} : D(\mathcal{A}) \subset R(A^*) \rightarrow R(A)$ ,  $\mathcal{A}^* : D(\mathcal{A}^*) \subset R(A) \rightarrow R(A^*)$ .



# 1st fundamental observations

recall

$$(i) \quad \exists c_A \in (0, \infty) \quad \forall x \in D(\mathcal{A}) \quad |x|_{H_0} \leq c_A |Ax|_{H_1}$$

$$(i^*) \quad \exists c_{A^*} \in (0, \infty) \quad \forall y \in D(\mathcal{A}^*) \quad |y|_{H_1} \leq c_{A^*} |A^*y|_{H_0}$$

'best' consts in (i) and (i\*) equal norms of the inv ops and Rayleigh quotients

$$c_A = |\mathcal{A}^{-1}|_{R(A), R(A^*)}$$

$$c_{A^*} = |(\mathcal{A}^*)^{-1}|_{R(A^*), R(A)}$$

$$\frac{1}{c_A} = \inf_{0 \neq x \in D(\mathcal{A})} \frac{|Ax|_{H_1}}{|x|_{H_0}}$$

$$\frac{1}{c_{A^*}} = \inf_{0 \neq y \in D(\mathcal{A}^*)} \frac{|A^*y|_{H_0}}{|y|_{H_1}}$$

Lemma (Friedrichs-Poincaré type const)

$$c_A = c_{A^*}$$





# 1st fundamental observations

## Lemma (cpt emb/cpt inv)

The following assertions are equivalent:

- (i)  $D(\mathcal{A}) \hookrightarrow H_0$  is compact.
- (i\*)  $D(\mathcal{A}^*) \hookrightarrow H_1$  is compact.
- (ii)  $\mathcal{A}^{-1} : R(\mathcal{A}) \rightarrow R(\mathcal{A}^*)$  is compact.
- (ii\*)  $(\mathcal{A}^*)^{-1} : R(\mathcal{A}^*) \rightarrow R(\mathcal{A})$  is compact.

## Lemma (Friedrichs-Poincaré type est/cl range/cont inv)

$\Downarrow$   $D(\mathcal{A}) \hookrightarrow H_0$  compact

- (i)  $\exists c_{\mathcal{A}} \in (0, \infty) \quad \forall x \in D(\mathcal{A}) \quad |x|_{H_0} \leq c_{\mathcal{A}} |Ax|_{H_1}$
- (i\*)  $\exists c_{\mathcal{A}^*} \in (0, \infty) \quad \forall y \in D(\mathcal{A}^*) \quad |y|_{H_1} \leq c_{\mathcal{A}^*} |A^*y|_{H_0}$
- (ii)  $R(\mathcal{A}) = R(\mathcal{A})$  is closed in  $H_1$ .
- (ii\*)  $R(\mathcal{A}^*) = R(\mathcal{A}^*)$  is closed in  $H_0$ .
- (iii)  $\mathcal{A}^{-1} : R(\mathcal{A}) \rightarrow D(\mathcal{A})$  is continuous and bijective.
- (iii\*)  $(\mathcal{A}^*)^{-1} : R(\mathcal{A}^*) \rightarrow D(\mathcal{A}^*)$  is continuous and bijective.

(i)-(iii\*) equi & the resp Helm deco hold &  $|\mathcal{A}^{-1}| = c_{\mathcal{A}} = c_{\mathcal{A}^*} = |(\mathcal{A}^*)^{-1}|$



## 2nd fundamental observations

So far no complex...

$$A_0 : D(A_0) \subset H_0 \rightarrow H_1, \quad A_1 : D(A_1) \subset H_1 \rightarrow H_2 \text{ (lddc)}$$

$$A_0^* : D(A_0^*) \subset H_1 \rightarrow H_0, \quad A_1^* : D(A_1^*) \subset H_2 \rightarrow H_1 \text{ (lddc)}$$

general complex ( $A_1 A_0 = 0$ , i.e.,  $R(A_0) \subset N(A_1)$  and  $R(A_1^*) \subset N(A_0^*)$ )

$$\boxed{\begin{array}{ccccccc} \dots & \begin{array}{c} \dots \\ \xrightarrow{\quad} \\ \dots \end{array} & H_0 & \begin{array}{c} A_0 \\ \xrightarrow{\quad} \\ A_0^* \end{array} & H_1 & \begin{array}{c} A_1 \\ \xrightarrow{\quad} \\ A_1^* \end{array} & H_2 & \begin{array}{c} \dots \\ \xrightarrow{\quad} \\ \dots \end{array} & \dots \end{array}}$$

recall Helmholtz deco

$$H_1 = \overline{R(A_0)} \oplus N(A_0^*)$$

$$\cap \quad \cup \quad \Rightarrow \text{(e.g.) } N(A_1) = \overline{R(A_0)} \oplus \underbrace{(N(A_1) \cap N(A_0^*))}_{=: N_1}$$

$$= N(A_1) \oplus \overline{R(A_1^*)}$$

$\Rightarrow$  refined Helmholtz deco

$$H_1 = \overline{R(A_0)} \oplus N_1 \oplus \overline{R(A_1^*)}$$



## 2nd fundamental observations

recall

$$D(A_1) = D(\mathcal{A}_1) \cap \overline{R(A_1^*)} \quad R(A_1) = R(\mathcal{A}_1) \quad R(A_1^*) = R(\mathcal{A}_1^*)$$

$$D(A_0^*) = D(\mathcal{A}_0^*) \cap \overline{R(A_0)} \quad R(A_0^*) = R(\mathcal{A}_0^*) \quad R(A_0) = R(\mathcal{A}_0)$$

cohomology group  $N_1 = N(A_1) \cap N(A_0^*)$

### Lemma (Helmholtz deco I)

$$H_1 = \overline{R(A_0)} \oplus N(A_0^*)$$

$$H_1 = \overline{R(A_1^*)} \oplus N(A_1)$$

$$D(A_0^*) = D(\mathcal{A}_0^*) \oplus N(A_0^*)$$

$$D(A_1) = D(\mathcal{A}_1) \oplus N(A_1)$$

$$N(A_1) = D(\mathcal{A}_0^*) \oplus N_1$$

$$N(A_0^*) = D(\mathcal{A}_1) \oplus N_1$$

$$D(A_1) = \overline{R(A_0)} \oplus (D(A_1) \cap N(A_0^*)) \quad D(A_0^*) = \overline{R(A_1^*)} \oplus (D(A_0^*) \cap N(A_1))$$

### Lemma (Helmholtz deco II)

$$H_1 = \overline{R(A_0)} \oplus N_1 \oplus \overline{R(A_1^*)}$$

$$D(A_1) = \overline{R(A_0)} \oplus N_1 \oplus D(\mathcal{A}_1)$$

$$D(A_0^*) = D(\mathcal{A}_0^*) \oplus N_1 \oplus \overline{R(A_1^*)}$$

$$D(A_1) \cap D(A_0^*) = D(\mathcal{A}_0^*) \oplus N_1 \oplus D(\mathcal{A}_1)$$



## 2nd fundamental observations

$$N_1 = N(\mathcal{A}_1) \cap N(\mathcal{A}_0^*) \quad D(\mathcal{A}_1) = D(\mathcal{A}_1) \cap \overline{R(\mathcal{A}_1^*)} \quad D(\mathcal{A}_0^*) = D(\mathcal{A}_0^*) \cap \overline{R(\mathcal{A}_0)}$$

### Lemma (cpt emb II)

The following assertions are equivalent:

- (i)  $D(\mathcal{A}_0) \overset{c}{\leftrightarrow} H_0$ ,  $D(\mathcal{A}_1) \overset{c}{\leftrightarrow} H_1$ , and  $N_1 \overset{c}{\leftrightarrow} H_1$  are compact.
- (ii)  $D(\mathcal{A}_1) \cap D(\mathcal{A}_0^*) \overset{c}{\leftrightarrow} H_1$  is compact.

In this case  $N_1 < \infty$ .

### Theorem (FA-ToolBox I)

⇔  $D(\mathcal{A}_1) \cap D(\mathcal{A}_0^*) \overset{c}{\leftrightarrow} H_1$  compact

- (i) all emb cpt, i.e.,  $D(\mathcal{A}_0) \overset{c}{\leftrightarrow} H_0$ ,  $D(\mathcal{A}_1) \overset{c}{\leftrightarrow} H_1$ ,  $D(\mathcal{A}_0^*) \overset{c}{\leftrightarrow} H_1$ ,  $D(\mathcal{A}_1^*) \overset{c}{\leftrightarrow} H_2$  cpt
- (ii) cohomology group  $N_1$  finite dim
- (iii) all ranges closed, i.e.,  $R(\mathcal{A}_0)$ ,  $R(\mathcal{A}_0^*)$ ,  $R(\mathcal{A}_1)$ ,  $R(\mathcal{A}_1^*)$  cl
- (iv) all Friedrichs-Poincaré type est hold
- (v) all Hodge-Helmholtz-Weyl type deco I & II hold with closed ranges



## 2nd fundamental observations

$$\text{complex} \quad \cdots \quad \begin{array}{c} \cdots \\ \xrightarrow{\quad} \\ \cdots \end{array} \quad H_0 \quad \begin{array}{c} A_0 \\ \xrightarrow{\quad} \\ A_0^* \end{array} \quad H_1 \quad \begin{array}{c} A_1 \\ \xrightarrow{\quad} \\ A_1^* \end{array} \quad H_2 \quad \begin{array}{c} \cdots \\ \xrightarrow{\quad} \\ \cdots \end{array} \quad \cdots$$

### Theorem (FA-ToolBox I (Friedrichs-Poincaré type est))

$$\Downarrow \quad \boxed{D(A_1) \cap D(A_0^*) \Leftrightarrow H_1 \text{ compact}} \quad \Rightarrow \quad \exists \quad |A_i^{-1}| = c_{A_i} = c_{A_i^*} = |(A_i^*)^{-1}| \in (0, \infty)$$

- (i)  $\forall x \in D(A_0) \quad |x|_{H_0} \leq c_{A_0} |A_0 x|_{H_1}$
- (i\*)  $\forall y \in D(A_0^*) \quad |y|_{H_1} \leq c_{A_0} |A_0^* y|_{H_0}$
- (ii)  $\forall y \in D(A_1) \quad |y|_{H_1} \leq c_{A_1} |A_1 y|_{H_2}$
- (ii\*)  $\forall z \in D(A_1^*) \quad |z|_{H_2} \leq c_{A_1} |A_1^* z|_{H_1}$
- (iii)  $\forall y \in D(A_1) \cap D(A_0^*) \quad |(1 - \pi_{N_1})y|_{H_1} \leq c_{A_1} |A_1 y|_{H_2} + c_{A_0} |A_0^* y|_{H_0}$

note  $\pi_{N_1} y \in N_1$  and  $(1 - \pi_{N_1})y \in N_1^\perp$

### Remark

enough  $R(A_0)$  and  $R(A_1)$  cl



## 2nd fundamental observations

$$\text{complex} \quad \dots \quad \begin{matrix} \dots \\ \rightleftarrows \\ \dots \end{matrix} \quad H_0 \quad \begin{matrix} A_0 \\ \rightleftarrows \\ A_0^* \end{matrix} \quad H_1 \quad \begin{matrix} A_1 \\ \rightleftarrows \\ A_1^* \end{matrix} \quad H_2 \quad \begin{matrix} \dots \\ \rightleftarrows \\ \dots \end{matrix} \quad \dots$$

Theorem (FA-ToolBox I (Helmholtz deco))

$$\Downarrow \quad \boxed{D(A_1) \cap D(A_0^*) \leftrightarrow H_1 \text{ compact}}$$

$$H_1 = R(A_0) \oplus N(A_0^*)$$

$$H_1 = R(A_1^*) \oplus N(A_1)$$

$$D(A_0^*) = D(\mathcal{A}_0^*) \oplus N(A_0^*)$$

$$D(A_1) = D(\mathcal{A}_1) \oplus N(A_1)$$

$$N(A_1) = D(\mathcal{A}_0^*) \oplus N_1$$

$$N(A_0^*) = D(\mathcal{A}_1) \oplus N_1$$

$$D(A_1) = R(A_0) \oplus (D(A_1) \cap N(A_0^*)) \quad D(A_0^*) = R(A_1^*) \oplus (D(A_0^*) \cap N(A_1))$$

$$H_1 = R(A_0) \oplus N_1 \oplus R(A_1^*)$$

$$D(A_1) = R(A_0) \oplus N_1 \oplus D(\mathcal{A}_1)$$

$$D(A_0^*) = D(\mathcal{A}_0^*) \oplus N_1 \oplus R(A_1^*)$$

$$D(A_1) \cap D(A_0^*) = D(\mathcal{A}_0^*) \oplus N_1 \oplus D(\mathcal{A}_1)$$

Remark

enough  $R(A_0)$  and  $R(A_1)$  cl



## Solving PDEs with Hilbert Complexes

(Static) First Order Systems



## (stat) first order system - solution theory

$$\text{complex} \quad \dots \begin{array}{c} \dots \\ \rightleftarrows \\ \dots \end{array} H_0 \begin{array}{c} A_0 \\ \rightleftarrows \\ A_0^* \end{array} H_1 \begin{array}{c} A_1 \\ \rightleftarrows \\ A_1^* \end{array} H_2 \begin{array}{c} \dots \\ \rightleftarrows \\ \dots \end{array} \dots$$

$$A_1 x = f$$

$$\dim N(A_1) = \infty$$

find  $x \in D(A_1) \cap D(A_0^*)$  such that the fos

$$\begin{array}{ll} A_1 x = f & (\text{rot } E = F) \\ A_0^* x = g & \text{think of } (-\text{div } E = g) \\ \pi_{N_1} x = k & (\pi_D E = K) \end{array}$$

$$\text{kernel} = \text{cohomology group} = N_1 = N(A_1) \cap N(A_0^*)$$

$$\text{trivially necessary} \quad f \in R(A_1) \quad g \in R(A_0^*) \quad k \in N_1$$

apply FA-ToolBox



# (stat) first order system - solution theory

$$\text{complex} \quad \dots \quad \begin{array}{c} \dots \\ \rightleftharpoons \\ \dots \end{array} \quad H_0 \quad \begin{array}{c} A_0 \\ \rightleftharpoons \\ A_0^* \end{array} \quad H_1 \quad \begin{array}{c} A_1 \\ \rightleftharpoons \\ A_1^* \end{array} \quad H_2 \quad \begin{array}{c} \dots \\ \rightleftharpoons \\ \dots \end{array} \quad \dots$$

$$\text{find } x \in D(A_1) \cap D(A_0^*) \text{ st fos} \quad A_1 x = f \quad A_0^* x = g \quad \pi_{N_1} x = k$$

## Theorem (FA-ToolBox II (solution theory))

$$\Downarrow \quad \boxed{D(A_1) \cap D(A_0^*) \leftrightarrow H_1 \text{ compact}}$$

$$\text{fos is uniq sol} \quad \leftrightarrow \quad f \in R(A_1) \quad g \in R(A_0^*) \quad k \in N_1$$

$$x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus N_1 = D(A_1) \cap D(A_0^*)$$

$$\boxed{x_f := \mathcal{A}_1^{-1} f} \in D(\mathcal{A}_1)$$

$$\boxed{x_g := (\mathcal{A}_0^*)^{-1} g} \in D(\mathcal{A}_0^*)$$

$$\text{dep cont on data} \quad |x|_{H_1} \leq |x_f|_{H_1} + |x_g|_{H_1} + |k|_{H_1} \leq c_{A_1} |f|_{H_2} + c_{A_0} |g|_{H_0} + |k|_{H_1}$$

moreover

$$\pi_{R(A_1^*)} x = x_f \quad \pi_{R(A_0)} x = x_g \quad \pi_{N_1} x = k \quad |x|_{H_1}^2 = |x_f|_{H_1}^2 + |x_g|_{H_1}^2 + |k|_{H_1}^2$$

## Remark

enough  $R(A_0)$  and  $R(A_1)$  cl



## (stat) first order system - a posteriori error estimates

problem:  $\boxed{\text{find } x \in D(A_1) \cap D(A_0^*) \text{ st } A_1 x = f \quad A_0^* x = g \quad \pi_{N_1} x = k}$

'very' non-conforming 'approximation' of  $x$ :  $\boxed{\tilde{x} \in H_1}$

def., dcmp. err.  $\boxed{e = x - \tilde{x}} = \pi_{R(A_0)} e + \pi_{N_1} e + \pi_{R(A_1^*)} e \in H_1 = R(A_0) \oplus N_1 \oplus R(A_1^*)$

### Theorem (sharp upper bounds)

Let  $\tilde{x} \in H_1$  and  $e = x - \tilde{x}$ . Then

$$|e|_{H_1}^2 = |\pi_{R(A_0)} e|_{H_1}^2 + |\pi_{N_1} e|_{H_1}^2 + |\pi_{R(A_1^*)} e|_{H_1}^2$$

$$|\pi_{R(A_0)} e|_{H_1} = \min_{\phi \in D(A_0^*)} (c_{A_0} |A_0^* \phi - g|_{H_0} + |\phi - \tilde{x}|_{H_1})$$

$\boxed{\text{reg } (A_0 A_0^* + 1)\text{-prbl in } D(A_0^*)}$

$$|\pi_{R(A_1^*)} e|_{H_1} = \min_{\varphi \in D(A_1)} (c_{A_1} |A_1 \varphi - f|_{H_2} + |\varphi - \tilde{x}|_{H_1})$$

$\boxed{\text{reg } (A_1^* A_1 + 1)\text{-prbl in } D(A_1)}$

$$|\pi_{N_1} e|_{H_1} = |\pi_{N_1} \tilde{x} - k|_{H_1} = \min_{\substack{\xi \in D(A_0) \\ \zeta \in D(A_1^*)}} |A_0 \xi + A_1^* \zeta + \tilde{x} - k|_{H_1}$$

$\boxed{\text{cpld } (A_0^* A_0) \text{-}(A_1 A_1^*)\text{-sys in } D(A_0) \text{-} D(A_1^*)}$

### Remark

Even  $\pi_{N_1} e = k - \pi_{N_1} \tilde{x}$  and the minima are attained at

$$\hat{\phi} = \pi_{R(A_0)} e + \tilde{x}, \quad \hat{\varphi} = \pi_{R(A_1^*)} e + \tilde{x}, \quad A_0 \hat{\xi} + A_1^* \hat{\zeta} = (\pi_{N_1} - 1) \tilde{x}.$$





# $A_0^*$ - $A_1$ -lemma (generalized global div-curl-lemma)

## Lemma (generalized $A_0^*$ - $A_1$ -lemma)

Let  $R(A_0)$  and  $R(A_1)$  be closed, and let  $N_1$  be finite dimensional. Moreover, let  $(x_n), (y_n) \subset H_1$  be bounded such that

- (i)  $\tilde{A}_1(x_n)$  is relatively compact in  $D(A_1^*)'$ ,
- (ii)  $\tilde{A}_0^*(y_n)$  is relatively compact in  $D(A_0)'$ .

$\Rightarrow \exists x, y \in H_1$  and subsequences st  $x_n \rightarrow x$  in  $H_1$  and  $y_n \rightarrow y$  in  $H_1$  as well as

$$\langle x_n, y_n \rangle_{H_1} \rightarrow \langle x, y \rangle_{H_1}.$$

proof uses key observation

## Lemma

Let  $R(A)$  be closed. For  $(x_n) \subset H_0$  the following statements are equivalent:

- (i)  $\tilde{A}x_n$  is relatively compact in  $D(A^*)'$ .
- (ii)  $\pi_{R(A^*)}x_n$  is relatively compact in  $R(A^*)$  resp.  $H_1$ .

If  $x_n \rightarrow x$  in  $H_1$ , then either of cond. (i) or (ii) implies  $\pi_{R(A^*)}x_n \rightarrow \pi_{R(A^*)}x$  in  $H_1$ .

nice results and joint work with Marcus Waurick



## Solving PDEs with Hilbert Complexes

Applications: FOS & SOS (First and Second Order Systems)



# classical de Rham complex in 3D ( $\nabla$ -rot-div-complex)

$\Omega \subset \mathbb{R}^3$  bounded weak Lipschitz domain,  $\partial\Omega = \Gamma = \overline{\Gamma_t \dot{\cup} \Gamma_n}$

(electro-magneto dynamics, Maxwell's equations)

$$\{0\} \begin{array}{c} \xrightarrow{\iota_{\{0\}}} \\ \xleftarrow{\pi_{\{0\}}} \end{array} L^2 \begin{array}{c} \xrightarrow{\dot{\nabla}} \\ \xleftarrow{-\operatorname{div}} \end{array} L^2 \begin{array}{c} \xrightarrow{\operatorname{rot}} \\ \xleftarrow{\operatorname{rot}} \end{array} L^2 \begin{array}{c} \xrightarrow{\operatorname{div}} \\ \xleftarrow{-\nabla} \end{array} L^2 \begin{array}{c} \xrightarrow{\pi_{\mathbb{R}}} \\ \xleftarrow{\iota_{\mathbb{R}}} \end{array} \mathbb{R}$$

mixed boundary conditions and inhomogeneous and anisotropic media

$$\{0\} \text{ or } \mathbb{R} \begin{array}{c} \xrightarrow{\iota} \\ \xleftarrow{\pi} \end{array} L^2 \begin{array}{c} \xrightarrow{\nabla_{\Gamma_t}} \\ \xleftarrow{-\operatorname{div}_{\Gamma_n} \varepsilon} \end{array} L^2_{\varepsilon} \begin{array}{c} \xrightarrow{\operatorname{rot}_{\Gamma_t}} \\ \xleftarrow{\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}} \end{array} L^2 \begin{array}{c} \xrightarrow{\operatorname{div}_{\Gamma_t}} \\ \xleftarrow{-\nabla_{\Gamma_n}} \end{array} L^2 \begin{array}{c} \xrightarrow{\pi} \\ \xleftarrow{\iota} \end{array} \mathbb{R} \text{ or } \{0\}$$



# classical de Rham complex in 3D ( $\nabla$ -rot-div-complex)

$\Omega \subset \mathbb{R}^3$  bounded weak Lipschitz domain,  $\partial\Omega = \Gamma = \overline{\Gamma_t \dot{\cup} \Gamma_n}$

(electro-magneto dynamics, Maxwell's equations with mixed boundary conditions)

$$\{0\} \text{ or } \mathbb{R} \xrightleftharpoons[\pi]{} L^2 \xrightleftharpoons[-\operatorname{div}_{\Gamma_n} \varepsilon]{} L^2_\varepsilon \xrightleftharpoons[\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}]{} L^2 \xrightleftharpoons[-\nabla_{\Gamma_n}]{} L^2 \xrightleftharpoons[\iota]{} \mathbb{R} \text{ or } \{0\}$$

related fos

$$\begin{array}{cccc|cccc} \nabla_{\Gamma_t} u = A & \text{in } \Omega & | & \operatorname{rot}_{\Gamma_t} E = J & \text{in } \Omega & | & \operatorname{div}_{\Gamma_t} H = k & \text{in } \Omega & | & \pi v = b & \text{in } \Omega \\ \pi u = a & \text{in } \Omega & | & -\operatorname{div}_{\Gamma_n} \varepsilon E = j & \text{in } \Omega & | & \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} H = K & \text{in } \Omega & | & -\nabla_{\Gamma_n} v = B & \text{in } \Omega \end{array}$$

related sos

$$\begin{array}{cccc|cccc} -\operatorname{div}_{\Gamma_n} \varepsilon \nabla_{\Gamma_t} u = j & \text{in } \Omega & | & \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} \operatorname{rot}_{\Gamma_t} E = K & \text{in } \Omega & | & -\nabla_{\Gamma_n} \operatorname{div}_{\Gamma_t} H = B & \text{in } \Omega \\ \pi u = a & \text{in } \Omega & | & -\operatorname{div}_{\Gamma_n} \varepsilon E = j & \text{in } \Omega & | & \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} H = K & \text{in } \Omega \end{array}$$

corresponding compact embeddings:

$$D(\nabla_{\Gamma_t}) \cap D(\pi) = D(\nabla_{\Gamma_t}) = H_{\Gamma_t}^1 \hookrightarrow L^2 \quad (\text{Rellich's selection theorem})$$

$$D(\operatorname{rot}_{\Gamma_t}) \cap D(-\operatorname{div}_{\Gamma_n} \varepsilon) = R_{\Gamma_t} \cap \varepsilon^{-1} D_{\Gamma_n} \hookrightarrow L^2_\varepsilon \quad (\text{Weck's selection theorem, '74})$$

$$D(\operatorname{div}_{\Gamma_t}) \cap D(\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}) = D_{\Gamma_t} \cap R_{\Gamma_n} \hookrightarrow L^2 \quad (\text{Weck's selection theorem, '74})$$

$$D(\nabla_{\Gamma_n}) \cap D(\pi) = D(\nabla_{\Gamma_n}) = H_{\Gamma_n}^1 \hookrightarrow L^2 \quad (\text{Rellich's selection theorem})$$

Weck's selection theorem for weak Lip. dom. and mixed bc: Bauer/Py/Schomburg ('16)

Weck's selection theorem (Weck '74, (Habil. '72) stimulated by Rolf Leis)

(Weber '80, Picard '84, Costabel '90, Witsch '93, Jochmann '97, Kuhn '99, Picard/Weck/Witsch '01, Py '96, '03, '06, '07, '08)



# classical de Rham complex in 3D ( $\nabla$ -rot-div-complex)

$$\begin{aligned}
 \operatorname{rot} E &= F && \text{in } \Omega \\
 -\operatorname{div} \varepsilon E &= g && \text{in } \Omega \\
 \nu \times E &= 0 && \text{at } \Gamma_t \\
 \nu \cdot \varepsilon E &= 0 && \text{at } \Gamma_n
 \end{aligned}$$

non-trivial kernel  $\mathcal{H}_{D,\varepsilon} = \{H \in L^2 : \operatorname{rot} H = 0, \operatorname{div} \varepsilon H = 0, \nu \times H|_{\Gamma_t} = 0, \nu \cdot \varepsilon H|_{\Gamma_n} = 0\}$   
 additional condition on Dirichlet/Neumann fields for uniqueness

$$\pi_D E = K \in \mathcal{H}_{D,\varepsilon}$$

$$\{0\} \text{ or } \mathbb{R} \xrightleftharpoons[\pi]{\iota} L^2 \xrightleftharpoons[-\operatorname{div}_{\Gamma_n} \varepsilon]{\nabla_{\Gamma_t}} L^2_\varepsilon \xrightleftharpoons[\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}]{\operatorname{rot}_{\Gamma_t}} L^2 \xrightleftharpoons[-\nabla_{\Gamma_n}]{\operatorname{div}_{\Gamma_t}} L^2 \xrightleftharpoons[\iota]{\pi} \mathbb{R} \text{ or } \{0\}$$

$$\dots \xrightleftharpoons[\dots]{\dots} H_{-1} \xrightleftharpoons[A_{-1}^*]{A_{-1}} H_0 \xrightleftharpoons[A_0^*]{A_0} H_1 \xrightleftharpoons[A_1^*]{A_1} H_2 \xrightleftharpoons[A_2^*]{A_2} H_3 \xrightleftharpoons[A_3^*]{A_3} H_4 \xrightleftharpoons[\dots]{\dots} \dots$$

$$\begin{array}{llll}
 \text{find } E \in R_{\Gamma_t}(\Omega) \cap \varepsilon^{-1} D_{\Gamma_n}(\Omega) & \text{st} & (\text{fos}) & \text{find } x \in D(A_1) \cap D(A_0^*) \text{ st} \\
 \operatorname{rot}_{\Gamma_t} E = F & & & A_1 x = f \\
 -\operatorname{div}_{\Gamma_n} \varepsilon E = g & & \text{translation} & A_0^* x = g \\
 \pi_{D/N} E = K & & & \pi_{N_1} x = k
 \end{array}$$





# classical de Rham complex in 3D ( $\nabla$ -rot-div-complex)

$c_{A_0} = c_{fp}$  (Friedrichs/Poincaré constant) and  $c_{A_1} = c_m$  (Maxwell constant)

**Lemma/Theorem**  $\Downarrow$   $D(A_1) \cap D(A_0^*) \Leftrightarrow L_\varepsilon^2(\Omega)$  compact

(i) all Friedrichs-Poincaré type est hold

$$\forall \varphi \in D(\mathcal{A}_0) \quad |\varphi|_{H_0} \leq c_{A_0} |A_0 \varphi|_{H_1} \quad \Leftrightarrow \quad \forall \varphi \in H_{\Gamma_t}^1 \quad |\varphi|_{L^2} \leq c_{fp} |\nabla \varphi|_{L_\varepsilon^2}$$

$$\forall \phi \in D(\mathcal{A}_0^*) \quad |\phi|_{H_1} \leq c_{A_0} |A_0^* \phi|_{H_0} \quad \Leftrightarrow \quad \forall \Phi \in \varepsilon^{-1} D_{\Gamma_n} \cap \nabla H_{\Gamma_t}^1 \quad |\Phi|_{L_\varepsilon^2} \leq c_{fp} |\operatorname{div} \varepsilon \Phi|_{L^2}$$

$$\forall \phi \in D(\mathcal{A}_1) \quad |\phi|_{H_1} \leq c_{A_1} |A_1 \phi|_{H_2} \quad \Leftrightarrow \quad \forall \Phi \in R_{\Gamma_t} \cap \varepsilon^{-1} \operatorname{rot} R_{\Gamma_n} \quad |\Phi|_{L_\varepsilon^2} \leq c_m |\operatorname{rot} \Phi|_{L^2}$$

$$\forall \psi \in D(\mathcal{A}_1^*) \quad |\psi|_{H_2} \leq c_{A_1} |A_1^* \psi|_{H_1} \quad \Leftrightarrow \quad \forall \Psi \in R_{\Gamma_n} \cap \operatorname{rot} R_{\Gamma_t} \quad |\Psi|_{L^2} \leq c_m |\operatorname{rot} \Psi|_{L_\varepsilon^2}$$

(ii) all ranges  $R(A_0) = \nabla H_{\Gamma_t}^1$ ,  $R(A_1) = \operatorname{rot} R_{\Gamma_t}$ ,  $R(A_0^*) = \operatorname{div} D_{\Gamma_n}$  are cl in  $L^2$

(iii) the inverse ops  $(\widetilde{\nabla}_{\Gamma_t})^{-1}$ ,  $(\widetilde{\operatorname{div}}_{\Gamma_n} \varepsilon)^{-1}$ ,  $(\widetilde{\operatorname{rot}}_{\Gamma_t})^{-1}$ ,  $(\widetilde{\varepsilon^{-1} \operatorname{rot}}_{\Gamma_n})^{-1}$  are cont, even cpt

(iv) all Helmholtz decomposition hold, e.g.,

$$H_1 = R(A_0) \oplus N_1 \oplus R(A_1^*) \quad \Leftrightarrow \quad L_\varepsilon^2 = \nabla H_{\Gamma_t}^1 \oplus_{L_\varepsilon^2} \mathcal{H}_{D,\varepsilon} \oplus_{L_\varepsilon^2} \varepsilon^{-1} \operatorname{rot} R_{\Gamma_n}$$

(v) solution theory

(vi) variational formulations

(vii) functional a posteriori error estimates

(viii) div-curl-lemma

(ix) ...



# classical de Rham complex in 3D ( $\nabla$ -rot-div-complex)

find  $E \in R_{\Gamma_t} \cap \varepsilon^{-1}D_{\Gamma_n}$  s.t. / think of  $x \in D(\mathcal{A}_1) \cap D(\mathcal{A}_0^*)$

$$\begin{array}{ll}
 \text{rot}_{\Gamma_t} E = F & \mathcal{A}_1 x = f \\
 \text{div}_{\Gamma_n} \varepsilon E = g & / \quad \text{think of} \quad \mathcal{A}_0^* x = g \\
 \pi_{\mathcal{H}_{D,\varepsilon}} E = K & \pi_{K_1} x = k
 \end{array}$$

sol is simply  $x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus K_1 = D(\mathcal{A}_1) \cap D(\mathcal{A}_0^*)$

with  $x_f := \mathcal{A}_1^{-1} f \in D(\mathcal{A}_1)$  and  $x_g := (\mathcal{A}_0^*)^{-1} g \in D(\mathcal{A}_0^*)$

i.e.,  $E = E_f + E_g + K$ , where

$$E_f := (\widetilde{\text{rot}}_{\Gamma_t})^{-1} F \in D(\widetilde{\text{rot}}_{\Gamma_t}) = R_{\Gamma_t} \cap \varepsilon^{-1} \text{rot} R_{\Gamma_n} = R_{\Gamma_t} \cap \varepsilon^{-1} D_{\Gamma_n,0} \cap \mathcal{H}_{D,\varepsilon}^\perp,$$

$$E_g := (\widetilde{\text{div}}_{\Gamma_n} \varepsilon)^{-1} g \in D(\widetilde{\text{div}}_{\Gamma_n} \varepsilon) = \varepsilon^{-1} D_{\Gamma_n} \cap \nabla H_{\Gamma_t}^1 = \varepsilon^{-1} D_{\Gamma_n} \cap R_{\Gamma_t,0} \cap \mathcal{H}_{D,\varepsilon}^\perp$$



# classical de Rham complex in 3D ( $\nabla$ -rot-div-complex)

## Theorem (sharp upper bounds)

Let  $\tilde{E} \in L^2_\varepsilon$  (very non-conforming approximation of  $E$ !) and  $e := E - \tilde{E}$ . Then

$$\begin{aligned}
 |e|_{L^2_\varepsilon}^2 &= |\pi_{R(\nabla_{\Gamma_t})} e|_{L^2_\varepsilon}^2 + |\pi_{R(\varepsilon^{-1} \text{rot}_{\Gamma_n})} e|_{L^2_\varepsilon}^2 + |\pi_{\mathcal{H}_{D,\varepsilon}} e|_{L^2_\varepsilon}^2 \\
 &= \min_{\Phi \in \varepsilon^{-1} D_{\Gamma_n}} (c_{fp} |\text{div } \varepsilon \Phi + g|_{L^2} + |\Phi - \tilde{E}|_{L^2_\varepsilon})^2 && \text{reg } (-\nabla_{\Gamma_t} \text{div}_{\Gamma_n} + 1)\text{-prbl in } D_{\Gamma_n} \\
 &\quad + \min_{\Phi \in R_{\Gamma_t}} (c_m |\text{rot } \Phi - F|_{L^2} + |\Phi - \tilde{E}|_{L^2_\varepsilon})^2 && \text{reg } (\text{rot}_{\Gamma_n} \text{rot}_{\Gamma_t} + 1)\text{-prbl in } R_{\Gamma_t} \\
 &\quad + \min_{\phi \in H^1_{\Gamma_t}, \Psi \in R_{\Gamma_n}} |\nabla \phi + \varepsilon^{-1} \text{rot } \Psi + \tilde{E} - K|_{L^2_\varepsilon}^2 && \text{cpld } (-\text{div}_{\Gamma_n} \nabla_{\Gamma_t})\text{-}(\text{rot}_{\Gamma_t} \text{rot}_{\Gamma_n})\text{-sys in } H^1_{\Gamma_t}\text{-}R_{\Gamma_n}
 \end{aligned}$$

## Remark

- $(\text{rot}_{\Gamma_t} \text{rot}_{\Gamma_n})$ -prbl needs saddle point formulation
- $\Omega$  top trv  $\Rightarrow \pi_D = 0$  and  $R_{\Gamma_t,0} = \nabla H^1_{\Gamma_t}$  and  $D_{\Gamma_n,0} = \text{rot } R_{\Gamma_n}$

$$\bullet \quad \Omega \text{ convex and } \varepsilon = \mu = 1 \text{ and } \Gamma_t = \Gamma \text{ or } \Gamma_n = \Gamma \Rightarrow c_f \leq c_m \leq c_p \leq \frac{\text{diam } \Omega}{\pi}$$



## div-curl-lemma

## Lemma (div-curl-lemma (global version))

*Assumptions:*

- (i)  $(E_n)$  bounded in  $L^2(\Omega)$
- (i')  $(H_n)$  bounded in  $L^2(\Omega)$
- (ii)  $(\operatorname{rot} E_n)$  bounded in  $L^2(\Omega)$
- (ii')  $(\operatorname{div} \varepsilon H_n)$  bounded in  $L^2(\Omega)$
- (iii)  $\nu \times E_n = 0$  on  $\Gamma_t$ , i.e.,  $E_n \in R_{\Gamma_t}(\Omega)$
- (iii')  $\nu \cdot \varepsilon H_n = 0$  on  $\Gamma_n$ , i.e.,  $H_n \in \varepsilon^{-1} D_{\Gamma_n}(\Omega)$

$\Rightarrow \exists E, H$  and subsequences st

$E_n \rightarrow E, \operatorname{rot} E_n \rightarrow \operatorname{rot} E$  and  $H_n \rightarrow H, \operatorname{div} H_n \rightarrow \operatorname{div} H$  in  $L^2(\Omega)$  and

$$\langle E_n, H_n \rangle_{L^2_\varepsilon(\Omega)} \rightarrow \langle E, H \rangle_{L^2_\varepsilon(\Omega)}$$

$\Rightarrow$  classical local version



## de Rham complex in ND or on Riemannian manifolds (d-complex)

$\Omega \subset \mathbb{R}^N$  bd w. Lip. dom. or  $\Omega$  Riemannian manifold with cpt cl. and Lip. boundary  $\Gamma$   
(generalized Maxwell equations)

$$\{0\} \begin{array}{c} \hookrightarrow \\ \xleftrightarrow{\pi_{\{0\}}} \end{array} L^{2,0} \begin{array}{c} \xrightarrow{d} \\ \xleftrightarrow{-\delta} \end{array} L^{2,1} \begin{array}{c} \xrightarrow{d} \\ \xleftrightarrow{-\delta} \end{array} \dots L^{2,q} \begin{array}{c} \xrightarrow{d} \\ \xleftrightarrow{-\delta} \end{array} L^{2,q+1} \dots L^{2,N-1} \begin{array}{c} \xrightarrow{d} \\ \xleftrightarrow{-\delta} \end{array} L^{2,N} \begin{array}{c} \xrightarrow{\pi_{\mathbb{R}}} \\ \xleftrightarrow{\iota_{\mathbb{R}}} \end{array} \mathbb{R}$$



## de Rham complex in ND or on Riemannian manifolds (d-complex)

$\Omega \subset \mathbb{R}^N$  bd w. Lip. dom. or  $\Omega$  Riemannian manifold with cpt cl. and Lip. boundary  $\Gamma$   
(generalized Maxwell equations)

$$\{0\} \text{ or } \mathbb{R} \xrightarrow{\frac{\iota}{\pi}} L^{2,0} \begin{array}{c} d_{\Gamma,t}^0 \\ \leftarrow \\ -\delta_{\Gamma,n}^1 \end{array} L^{2,1} \begin{array}{c} d_{\Gamma,t}^1 \\ \leftarrow \\ -\delta_{\Gamma,n}^2 \end{array} \dots L^{2,q} \begin{array}{c} d_{\Gamma,t}^q \\ \leftarrow \\ -\delta_{\Gamma,n}^{q+1} \end{array} L^{2,q+1} \dots L^{2,N-1} \begin{array}{c} d_{\Gamma,t}^{N-1} \\ \leftarrow \\ -\delta_{\Gamma,n}^N \end{array} L^{2,N} \xrightarrow{\frac{\pi}{\iota}} \mathbb{R} \text{ or } \{0\}$$

related fos

$$\begin{aligned} d_{\Gamma,t}^q E &= F && \text{in } \Omega \\ -\delta_{\Gamma,n}^q E &= G && \text{in } \Omega \end{aligned}$$

related sos

$$\begin{aligned} -\delta_{\Gamma,n}^{q+1} d_{\Gamma,t}^q E &= F && \text{in } \Omega \\ -\delta_{\Gamma,n}^q E &= G && \text{in } \Omega \end{aligned}$$

includes: EMS rot / div, Laplacian, rot rot, and more...  
corresponding compact embeddings:

$$D(d_{\Gamma,t}^q) \cap D(\delta_{\Gamma,n}^q) \hookrightarrow L^{2,q} \quad (\text{Weck's selection theorems, '74})$$

Weck's selection theorem for Lip. manifolds and mixed bc: Bauer/Py/Schomburg ('17)



# elasticity complex in 3D (sym $\nabla$ -Rot Rot $_{\mathbb{S}}^T$ -Div $_{\mathbb{S}}$ -complex)

$\Omega \subset \mathbb{R}^3$  bounded strong Lipschitz domain

$$\begin{array}{ccccccc}
 \{0\} & \begin{array}{c} \iota_{\{0\}} \\ \rightleftarrows \\ \pi_{\{0\}} \end{array} & L^2 & \begin{array}{c} \text{sym } \nabla \\ \rightleftarrows \\ -\text{Div}_{\mathbb{S}} \end{array} & L^2_{\mathbb{S}} & \begin{array}{c} \text{Rot Rot}_{\mathbb{S}}^T \\ \rightleftarrows \\ \text{Rot Rot}_{\mathbb{S}}^T \end{array} & L^2_{\mathbb{S}} & \begin{array}{c} \text{Div}_{\mathbb{S}} \\ \rightleftarrows \\ -\text{sym } \nabla \end{array} & L^2 & \begin{array}{c} \pi_{\text{RM}} \\ \rightleftarrows \\ \iota_{\text{RM}} \end{array} & \text{RM}
 \end{array}$$



# elasticity complex in 3D (sym $\nabla$ -Rot Rot $_{\mathbb{S}}^{\top}$ -Div $_{\mathbb{S}}$ -complex)

$\Omega \subset \mathbb{R}^3$  bounded strong Lipschitz domain

$$\{0\} \begin{array}{c} \xleftarrow{\iota_{\{0\}}} \\ \xrightarrow{\pi_{\{0\}}} \end{array} L^2 \begin{array}{c} \xleftarrow{\text{sym } \nabla} \\ \xrightarrow{-\text{Div}_{\mathbb{S}}} \end{array} L^2_{\mathbb{S}} \begin{array}{c} \xleftarrow{\text{Rot } \text{Rot}_{\mathbb{S}}^{\top}} \\ \xrightarrow{\text{Rot } \text{Rot}_{\mathbb{S}}^{\top}} \end{array} L^2_{\mathbb{S}} \begin{array}{c} \xleftarrow{\text{Div}_{\mathbb{S}}} \\ \xrightarrow{-\text{sym } \nabla} \end{array} L^2 \begin{array}{c} \xleftarrow{\pi_{\text{RM}}} \\ \xrightarrow{\iota_{\text{RM}}} \end{array} \text{RM}$$

related fos (Rot $^{\top}$ Rot $_{\mathbb{S}}^{\top}$ , Rot Rot $_{\mathbb{S}}^{\top}$  first order operators!)

$$\begin{array}{l|l|l|l} \text{sym } \nabla v = M & \text{in } \Omega & | & \text{Rot } \text{Rot}_{\mathbb{S}}^{\top} M = F & \text{in } \Omega & | & \text{Div}_{\mathbb{S}} N = g & \text{in } \Omega & | & \pi v = r & \text{in } \Omega \\ \pi v = 0 & \text{in } \Omega & | & -\text{Div}_{\mathbb{S}} M = f & \text{in } \Omega & | & \text{Rot } \text{Rot}_{\mathbb{S}}^{\top} N = G & \text{in } \Omega & | & -\text{sym } \nabla v = M & \text{in } \Omega \end{array}$$

related sos (Rot Rot $_{\mathbb{S}}^{\top}$  Rot $^{\top}$ Rot $_{\mathbb{S}}^{\top}$  second order operator!)

$$\begin{array}{l|l|l|l} -\text{Div}_{\mathbb{S}} \text{sym } \nabla v = f & \text{in } \Omega & | & \text{Rot } \text{Rot}_{\mathbb{S}}^{\top} \text{Rot } \text{Rot}_{\mathbb{S}}^{\top} M = G & \text{in } \Omega & | & -\text{sym } \nabla \text{Div}_{\mathbb{S}} N = M & \text{in } \Omega \\ \pi v = 0 & \text{in } \Omega & | & -\text{Div}_{\mathbb{S}} M = f & \text{in } \Omega & | & \text{Rot } \text{Rot}_{\mathbb{S}}^{\top} N = G & \text{in } \Omega \end{array}$$

corresponding compact embeddings:

$$D(\text{sym } \nabla) \cap D(\pi) = D(\nabla) = \dot{H}^1 \hookrightarrow L^2 \quad (\text{Rellich's selection theorem and Korn ineq.})$$

$$D(\text{Rot } \text{Rot}_{\mathbb{S}}^{\top}) \cap D(\text{Div}_{\mathbb{S}}) \hookrightarrow L^2_{\mathbb{S}} \quad (\text{new selection theorem})$$

$$D(\text{Div}_{\mathbb{S}}) \cap D(\text{Rot } \text{Rot}_{\mathbb{S}}^{\top}) \hookrightarrow L^2_{\mathbb{S}} \quad (\text{new selection theorem})$$

$$D(\pi) \cap D(\text{sym } \nabla) = D(\nabla) = H^1 \hookrightarrow L^2 \quad (\text{Rellich's selection theorem and Korn ineq.})$$

two new selection theorems for strong Lip. dom.: Py/Schomburg/Zulehner ('18)



# elasticity complex in 3D (sym $\nabla$ -Rot Rot $_{\mathbb{S}}^{\top}$ -Div $_{\mathbb{S}}$ -complex)

**Lemma/Theorem**  $\Downarrow$   $D(A_1) \cap D(A_0^*) \leftrightarrow H_1, \quad D(A_2) \cap D(A_1^*) \leftrightarrow H_2 \quad \text{cpt}$

(i) all Friedrichs-Poincaré type est hold

$$\text{est for } \mathcal{A}_0 \Leftrightarrow \forall \varphi \in D(\text{sym } \overset{\circ}{\nabla}) \cap R(\text{Div}_{\mathbb{S}}) = \dot{H}^1 \quad |\varphi|_{L^2} \leq c_0 |\text{sym } \nabla \varphi|_{L^2}$$

$$\text{est for } \mathcal{A}_0^* \Leftrightarrow \forall \Phi \in D(\text{Div}_{\mathbb{S}}) \cap R(\text{sym } \overset{\circ}{\nabla}) \quad |\Phi|_{L^2} \leq c_0 |\text{Div } \Phi|_{L^2}$$

$$\text{est for } \mathcal{A}_1 \Leftrightarrow \forall \Phi \in D(\text{Rot } \overset{\circ}{\text{Rot}}_{\mathbb{S}}^{\top}) \cap R(\text{Rot } \text{Rot}_{\mathbb{S}}^{\top}) \quad |\Phi|_{L^2} \leq c_1 |\text{Rot } \text{Rot}^{\top} \Phi|_{L^2}$$

$$\text{est for } \mathcal{A}_1^* \Leftrightarrow \forall \Phi \in D(\text{Rot } \text{Rot}_{\mathbb{S}}^{\top}) \cap R(\text{Rot } \overset{\circ}{\text{Rot}}_{\mathbb{S}}^{\top}) \quad |\Phi|_{L^2} \leq c_1 |\text{Rot } \text{Rot}^{\top} \Phi|_{L^2}$$

$$\text{est for } \mathcal{A}_2 \Leftrightarrow \forall \Phi \in D(\overset{\circ}{\text{Div}}_{\mathbb{S}}) \cap R(\text{sym } \nabla) \quad |\Phi|_{L^2} \leq c_2 |\text{Div } \Phi|_{L^2}$$

$$\text{est for } \mathcal{A}_2^* \Leftrightarrow \forall \varphi \in D(\text{sym } \nabla) \cap R(\overset{\circ}{\text{Div}}_{\mathbb{S}}) = H^1 \cap \text{RM}^{\perp} \quad |\varphi|_{L^2} \leq c_2 |\text{sym } \nabla \varphi|_{L^2}$$

(ii) all ranges  $R(A_n) = R(\mathcal{A}_n)$ ,  $R(A_n^*) = R(\mathcal{A}_n^*)$  are cl in  $L^2$

(iii) all inverse ops  $\mathcal{A}_n^{-1}$ ,  $(\mathcal{A}_n^*)^{-1}$  are cont, even cpt

(iv) all Helmholtz decomposition hold, e.g.,

$$H_1 = R(A_0) \oplus N_1 \oplus R(A_1^*) \Leftrightarrow L^2 = R(\text{sym } \overset{\circ}{\nabla}) \oplus_{L^2} \mathcal{H}_{D,\mathbb{S}} \oplus_{L^2} R(\text{Rot } \text{Rot}_{\mathbb{S}}^{\top})$$

(v) solution theories

(vi) variational formulations

(vii) functional a posteriori error estimates

(viii) div-curl-lemmas

(ix) ...

biharmonic / general relativity complex in 3D ( $\nabla\nabla$ -Rot<sub>S</sub>-Div<sub>T</sub>-complex)

$\Omega \subset \mathbb{R}^3$  bounded strong Lipschitz domain

$$\begin{array}{ccccccc}
 \{0\} & \begin{array}{c} \hookrightarrow_{\iota_{\{0\}}} \\ \leftarrow_{\pi_{\{0\}}} \end{array} & L^2 & \begin{array}{c} \nabla\nabla \\ \leftarrow_{\text{div Div}_S} \end{array} & L^2_S & \begin{array}{c} \mathring{\text{Rot}}_S \\ \leftarrow_{\text{sym Rot}_T} \end{array} & L^2_T & \begin{array}{c} \text{Div}_T \\ \leftarrow_{-\text{dev } \nabla} \end{array} & L^2 & \begin{array}{c} \pi_{RT} \\ \leftarrow_{\iota_{RT}} \end{array} & RT
 \end{array}$$

biharmonic / general relativity complex in 3D ( $\nabla\nabla$ -Rot $_{\mathbb{S}}$ -Div $_{\mathbb{T}}$ -complex)

$\Omega \subset \mathbb{R}^3$  bounded strong Lipschitz domain

$$\{0\} \begin{array}{c} \xleftrightarrow{\mathcal{L}\{0\}} \\ \xleftrightarrow{\pi\{0\}} \end{array} L^2 \begin{array}{c} \xleftrightarrow{\nabla\nabla} \\ \xleftrightarrow{\operatorname{div} \operatorname{Div}_{\mathbb{S}}} \end{array} L^2_{\mathbb{S}} \begin{array}{c} \xleftrightarrow{\operatorname{Rot}_{\mathbb{S}}} \\ \xleftrightarrow{\operatorname{sym} \operatorname{Rot}_{\mathbb{T}}} \end{array} L^2_{\mathbb{T}} \begin{array}{c} \xleftrightarrow{\operatorname{Div}_{\mathbb{T}}} \\ \xleftrightarrow{-\operatorname{dev} \nabla} \end{array} L^2 \begin{array}{c} \xleftrightarrow{\pi_{\operatorname{RT}}} \\ \xleftrightarrow{\mathcal{L}_{\operatorname{RT}}} \end{array} \operatorname{RT}$$

related fos ( $\nabla\nabla$ ,  $\operatorname{div} \operatorname{Div}_{\mathbb{S}}$  first order operators!)

$$\begin{array}{l} \nabla\nabla u = M \quad \text{in } \Omega \quad | \quad \operatorname{Rot}_{\mathbb{S}} M = F \quad \text{in } \Omega \quad | \quad \operatorname{Div}_{\mathbb{T}} N = g \quad \text{in } \Omega \quad | \quad \pi v = r \quad \text{in } \Omega \\ \pi u = 0 \quad \text{in } \Omega \quad | \quad \operatorname{div} \operatorname{Div}_{\mathbb{S}} M = f \quad \text{in } \Omega \quad | \quad \operatorname{sym} \operatorname{Rot}_{\mathbb{T}} N = G \quad \text{in } \Omega \quad | \quad -\operatorname{dev} \nabla v = T \quad \text{in } \Omega \end{array}$$

related sos ( $\operatorname{div} \operatorname{Div}_{\mathbb{S}} \nabla\nabla = \hat{\Delta}^2$  second order operator!)

$$\begin{array}{l} \operatorname{div} \operatorname{Div}_{\mathbb{S}} \nabla\nabla u = \hat{\Delta}^2 u = f \quad \text{in } \Omega \quad | \quad \operatorname{sym} \operatorname{Rot}_{\mathbb{T}} \operatorname{Rot}_{\mathbb{S}} M = G \quad \text{in } \Omega \quad | \quad -\operatorname{dev} \nabla \operatorname{Div}_{\mathbb{T}} N = T \quad \text{in } \Omega \\ \pi u = 0 \quad \text{in } \Omega \quad | \quad \operatorname{div} \operatorname{Div}_{\mathbb{S}} M = f \quad \text{in } \Omega \quad | \quad \operatorname{sym} \operatorname{Rot}_{\mathbb{T}} N = G \quad \text{in } \Omega \end{array}$$

corresponding compact embeddings:

$$D(\nabla\nabla) \cap D(\pi) = D(\nabla\nabla) = \dot{H}^2 \hookrightarrow L^2 \quad (\text{Rellich's selection theorem})$$

$$D(\operatorname{Rot}_{\mathbb{S}}) \cap D(\operatorname{div} \operatorname{Div}_{\mathbb{S}}) \hookrightarrow L^2_{\mathbb{S}} \quad (\text{new selection theorem})$$

$$D(\operatorname{Div}_{\mathbb{T}}) \cap D(\operatorname{sym} \operatorname{Rot}_{\mathbb{T}}) \hookrightarrow L^2_{\mathbb{T}} \quad (\text{new selection theorem})$$

$$D(\pi) \cap D(\operatorname{dev} \nabla) = D(\operatorname{dev} \nabla) = D(\nabla) = H^1 \hookrightarrow L^2 \quad (\text{Rellich's selection theorem and Korn type ineq.})$$

two new selection theorems for strong Lip. dom. and Korn Type ineq.: Py/Zulehner ('16)

biharmonic / general relativity complex in 3D ( $\nabla\nabla$ -Rot $_{\mathbb{S}}$ -Div $_{\mathbb{T}}$ -complex)

**Lemma/Theorem**  $\Downarrow$   $D(A_1) \cap D(A_0^*) \leftrightarrow H_1, \quad D(A_2) \cap D(A_1^*) \leftrightarrow H_2 \quad \text{cpt}$

(i) all Friedrichs-Poincaré type est hold

$$\text{est for } \mathcal{A}_0 \Leftrightarrow \forall \varphi \in D(\nabla\mathring{\nabla}) \cap R(\text{div Div}_{\mathbb{S}}) = \mathring{H}^2 \quad |\varphi|_{L^2} \leq c_0 |\nabla\nabla\varphi|_{L^2}$$

$$\text{est for } \mathcal{A}_0^* \Leftrightarrow \forall \Phi \in D(\text{div Div}_{\mathbb{S}}) \cap R(\nabla\mathring{\nabla}) \quad |\Phi|_{L^2} \leq c_0 |\text{div Div } \Phi|_{L^2}$$

$$\text{est for } \mathcal{A}_1 \Leftrightarrow \forall \Phi \in D(\mathring{\text{Rot}}_{\mathbb{S}}) \cap R(\text{sym Rot}_{\mathbb{T}}) \quad |\Phi|_{L^2} \leq c_1 |\text{Rot } \Phi|_{L^2}$$

$$\text{est for } \mathcal{A}_1^* \Leftrightarrow \forall \Phi \in D(\text{sym Rot}_{\mathbb{T}}) \cap R(\mathring{\text{Rot}}_{\mathbb{S}}) \quad |\Phi|_{L^2} \leq c_1 |\text{sym Rot } \Phi|_{L^2}$$

$$\text{est for } \mathcal{A}_2 \Leftrightarrow \forall \Phi \in D(\mathring{\text{Div}}_{\mathbb{T}}) \cap R(\text{dev } \nabla) \quad |\Phi|_{L^2} \leq c_2 |\text{Div } \Phi|_{L^2}$$

$$\text{est for } \mathcal{A}_2^* \Leftrightarrow \forall \varphi \in D(\text{dev } \nabla) \cap R(\mathring{\text{Div}}_{\mathbb{T}}) = H^1 \cap \text{RT}^{\perp} \quad |\varphi|_{L^2} \leq c_2 |\text{dev } \nabla\varphi|_{L^2}$$

(ii) all ranges  $R(A_n) = R(\mathcal{A}_n)$ ,  $R(A_n^*) = R(\mathcal{A}_n^*)$  are cl in  $L^2$

(iii) all inverse ops  $\mathcal{A}_n^{-1}$ ,  $(\mathcal{A}_n^*)^{-1}$  are cont, even cpt

(iv) all Helmholtz decomposition hold, e.g.,

$$H_1 = R(A_0) \oplus N_1 \oplus R(A_1^*) \quad \Leftrightarrow \quad L_{\mathbb{S}}^2 = R(\nabla\mathring{\nabla}) \oplus_{L_{\mathbb{S}}^2} \mathcal{H}_{D,\mathbb{S}} \oplus_{L_{\mathbb{S}}^2} R(\text{sym Rot}_{\mathbb{T}}),$$

$$H_2 = R(A_1) \oplus N_2 \oplus R(A_2^*) \quad \Leftrightarrow \quad L_{\mathbb{T}}^2 = R(\mathring{\text{Rot}}_{\mathbb{S}}) \oplus_{L_{\mathbb{T}}^2} \mathcal{H}_{N,\mathbb{T}} \oplus_{L_{\mathbb{T}}^2} R(\text{dev } \nabla)$$

(v)-(ix) solution theories, variational formulations, functional a posteriori error estimates, div-curl-lemmas, ...



## Solving PDEs with Hilbert Complexes

### APPENDIX: Literature

## literature (FA-ToolBox, complexes, a posteriori error estimates, ...)

some results of this talk:

- Py: *Solution Theory, Variational Formulations, and Functional a Posteriori Error Estimates for General First Order Systems with Applications to Electro-Magneto-Statics and More*, (NFAO) Numerical Functional Analysis and Optimization, 2019

# literature (complexes, Friedrichs type constants, Maxwell constants)

results of this talk:

- Py: *On Constants in Maxwell Inequalities for Bounded and Convex Domains*, Zapiski POMI/ (JMS)Journal of Mathematical Sciences (Springer New York), 2015
- Py: *On Maxwell's and Poincare's Constants*, (DCDS) Discrete and Continuous Dynamical Systems - Series S, 2015
- Py: *On the Maxwell Constants in 3D*, (M2AS) Mathematical Methods in the Applied Sciences, 2017
- Py: *On the Maxwell and Friedrichs/Poincaré Constants in ND*, (MZ) Mathematische Zeitschrift, 2019
  
- Py: ... some (so far) unpublished results



## literature (complexes, Friedrichs type constants, compact embeddings)

- Weck, N.: *Maxwell's boundary value problems on Riemannian manifolds with nonsmooth boundaries*,  
(JMA2) Journal of Mathematical Analysis and Applications, 1974 (1972)
- Picard, R.: *An elementary proof for a compact imbedding result in generalized electromagnetic theory*,  
(MZ) Mathematische Zeitschrift, 1984
- Witsch, K.-J.: *A remark on a compactness result in electromagnetic theory*,  
(M2AS) Mathematical Methods in the Applied Sciences, 1993

results of this talk:

- Bauer, S., Py, Schomburg, M.: *The Maxwell Compactness Property in Bounded Weak Lipschitz Domains with Mixed Boundary Conditions*,  
(SIMA) SIAM Journal on Mathematical Analysis, 2016
- Py, Zulehner, W.: *The divDiv-Complex and Applications to Biharmonic Equations*,  
(AA) Applicable Analysis, 2019
- Py, Zulehner, W.: *The Elasticity Complex*,  
submitted, 2019





## literature (div-curl-lemma)

original papers (local div-curl-lemma):

- Murat, F.: *Compacité par compensation*,  
Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 1978
- Tartar, L.: *Compensated compactness and applications to partial differential equations*,  
Nonlinear analysis and mechanics, Heriot-Watt symposium, 1979



## literature (div-curl-lemma)

recent papers (global div-curl-lemma,  $H^1$ -detour):

- Gloria, A., Neukamm, S., Otto, F.: *Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics*, (IM) Invent. Math., 2015
- Kozono, H., Yanagisawa, T.: *Global compensated compactness theorem for general differential operators of first order*, (ARMA) Arch. Ration. Mech. Anal., 2013
- Schweizer, B.: *On Friedrichs inequality, Helmholtz decomposition, vector potentials, and the div-curl lemma*, accepted preprint, 2018

recent papers (global div-curl-lemma, general results/this talk):

- Waurick, M.: *A Functional Analytic Perspective to the div-curl Lemma*, (JOP) J. Operator Theory, 2018
- Py: *A Global div-curl-Lemma for Mixed Boundary Conditions in Weak Lipschitz Domains and a Corresponding Generalized  $A_0^*$ - $A_1$ -Lemma in Hilbert Spaces*, (ANA) Analysis (Munich), 2019



## literature (full time-dependent Maxwell equations)

- Py, Picard, R.: *A Note on the Justification of the Eddy Current Model in Electrodynamics*,  
(M2AS) Mathematical Methods in the Applied Sciences, 2017
- Py, Picard, R., Trostorff, S., Waurick, M.: *On a Class of Degenerate Abstract Parabolic Problems and Applications to Some Eddy Current Models*,  
submitted, 2019

## literature (Maxwell's equations and more...)

books:



- Langer, U., Py, Repin, S. (Eds): *Maxwell's equations. Analysis and numerics*, Radon Series on Applied Mathematics, De Gruyter, July 2019
- Py: *Maxwell's Equations: Hilbert Space Methods for the Theory of Electromagnetism*, Radon Series on Applied Mathematics, De Gruyter, 2020

(last book: contains all results of this talk and more...)

## Solving PDEs with Hilbert Complexes

### APPENDIX: Solution Theory FOSs and Variational Formulations

## (stat) first order system - solution theory

$$\text{complex} \quad \dots \quad \begin{array}{c} \dots \\ \rightleftarrows \\ \dots \end{array} \quad H_0 \quad \begin{array}{c} A_0 \\ \rightleftarrows \\ A_0^* \end{array} \quad H_1 \quad \begin{array}{c} A_1 \\ \rightleftarrows \\ A_1^* \end{array} \quad H_2 \quad \begin{array}{c} \dots \\ \rightleftarrows \\ \dots \end{array} \quad \dots$$

$$\text{find } x \in D(A_1) \cap D(A_0^*) \text{ st fos} \quad A_1 x = f \quad A_0^* x = g \quad \pi_{N_1} x = k$$

## Theorem (FA-ToolBox II (solution theory))

$$\Downarrow \quad \boxed{D(A_1) \cap D(A_0^*) \leftrightarrow H_1 \text{ compact}}$$

$$\text{fos is uniq sol} \quad \Leftrightarrow \quad f \in R(A_1) \quad g \in R(A_0^*) \quad k \in N_1$$

$$x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus N_1 = D(A_1) \cap D(A_0^*)$$

$$\boxed{x_f := \mathcal{A}_1^{-1} f} \in D(\mathcal{A}_1)$$

$$\boxed{x_g := (\mathcal{A}_0^*)^{-1} g} \in D(\mathcal{A}_0^*)$$

$$\text{dep cont on data} \quad |x|_{H_1} \leq |x_f|_{H_1} + |x_g|_{H_1} + |k|_{H_1} \leq c_{A_1} |f|_{H_2} + c_{A_0} |g|_{H_0} + |k|_{H_1}$$

moreover

$$\pi_{R(A_1^*)} x = x_f \quad \pi_{R(A_0)} x = x_g \quad \pi_{N_1} x = k \quad |x|_{H_1}^2 = |x_f|_{H_1}^2 + |x_g|_{H_1}^2 + |k|_{H_1}^2$$

## Remark

enough  $R(A_0)$  and  $R(A_1)$  cl

## (stat) first order system - variational formulations

$$x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus N_1 = D(A_1) \cap D(A_0^*)$$

$$x_f := \mathcal{A}_1^{-1} f \in D(\mathcal{A}_1) = D(A_1) \cap R(A_1^*) = D(A_1) \cap N(A_0^*) \cap N_1^\perp$$

$$x_g := (\mathcal{A}_0^*)^{-1} g \in D(\mathcal{A}_0^*) = D(A_0^*) \cap R(A_0) = D(A_0^*) \cap N(A_1) \cap N_1^\perp$$

$A_1 x = f$	$A_1 x_f = f$	$A_1 x_g = 0$	$A_1 k = 0$
$A_0^* x = g$	$A_0^* x_f = 0$	$A_0^* x_g = g$	$A_0^* k = 0$
$\pi_{N_1} x = k$	$\pi_{N_1} x_f = 0$	$\pi_{N_1} x_g = 0$	$\pi_{N_1} k = k$

- option I: find  $x_f$  and  $x_g$  separately  $\Rightarrow x = x_f + x_g + k$
- option II: find  $x$  directly

## (stat) first order system - variational formulations I

finding

$$x_f := \mathcal{A}_1^{-1} f \in D(\mathcal{A}_1) = D(A_1) \cap \underbrace{R(A_1^*)}_{=R(\mathcal{A}_1^*)} = D(A_1) \cap N(A_0^*) \cap N_1^\perp$$

$$A_1 x_f = f$$

$$A_0^* x_f = 0$$

$$\pi_{N_1} x_f = 0$$

at least two options

- option Ia: multiply  $A_1 x_f = f$  by  $A_1 \xi \Rightarrow$

$$\forall \xi \in D(\mathcal{A}_1) \quad \langle A_1 x_f, A_1 \xi \rangle_{H_2} = \langle f, A_1 \xi \rangle_{H_2}$$

weak form of  $A_1^* A_1 x_f = A_1^* f$

- option Ib: repr  $x_f = A_1^* y_f$  with potential  $y_f = (\mathcal{A}_1^*)^{-1} x_f \in D(\mathcal{A}_1^*)$   
and mult by  $x_f$  by  $A_1^* \phi \Rightarrow$

$$\forall \phi \in D(\mathcal{A}_1^*) \quad \langle A_1^* y_f, A_1^* \phi \rangle_{H_1} = \langle x_f, A_1^* \phi \rangle_{H_1} = \langle A_1 x_f, \phi \rangle_{H_2} = \langle f, \phi \rangle_{H_2}$$

weak form of  $A_1 x_f = f$  and  $A_1 A_1^* y_f = f$

analogously for  $x_g$







## key tools to prove compact embeddings

crucial tool: compact embeddings

- localisation to top triv domains by partition of unity

fine for first order operators  $\checkmark$

tech prob for second order operators

sol: regular decompositions in  $H^{-1}$ , ...

- Helmholtz decompositions

- regular potentials

here is the hard analysis: weak/strong Lipschitz domains, mixed bc, ...

- boiling down to Rellich's selection theorem

## Solving PDEs with Hilbert Complexes

## APPENDIX: Regular Potentials and Regular Decompositions

joint work with

Ralf Hiptmair, Clemens Pechstein, Michael Schomburg, Walter Zulehner  
generalises and improves work/results of Costabel and McIntosh

## regular potentials

## Theorem (regular potentials)

Let  $(\Omega, \Gamma_t)$  be a bounded strong Lipschitz pair and  $k \geq 0$ . Then there exists a continuous linear operator

$$S_{d,k}^q : \mathring{H}_{\Gamma_t}^{k,q}(\Omega) \cap \mathring{D}_{\Gamma_t,0}^q(\Omega) \cap \mathcal{H}^q(\Omega)^\perp \longrightarrow \mathring{H}_{\Gamma_t}^{k+1,q-1}(\Omega),$$

such that  $d S_{d,k}^q = \text{id} |_{\mathring{H}_{\Gamma_t}^{k,q}(\Omega) \cap \mathring{D}_{\Gamma_t,0}^q(\Omega) \cap \mathcal{H}^q(\Omega)^\perp}$ . In particular,

$$\begin{aligned} \mathring{H}_{\Gamma_t}^{k,q}(\Omega) \cap \mathring{D}_{\Gamma_t,0}^q(\Omega) \cap \mathcal{H}^q(\Omega)^\perp &= \mathring{H}_{\Gamma_t}^{k,q}(\Omega) \cap d \mathring{D}_{\Gamma_t}^{q-1}(\Omega) \\ &= d \mathring{H}_{\Gamma_t}^{k+1,q-1}(\Omega) \\ &= d \mathring{D}_{\Gamma_t}^{k,q-1}(\Omega) \end{aligned}$$

and the regular  $\mathring{H}_{\Gamma_t}^{k+1,q-1}(\Omega)$ -potential depends continuously on the data. Especially, these spaces are closed subspaces of  $\mathring{H}^{k,q}(\Omega)$  and  $S_{d,k}^q$  is a right inverse to  $d$ .

## regular decompositions

## Theorem (regular decompositions)

Let  $(\Omega, \Gamma_t)$  be a bounded strong Lipschitz pair and  $k \geq 0$ . Then the regular decompositions

$$\begin{aligned} \mathring{D}_{\Gamma_t}^{k,q}(\Omega) &= \mathring{H}_{\Gamma_t}^{k+1,q}(\Omega) + d \mathring{H}_{\Gamma_t}^{k+1,q-1}(\Omega) \\ &\quad \cap \quad \cup \\ &= \mathcal{S}_{d,k}^{q+1} d \mathring{D}_{\Gamma_t}^{k,q}(\Omega) \dot{+} (\mathring{H}_{\Gamma_t}^{k,q}(\Omega) \cap \mathring{D}_{\Gamma_t,0}^q(\Omega)) \end{aligned}$$

hold with linear and continuous regular decomposition resp. potential operators, which can be defined explicitly by the orthonormal Helmholtz projectors and the operators  $\mathcal{S}_{d,k}^q$ .

## dual regular potentials and decompositions

Dual regular potentials and decompositions involving

$$\mathring{H}_{\Gamma_n}^{-k,q}(\Omega) = \mathring{H}_{\Gamma_t}^{k,q}(\Omega)'$$

can be proved by Banach space duality. E.g.:

- $\mathring{D}_{\Gamma_t}^{k,q}(\Omega)' = \mathring{\Delta}_{\Gamma_n}^{-k-1,q}(\Omega) := \{E' \in \mathring{H}_{\Gamma_n}^{-k-1,q}(\Omega) : \delta E' \in \mathring{H}_{\Gamma_n}^{-k-1,q-1}(\Omega)\}$
- dual ranges are closed
- dual Friedrichs/Poincaré type estimates, inf-sup condition, i.e.,

$$\delta^{-1} : \delta \mathring{H}_{\Gamma_n}^{-k,q}(\Omega) \longrightarrow (\mathring{d} \mathring{H}_{\Gamma_t}^{k+1,q-1}(\Omega))' \quad \text{cont}$$

$$\Updownarrow$$

$$\forall H' \in (\mathring{d} \mathring{H}_{\Gamma_t}^{k+1,q-1}(\Omega))' \quad \frac{1}{c} |H'|_{\mathring{H}_{\Gamma_n}^{-k,q}(\Omega)} \leq |\delta H'|_{\mathring{H}_{\Gamma_n}^{-k-1,q-1}(\Omega)} \leq c |H'|_{\mathring{H}_{\Gamma_n}^{-k,q}(\Omega)}$$

$$\Updownarrow$$

$$0 < \inf_{0 \neq H \in \mathring{H}_{\Gamma_t}^{k+1,q-1}(\Omega)} \sup_{0 \neq E \in \mathring{H}_{\Gamma_t}^{k+1,q-1}(\Omega)} \frac{\langle H, \mathring{d} E \rangle_{\mathring{H}_{\Gamma_t}^{k,q}(\Omega)}}{|H|_{\mathring{H}_{\Gamma_t}^{k,q}(\Omega)} |E|_{\mathring{H}_{\Gamma_t}^{k+1,q-1}(\Omega)}}$$



# dual regular potentials and decompositions

$\Omega$  top triv  $\Rightarrow$

$$d\mathring{H}_{\Gamma_t}^{k+1,q-1}(\Omega) = \mathring{H}_{\Gamma_t}^{k,q}(\Omega) \cap \mathring{D}_{\Gamma_t,0}^q(\Omega)$$

$$\delta\mathring{H}_{\Gamma_n}^{-k,q}(\Omega) = \mathring{\Delta}_{\Gamma_n,0}^{-k-1,q-1}(\Omega) = \{H' \in \mathring{H}_{\Gamma_n}^{-k-1,q-1}(\Omega) : \delta H' = 0\}$$

## Solving PDEs with Hilbert Complexes

APPENDIX: Friedrichs/Poincaré/Maxwell constants (numerics)

joint work with

Jan Valdman



# Friedrichs/Poincaré/Maxwell constants

assumption:  $\varepsilon = \mu = 1$  and  $\Gamma_t = \Gamma$ , i.e.,  $c_{fp} = c_f$  or  $\Gamma_n = \Gamma$ , i.e.,  $c_{fp} = c_p$

## Lemma (Maxwell-Poincaré constants)

$$\Omega \text{ convex and bounded} \quad \Rightarrow \quad c_m \leq c_p \leq \frac{\text{diam}\Omega}{\pi}$$

## Mild Conjecture (Maxwell-Poincaré constants)

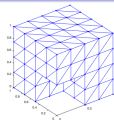
$$\Omega \text{ convex and bounded} \quad \Rightarrow \quad c_f \leq c_m \leq c_p \leq \frac{\text{diam}\Omega}{\pi}$$

## Theorem (FA-ToolBox / Friedrichs-Poincaré type estimates and constants)

$\forall \varphi \in D(\mathcal{A}_0)$	$ \varphi _{H_0} \leq c_{A_0}  A_0 \varphi _{H_1}$	$\Leftrightarrow$	$\forall \varphi \in H^1_{\Gamma}$	$ \varphi _{L^2} \leq c_f  \nabla \varphi _{L^2}$
$\forall \phi \in D(\mathcal{A}_0^*)$	$ \phi _{H_1} \leq c_{A_0}  A_0^* \phi _{H_0}$	$\Leftrightarrow$	$\forall \Phi \in D \cap \nabla H^1_{\Gamma}$	$ \Phi _{L^2} \leq c_f  \text{div } \Phi _{L^2}$
$\forall \phi \in D(\mathcal{A}_1)$	$ \phi _{H_1} \leq c_{A_1}  A_1 \phi _{H_2}$	$\Leftrightarrow$	$\forall \Phi \in R_{\Gamma} \cap \text{rot } R$	$ \Phi _{L^2} \leq c_m  \text{rot } \Phi _{L^2}$
$\forall \psi \in D(\mathcal{A}_1^*)$	$ \psi _{H_2} \leq c_{A_1}  A_1^* \psi _{H_1}$	$\Leftrightarrow$	$\forall \Psi \in R \cap \text{rot } R_{\Gamma}$	$ \Psi _{L^2} \leq c_m  \text{rot } \Psi _{L^2}$
$\forall \psi \in D(\mathcal{A}_2)$	$ \psi _{H_2} \leq c_{A_2}  A_2 \psi _{H_3}$	$\Leftrightarrow$	$\forall \Psi \in D_{\Gamma} \cap \nabla H^1$	$ \Psi _{L^2} \leq c_p  \text{div } \Psi _{L^2}$
$\forall \xi \in D(\mathcal{A}_2^*)$	$ \xi _{H_3} \leq c_{A_2}  A_2^* \xi _{H_2}$	$\Leftrightarrow$	$\forall \zeta \in H^1 \cap \mathbb{R}^{\perp}$	$ \zeta _{L^2} \leq c_p  \nabla \zeta _{L^2}$

## Friedrichs/Poincaré/Maxwell constants

surprise numerical tests show even for non-convex domains and mixed bc  
e.g., Fichera corner domain



## Conjecture (Maxwell-Poincaré constants)

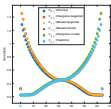
$$c_f \leq \min\{c_{fp}, c_{pf}\} \leq c_m \leq \max\{c_{fp}, c_{pf}\} \leq \sup_{\Gamma_t \neq \emptyset} \{c_{fp}\} < \infty$$

## Theorem (FA-ToolBox / Friedrichs-Poincaré type estimates and constants)

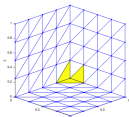
$\forall \varphi \in D(\mathcal{A}_0)$	$ \varphi _{H_0} \leq c_{A_0}  A_0 \varphi _{H_1}$	$\Leftrightarrow$	$\forall \varphi \in H_{\Gamma_t}^1$	$ \varphi _{L^2} \leq c_{fp}  \nabla \varphi _{L^2}$
$\forall \phi \in D(\mathcal{A}_0^*)$	$ \phi _{H_1} \leq c_{A_0}  A_0^* \phi _{H_0}$	$\Leftrightarrow$	$\forall \Phi \in D_{\Gamma_n} \cap \nabla H_{\Gamma_t}^1$	$ \Phi _{L^2} \leq c_{fp}  \operatorname{div} \Phi _{L^2}$
$\forall \phi \in D(\mathcal{A}_1)$	$ \phi _{H_1} \leq c_{A_1}  A_1 \phi _{H_2}$	$\Leftrightarrow$	$\forall \Phi \in R_{\Gamma_t} \cap \operatorname{rot} R_{\Gamma_n}$	$ \Phi _{L^2} \leq c_m  \operatorname{rot} \Phi _{L^2}$
$\forall \psi \in D(\mathcal{A}_1^*)$	$ \psi _{H_2} \leq c_{A_1}  A_1^* \psi _{H_1}$	$\Leftrightarrow$	$\forall \Psi \in R_{\Gamma_n} \cap \operatorname{rot} R_{\Gamma_t}$	$ \Psi _{L^2} \leq c_m  \operatorname{rot} \Psi _{L^2}$
$\forall \psi \in D(\mathcal{A}_2)$	$ \psi _{H_2} \leq c_{A_2}  A_2 \psi _{H_3}$	$\Leftrightarrow$	$\forall \Psi \in D_{\Gamma_t} \cap \nabla H_{\Gamma_n}^1$	$ \Psi _{L^2} \leq c_{pf}  \operatorname{div} \Psi _{L^2}$
$\forall \xi \in D(\mathcal{A}_2^*)$	$ \xi _{H_3} \leq c_{A_2}  A_2^* \xi _{H_2}$	$\Leftrightarrow$	$\forall \zeta \in H_{\Gamma_n}^1$	$ \zeta _{L^2} \leq c_{pf}  \nabla \zeta _{L^2}$



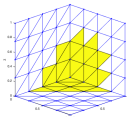
## Friedrichs/Poincaré/Maxwell constants



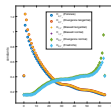
2D unit square



3D unit cube



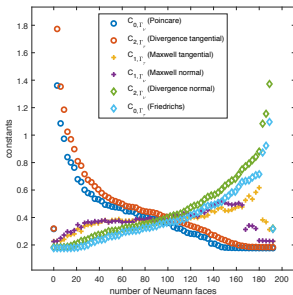
3D Fichera corner



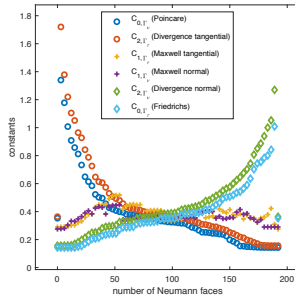
2D L-shape

## Conjecture (Maxwell-Poincaré constants)

$$c_f \leq \min\{c_{fp}, c_{pf}\} \leq c_m \leq \max\{c_{fp}, c_{pf}\} \leq \sup_{\Gamma_t \neq \emptyset} \{c_{fp}\} < \infty$$



3D unit cube



3D Fichera corner domain

## Solving PDEs with Hilbert Complexes

### APPENDIX: A Posteriori Error Estimates for BEM (Boundary Element Method)

joint work with

Stefan Kurz, Dirk Praetorius, Sergey Repin, Daniel Sebastian

## functional a posteriori error estimates for BEM

problem: num approx with BEM

$$\Delta u = 0 \quad \text{in } \Omega, \quad u|_{\Gamma} = g \quad \text{on } \Gamma.$$

functional a posteriori error estimates: num approx with FEM

$$\max_{\substack{E \in L^2(\Omega) \\ \operatorname{div} E = 0}} \left( 2 \langle n \cdot E, g - \tilde{u}|_{\Gamma} \rangle_{H^{-1/2}(\Gamma)} - |E|_{L^2(\Omega)}^2 \right) = |\nabla(u - \tilde{u})|_{L^2(\Omega)}^2 = \min_{\substack{v \in H^1(\Omega) \\ v|_{\Gamma} = g - \tilde{u}|_{\Gamma}}} |\nabla v|_{L^2(\Omega)}^2$$

natural energy norm ( $H^1(\Omega)$ -volume norm)

idea: compute upper and lower bounds in a thin boundary layer using FEM

## functional a posteriori error estimates for BEM

$$\max_{\substack{E \in L^2(\Omega) \\ \operatorname{div} E = 0}} (2 \langle n \cdot E, g - \tilde{u}|_{\Gamma} \rangle_{H^{-1/2}(\Gamma)} - |E|_{L^2(\Omega)}^2) = |\nabla(u - \tilde{u})|_{L^2(\Omega)}^2 = \min_{\substack{v \in H^1(\Omega) \\ v|_{\Gamma} = g - \tilde{u}|_{\Gamma}}} |\nabla v|_{L^2(\Omega)}^2$$

minimiser of upper bound  $\bar{v} = u - \tilde{u}$ : standard Dirichlet-Laplacian

$$\Delta v = 0 \quad \text{in } \Omega, \quad v|_{\Gamma} = g - \tilde{u}|_{\Gamma} \quad \text{on } \Gamma$$

exact solution is  $v = \bar{v} \Rightarrow$  standard FEM on boundary layer for  $v$

maximiser of lower bound  $\underline{E} = \nabla \bar{v} = \nabla(u - \tilde{u})$ : Neumann-type-Laplacian

$$\Delta v = 0 \quad \text{in } \Omega, \quad n \cdot \nabla v|_{\Gamma} = \langle g - \tilde{u}|_{\Gamma}, n \cdot \nabla(\widehat{\cdot})|_{\Gamma} \rangle \quad \text{in } H^{-1/2}(\Gamma)$$

(here  $\widehat{(\cdot)}$  harmonic extension and  $n \cdot \nabla(\widehat{\cdot})|_{\Gamma}$  Dirichlet2Neumann operator)

exact solution is  $v = \bar{v}$  and  $\nabla v = \underline{E} \Rightarrow$  non-standard FEM on bd layer for  $E$

$\Rightarrow$  saddle point formulation (mixed/dual Laplacian)

Find  $(E, v) \in H(\operatorname{div}, \Omega) \times L^2(\Omega)$  s.t. for all  $(\Phi, \varphi) \in H(\operatorname{div}, \Omega) \times L^2(\Omega)$

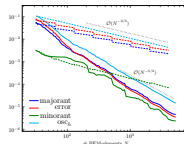
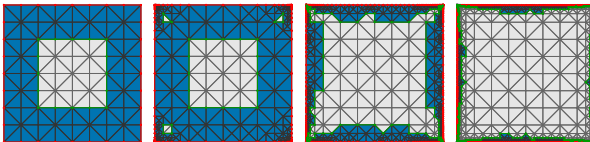
$$\langle E, \Phi \rangle_{L^2(\Omega)} + \langle \operatorname{div} \Phi, v \rangle_{L^2(\Omega)} = \langle n \cdot \Phi, g - \tilde{u}|_{\Gamma} \rangle_{H^{-1/2}(\Gamma)},$$

$$\langle \operatorname{div} E, \varphi \rangle_{L^2(\Omega)} = 0$$

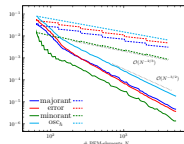
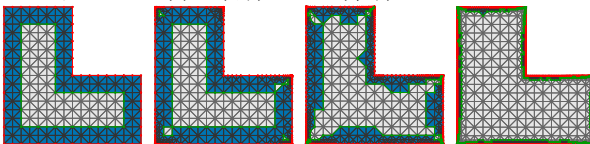
unique sol  $(E, v) = (\underline{E}, \bar{v})$

## functional a posteriori error estimates for BEM - some pics

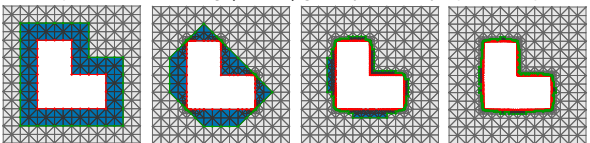
$\Omega$ : unit square,  $u(x) = \cosh(x_1)\cos(x_2)$ , known smooth solution  $u$



$\Omega$ : L-shaped domain,  $u(x) = u(r, \varphi) = r^{2/3} \cos(2/3\varphi)$ , known non-smooth solution  $u$



$\Omega$ : L-shaped exterior domain,  $g$  (bd data) given by double-layer potential operator, unknown exact solution  $u$



oscillatory error

upper bound

exact error

lower bound

convergence rates

adaptive mesh-ref with Dörfler marking (solid lines) vs. unif mesh-ref (dashed lines)

## Solving PDEs with Hilbert Complexes

### APPENDIX: Electro-Magneto-Static Optimal Control with A Posteriori Error Estimates

joint work with

Irwin Yousept



# Analysis

- $\emptyset \neq \omega \subset \Omega$ : beschränkte Lipschitz Gebiete mit Rändern  $\gamma$  und  $\Gamma$ .
- $\varepsilon, \mu : \Omega \rightarrow \mathbb{R}^{3 \times 3}$ :  $L^\infty$ -Materialeigenschaften des Mediums

Aufgabenstellung: (klassische Formulierungen)

gegeben:  $\widehat{H} \in L^2(\Omega)$  erwünschtes magnetisches Feld,  $\widehat{j} \in L^2(\omega)$  erwünschte Kontrolle,  $\widehat{J} \in L^2(\Omega)$  externe Stromdichte

gesucht:  $\bar{j}$  optimale Stromdichte in  $\omega$ , Lösung des Optimierungsproblems

$$F(\bar{j}) = \min_{j \in \mathcal{J}} F(j), \quad F(j) := \|H(j) - \widehat{H}\|_{L^2_\mu(\Omega)}^2 + \kappa \|j - \widehat{j}\|_{L^2_\varepsilon(\omega)}^2 \quad (1)$$

unter der Nebenbedingung des magnetostatischen Problems (MSP) für  $H = H(j)$

$$\begin{aligned} \operatorname{rot} H &= \varepsilon \pi(\zeta j + J) && \text{in } \Omega, \\ \operatorname{div} \mu H &= 0 && \text{in } \Omega, \\ n \cdot \mu H &= 0 && \text{auf } \Gamma, \\ \mu H &\perp_{L^2(\Omega)} \mathcal{H}_N(\Omega). \end{aligned}$$

Hier: Neumann Felder  $\mathcal{H}_N(\Omega) = \{E \in L^2(\Omega) : \operatorname{rot} E = 0, \operatorname{div} \mu H = 0, n \cdot \mu H|_\Gamma = 0\}$ ,  
 $\zeta$  Nullerweiterung,  $\pi$  Orthonormalprojektor auf Wertebereich von  $\varepsilon^{-1}$ -Rotationen,  
 $\mathcal{J}$  nicht-trivialer und abg Teilraum vom  $L^2_\varepsilon(\omega)$ , z.B.  $\mathcal{J} = \varepsilon^{-1} \mathring{D}_0(\omega) \subset_\zeta \varepsilon^{-1} \mathring{D}_0(\Omega)$

# Analysis

Finde  $\bar{j}$

$$F(\bar{j}) = \min_{j \in \mathcal{J}} F(j), \quad F(j) = \|H(j) - \widehat{H}\|_{L^2_\mu(\Omega)}^2 + \kappa \|j - \widehat{j}\|_{L^2_\varepsilon(\omega)}^2$$

NB (MSP) für  $H = H(j)$

$$\begin{aligned} \operatorname{rot} H &= \varepsilon \pi(\zeta j + J) && \text{in } \Omega, \\ \operatorname{div} \mu H &= 0 && \text{in } \Omega, \\ n \cdot \mu H &= 0 && \text{auf } \Gamma, \\ \mu H &\perp_{L^2(\Omega)} \mathcal{H}_N(\Omega). \end{aligned}$$

Idee: FA-ToolBox ... PDGI (MSP für  $H$ )

↪ Operator  $A$  ( $AH$ )

↪ reduzierter Operator  $\mathcal{A} = A|_{N(A)^\perp}$  ( $\mathcal{A}H$ ) mit allen NB  
(div, RB, Kern)

## Analysis

 $H \rightsquigarrow x$ alle rechten Seiten  $\rightsquigarrow f$ MSP  $\rightsquigarrow A$  bzw.  $\mathcal{A}$ 

$$Ax = f$$

Dann: Finde  $\bar{j}$ 

$$F(\bar{j}) = \min_{j \in \mathcal{J}} F(j), \quad F(j) = \|H(j) - \widehat{H}\|_{L^2_\mu(\Omega)}^2 + \kappa \|j - \widehat{j}\|_{L^2_\varepsilon(\omega)}^2$$

NB (MSP) für  $H = H(j)$ 

$$\rightsquigarrow H = H(j) = \mathcal{A}^{-1}\pi(\zeta j + J) \quad (\text{affin linear})$$

 $\Rightarrow$  Finde  $\bar{j}$ 

$$F(\bar{j}) = \min_{j \in \mathcal{J}} F(j), \quad F(j) = \|\mathcal{A}^{-1}\pi(\zeta j + J) - \widehat{H}\|_{L^2_\mu(\Omega)}^2 + \kappa \|j - \widehat{j}\|_{L^2_\varepsilon(\omega)}^2$$

# Analysis

↪ Tafel

# Analysis

Finde  $\bar{j}$

$$F(\bar{j}) = \min_{j \in \mathcal{J}} F(j), \quad F(j) = \|\mathcal{A}^{-1}\pi(\zeta j + J) - \widehat{H}\|_{L^2_\mu(\Omega)}^2 + \kappa \|j - \widehat{j}\|_{L^2_\varepsilon(\omega)}^2$$

... ein wenig FA ...  $\Rightarrow$

Die eindeutige Lösung  $\bar{j}$  is gegeben durch

die eindeutige Lösung  $(\bar{j}, \bar{E}, \bar{H}) \in \mathcal{J} \times D(\mathcal{A}) \times D(\mathcal{A}^*)$  des Systems

$$\bar{j} = \widehat{j} - \frac{1}{\kappa} \pi_\omega \zeta^* \bar{E}, \quad (\text{Opt. Kontr. durch adj. Zustand } \bar{E})$$

$$\bar{E} = \mathcal{A}^{-1}(\bar{H} - \widehat{H}), \quad (\text{Adjungierte Zustandsgleichung})$$

$$\bar{H} = (\mathcal{A}^*)^{-1}\pi(\zeta \bar{j} + J), \quad (\text{Zustandsgleichung})$$

Numerik  $\Rightarrow$  variationelle Formulierung:  $\bar{j} = \widehat{j} - \frac{1}{\kappa} \pi_\omega \zeta^* \bar{E}$  und  $\bar{E} \in D(\mathcal{A})$  löst

$$\forall \Phi \in D(\mathcal{A}) \quad \langle \mathcal{A}\bar{E}, \mathcal{A}\Phi \rangle_{L^2_\mu(\Omega)} + \frac{1}{\kappa} \langle \pi_\omega \zeta^* \bar{E}, \pi_\omega \zeta^* \Phi \rangle_{L^2_\varepsilon(\omega)} = \langle \widehat{\zeta j} + J, \Phi \rangle_{L^2_\varepsilon(\omega)} - \langle \widehat{H}, \mathcal{A}\Phi \rangle_{L^2_\mu(\Omega)}$$

# Analysis und ein wenig Numerik

variationelle Formulierung:  $\bar{j} = \widehat{j} - \frac{1}{\kappa} \pi_\omega \zeta^* \bar{E}$  und  $\bar{E} \in D(\mathcal{A})$  löst

$$\forall \Phi \in D(\mathcal{A}) \quad \langle A\bar{E}, A\Phi \rangle_{L^2_\mu(\Omega)} + \frac{1}{\kappa} \langle \pi_\omega \zeta^* \bar{E}, \pi_\omega \zeta^* \Phi \rangle_{L^2_\varepsilon(\omega)} = \langle \widehat{j} + J, \Phi \rangle_{L^2_\varepsilon(\Omega)} - \langle \widehat{H}, A\Phi \rangle_{L^2_\mu(\Omega)}$$

↑

degeneriert  $\Rightarrow$  Sattelpunktformulierung

$$N(A) = H_0(\text{rot}, \Omega) \supset \nabla H_0^1(\Omega)$$

$\Rightarrow$  Finde

$(\bar{E}, \bar{u}, \bar{v}) \in H_0(\text{rot}, \Omega) \times H_0^1(\Omega) \times H^1(\omega)$ , so dass für alle

$(\Phi, \varphi, \phi) \in H_0(\text{rot}, \Omega) \times H_0^1(\Omega) \times H^1(\omega)$

$$\begin{aligned} & \langle \mu^{-1} \text{rot } \bar{E}, \text{rot } \Phi \rangle_{L^2(\Omega)} + \frac{1}{\kappa} \langle \varepsilon \zeta^* \bar{E}, \zeta^* \Phi \rangle_{L^2(\omega)} \\ & + \langle \varepsilon \Phi, \nabla \bar{u} \rangle_{L^2(\Omega)} + \frac{1}{\kappa} \langle \varepsilon \zeta^* \Phi, \nabla \bar{v} \rangle_{L^2(\omega)} = \langle \varepsilon (\widehat{j} + J), \Phi \rangle_{L^2(\Omega)} - \langle \widehat{H}, \text{rot } \Phi \rangle_{L^2(\Omega)}, \\ & \langle \varepsilon \bar{E}, \nabla \varphi \rangle_{L^2(\Omega)} = 0, \\ & \langle \varepsilon \zeta^* \bar{E}, \nabla \phi \rangle_{L^2(\omega)} + \langle \varepsilon \nabla \bar{v}, \nabla \phi \rangle_{L^2(\omega)} = 0 \end{aligned}$$



# Analysis und ein wenig Numerik

... nun funktionale a posteriori Fehlerschätzer im Stile von Repin ...

# ein wenig Numerik

## Gebiete

$$\begin{aligned}\Omega &:= (-0.5, 1)^3, & \Xi &:= (-0.5, 0) \times (-0.5, 0) \times (-0.5, 1), \\ \omega &:= (0, 0.5)^3, & \Theta &:= (0, 0.5) \times (0, 0.5) \times (-0.5, 1)\end{aligned}$$

$\varepsilon := 1$ ,  $\kappa := 1$ , stückweise konstante magnetische Permeabilität  $\mu := \begin{cases} 10 & \text{in } \Xi, \\ 1 & \text{sonst.} \end{cases}$

Konstruktion analytischer Lösungen:

$$E(x) := \frac{\mu^2(x)}{8\pi^2} \sin^2(2\pi x_1) \sin^2(2\pi x_2) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad x \in \Omega$$

$$\vec{j}(x) := 100 \begin{bmatrix} \sin(2\pi x_1) \cos(2\pi x_2) \\ -\cos(2\pi x_1) \sin(2\pi x_2) \\ 0 \end{bmatrix}, \quad x \in \omega$$

$$\vec{E} := \chi_{\Omega \setminus \Theta} E, \quad \vec{H} := \mu^{-1} \operatorname{rot} E, \quad \widehat{H} := \chi_{\Theta} \vec{H}, \quad \widehat{j} := \vec{j}, \quad J := \operatorname{rot} \vec{H} - \chi_{\Omega} \vec{j}$$

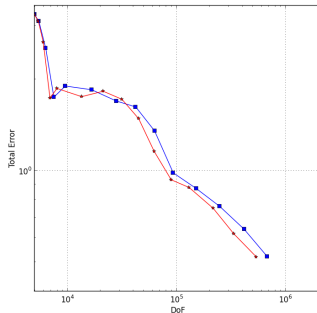
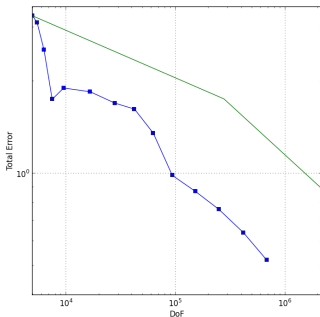


## ein wenig Numerik

Gesamtfehler bei uniformer Gitterverfeinerung (**grüne Linie**)

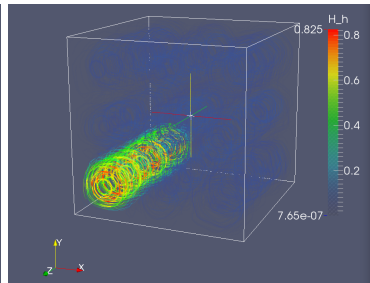
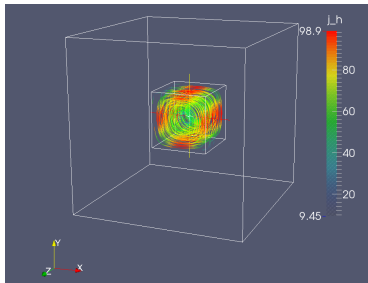
Gesamtfehler bei adaptiver Gitterverf basierend auf dem exakten Fehler (**rote Linie**)

GF bei adaptiver Gitterverf bas auf unserem a posteriori Fehlerschätzer (**blaue Linie**)



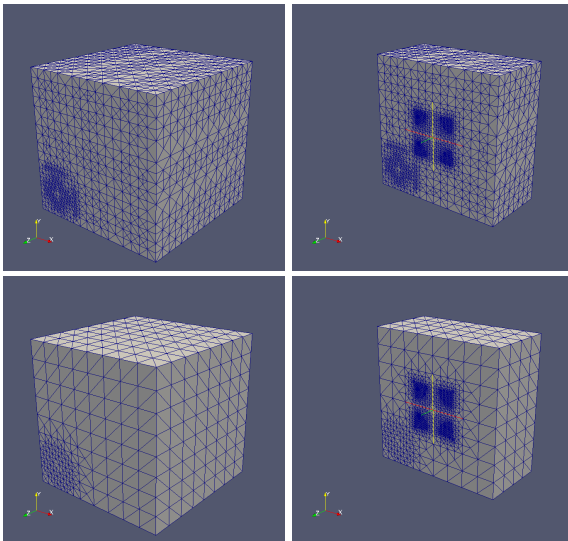
## ein wenig Numerik

berechnete optimale Kontrolle (linkes Bildchen)  
berechnetes optimales Magnetfeld (rechtes Bildchen)  
(alles auf dem feinsten adaptiven Gitter)



## ein wenig Numerik

adaptives Gitter resul aus unserem Fehlerschätzer (oben) und exaktem Fehler (unten)





COMMERCIALS

... the world is full of complexes ... ;-)

⇒ relaxing at (and you're all invited!)

## AANMPDE<sub>JS</sub>13 2020

13th Workshop on **Analysis and Advanced Numerical Methods for Partial Differential Equations** (not only) for **Junior Scientists**

<https://www.uni-due.de/maxwell/aanmpde13/>

September 28 - October 2 2020  
Pitsidia, Crete, Greece

INVITED

KEY NOTE SPEAKERS:

Joachim Schöberl (Wien)  
Carsten Trunk (Ilmenau)  
Ragnar Winther (Oslo)

ORGANIZERS: Johannes Kraus, Dirk Pauly, Sergey Repin, Marcus Waurick

