

On the Maxwell Constants in 3D

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Sobolev Spaces

- $\Omega \subset \mathbb{R}^3$ bounded domain with Lipschitz (or weaker) boundary $\Gamma = \partial\Omega$
crucial assumption MCP: embedding $\mathring{R}(\Omega) \cap D(\Omega) \hookrightarrow L^2(\Omega)$ compact
- Sobolev spaces for $\text{rot} = \text{curl}$

$$R(\Omega) := \{E \in L^2(\Omega) : \text{rot } E \in L^2(\Omega)\} \quad (= H(\text{curl}; \Omega))$$

$$R_0(\Omega) := \{E \in R(\Omega) : \text{rot } E = 0\}$$

$$\mathring{R}(\Omega) := \overline{\mathring{C}^\infty(\Omega)}^{R(\Omega)} = \{E \in R(\Omega) : \tau E = 0\}$$

$$\mathring{R}_0(\Omega) := \mathring{R}(\Omega) \cap R_0(\Omega)$$

analogously for div

$$D(\Omega), \quad D_0(\Omega), \quad \mathring{D}(\Omega), \quad \mathring{D}_0(\Omega) \quad (= H(\text{div}; \Omega))$$

and Dirichlet resp. Neumann fields $\mathcal{H}_D(\Omega)$ resp. $\mathcal{H}_N(\Omega)$

$$\begin{aligned} \mathcal{H}_D(\Omega) &:= \mathring{R}_0(\Omega) \cap D_0(\Omega) \quad (\text{finite dimensional by compact embedding}) \\ &= \{E \in L^2(\Omega) : \text{rot } E = 0, \text{div } E = 0, \tau E = 0\} \end{aligned}$$

Estimates for the Maxwell Constants

open problem: estimates for the Maxwell constants c_m in 3D?

$$\forall E \in \mathring{R}(\Omega) \cap D(\Omega) \cap \mathcal{H}_D(\Omega)^\perp \quad |E|_{L^2(\Omega)} \leq c_{m,t} (|\operatorname{rot} E|_{L^2(\Omega)}^2 + |\operatorname{div} E|_{L^2(\Omega)}^2)^{1/2}$$

$$\forall E \in R(\Omega) \cap \mathring{D}(\Omega) \cap \mathcal{H}_N(\Omega)^\perp \quad |E|_{L^2(\Omega)} \leq c_{m,n} (|\operatorname{rot} E|_{L^2(\Omega)}^2 + |\operatorname{div} E|_{L^2(\Omega)}^2)^{1/2}$$

question:

$$? \leq c_{m,t}, c_{m,n} \leq ?$$

in 2D well known

$$\mathring{c}_p \leq c_{m,t}, c_{m,n} \leq c_p \quad \text{even} \quad \mathring{c}_p < c_{m,t} = c_{m,n} = c_p$$

with Poincaré constants

$$\forall u \in \mathring{H}^1(\Omega) \quad |u|_{L^2(\Omega)} \leq \mathring{c}_p |\nabla u|_{L^2(\Omega)} \quad (\mathring{c}_p \leq d, \text{ if } \Omega \text{ bd in one dir., trivial})$$

$$\forall u \in H^1(\Omega) \cap \mathbb{R}^\perp \quad |u|_{L^2(\Omega)} \leq c_p |\nabla u|_{L^2(\Omega)} \quad (c_p \leq \operatorname{diam}(\Omega)/\pi, \text{ if } \Omega \text{ bd \& convex,}$$

main result

'60 Payne & Weinberger)

Theorem ('13 DP)

$$\Omega \subset \mathbb{R}^3 \text{ bounded \& convex} \quad \Rightarrow \quad \mathring{c}_p \leq c_{m,t} \leq c_{m,n} = c_p \leq \operatorname{diam}(\Omega)/\pi$$

note always

$$\mathring{c}_p = \frac{1}{\sqrt{\lambda_1}} < \frac{1}{\sqrt{\mu_2}} = c_p$$

Step 1: Problem Reduction by Helmholtz Decomposition (as usual)

reminder

$$\forall E \in \mathring{R}(\Omega) \cap D(\Omega) \cap \mathcal{H}_D(\Omega)^\perp \quad |E|_{L^2(\Omega)} \leq c_{m,t} (|\operatorname{rot} E|_{L^2(\Omega)}^2 + |\operatorname{div} E|_{L^2(\Omega)}^2)^{1/2}$$

$$\forall E \in R(\Omega) \cap \mathring{D}(\Omega) \cap \mathcal{H}_N(\Omega)^\perp \quad |E|_{L^2(\Omega)} \leq c_{m,n} (|\operatorname{rot} E|_{L^2(\Omega)}^2 + |\operatorname{div} E|_{L^2(\Omega)}^2)^{1/2}$$

Helmholtz decomposition \Rightarrow splits 2 problems into 4 'nicer' problems

$$\forall E \in D(\Omega) \cap \underbrace{\mathring{R}_0(\Omega) \cap \mathcal{H}_D(\Omega)^\perp}_{=\nabla \mathring{H}^1(\Omega)} \quad |E|_{L^2(\Omega)} \leq c_{m,t,\operatorname{div}} |\operatorname{div} E|_{L^2(\Omega)}$$

$$\forall E \in \mathring{R}(\Omega) \cap \underbrace{D_0(\Omega) \cap \mathcal{H}_D(\Omega)^\perp}_{=\operatorname{rot} R(\Omega)} \quad |E|_{L^2(\Omega)} \leq c_{m,t,\operatorname{rot}} |\operatorname{rot} E|_{L^2(\Omega)}$$

$$\forall E \in \mathring{D}(\Omega) \cap \underbrace{R_0(\Omega) \cap \mathcal{H}_N(\Omega)^\perp}_{=\nabla H^1(\Omega)} \quad |E|_{L^2(\Omega)} \leq c_{m,n,\operatorname{div}} |\operatorname{div} E|_{L^2(\Omega)}$$

$$\forall E \in R(\Omega) \cap \underbrace{\mathring{D}_0(\Omega) \cap \mathcal{H}_N(\Omega)^\perp}_{=\operatorname{rot} \mathring{R}(\Omega)} \quad |E|_{L^2(\Omega)} \leq c_{m,n,\operatorname{rot}} |\operatorname{rot} E|_{L^2(\Omega)}$$

How to do Step 1 (Helmholtz Decomposition)

e.g. tangential case

Helmholtz decomposition

$$\begin{aligned} L^2(\Omega) &= \underbrace{\nabla \mathring{H}^1(\Omega)}_{\cap} \oplus \underbrace{D_0(\Omega)}_{\cup} \\ &= \underbrace{\mathring{R}_0(\Omega)}_{\cap} \oplus \underbrace{\text{rot } R(\Omega)}_{\cup} \end{aligned}$$

$$\Rightarrow L^2(\Omega) = \nabla \mathring{H}^1(\Omega) \oplus \mathcal{H}_D(\Omega) \oplus \text{rot } R(\Omega), \quad \mathcal{H}_D(\Omega) = \mathring{R}_0(\Omega) \cap D_0(\Omega)$$

pick some $E \in \mathring{R}(\Omega) \cap D(\Omega) \cap \mathcal{H}_D(\Omega)^\perp$

$$\Rightarrow E = E_\nabla \oplus E_{\text{rot}} \in (\nabla \mathring{H}^1(\Omega) \cap D(\Omega)) \oplus (\text{rot } R(\Omega) \cap \mathring{R}(\Omega))$$

$$\text{as well as } \text{rot } E_{\text{rot}} = \text{rot } E \quad \text{and} \quad \text{div } E_\nabla = \text{div } E$$

$$\begin{aligned} \Rightarrow |E|_{L^2(\Omega)}^2 &= |E_\nabla|_{L^2(\Omega)}^2 + |E_{\text{rot}}|_{L^2(\Omega)}^2 \\ &\leq c_{\text{m,t,div}}^2 |\text{div } E_\nabla|_{L^2(\Omega)}^2 + c_{\text{m,t,rot}}^2 |\text{rot } E_{\text{rot}}|_{L^2(\Omega)}^2 \\ &\leq \max\{c_{\text{m,t,div}}, c_{\text{m,t,rot}}\}^2 (|\text{div } E|_{L^2(\Omega)}^2 + |\text{rot } E|_{L^2(\Omega)}^2) \end{aligned}$$

Step 2: First Results

reminder

$$\forall E \in \mathring{R}(\Omega) \cap D(\Omega) \cap \mathcal{H}_D(\Omega)^\perp \quad |E|_{L^2(\Omega)} \leq c_{m,t} (|\operatorname{rot} E|_{L^2(\Omega)}^2 + |\operatorname{div} E|_{L^2(\Omega)}^2)^{1/2}$$

$$\forall E \in R(\Omega) \cap \mathring{D}(\Omega) \cap \mathcal{H}_N(\Omega)^\perp \quad |E|_{L^2(\Omega)} \leq c_{m,n} (|\operatorname{rot} E|_{L^2(\Omega)}^2 + |\operatorname{div} E|_{L^2(\Omega)}^2)^{1/2}$$

$$\forall E \in D(\Omega) \cap \mathring{\nabla}H^1(\Omega) \quad |E|_{L^2(\Omega)} \leq c_{m,t,\operatorname{div}} |\operatorname{div} E|_{L^2(\Omega)}$$

$$\forall E \in \mathring{R}(\Omega) \cap \operatorname{rot} R(\Omega) \quad |E|_{L^2(\Omega)} \leq c_{m,t,\operatorname{rot}} |\operatorname{rot} E|_{L^2(\Omega)}$$

$$\forall E \in \mathring{D}(\Omega) \cap \nabla H^1(\Omega) \quad |E|_{L^2(\Omega)} \leq c_{m,n,\operatorname{div}} |\operatorname{div} E|_{L^2(\Omega)}$$

$$\forall E \in R(\Omega) \cap \operatorname{rot} \mathring{R}(\Omega) \quad |E|_{L^2(\Omega)} \leq c_{m,n,\operatorname{rot}} |\operatorname{rot} E|_{L^2(\Omega)}$$

trivially: $c_{m,t,\operatorname{rot}}, c_{m,t,\operatorname{div}} \leq c_{m,t}$ and $c_{m,n,\operatorname{rot}}, c_{m,n,\operatorname{div}} \leq c_{m,n}$

trivially: $c_{m,t} \leq \max\{c_{m,t,\operatorname{rot}}, c_{m,t,\operatorname{div}}\}$ and $c_{m,n} \leq \max\{c_{m,n,\operatorname{rot}}, c_{m,n,\operatorname{div}}\}$ (Helmholtz)

trivially: $c_{m,t} = \max\{c_{m,t,\operatorname{rot}}, c_{m,t,\operatorname{div}}\}$ and $c_{m,n} = \max\{c_{m,n,\operatorname{rot}}, c_{m,n,\operatorname{div}}\}$

Lemma

$$c_{m,t,\operatorname{div}} = \mathring{c}_p \quad c_{m,n,\operatorname{div}} = c_p \quad c_{m,t,\operatorname{rot}} = c_{m,n,\operatorname{rot}}$$

remains to estimate

$$c_{m,\operatorname{rot}} := c_{m,t,\operatorname{rot}} = c_{m,n,\operatorname{rot}}$$

Step 3: Main Results

reminder

$$\forall E \in \mathring{R}(\Omega) \cap D(\Omega) \cap \mathcal{H}_D(\Omega)^\perp \quad |E|_{L^2(\Omega)} \leq c_{m,t} (|\operatorname{rot} E|_{L^2(\Omega)}^2 + |\operatorname{div} E|_{L^2(\Omega)}^2)^{1/2}$$

$$\forall E \in R(\Omega) \cap \mathring{D}(\Omega) \cap \mathcal{H}_N(\Omega)^\perp \quad |E|_{L^2(\Omega)} \leq c_{m,n} (|\operatorname{rot} E|_{L^2(\Omega)}^2 + |\operatorname{div} E|_{L^2(\Omega)}^2)^{1/2}$$

$$\forall E \in D(\Omega) \cap \mathring{\nabla}H^1(\Omega) \quad |E|_{L^2(\Omega)} \leq \mathring{c}_p |\operatorname{div} E|_{L^2(\Omega)}$$

$$\forall E \in \mathring{R}(\Omega) \cap \operatorname{rot} R(\Omega) \quad |E|_{L^2(\Omega)} \leq c_{m,\operatorname{rot}} |\operatorname{rot} E|_{L^2(\Omega)}$$

$$\forall E \in \mathring{D}(\Omega) \cap \nabla H^1(\Omega) \quad |E|_{L^2(\Omega)} \leq c_p |\operatorname{div} E|_{L^2(\Omega)}$$

$$\forall E \in R(\Omega) \cap \operatorname{rot} \mathring{R}(\Omega) \quad |E|_{L^2(\Omega)} \leq c_{m,\operatorname{rot}} |\operatorname{rot} E|_{L^2(\Omega)}$$

trivially: $c_{m,t} = \max\{c_{m,\operatorname{rot}}, \mathring{c}_p\}$ and $c_{m,n} = \max\{c_{m,\operatorname{rot}}, c_p\}$

remains to estimate

$c_{m,\operatorname{rot}}$

Theorem ('13 DP)

Let Ω be bounded and convex. Then $c_{m,\operatorname{rot}} \leq c_p$. Moreover, $\mathring{c}_p \leq c_{m,t} \leq c_{m,n} = c_p$.

equivalent formulation for eigenvalues

Proof of First Theorem

Proof ... by some functional analysis ...

- $A : D(A) \subset H_1 \rightarrow H_2$ lin., dens. def., closed with adjoint $A^* : D(A^*) \subset H_2 \rightarrow H_1$
- assume $D(A) \cap \overline{R(A^*)} \hookrightarrow H_1$ compact! note $H_1 = \overline{R(A^*)} \oplus N(A)$
- define $M := \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$ and note $M^2 := \begin{bmatrix} A^*A & 0 \\ 0 & AA^* \end{bmatrix}$

$\Rightarrow M, M^2, A^*A, AA^*$ s.a. with compact resolvent (red. op.) \Rightarrow pure point spectra

$$\sigma_p(M) = \pm \sqrt{\sigma_p(A^*A)} = \pm \sqrt{\sigma_p(AA^*)} = \pm \{\kappa_1, \kappa_2, \dots\}, \quad 0 \leq \kappa_n \nearrow \infty$$

looking at first resp. second eigenvalues \Rightarrow

Lemma (school of Rolf Leis: R. Leis & R. Picard, N. Weck, K.-J. Witsch, ...)

$$\inf_{0 \neq u \in D(A) \cap R(A^*)} \frac{\|Au\|_{H_2}^2}{\|u\|_{H_1}^2} = \inf_{0 \neq v \in D(A^*) \cap R(A)} \frac{\|A^*v\|_{H_1}^2}{\|v\|_{H_2}^2}$$

Proof of First Theorem...

reminder

$A : D(A) \subset H_1 \rightarrow H_2$ lin., dens. def., closed, adjoint $A^* : D(A^*) \subset H_2 \rightarrow H_1$

$D(A) \cap R(A^*) \hookrightarrow H_1$ compact

$(R(A^*) = N(A)^\perp$ and $R(A), R(A^*)$ closed and $\forall u \in D(A) \cap R(A^*) : |u|_{H_1} \leq c|Au|_{H_2}$)

$$\inf_{0 \neq u \in D(A) \cap R(A^*)} \frac{\|Au\|_{H_2}^2}{\|u\|_{H_1}^2} = \inf_{0 \neq v \in D(A^*) \cap R(A)} \frac{\|A^*v\|_{H_1}^2}{\|v\|_{H_2}^2}$$

especially

$A := \overset{\circ}{\nabla} : \overset{\circ}{H}^1(\Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega)$, $A^* := -\operatorname{div} : D(\Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega)$

$R(A^*) = N(A)^\perp = \{0\}^\perp = L^2(\Omega)$

$\overset{\circ}{H}^1(\Omega) \cap L^2(\Omega) = \overset{\circ}{H}^1(\Omega) \hookrightarrow L^2(\Omega)$ compact by Rellich's selection theorem

$$\frac{1}{\overset{\circ}{C}_p^2} = \lambda_1 = \inf_{0 \neq u \in \overset{\circ}{H}^1(\Omega)} \frac{|\nabla u|_{L^2(\Omega)}^2}{|u|_{L^2(\Omega)}^2} = \inf_{0 \neq E \in D(\Omega) \cap \overset{\circ}{\nabla} \overset{\circ}{H}^1(\Omega)} \frac{|\operatorname{div} E|_{L^2(\Omega)}^2}{|E|_{L^2(\Omega)}^2} = \frac{1}{C_{m,t,\operatorname{div}}^2}$$

$\Rightarrow C_{m,t,\operatorname{div}} = \overset{\circ}{C}_p$ known!

Proof of First Theorem.....

reminder

$A : D(A) \subset H_1 \rightarrow H_2$ lin., dens. def., closed, adjoint $A^* : D(A^*) \subset H_2 \rightarrow H_1$

$D(A) \cap R(A^*) \hookrightarrow H_1$ compact

$(R(A^*) = N(A)^\perp$ and $R(A), R(A^*)$ closed and $\forall u \in D(A) \cap R(A^*) : |u|_{H_1} \leq c|Au|_{H_2}$)

$$\inf_{0 \neq u \in D(A) \cap R(A^*)} \frac{\|Au\|_{H_2}^2}{\|u\|_{H_1}^2} = \inf_{0 \neq v \in D(A^*) \cap R(A)} \frac{\|A^*v\|_{H_1}^2}{\|v\|_{H_2}^2}$$

especially

$A := \nabla : H^1(\Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega)$, $A^* := -\operatorname{div} : \mathring{D}(\Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega)$

$R(A^*) = N(A)^\perp = \mathbb{R}^\perp$

$H^1(\Omega) \cap \mathbb{R}^\perp \subset H^1(\Omega) \hookrightarrow L^2(\Omega)$ compact by Rellich's selection theorem

$$\frac{1}{c_p^2} = \mu_2 = \inf_{0 \neq u \in H^1(\Omega) \cap \mathbb{R}^\perp} \frac{|\nabla u|_{L^2(\Omega)}^2}{|u|_{L^2(\Omega)}^2} = \inf_{0 \neq E \in \mathring{D}(\Omega) \cap \nabla H^1(\Omega)} \frac{|\operatorname{div} E|_{L^2(\Omega)}^2}{|E|_{L^2(\Omega)}^2} = \frac{1}{c_{m,n,\operatorname{div}}^2}$$

$\Rightarrow c_{m,n,\operatorname{div}} = c_p$ known!

Proof of First Theorem.....

reminder

$A : D(A) \subset H_1 \rightarrow H_2$ lin., dens. def., closed, adjoint $A^* : D(A^*) \subset H_2 \rightarrow H_1$

$D(A) \cap R(A^*) \hookrightarrow H_1$ compact

$(R(A^*) = N(A)^\perp$ and $R(A), R(A^*)$ closed and $\forall u \in D(A) \cap R(A^*) : |u|_{H_1} \leq c|Au|_{H_2}$)

$$\inf_{0 \neq u \in D(A) \cap R(A^*)} \frac{\|Au\|_{H_2}^2}{\|u\|_{H_1}^2} = \inf_{0 \neq v \in D(A^*) \cap R(A)} \frac{\|A^*v\|_{H_1}^2}{\|v\|_{H_2}^2}$$

especially

$A := \mathring{\text{rot}} : \mathring{R}(\Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega)$, $A^* := \text{rot} : R(\Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega)$

$R(A^*) = \text{rot } R(\Omega)$

$\mathring{R}(\Omega) \cap \text{rot } R(\Omega) \subset \mathring{R}(\Omega) \cap D(\Omega) \hookrightarrow L^2(\Omega)$ compact by MCP

$$\frac{1}{c_{m,t,\text{rot}}^2} = \kappa_2 = \inf_{0 \neq E \in \mathring{R}(\Omega) \cap \text{rot } R(\Omega)} \frac{|\text{rot } E|_{L^2(\Omega)}^2}{|E|_{L^2(\Omega)}^2} = \inf_{0 \neq E \in R(\Omega) \cap \text{rot } \mathring{R}(\Omega)} \frac{|\text{rot } E|_{L^2(\Omega)}^2}{|E|_{L^2(\Omega)}^2} = \frac{1}{c_{m,n,\text{rot}}^2}$$

$\Rightarrow c_{m,t,\text{rot}} = c_{m,n,\text{rot}} := c_{m,\text{rot}} \quad \square$

\Rightarrow remains to estimate only one constant $c_{m,\text{rot}}!$

Proof of Second (Main) Theorem

Proof

crucial estimate for convex domains

Lemma ('98 C. Amrouche, C. Bernardi, M. Dauge, V. Girault)

$\Omega \subset \mathbb{R}^3$ *bd and convex*. Then $E \in \mathring{R}(\Omega) \cap D(\Omega), R(\Omega) \cap \mathring{D}(\Omega) \subset H^1(\Omega)$ *continuous and*

$$|\nabla E|_{L^2(\Omega)}^2 \leq 1 \cdot (|\operatorname{rot} E|_{L^2(\Omega)}^2 + |\operatorname{div} E|_{L^2(\Omega)}^2).$$

related, earlier, partial results by

J. Kadlec ('64), R. Leis ('68), P. Grisvard ('72, '85), J. Saranen ('82), J.-C. Nédélec ('82), V. Girault & P.-A. Raviart ('86), M. Costabel ('91)

\Rightarrow pick $E \in R(\Omega) \cap \operatorname{rot} \mathring{R}(\Omega) = R(\Omega) \cap \mathring{D}_0(\Omega) \cap \mathcal{H}_N(\Omega)^\perp = R(\Omega) \cap \mathring{D}_0(\Omega)$ (Ω convex)

$\Rightarrow \langle E, a \rangle_{L^2(\Omega)} = \langle \operatorname{rot} H, a \rangle_{L^2(\Omega)} = 0$ for all $a \in \mathbb{R}^3$ since $H \in \mathring{R}(\Omega)$

$\Rightarrow E \in H^1(\Omega) \cap (\mathbb{R}^3)^\perp$

$\Rightarrow |E|_{L^2(\Omega)} \stackrel{\text{Poincaré}}{\leq} c_p |\nabla E|_{L^2(\Omega)} \stackrel{\text{Lemma}}{\leq} 1 \cdot c_p |\operatorname{rot} E|_{L^2(\Omega)}$

$\Rightarrow c_{m,\operatorname{rot}} \leq c_p$

very simple!!! \square

Last Slide!

Merci / Thank You

more results:

- also inhomogeneous media, i.e., $\varepsilon \neq \text{id}$, $\mu \neq \text{id}$ non-smooth
- also *ND*-case, i.e., $\Omega \subset \mathbb{R}^N$ with differential forms, same result
- also non-convex polygons (not too pointy) or combinations

applications:

- functional a posteriori error estimates for problems with rot
- preconditioning in numerical algorithms with rot
- ...