Static Solution Theory

(or: Solving PDEs with Hilbert Complexes and Compact Embeddings)

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Solving PDEs with Hilbert Complexes

FA-ToolBox

general observations

$$Ax = f$$

general observations

$$\mathrm{A}x = f$$

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ (lin, dd, cl) and H_0, H_1 Hilbert spaces

question: How to solve?

general observations

$$Ax = f$$

$A : D(A) \subset H_0 \to H_1$ (lin, dd, cl)

solution theory in the sense of Hadamard

- existence $\Leftrightarrow$ $f \in R(A)$
- uniqueness $\Leftrightarrow$ A inj $\Leftrightarrow$ $N(A) = \{0\}$ $\Leftrightarrow$ A^{-1} exists
- cont dep on f $\Leftrightarrow$ A^{-1} cont

$\Rightarrow$ $x = A^{-1}f \in D(A)$ and cont estimate (Friedrichs/Poincaré type estimate)

$$|x|_{H_0} = |A^{-1}f|_{H_0} \leq c_A|f|_{H_1} = c_A|Ax|_{H_1}$$

$\Rightarrow$ best constant $c_A = |A^{-1}|_{R(A),H_0}$

general observations

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$

$\mathrm{A}^* : D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0$ Hilbert space adjoint

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$\mathrm{H}_1 = \overline{R(\mathrm{A})} \oplus N(\mathrm{A}^*), \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

$$\mathrm{A}x = f$$

solution theory in the sense of Hadamard

- existence $\Leftrightarrow$ $f \in R(\mathrm{A}) = N(\mathrm{A}^*)^\perp$ (Fredholm alt, if $R(\mathrm{A})$ cl)
- uniqueness $\Leftrightarrow$ A inj $\Leftrightarrow$ $N(\mathrm{A}) = \{0\}$ $\Leftrightarrow$ A^{-1} exists
- cont dep on f $\Leftrightarrow$ A^{-1} cont $\Leftrightarrow$ $R(\mathrm{A})$ cl (cl range theo)

fund range cond: $\boxed{R(\mathrm{A}) = \overline{R(\mathrm{A})} \text{ closed}}$ (must hold $\rightsquigarrow$ right setting!)

kernel cond: $\boxed{N(\mathrm{A}) = \{0\}}$ $\big($fails in gen $\rightsquigarrow$ proj onto $N(\mathrm{A})^\perp = \overline{R(\mathrm{A}^*)} = R(\mathrm{A}^*)\big)$

FA-ToolBox for linear (first order) problems/systems

$$Ax = f$$

general theory

- solution theory
- closed ranges
- Friedrichs/Poincaré estimates and constants
- Helmholtz/Hodge/Weyl decompositions
- compact embeddings
- continuous and compact inverse operators
- ~~regular potentials and regular decompositions (to show compact embeddings)~~
- ~~variational formulations~~
- ~~generalized div-curl-lemma~~
- ~~index theorems~~
- ~~dimensions and bases of cohomology groups~~
- ~~functional a posteriori error estimates~~
- ...

idea: solve problem with general and simple lin fa ($\Rightarrow$ FA-ToolBox) ...

literature: many parts probably very well known for ages, but hard to find ...

(Friedrichs, Weyl, Hörmander, Fredholm, von Neumann, Riesz, Banach, ... ?)

Why not rediscover and extend/modify for our purposes?

1st fundamental observations

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ lddc, $\quad \mathrm{A}^* : D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0$ Hilbert space adjoint

$(\mathrm{A}, \mathrm{A}^*)$ dual pair as $(\mathrm{A}^*)^* = \overline{\mathrm{A}} = \mathrm{A}$

A, A^* may not be inj

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$\mathrm{H}_1 = N(\mathrm{A}^*) \oplus \overline{R(\mathrm{A})} \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

reduced operators restr to $N(\mathrm{A})^\perp$ and $N(\mathrm{A}^*)^\perp$

$$\boxed{\mathcal{A} := \mathrm{A}|_{N(\mathrm{A})^\perp} = \mathrm{A}|_{\overline{R(\mathrm{A}^*)}}} \qquad \boxed{\mathcal{A}^* := \mathrm{A}^*|_{N(\mathrm{A}^*)^\perp} = \mathrm{A}^*|_{\overline{R(\mathrm{A})}}}$$

$\mathcal{A}$, $\mathcal{A}^*$ inj $\quad \Rightarrow \quad \mathcal{A}^{-1}$, $(\mathcal{A}^*)^{-1}$ ex

1st fundamental observations

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1, \quad \mathrm{A}^* : D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0$ lddc $\qquad (\mathrm{A}, \mathrm{A}^*)$ dual pair

$$\mathrm{H}_1 = N(\mathrm{A}^*) \oplus \overline{R(\mathrm{A})} \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

more precisely

$$\mathcal{A} := \mathrm{A}|_{\overline{R(\mathrm{A}^*)}} : D(\mathcal{A}) \subset \overline{R(\mathrm{A}^*)} \to \overline{R(\mathrm{A})}, \qquad D(\mathcal{A}) := D(\mathrm{A}) \cap N(\mathrm{A})^{\perp} = D(\mathrm{A}) \cap \overline{R(\mathrm{A}^*)}$$

$$\mathcal{A}^* := \mathrm{A}^*|_{\overline{R(\mathrm{A})}} : D(\mathcal{A}^*) \subset \overline{R(\mathrm{A})} \to \overline{R(\mathrm{A}^*)}, \quad D(\mathcal{A}^*) := D(\mathrm{A}^*) \cap N(\mathrm{A}^*)^{\perp} = D(\mathrm{A}^*) \cap \overline{R(\mathrm{A})}$$

$(\mathcal{A}, \mathcal{A}^*)$ dual pair $\quad$ and $\quad \mathcal{A}, \mathcal{A}^*$ inj $\quad \Rightarrow$

inverse ops exist (and bij)

$$\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A}) \qquad (\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$$

refined decompositions

$$D(\mathrm{A}) = N(\mathrm{A}) \oplus D(\mathcal{A}) \qquad D(\mathrm{A}^*) = N(\mathrm{A}^*) \oplus D(\mathcal{A}^*)$$

$\Rightarrow$

$$R(\mathrm{A}) = R(\mathcal{A}) \qquad R(\mathrm{A}^*) = R(\mathcal{A}^*)$$

1st fundamental observations

closed range theorem & closed graph theorem $\Rightarrow$

Lemma (Friedrichs-Poincaré type est/cl range/cont inv)

The following assertions are equivalent:

- **(i)** $\exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathrm{H}_0} \leq c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$
- **(i*)** $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$
- **(ii)** $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_1.
- **(ii*)** $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_0.
- **(iii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is continuous and bijective.*
- **(iii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is continuous and bijective.*

1st fundamental observations

recall

(i) $\exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathrm{H}_0} \leq c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$

(i*) $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$

'best' consts in **(i)** and **(i*)** equal norms of the inv ops and Rayleigh quotients

$$c_{\mathrm{A}} = |\mathcal{A}^{-1}|_{R(\mathrm{A}),R(\mathrm{A}^*)} \qquad c_{\mathrm{A}^*} = |(\mathcal{A}^*)^{-1}|_{R(\mathrm{A}^*),R(\mathrm{A})}$$

$$\frac{1}{c_{\mathrm{A}}} = \inf_{0 \neq x \in D(\mathcal{A})} \frac{|\mathrm{A}x|_{\mathrm{H}_1}}{|x|_{\mathrm{H}_0}} \qquad \frac{1}{c_{\mathrm{A}^*}} = \inf_{0 \neq y \in D(\mathcal{A}^*)} \frac{|\mathrm{A}^* y|_{\mathrm{H}_0}}{|y|_{\mathrm{H}_1}}$$

Lemma (Friedrichs-Poincaré type const)

$c_{\mathrm{A}} = c_{\mathrm{A}^*}$

1st fundamental observations

Lemma (Friedrichs-Poincaré type est/cl range/cont inv)

$\Downarrow$ $D(\mathcal{A}) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_0$ *compact*

(i) $\exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathrm{H}_0} \le c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$

(i*) $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathrm{H}_1} \le c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$

(ii) $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_1.

(ii*) $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_0.

(iii) $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is continuous and bijective.*

(iii*) $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is continuous and bijective.*

(i)-(iii*) equi & the resp Helm deco hold & $|\mathcal{A}^{-1}| = c_{\mathrm{A}} = c_{\mathrm{A}^*} = |(\mathcal{A}^*)^{-1}|$

Lemma (cpt emb/cpt inv)

The following assertions are equivalent:

(i) $D(\mathcal{A}) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_0$ *is compact.*

(i*) $D(\mathcal{A}^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1$ *is compact.*

(ii) $\mathcal{A}^{-1} : R(\mathrm{A}) \to R(\mathrm{A}^*)$ *is compact.*

(ii*) $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to R(\mathrm{A})$ *is compact.*

2nd fundamental observations

So far no complex...

$\mathrm{A}_0 : D(\mathrm{A}_0) \subset \mathrm{H}_0 \to \mathrm{H}_1, \quad \mathrm{A}_1 : D(\mathrm{A}_1) \subset \mathrm{H}_1 \to \mathrm{H}_2$ (lddc)
$\mathrm{A}_0^* : D(\mathrm{A}_0^*) \subset \mathrm{H}_1 \to \mathrm{H}_0, \quad \mathrm{A}_1^* : D(\mathrm{A}_1^*) \subset \mathrm{H}_2 \to \mathrm{H}_1$ (lddc)

general complex ($\boxed{\mathrm{A}_1\mathrm{A}_0 = 0}$, i.e., $R(\mathrm{A}_0) \subset N(\mathrm{A}_1)$ and $R(\mathrm{A}_1^*) \subset N(\mathrm{A}_0^*)$)

$$\boxed{\cdots \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad \mathrm{H}_0 \quad \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \quad \mathrm{H}_1 \quad \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \quad \mathrm{H}_2 \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad \cdots}$$

recall Helmholtz deco

$$\begin{array}{rcccl} \mathrm{H}_1 = & \overline{R(\mathrm{A}_0)} & \oplus & N(\mathrm{A}_0^*) \\ & \cap & & \cup \\ = & N(\mathrm{A}_1) & \oplus & \overline{R(\mathrm{A}_1^*)} \end{array} \qquad \Rightarrow \quad \text{(e.g.)} \ N(\mathrm{A}_1) = \overline{R(\mathrm{A}_0)} \oplus \big(\underbrace{N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)}_{=:N_1} \big)$$

$\Rightarrow$ refined Helmholtz deco

$$\boxed{\mathrm{H}_1 = \overline{R(\mathrm{A}_0)} \oplus N_1 \oplus \overline{R(\mathrm{A}_1^*)}}$$

2nd fundamental observations

$$N_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \qquad D(\mathrm{A}_1) = D(\mathcal{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \qquad D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \cap \overline{R(\mathrm{A}_0)}$$

Lemma (cpt emb II)

The following assertions are equivalent:

(i) $D(\mathcal{A}_0) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_0$, $D(\mathcal{A}_1) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$, *and* $N_1 \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$ *are compact.*

(ii) $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$ *is compact.*

In this case $N_1 < \infty$.

Theorem (FA-ToolBox I)

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1 \text{ compact}}$

(i) *all emb cpt, i.e.,* $D(\mathcal{A}_0) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_0$, $D(\mathcal{A}_1) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$, $D(\mathcal{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$, $D(\mathcal{A}_1^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_2$ *cpt*

(ii) *cohomology group* N_1 *finite dim*

(iii) *all ranges closed, i.e.,* $R(\mathrm{A}_0)$, $R(\mathrm{A}_0^*)$, $R(\mathrm{A}_1)$, $R(\mathrm{A}_1^*)$ *cl*

(iv) *all Friedrichs-Poincaré type est hold*

(v) *all Hodge-Helmholtz-Weyl type deco I & II hold with closed ranges*

2nd fundamental observations

complex $\cdots \overset{\cdots}{\underset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \overset{\mathrm{A}_0}{\underset{\mathrm{A}_0^*}{\rightleftarrows}} \mathrm{H}_1 \overset{\mathrm{A}_1}{\underset{\mathrm{A}_1^*}{\rightleftarrows}} \mathrm{H}_2 \overset{\cdots}{\underset{\cdots}{\rightleftarrows}} \cdots$

Theorem (FA-ToolBox I (Friedrichs-Poincaré type est))

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1 \textit{ compact}}$ $\Rightarrow$ $\exists$ $|\mathcal{A}_i^{-1}| = c_{\mathrm{A}_i} = c_{\mathrm{A}_i^*} = |(\mathcal{A}_i^*)^{-1}| \in (0, \infty)$

(i) $\forall\, x \in D(\mathcal{A}_0)$ $\quad |x|_{\mathrm{H}_0} \leq c_{\mathrm{A}_0} |\mathrm{A}_0 x|_{\mathrm{H}_1}$

(i*) $\forall\, y \in D(\mathcal{A}_0^*)$ $\quad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}_0} |\mathrm{A}_0^* y|_{\mathrm{H}_0}$

(ii) $\forall\, y \in D(\mathcal{A}_1)$ $\quad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |\mathrm{A}_1 y|_{\mathrm{H}_2}$

(ii*) $\forall\, z \in D(\mathcal{A}_1^*)$ $\quad |z|_{\mathrm{H}_2} \leq c_{\mathrm{A}_1} |\mathrm{A}_1^* z|_{\mathrm{H}_1}$

(iii) $\forall\, y \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ $\quad |(1 - \pi_{N_1}) y|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |\mathrm{A}_1 y|_{\mathrm{H}_2} + c_{\mathrm{A}_0} |\mathrm{A}_0^* y|_{\mathrm{H}_0}$

note $\pi_{N_1} y \in N_1$ and $(1 - \pi_{N_1}) y \in N_1^{\perp}$

Remark

enough $R(\mathrm{A}_0)$ *and* $R(\mathrm{A}_1)$ *cl*

Solving PDEs with Hilbert Complexes

(Static) First Order Systems

(stat) first order system - solution theory

complex $\quad \cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

$$\boxed{\mathrm{A}_1 x = f} \qquad \dim N(\mathrm{A}_1) = \infty$$

find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ such that the fos

$$\begin{aligned} \mathrm{A}_1 x &= f \\ \mathrm{A}_0^* x &= g \\ \pi_{N_1} x &= k \end{aligned} \qquad \text{think of} \qquad \begin{aligned} (\mathring{\mathrm{rot}} E &= F) \\ (-\operatorname{div} E &= g) \\ (\pi_{\mathrm{D}} E &= K) \end{aligned}$$

kernel = cohomology group = $N_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)$

trivially necessary $\quad f \in R(\mathrm{A}_1) \quad g \in R(\mathrm{A}_0^*) \quad k \in N_1$

apply FA-ToolBox

(stat) first order system - solution theory

complex $\cdots \overset{\cdots}{\underset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \overset{\mathrm{A}_0}{\underset{\mathrm{A}_0^*}{\rightleftarrows}} \mathrm{H}_1 \overset{\mathrm{A}_1}{\underset{\mathrm{A}_1^*}{\rightleftarrows}} \mathrm{H}_2 \overset{\cdots}{\underset{\cdots}{\rightleftarrows}} \cdots$

find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ st fos $\boxed{\mathrm{A}_1 x = f \qquad \mathrm{A}_0^* x = g \qquad \pi_{N_1} x = k}$

Theorem (FA-ToolBox II (solution theory))

$\Downarrow \quad \boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1 \textit{ compact}}$

fos is uniq sol $\Leftrightarrow \quad f \in R(\mathrm{A}_1) \qquad g \in R(\mathrm{A}_0^*) \qquad k \in N_1$

$$x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus N_1 = D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$$

$$\boxed{x_f := \mathcal{A}_1^{-1} f} \in D(\mathcal{A}_1)$$

$$\boxed{x_g := (\mathcal{A}_0^*)^{-1} g} \in D(\mathcal{A}_0^*)$$

dep cont on data $|x|_{\mathrm{H}_1} \leq |x_f|_{\mathrm{H}_1} + |x_g|_{\mathrm{H}_1} + |k|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |f|_{\mathrm{H}_2} + c_{\mathrm{A}_0} |g|_{\mathrm{H}_0} + |k|_{\mathrm{H}_1}$

moreover

$\pi_{R(\mathrm{A}_1^*)} x = x_f \qquad \pi_{R(\mathrm{A}_0)} x = x_g \qquad \pi_{N_1} x = k \qquad |x|_{\mathrm{H}_1}^2 = |x_f|_{\mathrm{H}_1}^2 + |x_g|_{\mathrm{H}_1}^2 + |k|_{\mathrm{H}_1}^2$

Remark

enough $R(\mathrm{A}_0)$ *and* $R(\mathrm{A}_1)$ *cl*

Solving PDEs with Hilbert Complexes

Applications: FOS & SOS (First and Second Order Systems)

classical de Rham complex in 3D (∇-rot-div-complex)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot{\cup} \Gamma_n}$

(electro-magneto dynamics, Maxwell's equations)

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{div}}{\overset{\mathring{\nabla}}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{rot}}{\overset{\mathring{\operatorname{rot}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\nabla}{\overset{\mathring{\operatorname{div}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}} \mathbb{R}$$

mixed boundary conditions and inhomogeneous and anisotropic media

$$\{0\} \text{ or } \mathbb{R} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{div}_{\Gamma_n} \varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_\varepsilon \underset{\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathbb{R} \text{ or } \{0\}$$

classical de Rham complex in 3D (∇-rot-div-complex)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot{\cup} \Gamma_n}$

(electro-magneto dynamics, Maxwell's equations with mixed boundary conditions)

$$\{0\} \text{ or } \mathbb{R} \;\underset{\pi}{\overset{\iota}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\operatorname{div}_{\Gamma_n}\varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2_\varepsilon \;\underset{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota}{\overset{\pi}{\rightleftarrows}}\; \mathbb{R} \text{ or } \{0\}$$

related fos

$\nabla_{\Gamma_t} u = A$ in Ω	$\operatorname{rot}_{\Gamma_t} E = J$ in Ω	$\operatorname{div}_{\Gamma_t} H = k$ in Ω	$\pi v = b$ in Ω
$\pi u = a$ in Ω	$-\operatorname{div}_{\Gamma_n} \varepsilon E = j$ in Ω	$\varepsilon^{-1} \operatorname{rot}_{\Gamma_n} H = K$ in Ω	$-\nabla_{\Gamma_n} v = B$ in Ω

related sos

$-\operatorname{div}_{\Gamma_n} \varepsilon \nabla_{\Gamma_t} u = j$ in Ω	$\varepsilon^{-1} \operatorname{rot}_{\Gamma_n} \operatorname{rot}_{\Gamma_t} E = K$ in Ω	$-\nabla_{\Gamma_n} \operatorname{div}_{\Gamma_t} H = B$ in Ω
$\pi u = a$ in Ω	$-\operatorname{div}_{\Gamma_n} \varepsilon E = j$ in Ω	$\varepsilon^{-1} \operatorname{rot}_{\Gamma_n} H = K$ in Ω

corresponding compact embeddings:

$$D(\nabla_{\Gamma_t}) \cap D(\pi) = D(\nabla_{\Gamma_t}) = \mathsf{H}^1_{\Gamma_t} \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

$$D(\operatorname{rot}_{\Gamma_t}) \cap D(-\operatorname{div}_{\Gamma_n}\varepsilon) = \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n} \hookrightarrow \mathsf{L}^2_\varepsilon \qquad \text{(Weck's selection theorem, '74)}$$

$$D(\operatorname{div}_{\Gamma_t}) \cap D(\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}) = \mathsf{D}_{\Gamma_t} \cap \mathsf{R}_{\Gamma_n} \hookrightarrow \mathsf{L}^2 \qquad \text{(Weck's selection theorem, '74)}$$

$$D(\nabla_{\Gamma_n}) \cap D(\pi) = D(\nabla_{\Gamma_n}) = \mathsf{H}^1_{\Gamma_n} \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

Weck's selection theorem for weak Lip. dom. and mixed bc: Bauer/Py/Schomburg ('16)

Weck's selection theorem (Weck '74, (Habil. '72) stimulated by Rolf Leis)
(Weber '80, Picard '84, Costabel '90, Witsch '93, Jochmann '97, Kuhn '99, Picard/Weck/Witsch '01, Py '96, '03, '06, '07, '08)

classical de Rham complex in 3D (∇-rot-div-complex)

$$\begin{aligned} \operatorname{rot} E &= F && \text{in } \Omega \\ -\operatorname{div} \varepsilon E &= g && \text{in } \Omega \\ \nu \times E &= 0 && \text{at } \Gamma_t \\ \nu \cdot \varepsilon E &= 0 && \text{at } \Gamma_n \end{aligned}$$

non-trivial kernel $\mathcal{H}_{\mathrm{D},\varepsilon} = \{H \in \mathrm{L}^2 : \operatorname{rot} H = 0, \operatorname{div} \varepsilon H = 0, \nu \times H|_{\Gamma_t} = 0, \nu \cdot \varepsilon H|_{\Gamma_n} = 0\}$
additional condition on Dirichlet/Neumann fields for uniqueness

$$\pi_{\mathrm{D}} E = K \in \mathcal{H}_{\mathrm{D},\varepsilon}$$

$$\{0\} \text{ or } \mathbb{R} \;\underset{\pi}{\overset{\iota}{\rightleftarrows}}\; \mathrm{L}^2 \;\underset{-\operatorname{div}_{\Gamma_n} \varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}}\; \mathrm{L}^2_{\varepsilon} \;\underset{\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}}\; \mathrm{L}^2 \;\underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}}\; \mathrm{L}^2 \;\underset{\iota}{\overset{\pi}{\rightleftarrows}}\; \mathbb{R} \text{ or } \{0\}$$

$$\cdots \;\underset{\cdots}{\overset{\cdots}{\rightleftarrows}}\; \mathrm{H}_{-1} \;\underset{\mathrm{A}_{-1}^*}{\overset{\mathrm{A}_{-1}}{\rightleftarrows}}\; \mathrm{H}_0 \;\underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}}\; \mathrm{H}_1 \;\underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}}\; \mathrm{H}_2 \;\underset{\mathrm{A}_2^*}{\overset{\mathrm{A}_2}{\rightleftarrows}}\; \mathrm{H}_3 \;\underset{\mathrm{A}_3^*}{\overset{\mathrm{A}_3}{\rightleftarrows}}\; \mathrm{H}_4 \;\underset{\cdots}{\overset{\cdots}{\rightleftarrows}}\; \cdots$$

find $E \in \mathrm{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\mathrm{D}_{\Gamma_n}(\Omega)$ st (fos) find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ st

$$\begin{aligned} \operatorname{rot}_{\Gamma_t} E &= F \\ -\operatorname{div}_{\Gamma_n} \varepsilon E &= g \\ \pi_{\mathrm{D/N}} E &= K \end{aligned} \qquad \text{translation} \qquad \begin{aligned} \mathrm{A}_1 x &= f \\ \mathrm{A}_0^* x &= g \\ \pi_{N_1} x &= k \end{aligned}$$

classical de Rham complex in 3D (∇-rot-div-complex)

$c_{\mathrm{A}_0} = c_{\mathrm{fp}}$ (Friedrichs/Poincaré constant) and $c_{\mathrm{A}_1} = c_{\mathrm{m}}$ (Maxwell constant)

Lemma/Theorem $\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{L}^2_\varepsilon(\Omega) \text{ compact}}$

(i) all Friedrichs-Poincaré type est hold

$$\begin{array}{lllll}
\forall\, \varphi \in D(\mathcal{A}_0) & |\varphi|_{\mathrm{H}_0} \le c_{\mathrm{A}_0} |\mathrm{A}_0 \varphi|_{\mathrm{H}_1} & \Leftrightarrow & \forall\, \varphi \in \mathrm{H}^1_{\Gamma_t} & |\varphi|_{\mathrm{L}^2} \le c_{\mathrm{fp}} |\nabla \varphi|_{\mathrm{L}^2_\varepsilon} \\
\forall\, \phi \in D(\mathcal{A}_0^*) & |\phi|_{\mathrm{H}_1} \le c_{\mathrm{A}_0} |\mathrm{A}_0^* \phi|_{\mathrm{H}_0} & \Leftrightarrow & \forall\, \Phi \in \varepsilon^{-1} \mathrm{D}_{\Gamma_n} \cap \nabla \mathrm{H}^1_{\Gamma_t} & |\Phi|_{\mathrm{L}^2_\varepsilon} \le c_{\mathrm{fp}} |\operatorname{div} \varepsilon \Phi|_{\mathrm{L}^2} \\
\forall\, \phi \in D(\mathcal{A}_1) & |\phi|_{\mathrm{H}_1} \le c_{\mathrm{A}_1} |\mathrm{A}_1 \phi|_{\mathrm{H}_2} & \Leftrightarrow & \forall\, \Phi \in \mathrm{R}_{\Gamma_t} \cap \varepsilon^{-1} \operatorname{rot} \mathrm{R}_{\Gamma_n} & |\Phi|_{\mathrm{L}^2_\varepsilon} \le c_{\mathrm{m}} |\operatorname{rot} \Phi|_{\mathrm{L}^2} \\
\forall\, \psi \in D(\mathcal{A}_1^*) & |\psi|_{\mathrm{H}_2} \le c_{\mathrm{A}_1} |\mathrm{A}_1^* \psi|_{\mathrm{H}_1} & \Leftrightarrow & \forall\, \Psi \in \mathrm{R}_{\Gamma_n} \cap \operatorname{rot} \mathrm{R}_{\Gamma_t} & |\Psi|_{\mathrm{L}^2} \le c_{\mathrm{m}} |\operatorname{rot} \Psi|_{\mathrm{L}^2_\varepsilon}
\end{array}$$

(ii) all ranges $R(\mathrm{A}_0) = \nabla \mathrm{H}^1_{\Gamma_t}$, $R(\mathrm{A}_1) = \operatorname{rot} \mathrm{R}_{\Gamma_t}$, $R(\mathrm{A}_0^*) = \operatorname{div} \mathrm{D}_{\Gamma_n}$ are cl in L^2

(iii) the inverse ops $(\widetilde{\nabla}_{\Gamma_t})^{-1}$, $(\widetilde{\operatorname{div}_{\Gamma_n} \varepsilon})^{-1}$, $(\widetilde{\operatorname{rot}}_{\Gamma_t})^{-1}$, $(\widetilde{\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}})^{-1}$ are cont, even cpt

(iv) all Helmholtz decomposition hold, e.g.,

$$\mathrm{H}_1 = R(\mathrm{A}_0) \oplus N_1 \oplus R(\mathrm{A}_1^*) \quad \Leftrightarrow \quad \mathrm{L}^2_\varepsilon = \nabla \mathrm{H}^1_{\Gamma_t} \oplus_{\mathrm{L}^2_\varepsilon} \mathcal{H}_{\mathrm{D},\varepsilon} \oplus_{\mathrm{L}^2_\varepsilon} \varepsilon^{-1} \operatorname{rot} \mathrm{R}_{\Gamma_n}$$

(v) solution theory

(vi) ...

classical de Rham complex in 3D (∇-rot-div-complex)

find $E \in \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n}$ s.t. / think of $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$

$$\begin{aligned} \mathrm{rot}_{\Gamma_t} E &= F \\ \mathrm{div}_{\Gamma_n} \varepsilon E &= g \\ \pi_{\mathcal{H}_{\mathrm{D},\varepsilon}} E &= K \end{aligned} \qquad / \quad \text{think of} \qquad \begin{aligned} \mathrm{A}_1 x &= f \\ \mathrm{A}_0^* x &= g \\ \pi_{K_1} x &= k \end{aligned}$$

sol is simply $x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus K_1 = D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$

with $\boxed{x_f := \mathcal{A}_1^{-1} f} \in D(\mathcal{A}_1)$ and $\boxed{x_g := (\mathcal{A}_0^*)^{-1} g} \in D(\mathcal{A}_0^*)$

i.e., $E = E_F + E_g + K$, where

$$\boxed{E_F := (\widetilde{\mathrm{rot}}_{\Gamma_t})^{-1} F} \in D(\widetilde{\mathrm{rot}}_{\Gamma_t}) = \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1} \,\mathrm{rot}\, \mathsf{R}_{\Gamma_n} = \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1} \mathsf{D}_{\Gamma_n,0} \cap \mathcal{H}_{\mathrm{D},\varepsilon}^{\perp},$$

$$\boxed{E_g := (\widetilde{\mathrm{div}}_{\Gamma_n} \varepsilon)^{-1} g} \in D(\widetilde{\mathrm{div}}_{\Gamma_n} \varepsilon) = \varepsilon^{-1} \mathsf{D}_{\Gamma_n} \cap \nabla \mathsf{H}^1_{\Gamma_t} = \varepsilon^{-1} \mathsf{D}_{\Gamma_n} \cap \mathsf{R}_{\Gamma_t,0} \cap \mathcal{H}_{\mathrm{D},\varepsilon}^{\perp}$$

Solving PDEs with Hilbert Complexes

APPENDIX: Friedrichs/Poincaré/Maxwell constants (numerics)

joint work with Jan Valdman

Friedrichs/Poincaré/Maxwell constants

assumption: $\varepsilon = \mu = 1$ and $\boxed{\Gamma_t = \Gamma, \text{ i.e., } c_{\mathrm{fp}} = c_{\mathrm{f}}}$ or $\boxed{\Gamma_n = \Gamma, \text{ i.e., } c_{\mathrm{fp}} = c_{\mathrm{p}}}$

Lemma (Maxwell-Poincaré constants)

Ω *convex and bounded* $\Rightarrow$ $\boxed{c_{\mathrm{m}} \le c_{\mathrm{p}} \le \frac{\mathrm{diam}_\Omega}{\pi}}$

Mild Conjecture (Maxwell-Poincaré constants)

Ω convex and bounded $\Rightarrow$ $c_{\mathrm{f}} \le c_{\mathrm{m}} \le c_{\mathrm{p}} \le \frac{\mathrm{diam}_\Omega}{\pi}$

Theorem (FA-ToolBox / Friedrichs-Poincaré type estimates and constants)

$$\forall\, \varphi \in D(\mathcal{A}_0) \quad |\varphi|_{\mathrm{H}_0} \le c_{\mathrm{A}_0} |\mathrm{A}_0\varphi|_{\mathrm{H}_1} \quad \Leftrightarrow \quad \forall\, \varphi \in \mathrm{H}^1_\Gamma \quad |\varphi|_{\mathrm{L}^2} \le c_{\mathrm{f}} |\nabla\varphi|_{\mathrm{L}^2}$$

$$\forall\, \phi \in D(\mathcal{A}_0^*) \quad |\phi|_{\mathrm{H}_1} \le c_{\mathrm{A}_0} |\mathrm{A}_0^*\phi|_{\mathrm{H}_0} \quad \Leftrightarrow \quad \forall\, \Phi \in \mathrm{D} \cap \nabla\mathrm{H}^1_\Gamma \quad |\Phi|_{\mathrm{L}^2} \le c_{\mathrm{f}} |\operatorname{div}\Phi|_{\mathrm{L}^2}$$

$$\forall\, \phi \in D(\mathcal{A}_1) \quad |\phi|_{\mathrm{H}_1} \le c_{\mathrm{A}_1} |\mathrm{A}_1\phi|_{\mathrm{H}_2} \quad \Leftrightarrow \quad \forall\, \Phi \in \mathrm{R}_\Gamma \cap \operatorname{rot}\mathrm{R} \quad |\Phi|_{\mathrm{L}^2} \le c_{\mathrm{m}} |\operatorname{rot}\Phi|_{\mathrm{L}^2}$$

$$\forall\, \psi \in D(\mathcal{A}_1^*) \quad |\psi|_{\mathrm{H}_2} \le c_{\mathrm{A}_1} |\mathrm{A}_1^*\psi|_{\mathrm{H}_1} \quad \Leftrightarrow \quad \forall\, \Psi \in \mathrm{R} \cap \operatorname{rot}\mathrm{R}_\Gamma \quad |\Psi|_{\mathrm{L}^2} \le c_{\mathrm{m}} |\operatorname{rot}\Psi|_{\mathrm{L}^2}$$

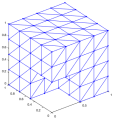
$$\forall\, \psi \in D(\mathcal{A}_2) \quad |\psi|_{\mathrm{H}_2} \le c_{\mathrm{A}_2} |\mathrm{A}_2\psi|_{\mathrm{H}_3} \quad \Leftrightarrow \quad \forall\, \Psi \in \mathrm{D}_\Gamma \cap \nabla\mathrm{H}^1 \quad |\Psi|_{\mathrm{L}^2} \le c_{\mathrm{p}} |\operatorname{div}\Psi|_{\mathrm{L}^2}$$

$$\forall\, \xi \in D(\mathcal{A}_2^*) \quad |\xi|_{\mathrm{H}_3} \le c_{\mathrm{A}_2} |\mathrm{A}_2^*\xi|_{\mathrm{H}_2} \quad \Leftrightarrow \quad \forall\, \zeta \in \mathrm{H}^1 \cap \mathbb{R}^\perp \quad |\zeta|_{\mathrm{L}^2} \le c_{\mathrm{p}} |\nabla\zeta|_{\mathrm{L}^2}$$

Friedrichs/Poincaré/Maxwell constants

surprise numerical tests show even for non-convex domains and mixed bc

e.g., Fichera corner domain



Conjecture (Maxwell-Poincaré constants)

$$c_{\mathrm{f}} \leq \min\{c_{\mathrm{fp}}, c_{\mathrm{pf}}\} \leq c_{\mathrm{m}} \leq \max\{c_{\mathrm{fp}}, c_{\mathrm{pf}}\} \leq \sup_{\Gamma_t \neq \emptyset}\{c_{\mathrm{fp}}\} < \infty$$

Theorem (FA-ToolBox / Friedrichs-Poincaré type estimates and constants)

$$\forall\, \varphi \in D(\mathcal{A}_0) \quad |\varphi|_{\mathrm{H}_0} \leq c_{\mathrm{A}_0} |\mathrm{A}_0 \varphi|_{\mathrm{H}_1} \quad \Leftrightarrow \quad \forall\, \varphi \in \mathrm{H}^1_{\Gamma_t} \quad |\varphi|_{\mathrm{L}^2} \leq c_{\mathrm{fp}} |\nabla \varphi|_{\mathrm{L}^2}$$

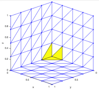
$$\forall\, \phi \in D(\mathcal{A}_0^*) \quad |\phi|_{\mathrm{H}_1} \leq c_{\mathrm{A}_0} |\mathrm{A}_0^* \phi|_{\mathrm{H}_0} \quad \Leftrightarrow \quad \forall\, \Phi \in \mathrm{D}_{\Gamma_n} \cap \nabla \mathrm{H}^1_{\Gamma_t} \quad |\Phi|_{\mathrm{L}^2} \leq c_{\mathrm{fp}} |\operatorname{div} \Phi|_{\mathrm{L}^2}$$

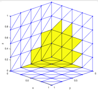
$$\forall\, \phi \in D(\mathcal{A}_1) \quad |\phi|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |\mathrm{A}_1 \phi|_{\mathrm{H}_2} \quad \Leftrightarrow \quad \forall\, \Phi \in \mathrm{R}_{\Gamma_t} \cap \operatorname{rot} \mathrm{R}_{\Gamma_n} \quad |\Phi|_{\mathrm{L}^2} \leq c_{\mathrm{m}} |\operatorname{rot} \Phi|_{\mathrm{L}^2}$$

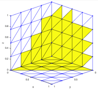
$$\forall\, \psi \in D(\mathcal{A}_1^*) \quad |\psi|_{\mathrm{H}_2} \leq c_{\mathrm{A}_1} |\mathrm{A}_1^* \psi|_{\mathrm{H}_1} \quad \Leftrightarrow \quad \forall\, \Psi \in \mathrm{R}_{\Gamma_n} \cap \operatorname{rot} \mathrm{R}_{\Gamma_t} \quad |\Psi|_{\mathrm{L}^2} \leq c_{\mathrm{m}} |\operatorname{rot} \Psi|_{\mathrm{L}^2}$$

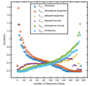
$$\forall\, \psi \in D(\mathcal{A}_2) \quad |\psi|_{\mathrm{H}_2} \leq c_{\mathrm{A}_2} |\mathrm{A}_2 \psi|_{\mathrm{H}_3} \quad \Leftrightarrow \quad \forall\, \Psi \in \mathrm{D}_{\Gamma_t} \cap \nabla \mathrm{H}^1_{\Gamma_n} \quad |\Psi|_{\mathrm{L}^2} \leq c_{\mathrm{pf}} |\operatorname{div} \Psi|_{\mathrm{L}^2}$$

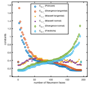
$$\forall\, \xi \in D(\mathcal{A}_2^*) \quad |\xi|_{\mathrm{H}_3} \leq c_{\mathrm{A}_2} |\mathrm{A}_2^* \xi|_{\mathrm{H}_2} \quad \Leftrightarrow \quad \forall\, \zeta \in \mathrm{H}^1_{\Gamma_n} \quad |\zeta|_{\mathrm{L}^2} \leq c_{\mathrm{pf}} |\nabla \zeta|_{\mathrm{L}^2}$$



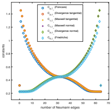




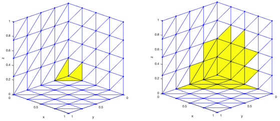




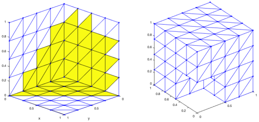
Friedrichs/Poincaré/Maxwell constants



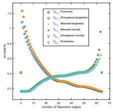
2D unit square



3D unit cube



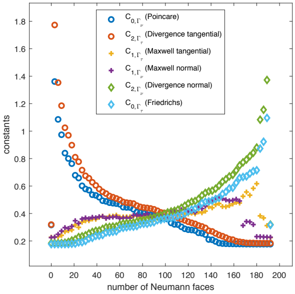
3D Fichera corner



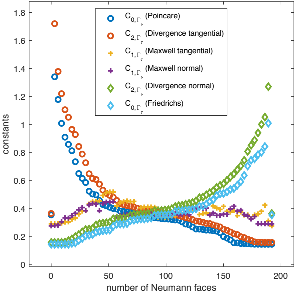
2D L-shape

Conjecture (Maxwell-Poincaré constants)

$$c_{\mathrm{f}} \leq \min\{c_{\mathrm{fp}}, c_{\mathrm{pf}}\} \leq c_{\mathrm{m}} \leq \max\{c_{\mathrm{fp}}, c_{\mathrm{pf}}\} \leq \sup_{\Gamma_t \neq \emptyset}\{c_{\mathrm{fp}}\} < \infty$$



3D unit cube



3D Fichera corner domain

Solving PDEs with Hilbert Complexes

APPENDIX: More Complexes

elasticity complex in 3D (sym ∇-Rot $\operatorname{Rot}_{\mathbb{S}}^{\top}$-$\operatorname{Div}_{\mathbb{S}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{Div}_{\mathbb{S}}}{\overset{\operatorname{sym}\mathring{\nabla}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}}{\overset{\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^{\top}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{-\operatorname{sym}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathsf{RM}}}{\overset{\pi_{\mathsf{RM}}}{\rightleftarrows}} \mathsf{RM}$$

elasticity complex in 3D (sym ∇-Rot $\mathrm{Rot}_{\mathbb{S}}^{\top}$-$\mathrm{Div}_{\mathbb{S}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \overset{\iota_{\{0\}}}{\underset{\pi_{\{0\}}}{\rightleftarrows}} \mathrm{L}^2 \overset{\mathring{\operatorname{sym}} \nabla}{\underset{-\operatorname{Div}_{\mathbb{S}}}{\rightleftarrows}} \mathrm{L}^2_{\mathbb{S}} \overset{\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^{\top}}{\underset{\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}}{\rightleftarrows}} \mathrm{L}^2_{\mathbb{S}} \overset{\mathring{\operatorname{Div}}_{\mathbb{S}}}{\underset{-\operatorname{sym}\nabla}{\rightleftarrows}} \mathrm{L}^2 \overset{\pi_{\mathrm{RM}}}{\underset{\iota_{\mathrm{RM}}}{\rightleftarrows}} \mathrm{RM}$$

related fos ($\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^{\top}$, $\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}$ first order operators!)

$$\begin{array}{rl|rl|rl|rl} \mathring{\operatorname{sym}}\nabla v = M & \text{in } \Omega & \operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^{\top} M = F & \text{in } \Omega & \mathring{\operatorname{Div}}_{\mathbb{S}} N = g & \text{in } \Omega & \pi v = r & \text{in } \Omega \\ \pi v = 0 & \text{in } \Omega & -\operatorname{Div}_{\mathbb{S}} M = f & \text{in } \Omega & \operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top} N = G & \text{in } \Omega & -\operatorname{sym}\nabla v = M & \text{in } \Omega \end{array}$$

related sos ($\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^{\top}$ second order operator!)

$$\begin{array}{rl|rl|rl} -\operatorname{Div}_{\mathbb{S}}\mathring{\operatorname{sym}}\nabla v = f & \text{in } \Omega & \operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^{\top} M = G & \text{in } \Omega & -\operatorname{sym}\nabla\mathring{\operatorname{Div}}_{\mathbb{S}} N = M & \text{in } \Omega \\ \pi v = 0 & \text{in } \Omega & -\operatorname{Div}_{\mathbb{S}} M = f & \text{in } \Omega & \operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top} N = G & \text{in } \Omega \end{array}$$

corresponding compact embeddings:

$$D(\mathring{\operatorname{sym}}\nabla) \cap D(\pi) = D(\mathring{\nabla}) = \mathring{\mathrm{H}}^1 \hookrightarrow \mathrm{L}^2$$ (Rellich's selection theorem and Korn ineq.)

$$D(\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^{\top}) \cap D(\operatorname{Div}_{\mathbb{S}}) \hookrightarrow \mathrm{L}^2_{\mathbb{S}}$$ (new selection theorem)

$$D(\mathring{\operatorname{Div}}_{\mathbb{S}}) \cap D(\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}) \hookrightarrow \mathrm{L}^2_{\mathbb{S}}$$ (new selection theorem)

$$D(\pi) \cap D(\operatorname{sym}\nabla) = D(\nabla) = \mathrm{H}^1 \hookrightarrow \mathrm{L}^2$$ (Rellich's selection theorem and Korn ineq.)

two new selection theorems for strong Lip. dom.: Py/Schomburg/Zulehner ('18)

elasticity complex in 3D (sym ∇-Rot $\mathrm{Rot}_{\mathbb{S}}^{\top}$-$\mathrm{Div}_{\mathbb{S}}$-complex)

Lemma/Theorem $\Downarrow$ $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1, \quad D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_2$ cpt

(i) all Friedrichs-Poincaré type est hold

est for $\mathcal{A}_0$	$\Leftrightarrow$	$\forall\, \varphi \in D(\mathrm{sym}\,\mathring{\nabla}) \cap R(\mathrm{Div}_{\mathbb{S}}) = \mathring{\mathrm{H}}^1$	$\lvert\varphi\rvert_{\mathrm{L}^2} \le c_0 \lvert \mathrm{sym}\,\nabla\varphi\rvert_{\mathrm{L}^2}$
est for $\mathcal{A}_0^*$	$\Leftrightarrow$	$\forall\, \Phi \in D(\mathrm{Div}_{\mathbb{S}}) \cap R(\mathrm{sym}\,\mathring{\nabla})$	$\lvert\Phi\rvert_{\mathrm{L}^2} \le c_0 \lvert \mathrm{Div}\,\Phi\rvert_{\mathrm{L}^2}$
est for $\mathcal{A}_1$	$\Leftrightarrow$	$\forall\, \Phi \in D(\mathrm{Rot}\,\mathring{\mathrm{Rot}}_{\mathbb{S}}^{\top}) \cap R(\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top})$	$\lvert\Phi\rvert_{\mathrm{L}^2} \le c_1 \lvert \mathrm{Rot}\,\mathrm{Rot}^{\top}\,\Phi\rvert_{\mathrm{L}^2}$
est for $\mathcal{A}_1^*$	$\Leftrightarrow$	$\forall\, \Phi \in D(\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top}) \cap R(\mathrm{Rot}\,\mathring{\mathrm{Rot}}_{\mathbb{S}}^{\top})$	$\lvert\Phi\rvert_{\mathrm{L}^2} \le c_1 \lvert \mathrm{Rot}\,\mathrm{Rot}^{\top}\,\Phi\rvert_{\mathrm{L}^2}$
est for $\mathcal{A}_2$	$\Leftrightarrow$	$\forall\, \Phi \in D(\mathring{\mathrm{Div}}_{\mathbb{S}}) \cap R(\mathrm{sym}\,\nabla)$	$\lvert\Phi\rvert_{\mathrm{L}^2} \le c_2 \lvert \mathrm{Div}\,\Phi\rvert_{\mathrm{L}^2}$
est for $\mathcal{A}_2^*$	$\Leftrightarrow$	$\forall\, \varphi \in D(\mathrm{sym}\,\nabla) \cap R(\mathring{\mathrm{Div}}_{\mathbb{S}}) = \mathrm{H}^1 \cap \mathrm{RM}^{\perp}$	$\lvert\varphi\rvert_{\mathrm{L}^2} \le c_2 \lvert \mathrm{sym}\,\nabla\varphi\rvert_{\mathrm{L}^2}$

(ii) all ranges $R(\mathrm{A}_n) = R(\mathcal{A}_n), \quad R(\mathrm{A}_n^*) = R(\mathcal{A}_n^*)$ are cl in L^2

(iii) all inverse ops $\mathcal{A}_n^{-1}$, $(\mathcal{A}_n^*)^{-1}$ are cont, even cpt

(iv) all Helmholtz decomposition hold, e.g.,

$$\mathrm{H}_1 = R(\mathrm{A}_0) \oplus N_1 \oplus R(\mathrm{A}_1^*) \quad \Leftrightarrow \quad \mathrm{L}^2 = R(\mathrm{sym}\,\mathring{\nabla}) \oplus_{\mathrm{L}^2} \mathcal{H}_{\mathrm{D},\mathbb{S}} \oplus_{\mathrm{L}^2} R(\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top})$$

(v) solution theories

(vi) variational formulations

(vii) functional a posteriori error estimates

(viii) div-curl-lemmas

(ix) ...

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\operatorname{Rot}_{\mathbb{S}}$-$\operatorname{Div}_{\mathbb{T}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\nabla\nabla}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}} \underset{-\operatorname{dev}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathsf{RT}}}{\overset{\pi_{\mathsf{RT}}}{\rightleftarrows}} \mathsf{RT}$$

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\mathrm{Rot}_{\mathbb{S}}$-$\mathrm{Div}_{\mathbb{T}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathrm{L}^2 \underset{\operatorname{div}\mathrm{Div}_{\mathbb{S}}}{\overset{\mathring{\nabla}\nabla}{\rightleftarrows}} \mathrm{L}^2_{\mathbb{S}} \underset{\operatorname{sym}\mathrm{Rot}_{\mathbb{T}}}{\overset{\mathring{\mathrm{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \mathrm{L}^2_{\mathbb{T}} \underset{-\operatorname{dev}\nabla}{\overset{\mathring{\mathrm{Div}}_{\mathbb{T}}}{\rightleftarrows}} \mathrm{L}^2 \underset{\iota_{\mathrm{RT}}}{\overset{\pi_{\mathrm{RT}}}{\rightleftarrows}} \mathrm{RT}$$

related fos ($\mathring{\nabla}\nabla$, $\operatorname{div}\mathrm{Div}_{\mathbb{S}}$ first order operators!)

$\mathring{\nabla}\nabla u = M$	in Ω	|	$\mathring{\mathrm{Rot}}_{\mathbb{S}} M = F$	in Ω	|	$\mathring{\mathrm{Div}}_{\mathbb{T}} N = g$	in Ω	|	$\pi v = r$	in Ω
$\pi u = 0$	in Ω	|	$\operatorname{div}\mathrm{Div}_{\mathbb{S}} M = f$	in Ω	|	$\operatorname{sym}\mathrm{Rot}_{\mathbb{T}} N = G$	in Ω	|	$-\operatorname{dev}\nabla v = T$	in Ω

related sos ($\operatorname{div}\mathrm{Div}_{\mathbb{S}}\mathring{\nabla}\nabla = \mathring{\Delta}^2$ second order operator!)

$\operatorname{div}\mathrm{Div}_{\mathbb{S}}\mathring{\nabla}\nabla u = \mathring{\Delta}^2 u = f$	in Ω	|	$\operatorname{sym}\mathrm{Rot}_{\mathbb{T}}\mathring{\mathrm{Rot}}_{\mathbb{S}} M = G$	in Ω	|	$-\operatorname{dev}\nabla\mathring{\mathrm{Div}}_{\mathbb{T}} N = T$	in Ω
$\pi u = 0$	in Ω	|	$\operatorname{div}\mathrm{Div}_{\mathbb{S}} M = f$	in Ω	|	$\operatorname{sym}\mathrm{Rot}_{\mathbb{T}} N = G$	in Ω

corresponding compact embeddings:

$$D(\mathring{\nabla}\nabla) \cap D(\pi) = D(\mathring{\nabla}\nabla) = \mathring{\mathrm{H}}^2 \hookrightarrow \mathrm{L}^2 \qquad \text{(Rellich's selection theorem)}$$

$$D(\mathring{\mathrm{Rot}}_{\mathbb{S}}) \cap D(\operatorname{div}\mathrm{Div}_{\mathbb{S}}) \hookrightarrow \mathrm{L}^2_{\mathbb{S}} \qquad \text{(new selection theorem)}$$

$$D(\mathring{\mathrm{Div}}_{\mathbb{T}}) \cap D(\operatorname{sym}\mathrm{Rot}_{\mathbb{T}}) \hookrightarrow \mathrm{L}^2_{\mathbb{T}} \qquad \text{(new selection theorem)}$$

$$D(\pi) \cap D(\operatorname{dev}\nabla) = D(\operatorname{dev}\nabla) = D(\nabla) = \mathrm{H}^1 \hookrightarrow \mathrm{L}^2 \qquad \text{(Rellich's selection theorem and Korn type ineq.)}$$

two new selection theorems for strong Lip. dom. and Korn Type ineq.: Py/Zulehner ('16)

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\mathrm{Rot}_{\mathbb{S}}$-$\mathrm{Div}_{\mathbb{T}}$-complex)

Lemma/Theorem $\Downarrow$ $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1, \quad D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_2 \quad \text{cpt}$

(i) all Friedrichs-Poincaré type est hold

est for $\mathcal{A}_0$	$\Leftrightarrow$	$\forall\, \varphi \in D(\mathring{\nabla}\nabla) \cap R(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) = \mathring{\mathrm{H}}^2$	$\lvert\varphi\rvert_{\mathrm{L}^2} \le c_0 \lvert\nabla\nabla\varphi\rvert_{\mathrm{L}^2}$
est for $\mathcal{A}_0^*$	$\Leftrightarrow$	$\forall\, \Phi \in D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \cap R(\mathring{\nabla}\nabla)$	$\lvert\Phi\rvert_{\mathrm{L}^2} \le c_0 \lvert\operatorname{div}\operatorname{Div}\Phi\rvert_{\mathrm{L}^2}$
est for $\mathcal{A}_1$	$\Leftrightarrow$	$\forall\, \Phi \in D(\mathring{\mathrm{Rot}}_{\mathbb{S}}) \cap R(\operatorname{sym}\mathrm{Rot}_{\mathbb{T}})$	$\lvert\Phi\rvert_{\mathrm{L}^2} \le c_1 \lvert\mathrm{Rot}\,\Phi\rvert_{\mathrm{L}^2}$
est for $\mathcal{A}_1^*$	$\Leftrightarrow$	$\forall\, \Phi \in D(\operatorname{sym}\mathrm{Rot}_{\mathbb{T}}) \cap R(\mathring{\mathrm{Rot}}_{\mathbb{S}})$	$\lvert\Phi\rvert_{\mathrm{L}^2} \le c_1 \lvert\operatorname{sym}\mathrm{Rot}\,\Phi\rvert_{\mathrm{L}^2}$
est for $\mathcal{A}_2$	$\Leftrightarrow$	$\forall\, \Phi \in D(\mathring{\mathrm{Div}}_{\mathbb{T}}) \cap R(\operatorname{dev}\nabla)$	$\lvert\Phi\rvert_{\mathrm{L}^2} \le c_2 \lvert\mathrm{Div}\,\Phi\rvert_{\mathrm{L}^2}$
est for $\mathcal{A}_2^*$	$\Leftrightarrow$	$\forall\, \varphi \in D(\operatorname{dev}\nabla) \cap R(\mathring{\mathrm{Div}}_{\mathbb{T}}) = \mathrm{H}^1 \cap \mathrm{RT}^\perp$	$\lvert\varphi\rvert_{\mathrm{L}^2} \le c_2 \lvert\operatorname{dev}\nabla\varphi\rvert_{\mathrm{L}^2}$

(ii) all ranges $R(\mathrm{A}_n) = R(\mathcal{A}_n), \quad R(\mathrm{A}_n^*) = R(\mathcal{A}_n^*)$ are cl in L^2

(iii) all inverse ops $\mathcal{A}_n^{-1}$, $(\mathcal{A}_n^*)^{-1}$ are cont, even cpt

(iv) all Helmholtz decomposition hold, e.g.,

$$\mathrm{H}_1 = R(\mathrm{A}_0) \oplus N_1 \oplus R(\mathrm{A}_1^*) \quad \Leftrightarrow \quad \mathrm{L}^2_{\mathbb{S}} = R(\mathring{\nabla}\nabla) \oplus_{\mathrm{L}^2_{\mathbb{S}}} \mathcal{H}_{\mathrm{D},\mathbb{S}} \oplus_{\mathrm{L}^2_{\mathbb{S}}} R(\operatorname{sym}\mathrm{Rot}_{\mathbb{T}}),$$

$$\mathrm{H}_2 = R(\mathrm{A}_1) \oplus N_2 \oplus R(\mathrm{A}_2^*) \quad \Leftrightarrow \quad \mathrm{L}^2_{\mathbb{T}} = R(\mathring{\mathrm{Rot}}_{\mathbb{S}}) \oplus_{\mathrm{L}^2_{\mathbb{T}}} \mathcal{H}_{\mathrm{N},\mathbb{T}} \oplus_{\mathrm{L}^2_{\mathbb{T}}} R(\operatorname{dev}\nabla)$$

(v)-(ix) solution theories, variational formulations, functional a posteriori error estimates, div-curl-lemmas, . . .

Solving PDEs with Hilbert Complexes

APPENDIX: Literature

literature (FA-ToolBox, complexes, a posteriori error estimates, ...)

some results of this talk:

- Py: *Solution Theory, Variational Formulations, and Functional a Posteriori Error Estimates for General First Order Systems with Applications to Electro-Magneto-Statics and More*, (NFAO) Numerical Functional Analysis and Optimization, 2020

literature (complexes, Friedrichs type constants, Maxwell constants)

results of this talk:

- Py: *On Constants in Maxwell Inequalities for Bounded and Convex Domains*, Zapiski POMI/ (JMS)Journal of Mathematical Sciences (Springer New York), 2015
- Py: *On Maxwell's and Poincare's Constants*, (DCDS) Discrete and Continuous Dynamical Systems - Series S, 2015
- Py: *On the Maxwell Constants in 3D*, (M2AS) Mathematical Methods in the Applied Sciences, 2017
- Py: *On the Maxwell and Friedrichs/Poincaré Constants in ND*, (MZ) Mathematische Zeitschrift, 2019

- Py: *. . . some (so far) unpublished results*

literature (complexes, Friedrichs type constants, compact embeddings)

- Weck, N.: *Maxwell's boundary value problems on Riemannian manifolds with nonsmooth boundaries*,
  (JMA2) Journal of Mathematical Analysis and Applications, 1974 (1972)
- Picard, R.: *An elementary proof for a compact imbedding result in generalized electromagnetic theory*,
  (MZ) Mathematische Zeitschrift, 1984
- Witsch, K.-J.: *A remark on a compactness result in electromagnetic theory*,
  (M2AS) Mathematical Methods in the Applied Sciences, 1993

results of this talk:

- Bauer, S., Py, Schomburg, M.: *The Maxwell Compactness Property in Bounded Weak Lipschitz Domains with Mixed Boundary Conditions*,
  (SIMA) SIAM Journal on Mathematical Analysis, 2016
- Py, Zulehner, W.: *The divDiv-Complex and Applications to Biharmonic Equations*,
  (AA) Applicable Analysis, 2020
- Py, Zulehner, W.: *The Elasticity Complex*,
  submitted, 2020

COMMERCIALS

...the world is full of complexes ... ;-)

⇒ relaxing at (and you're all invited!)

AANMPDE$_{\text{JS}}$13 2020

13th Workshop on **A**nalysis and **A**dvanced **N**umerical **M**ethods for **P**artial **D**ifferential **E**quations (not only) for **J**unior **S**cientists	INVITED KEY NOTE SPEAKERS:
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