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Dirk Pauly

$\Omega \subset \mathbb{R}^3$

$$\operatorname{curl}_0 E = F, \qquad \operatorname{div} E = g$$

$$n \times E\big|_{\partial\Omega} = 0 \qquad\qquad n \cdot E\big|_{\partial\Omega} = 0$$

$$E = E_F + E_g$$

$$\operatorname{curl}_0 E_F = F, \qquad \operatorname{div} E_F = 0$$

$$\operatorname{curl}_0 E_g = 0, \qquad \operatorname{div} E_g = g$$

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① $\operatorname{curl}_0 E_F = F$

② $\operatorname{div} E_g = g$

$$\mathcal{H}_D := \{ E : \operatorname{curl}_0 E = 0 \wedge \operatorname{div} E = 0 \}$$

$$F \in R(\mathrm{curl}_0) \quad \wedge \quad g \in R(\mathrm{div})$$

$$E_F := \widetilde{\mathrm{curl}_0}^{\,-1} F \quad ; \quad \widetilde{\mathrm{curl}_0} \text{ inj.}$$

$$E_g := \widetilde{\mathrm{div}}^{\,-1} g \quad ; \quad \widetilde{\mathrm{div}} \text{ inj.}$$

notes $E_F \in N(\mathrm{curl}_0)^\perp = \overline{R(\mathrm{curl}_0^*)} = \overline{R(\mathrm{curl})}$

$E_g \in N(\mathrm{div})^\perp = \overline{R(\mathrm{div}^*)} = \overline{R(\nabla_0)}$

$$\boxed{R(\mathrm{curl}) \text{ closed}, \quad R(\nabla_0) \text{ closed}}$$

$$\left.\begin{array}{l} E_F \in \overline{R(\mathrm{curl})} \subset N(\mathrm{div}) \\ E_g \in \overline{R(\nabla_0)} \subset N(\mathrm{curl}_0) \end{array}\right\} \text{ complex property}$$

de Rham complex

$$\begin{tikzcd} L^2 \arrow[r, shift left, "\nabla_0"] & L^{2,3} \arrow[l, shift left, "\nabla_0^* = -\mathrm{div}"] \arrow[r, shift left, "\mathrm{curl}_0"] & L^{2,3} \arrow[l, shift left, "\mathrm{curl}_0^* = \mathrm{curl}"] \arrow[r, shift left, "\mathrm{div}_0"] & L^2 \arrow[l, shift left, "\mathrm{div}_0^* = -\nabla"] \end{tikzcd}$$

Hilbert complex

$$H_0 \underset{A_0^*}{\overset{A_0\,(\text{lddc})}{\rightleftarrows}} H_1 \underset{A_1^*}{\overset{A_1}{\rightleftarrows}} H_2$$

$$\boxed{A_1 x = f, \qquad A_0^* x = g}$$

complex prop.: $\boxed{A_1 A_0 = 0}$

$$\boxed{D(A_1) \cap D(A_0^*) \overset{\text{cpt}}{\hookrightarrow} H_1}$$

$$\boxed{\begin{aligned} &D(\mathrm{curl}_0) \cap D(\mathrm{div}) \hookrightarrow L^{2,3} \\ &= H_0(\mathrm{curl}) \cap H(\mathrm{div}) \end{aligned}}$$