FA-Toolbox: Part 2
Solution Theory and A Posteriori Error Estimates for Maxwell Type Problems

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ICIAM 2019

IMS: Modeling, Simulation and Optimization in Electrical Engineering

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Valencia, July 16, 2019

classical de Rham complex in 3D (∇-rot-div-complex)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot\cup \Gamma_n}$

(electro-magnetics, Maxwell's equations)

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{div}}{\overset{\mathring{\nabla}}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{rot}}{\overset{\mathring{\operatorname{rot}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\nabla}{\overset{\mathring{\operatorname{div}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}} \mathbb{R}$$

mixed boundary conditions and inhomogeneous and anisotropic media

$$\{0\} \text{ or } \mathbb{R} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{div}_{\Gamma_n}\varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_{\varepsilon} \underset{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}}{\overset{\mu^{-1}\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_{\mu} \underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}\mu}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathbb{R} \text{ or } \{0\}$$

for this talk: $\varepsilon = \mu = 1$ ($=$ id) and no mixed boundary conditions
for all appearing complexes

de Rham complex in ND or on Riemannian manifolds (d-complex)

$\Omega \subset \mathbb{R}^N$ bd w. Lip. dom. or Ω Riemannian manifold with cpt cl. and Lip. boundary Γ

(generalized Maxwell equations $\boxed{\text{the mother of all complexes}}$)

$$\{0\} \;\underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}}\; \mathsf{L}^{2,0} \;\underset{-\,\delta}{\overset{\mathring{\mathsf{d}}}{\rightleftarrows}}\; \mathsf{L}^{2,1} \;\underset{-\,\delta}{\overset{\mathring{\mathsf{d}}}{\rightleftarrows}}\; \cdots \;\boxed{\mathsf{L}^{2,q} \;\underset{-\,\delta}{\overset{\mathring{\mathsf{d}}}{\rightleftarrows}}\; \mathsf{L}^{2,q+1}}\; \cdots \;\mathsf{L}^{2,N-1} \;\underset{-\,\delta}{\overset{\mathring{\mathsf{d}}}{\rightleftarrows}}\; \mathsf{L}^{2,N} \;\underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}}\; \mathbb{R}$$

elasticity complex in 3D ($\operatorname{sym}\nabla$-$\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}$-$\operatorname{Div}_{\mathbb{S}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\operatorname{sym}}\nabla}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}}{\overset{\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^{\top}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{-\operatorname{sym}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathsf{RM}}}{\overset{\pi_{\mathsf{RM}}}{\rightleftarrows}} \mathsf{RM}$$

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\mathrm{Rot}_{\mathbb{S}}$-$\mathrm{Div}_{\mathbb{T}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\nabla\nabla}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}} \underset{-\operatorname{dev}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathsf{RT}}}{\overset{\pi_{\mathsf{RT}}}{\rightleftarrows}} \mathsf{RT}$$

general complex

$$A_0 : D(A_0) \subset H_0 \to H_1, \qquad A_1 : D(A_1) \subset H_1 \to H_2 \qquad \text{(lddc)}$$

$$A_0^* : D(A_0^*) \subset H_1 \to H_0, \qquad A_1^* : D(A_1^*) \subset H_2 \to H_1$$

general complex property $\boxed{A_1 A_0 = 0}$,

i.e., $R(A_0) \subset N(A_1)$ and/or eq $R(A_1^*) \subset N(A_0^*)$

$$\cdots \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad H_0 \quad \underset{A_0^*}{\overset{A_0}{\rightleftarrows}} \quad H_1 \quad \underset{A_1^*}{\overset{A_1}{\rightleftarrows}} \quad H_2 \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad \cdots$$

general observations

$$Ax = f$$

general theory

- $\boxed{\text{compact embeddings}}$

  $\Downarrow$
- closed ranges

  $\Downarrow$
- solution theory
- Friedrichs/Poincaré estimates and constants
- Helmholtz/Hodge/Weyl decompositions
- continuous and compact inverse operators
- variational formulations
- functional a posteriori error estimates
- generalized div-curl-lemma
- ...

idea: solve problem with general and simple linear functional analysis

general observations

$$\mathrm{A}x = f$$

let's say $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ linear and H_0, H_1 Hilbert spaces

question: How to solve?

general observations

$$\mathrm{A}x = f$$

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ linear

solution theory in the sense of Hadamard

- existence $\Leftrightarrow$ $f \in R(\mathrm{A})$
- uniqueness $\Leftrightarrow$ A inj $\Leftrightarrow$ $N(\mathrm{A}) = \{0\}$ $\Leftrightarrow$ A^{-1} exists
- cont dep on f $\Leftrightarrow$ A^{-1} cont

$\Rightarrow$ $x = \mathrm{A}^{-1} f \in D(\mathrm{A})$ and cont estimate (Friedrichs/Poincaré type estimate)

$$|x|_{\mathrm{H}_0} = |\mathrm{A}^{-1} f|_{\mathrm{H}_0} \leq c_{\mathrm{A}} |f|_{\mathrm{H}_1} = c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$$

$\Rightarrow$ best constant $c_{\mathrm{A}} = |\mathrm{A}^{-1}|_{R(\mathrm{A}),\mathrm{H}_0}$ $\quad |\mathrm{A}^{-1}|_{R(\mathrm{A}),D(\mathrm{A})} = (c_{\mathrm{A}}^2 + 1)^{1/2}$

general observations

$\mathrm{A}: D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$

$\mathrm{A}^*: D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0$ Hilbert space adjoint

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$\mathrm{H}_1 = \overline{R(\mathrm{A})} \oplus N(\mathrm{A}^*) \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

$$\mathrm{A}x = f$$

solution theory in the sense of Hadamard

- existence $\Leftrightarrow$ $f \in R(\mathrm{A}) = N(\mathrm{A}^*)^\perp$
- uniqueness $\Leftrightarrow$ A inj $\Leftrightarrow$ $N(\mathrm{A}) = \{0\}$ $\Leftrightarrow$ A^{-1} exists
- cont dep on f $\Leftrightarrow$ A^{-1} cont $\Leftrightarrow$ $R(\mathrm{A})$ cl (cl range theo)

fund range cond: $\boxed{R(\mathrm{A}) = \overline{R(\mathrm{A})} \text{ closed}}$ (must hold $\rightsquigarrow$ right setting!)

kernel cond: $\boxed{N(\mathrm{A}) = \{0\}}$ $\big($fails in gen $\rightsquigarrow$ proj onto $N(\mathrm{A})^\perp = \overline{R(\mathrm{A}^*)}\big)$

general observations

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$\mathrm{H}_1 = \overline{R(\mathrm{A})} \oplus N(\mathrm{A}^*) \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

remarkable observations

- time-dependent problems are simple
  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H} \to \mathrm{H}$, $\boxed{\mathrm{A} = \partial_t + \mathrm{T}}$ (gen T skew-sa, or alt lsast Re T ≥ 0)

  $$N(\mathrm{A}) = \{0\} \qquad N(\mathrm{A}^*) = \{0\} \qquad R(\mathrm{A}) \text{ (cl) } = N(\mathrm{A}^*)^{\perp} = \mathrm{H}$$

- time-harmonic problems are more complicated
  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H} \to \mathrm{H}$, $\boxed{\mathrm{A} = -\omega + \mathrm{T}}$

  $$N(\mathrm{A}),\ N(\mathrm{A}^*) \text{ (fin dim)} \qquad R(\mathrm{A}) \text{ (cl, fin co-dim) } = N(\mathrm{A}^*)^{\perp}$$

  (Fredholm alternative)

- stat problems are most complicated
  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$, $\boxed{\mathrm{A} = 0 + \mathrm{T}}$

  $$\dim N(\mathrm{A}) = \dim N(\mathrm{A}^*) = \infty \text{ (possibly)} \qquad R(\mathrm{A}) \text{ (cl, infin co-dim) } = N(\mathrm{A}^*)^{\perp}$$

1st fundamental observations

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ lddc, $\quad \mathrm{A}^* : D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0$ Hilbert space adjoint

$(\mathrm{A}, \mathrm{A}^*)$ dual pair as $(\mathrm{A}^*)^* = \overline{\mathrm{A}} = \mathrm{A}$

A, A^* may not be inj

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$\mathrm{H}_1 = N(\mathrm{A}^*) \oplus \overline{R(\mathrm{A})} \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

reduced operators restr to $N(\mathrm{A})^\perp$ and $N(\mathrm{A}^*)^\perp$

$$\mathcal{A} := \mathrm{A}|_{N(\mathrm{A})^\perp} = \mathrm{A}|_{\overline{R(\mathrm{A}^*)}} \qquad \mathcal{A}^* := \mathrm{A}^*|_{N(\mathrm{A}^*)^\perp} = \mathrm{A}^*|_{\overline{R(\mathrm{A})}}$$

$\mathcal{A}$, $\mathcal{A}^*$ inj $\quad \Rightarrow \quad \mathcal{A}^{-1}$, $(\mathcal{A}^*)^{-1}$ ex

1st fundamental observations

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1, \quad \mathrm{A}^* : D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0$ lddc $\qquad (\mathrm{A}, \mathrm{A}^*)$ dual pair

$$\mathrm{H}_1 = N(\mathrm{A}^*) \oplus \overline{R(\mathrm{A})} \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

more precisely

$$\mathcal{A} := \mathrm{A}|_{\overline{R(\mathrm{A}^*)}} : D(\mathcal{A}) \subset \overline{R(\mathrm{A}^*)} \to \overline{R(\mathrm{A})}, \qquad D(\mathcal{A}) := D(\mathrm{A}) \cap N(\mathrm{A})^{\perp} = D(\mathrm{A}) \cap \overline{R(\mathrm{A}^*)}$$

$$\mathcal{A}^* := \mathrm{A}^*|_{\overline{R(\mathrm{A})}} : D(\mathcal{A}^*) \subset \overline{R(\mathrm{A})} \to \overline{R(\mathrm{A}^*)}, \quad D(\mathcal{A}^*) := D(\mathrm{A}^*) \cap N(\mathrm{A}^*)^{\perp} = D(\mathrm{A}^*) \cap \overline{R(\mathrm{A})}$$

$(\mathcal{A}, \mathcal{A}^*)$ dual pair and $\mathcal{A}$, $\mathcal{A}^*$ inj $\Rightarrow$

inverse ops exist (and bij)

$$\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A}) \qquad (\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$$

refined decompositions

$$D(\mathrm{A}) = N(\mathrm{A}) \oplus D(\mathcal{A}) \qquad D(\mathrm{A}^*) = N(\mathrm{A}^*) \oplus D(\mathcal{A}^*)$$

$\Rightarrow$

$$R(\mathrm{A}) = R(\mathcal{A}) \qquad R(\mathrm{A}^*) = R(\mathcal{A}^*)$$

1st fundamental observations

closed range theorem & closed graph theorem $\Rightarrow$

Lemma (Friedrichs-Poincaré type est/cl range/cont inv)

The following assertions are equivalent:

- **(i)** $\exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathrm{H}_0} \le c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$
- **(i*)** $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathrm{H}_1} \le c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$
- **(ii)** $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_1.
- **(ii*)** $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_0.
- **(iii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is continuous and bijective.*
- **(iii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is continuous and bijective.*

In case that one of the latter assertions is true, e.g., (ii), $R(\mathrm{A})$ *is closed, we have*

$$\begin{aligned} \mathrm{H}_0 &= N(\mathrm{A}) \oplus R(\mathrm{A}^*) & \mathrm{H}_1 &= N(\mathrm{A}^*) \oplus R(\mathrm{A}) \\ D(\mathrm{A}) &= N(\mathrm{A}) \oplus D(\mathcal{A}) & D(\mathrm{A}^*) &= N(\mathrm{A}^*) \oplus D(\mathcal{A}^*) \\ D(\mathcal{A}) &= D(\mathrm{A}) \cap R(\mathrm{A}^*) & D(\mathcal{A}^*) &= D(\mathrm{A}^*) \cap R(\mathrm{A}) \end{aligned}$$

and $\mathcal{A} : D(\mathcal{A}) \subset R(\mathrm{A}^*) \to R(\mathrm{A}), \quad \mathcal{A}^* : D(\mathcal{A}^*) \subset R(\mathrm{A}) \to R(\mathrm{A}^*).$

Note: trivial equivalence to inf-sup condition

1st fundamental observations

recall

$$\textbf{(i)}\quad \exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \quad |x|_{\mathrm{H}_0} \le c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$$
$$\textbf{(i}^*\textbf{)}\quad \exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \quad |y|_{\mathrm{H}_1} \le c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$$

'best' consts in **(i)** and **(i*)** equal norms of the inv ops and Rayleigh quotients

$$c_{\mathrm{A}} = |\mathcal{A}^{-1}|_{R(\mathrm{A}),R(\mathrm{A}^*)} \qquad c_{\mathrm{A}^*} = |(\mathcal{A}^*)^{-1}|_{R(\mathrm{A}^*),R(\mathrm{A})}$$

$$\frac{1}{c_{\mathrm{A}}} = \inf_{0\neq x\in D(\mathcal{A})} \frac{|\mathrm{A}x|_{\mathrm{H}_1}}{|x|_{\mathrm{H}_0}} \qquad \frac{1}{c_{\mathrm{A}^*}} = \inf_{0\neq y\in D(\mathcal{A}^*)} \frac{|\mathrm{A}^* y|_{\mathrm{H}_0}}{|y|_{\mathrm{H}_1}}$$

Lemma (Friedrichs-Poincaré type const)

$c_{\mathrm{A}} = c_{\mathrm{A}^*}$

1st fundamental observations

Lemma (cpt emb/cpt inv)

The following assertions are equivalent:

(i) $D(\mathcal{A}) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_0$ *is compact.*

(i*) $D(\mathcal{A}^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1$ *is compact.*

(ii) $\mathcal{A}^{-1} : R(\mathrm{A}) \to R(\mathrm{A}^*)$ *is compact.*

(ii*) $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to R(\mathrm{A})$ *is compact.*

Lemma (Friedrichs-Poincaré type est/cl range/cont inv)

$\Downarrow$ $\boxed{D(\mathcal{A}) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_0 \text{ compact}}$

(i) $\exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathrm{H}_0} \le c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$

(i*) $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathrm{H}_1} \le c_{\mathrm{A}^*} |\mathrm{A}^*y|_{\mathrm{H}_0}$

(ii) $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_1.

(ii*) $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_0.

(iii) $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is continuous and bijective.*

(iii*) $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is continuous and bijective.*

(i)-(iii*) *equi* & *the resp Helm deco hold* & $|\mathcal{A}^{-1}| = c_{\mathrm{A}} = c_{\mathrm{A}^*} = |(\mathcal{A}^*)^{-1}|$

2nd fundamental observations

So far no complex...

$$\mathrm{A}_0 : D(\mathrm{A}_0) \subset \mathrm{H}_0 \to \mathrm{H}_1, \quad \mathrm{A}_1 : D(\mathrm{A}_1) \subset \mathrm{H}_1 \to \mathrm{H}_2 \text{ (lddc)}$$
$$\mathrm{A}_0^* : D(\mathrm{A}_0^*) \subset \mathrm{H}_1 \to \mathrm{H}_0, \quad \mathrm{A}_1^* : D(\mathrm{A}_1^*) \subset \mathrm{H}_2 \to \mathrm{H}_1 \text{ (lddc)}$$

general complex ($\mathrm{A}_1\mathrm{A}_0 = 0$, i.e., $R(\mathrm{A}_0) \subset N(\mathrm{A}_1)$ and $R(\mathrm{A}_1^*) \subset N(\mathrm{A}_0^*)$)

$$\cdots \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad \mathrm{H}_0 \quad \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \quad \mathrm{H}_1 \quad \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \quad \mathrm{H}_2 \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad \cdots$$

recall Helmholtz deco

$$\begin{array}{rcl} \mathrm{H}_1 = \overline{R(\mathrm{A}_0)} & \oplus & N(\mathrm{A}_0^*) \\ \bigcap & & \bigcup \\ = N(\mathrm{A}_1) & \oplus & \overline{R(\mathrm{A}_1^*)} \end{array} \qquad \Rightarrow \text{ (e.g.) } N(\mathrm{A}_1) = \overline{R(\mathrm{A}_0)} \oplus \Big(\underbrace{N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)}_{=:K_1}\Big)$$

$\Rightarrow$ refined Helmholtz deco

$$\mathrm{H}_1 = \overline{R(\mathrm{A}_0)} \oplus K_1 \oplus \overline{R(\mathrm{A}_1^*)}$$

2nd fundamental observations

recall

$$D(\mathrm{A}_1) = D(\mathcal{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \qquad R(\mathrm{A}_1) = R(\mathcal{A}_1) \qquad R(\mathrm{A}_1^*) = R(\mathcal{A}_1^*)$$
$$D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \cap \overline{R(\mathrm{A}_0)} \qquad R(\mathrm{A}_0^*) = R(\mathcal{A}_0^*) \qquad R(\mathrm{A}_0) = R(\mathcal{A}_0)$$

cohomology group $K_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)$

Lemma (Helmholtz deco I)

$$\begin{aligned} \mathrm{H}_1 &= \overline{R(\mathrm{A}_0)} \oplus N(\mathrm{A}_0^*) & \mathrm{H}_1 &= \overline{R(\mathrm{A}_1^*)} \oplus N(\mathrm{A}_1) \\ D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus N(\mathrm{A}_0^*) & D(\mathrm{A}_1) &= D(\mathcal{A}_1) \oplus N(\mathrm{A}_1) \\ N(\mathrm{A}_1) &= D(\mathcal{A}_0^*) \oplus K_1 & N(\mathrm{A}_0^*) &= D(\mathcal{A}_1) \oplus K_1 \\ D(\mathrm{A}_1) &= \overline{R(\mathrm{A}_0)} \oplus \big(D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)\big) & D(\mathrm{A}_0^*) &= \overline{R(\mathrm{A}_1^*)} \oplus \big(D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1)\big) \end{aligned}$$

Lemma (Helmholtz deco II)

$$\begin{aligned} \mathrm{H}_1 &= \overline{R(\mathrm{A}_0)} \oplus K_1 \oplus \overline{R(\mathrm{A}_1^*)} \\ D(\mathrm{A}_1) &= \overline{R(\mathrm{A}_0)} \oplus K_1 \oplus D(\mathcal{A}_1) \\ D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus K_1 \oplus \overline{R(\mathrm{A}_1^*)} \\ D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus K_1 \oplus D(\mathcal{A}_1) \end{aligned}$$

2nd fundamental observations

$$K_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \qquad D(\mathrm{A}_1) = D(\mathcal{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \qquad D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \cap \overline{R(\mathrm{A}_0)}$$

Lemma (cpt emb II)

The following assertions are equivalent:

(i) $D(\mathcal{A}_0) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_0$, $D(\mathcal{A}_1) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$, *and* $K_1 \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$ *are compact.*

(ii) $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$ *is compact.*

In this case $K_1 < \infty$.

Theorem (fa-toolbox I)

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1 \textit{ compact}}$

(i) *all emb cpt, i.e.,* $D(\mathcal{A}_0) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_0$, $D(\mathcal{A}_1) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$, $D(\mathcal{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$, $D(\mathcal{A}_1^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_2$ *cpt*

(ii) *cohomology group* K_1 *finite dim*

(iii) *all ranges closed, i.e.,* $R(\mathrm{A}_0)$, $R(\mathrm{A}_0^*)$, $R(\mathrm{A}_1)$, $R(\mathrm{A}_1^*)$ *cl*

(iv) *all Friedrichs-Poincaré type est hold*

(v) *all Hodge-Helmholtz-Weyl type deco I & II hold with closed ranges*

2nd fundamental observations

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

Theorem (fa-toolbox I (Friedrichs-Poincaré type est))

$\Downarrow \quad \boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1 \text{ compact}} \quad \Rightarrow \quad \exists \quad |\mathcal{A}_i^{-1}| = c_{\mathrm{A}_i} = c_{\mathrm{A}_i^*} = |(\mathcal{A}_i^*)^{-1}| \in (0, \infty)$

(i) $\forall\, x \in D(\mathcal{A}_0) \qquad |x|_{\mathrm{H}_0} \leq c_{\mathrm{A}_0} |\mathrm{A}_0 x|_{\mathrm{H}_1}$

(i*) $\forall\, y \in D(\mathcal{A}_0^*) \qquad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}_0} |\mathrm{A}_0^* y|_{\mathrm{H}_0}$

(ii) $\forall\, y \in D(\mathcal{A}_1) \qquad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |\mathrm{A}_1 y|_{\mathrm{H}_2}$

(ii*) $\forall\, z \in D(\mathcal{A}_1^*) \qquad |z|_{\mathrm{H}_2} \leq c_{\mathrm{A}_1} |\mathrm{A}_1^* z|_{\mathrm{H}_1}$

(iii) $\forall\, y \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \qquad |(1 - \pi_{K_1}) y|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |\mathrm{A}_1 y|_{\mathrm{H}_2} + c_{\mathrm{A}_0} |\mathrm{A}_0^* y|_{\mathrm{H}_0}$

note $\pi_{K_1} y \in K_1$ and $(1 - \pi_{K_1}) y \in K_1^{\perp}$

Remark

enough $R(\mathrm{A}_0)$ *and* $R(\mathrm{A}_1)$ *cl*

2nd fundamental observations

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

Theorem (fa-toolbox I (Helmholtz deco))

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1 \textit{ compact}}$

$$\begin{aligned}
\mathrm{H}_1 &= R(\mathrm{A}_0) \oplus N(\mathrm{A}_0^*) & \mathrm{H}_1 &= R(\mathrm{A}_1^*) \oplus N(\mathrm{A}_1)\\
D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus N(\mathrm{A}_0^*) & D(\mathrm{A}_1) &= D(\mathcal{A}_1) \oplus N(\mathrm{A}_1)\\
N(\mathrm{A}_1) &= D(\mathcal{A}_0^*) \oplus K_1 & N(\mathrm{A}_0^*) &= D(\mathcal{A}_1) \oplus K_1\\
D(\mathrm{A}_1) &= R(\mathrm{A}_0) \oplus \big(D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)\big) & D(\mathrm{A}_0^*) &= R(\mathrm{A}_1^*) \oplus \big(D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1)\big)
\end{aligned}$$

$$\begin{aligned}
\mathrm{H}_1 &= R(\mathrm{A}_0) \oplus K_1 \oplus R(\mathrm{A}_1^*)\\
D(\mathrm{A}_1) &= R(\mathrm{A}_0) \oplus K_1 \oplus D(\mathcal{A}_1)\\
D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus K_1 \oplus R(\mathrm{A}_1^*)\\
D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus K_1 \oplus D(\mathcal{A}_1)
\end{aligned}$$

Remark

enough $R(\mathrm{A}_0)$ and $R(\mathrm{A}_1)$ cl

(stat) first order system - solution theory

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

$\boxed{\mathrm{A}_1 x = f}$ $\qquad \dim N(\mathrm{A}_1) = \infty$

find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ such that the fos

$$\begin{aligned} \mathrm{A}_1 x &= f \\ \mathrm{A}_0^* x &= g \\ \pi_{K_1} x &= k \end{aligned} \qquad \text{think of} \qquad \begin{aligned} (\mathring{\mathrm{rot}} E &= F) \\ (-\operatorname{div} E &= g) \\ (\pi_{\mathrm{D}} E &= K) \end{aligned}$$

kernel = cohomology group = $K_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)$

trivially necessary $\quad f \in R(\mathrm{A}_1) \quad g \in R(\mathrm{A}_0^*) \quad k \in K_1$

$\boxed{\text{apply fa-toolbox}}$

(stat) first order system - solution theory

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ st fos $\qquad \mathrm{A}_1 x = f \qquad \mathrm{A}_0^* x = g \qquad \pi_{K_1} x = k$

Theorem (fa-toolbox II (solution theory))

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1 \textit{ compact}}$

fos is uniq sol $\Leftrightarrow$ $f \in R(\mathrm{A}_1) \qquad g \in R(\mathrm{A}_0^*) \qquad k \in K_1$

$$x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus K_1 = D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$$

$$\boxed{x_f := \mathcal{A}_1^{-1} f} \in D(\mathcal{A}_1)$$

$$\boxed{x_g := (\mathcal{A}_0^*)^{-1} g} \in D(\mathcal{A}_0^*)$$

dep cont on data $|x|_{\mathrm{H}_1} \leq |x_f|_{\mathrm{H}_1} + |x_g|_{\mathrm{H}_1} + |k|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |f|_{\mathrm{H}_2} + c_{\mathrm{A}_0} |g|_{\mathrm{H}_0} + |k|_{\mathrm{H}_1}$

moreover

$$\pi_{R(\mathrm{A}_1^*)} x = x_f \qquad \pi_{R(\mathrm{A}_0)} x = x_g \qquad \pi_{K_1} x = k \qquad |x|_{\mathrm{H}_1}^2 = |x_f|_{\mathrm{H}_1}^2 + |x_g|_{\mathrm{H}_1}^2 + |k|_{\mathrm{H}_1}^2$$

Remark

enough $R(\mathrm{A}_0)$ *and* $R(\mathrm{A}_1)$ *cl*

(stat) first order system - variational formulations

$$x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus K_1 = D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$$

$$x_f := \mathcal{A}_1^{-1} f \in D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap R(\mathrm{A}_1^*) = D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \cap K_1^\perp$$

$$x_g := (\mathcal{A}_0^*)^{-1} g \in D(\mathcal{A}_0^*) = D(\mathrm{A}_0^*) \cap R(\mathrm{A}_0) = D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1) \cap K_1^\perp$$

$$\begin{aligned} \mathrm{A}_1 x &= f \\ \mathrm{A}_0^* x &= g \\ \pi_{K_1} x &= k \end{aligned} \qquad \begin{aligned} \mathrm{A}_1 x_f &= f \\ \mathrm{A}_0^* x_f &= 0 \\ \pi_{K_1} x_f &= 0 \end{aligned} \qquad \begin{aligned} \mathrm{A}_1 x_g &= 0 \\ \mathrm{A}_0^* x_g &= g \\ \pi_{K_1} x_g &= 0 \end{aligned} \qquad \begin{aligned} \mathrm{A}_1 k &= 0 \\ \mathrm{A}_0^* k &= 0 \\ \pi_{K_1} k &= k \end{aligned}$$

- option I: find x_f and x_g separately $\Rightarrow$ $x = x_f + x_g + k$
- option II: find x directly

(stat) first order system - variational formulations I

finding

$$x_f := \mathcal{A}_1^{-1} f \in D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap \underbrace{R(\mathrm{A}_1^*)}_{=R(\mathcal{A}_1^*)} = D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \cap K_1^{\perp}$$

$$\begin{aligned} \mathrm{A}_1 x_f &= f \\ \mathrm{A}_0^* x_f &= 0 \\ \pi_{K_1} x_f &= 0 \end{aligned}$$

at least two options

- option Ia: multiply $\mathrm{A}_1 x_f = f$ by $\mathrm{A}_1 \xi$ $\Rightarrow$

  $$\forall\, \xi \in D(\mathcal{A}_1) \qquad \langle \mathrm{A}_1 x_f, \mathrm{A}_1 \xi \rangle_{\mathrm{H}_2} = \langle f, \mathrm{A}_1 \xi \rangle_{\mathrm{H}_2}$$

  weak form of $\boxed{\mathrm{A}_1^* \mathrm{A}_1 x_f = \mathrm{A}_1^* f}$
- option Ib: repr $x_f = \mathrm{A}_1^* y_f$ with potential $y_f = (\mathcal{A}_1^*)^{-1} x_f \in D(\mathcal{A}_1^*)$
  and mult by x_f by $\mathrm{A}_1^* \phi$ $\Rightarrow$

  $$\forall\, \phi \in D(\mathcal{A}_1^*) \qquad \langle \mathrm{A}_1^* y_f, \mathrm{A}_1^* \phi \rangle_{\mathrm{H}_1} = \langle x_f, \mathrm{A}_1^* \phi \rangle_{\mathrm{H}_1} = \langle \mathrm{A}_1 x_f, \phi \rangle_{\mathrm{H}_2} = \langle f, \phi \rangle_{\mathrm{H}_2}$$

  weak form of $\boxed{\mathrm{A}_1 x_f = f}$ and $\boxed{\mathrm{A}_1 \mathrm{A}_1^* y_f = f}$

analogously for x_g

(stat) first order system - a posteriori error estimates

problem: $\boxed{\text{find} \quad x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \quad \text{st} \quad \mathrm{A}_1 x = f \quad \mathrm{A}_0^* x = g \quad \pi_{K_1} x = k}$

'very' non-conforming 'approximation' of x: $\boxed{\tilde{x} \in \mathrm{H}_1}$

def., dcmp. err. $\boxed{e = x - \tilde{x}} = \pi_{R(\mathrm{A}_0)} e + \pi_{K_1} e + \pi_{R(\mathrm{A}_1^*)} e \in \mathrm{H}_1 = R(\mathrm{A}_0) \oplus K_1 \oplus R(\mathrm{A}_1^*)$

Theorem (sharp upper bounds)

Let $\tilde{x} \in \mathrm{H}_1$ *and* $e = x - \tilde{x}$. *Then*

$$|e|_{\mathrm{H}_1}^2 = |\pi_{R(\mathrm{A}_0)} e|_{\mathrm{H}_1}^2 + |\pi_{K_1} e|_{\mathrm{H}_1}^2 + |\pi_{R(\mathrm{A}_1^*)} e|_{\mathrm{H}_1}^2$$

$$|\pi_{R(\mathrm{A}_0)} e|_{\mathrm{H}_1} = \min_{\phi \in D(\mathrm{A}_0^*)} \big(c_{\mathrm{A}_0} |\mathrm{A}_0^* \phi - g|_{\mathrm{H}_0} + |\phi - \tilde{x}|_{\mathrm{H}_1} \big)$$

$\boxed{\textit{reg } (\mathrm{A}_0 \mathrm{A}_0^* + 1)\textit{-prbl in } D(\mathrm{A}_0^*)}$

$$|\pi_{R(\mathrm{A}_1^*)} e|_{\mathrm{H}_1} = \min_{\varphi \in D(\mathrm{A}_1)} \big(c_{\mathrm{A}_1} |\mathrm{A}_1 \varphi - f|_{\mathrm{H}_2} + |\varphi - \tilde{x}|_{\mathrm{H}_1} \big)$$

$\boxed{\textit{reg } (\mathrm{A}_1^* \mathrm{A}_1 + 1)\textit{-prbl in } D(\mathrm{A}_1)}$

$$|\pi_{K_1} e|_{\mathrm{H}_1} = |\pi_{K_1} \tilde{x} - k|_{\mathrm{H}_1} = \min_{\substack{\xi \in D(\mathrm{A}_0) \\ \zeta \in D(\mathrm{A}_1^*)}} |\mathrm{A}_0 \xi + \mathrm{A}_1^* \zeta + \tilde{x} - k|_{\mathrm{H}_1}$$

$\boxed{\textit{cpld } (\mathrm{A}_0^* \mathrm{A}_0)\text{-}(\mathrm{A}_1 \mathrm{A}_1^*)\textit{-sys in } D(\mathcal{A}_0)\text{-}D(\mathcal{A}_1^*)}$

Remark

Even $\pi_{K_1} e = k - \pi_{K_1} \tilde{x}$ *and the minima are attained at*

$$\hat{\phi} = \pi_{R(\mathrm{A}_0)} e + \tilde{x}, \qquad \hat{\varphi} = \pi_{R(\mathrm{A}_1^*)} e + \tilde{x}, \qquad \mathrm{A}_0 \hat{\xi} + \mathrm{A}_1^* \hat{\zeta} = (\pi_{K_1} - 1) \tilde{x}.$$

A_0^*-A_1-lemma (generalized global div-curl-lemma)

Lemma (A_0^*-A_1-lemma)

Let $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathrm{H}_1$ be compact, and

(i) *(x_n) bounded in $D(\mathrm{A}_1)$,*

(ii) *(y_n) bounded in $D(\mathrm{A}_0^*)$.*

$\Rightarrow \exists\ x \in D(\mathrm{A}_1)$, $y \in D(\mathrm{A}_0^*)$ and subsequences st

$x_n \rightharpoonup x$ in $D(\mathrm{A}_1)$ and $y_n \rightharpoonup y$ in $D(\mathrm{A}_0^*)$ as well as

$$\langle x_n, y_n \rangle_{\mathrm{H}_1} \to \langle x, y \rangle_{\mathrm{H}_1}.$$

A_0^*-A_1-lemma (generalized global div-curl-lemma)

Lemma (generalized A_0^*-A_1-lemma)

Let $R(\mathrm{A}_0)$ and $R(\mathrm{A}_1$ be closed, and let K_1 be finite dimensional. Moreover, let $(x_n), (y_n) \subset \mathrm{H}_1$ be bounded such that

(i) $\widetilde{\mathrm{A}}_1 x_n$ *is relatively compact in* $D(\mathrm{A}_1^*)'$,

(ii) $\widetilde{\mathrm{A}}_0^* y_n$ *is relatively compact in* $D(\mathrm{A}_0)'$.

$\Rightarrow \exists\ x, y \in \mathrm{H}_1$ *and subsequences st* $x_n \rightharpoonup x$ *in* H_1 *and* $y_n \rightharpoonup y$ *in* H_1 *as well as*

$$\langle x_n, y_n \rangle_{\mathrm{H}_1} \to \langle x, y \rangle_{\mathrm{H}_1}.$$

proof uses key observation

Lemma

Let $R(\mathrm{A})$ be closed. For $(x_n) \subset \mathrm{H}_0$ the following statements are equivalent:

(i) $\widetilde{\mathrm{A}} x_n$ *is relatively compact in* $D(\mathrm{A}^*)'$.

(ii) $\pi_{R(\mathrm{A}^*)} x_n$ *is relatively compact in* $R(\mathrm{A}^*)$ *resp.* H_1.

If $x_n \rightharpoonup x$ *in* H_1, *then either of cond. (i) or (ii) implies* $\pi_{R(\mathrm{A}^*)} x_n \to \pi_{R(\mathrm{A}^*)} x$ *in* H_1.

nice results (and joint work/communication with) Marcus Waurick

classical de Rham complex in 3D (∇-rot-div-complex)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot{\cup} \Gamma_n}$

(electro-magneto dynamics, Maxwell's equations)

$$\{0\} \quad \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{-\operatorname{div}}{\overset{\mathring{\nabla}}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{\operatorname{rot}}{\overset{\mathring{\operatorname{rot}}}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{-\nabla}{\overset{\mathring{\operatorname{div}}}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}} \quad \mathbb{R}$$

mixed boundary conditions and inhomogeneous and anisotropic media

$$\{0\} \text{ or } \mathbb{R} \quad \underset{\pi}{\overset{\iota}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{-\operatorname{div}_{\Gamma_n}\varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}} \quad \mathsf{L}^2_\varepsilon \quad \underset{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{\iota}{\overset{\pi}{\rightleftarrows}} \quad \mathbb{R} \text{ or } \{0\}$$

classical de Rham complex in 3D (∇-rot-div-complex)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot\cup \Gamma_n}$

(electro-magneto dynamics, Maxwell's equations with mixed boundary conditions)

$$\{0\} \text{ or } \mathbb{R} \;\underset{\pi}{\overset{\iota}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\operatorname{div}_{\Gamma_n}\varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2_\varepsilon \;\underset{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota}{\overset{\pi}{\rightleftarrows}}\; \mathbb{R} \text{ or } \{0\}$$

related fos

$\nabla_{\Gamma_t} u = A$ in Ω	$\operatorname{rot}_{\Gamma_t} E = J$ in Ω	$\operatorname{div}_{\Gamma_t} H = k$ in Ω	$\pi v = b$ in Ω
$\pi u = a$ in Ω	$-\operatorname{div}_{\Gamma_n} \varepsilon E = j$ in Ω	$\varepsilon^{-1}\operatorname{rot}_{\Gamma_n} H = K$ in Ω	$-\nabla_{\Gamma_n} v = B$ in Ω

related sos

$-\operatorname{div}_{\Gamma_n} \varepsilon \nabla_{\Gamma_t} u = j$ in Ω	$\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}\operatorname{rot}_{\Gamma_t} E = K$ in Ω	$-\nabla_{\Gamma_n}\operatorname{div}_{\Gamma_t} H = B$ in Ω
$\pi u = a$ in Ω	$-\operatorname{div}_{\Gamma_n} \varepsilon E = j$ in Ω	$\varepsilon^{-1}\operatorname{rot}_{\Gamma_n} H = K$ in Ω

corresponding compact embeddings:

$$D(\nabla_{\Gamma_t}) \cap D(\pi) = D(\nabla_{\Gamma_t}) = \mathsf{H}^1_{\Gamma_t} \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

$$D(\operatorname{rot}_{\Gamma_t}) \cap D(-\operatorname{div}_{\Gamma_n}\varepsilon) = \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n} \hookrightarrow \mathsf{L}^2_\varepsilon \qquad \text{(Weck's selection theorem, '72)}$$

$$D(\operatorname{div}_{\Gamma_t}) \cap D(\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}) = \mathsf{D}_{\Gamma_t} \cap \mathsf{R}_{\Gamma_n} \hookrightarrow \mathsf{L}^2 \qquad \text{(Weck's selection theorem, '72)}$$

$$D(\nabla_{\Gamma_n}) \cap D(\pi) = D(\nabla_{\Gamma_n}) = \mathsf{H}^1_{\Gamma_n} \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

Weck's selection theorem for weak Lip. dom. and mixed bc: Bauer/Py/Schomburg ('16)

Weck's selection theorem (Weck '74, (Habil. '72) stimulated by Rolf Leis)

(Weber '80, Picard '84, Costabel '90, Witsch '93, Jochmann '97, Fernandes/Gilardi '97, Kuhn '99, Picard/Weck/Witsch '01, Py '96, '03, '06, '07, '08)

classical de Rham complex in 3D (∇-rot-div-complex)

$$\begin{aligned} \operatorname{rot} E &= F && \text{in } \Omega \\ -\operatorname{div} \varepsilon E &= g && \text{in } \Omega \\ \nu \times E &= 0 && \text{at } \Gamma_t \\ \nu \cdot \varepsilon E &= 0 && \text{at } \Gamma_n \end{aligned}$$

non-trivial kernel $\mathcal{H}_{\mathrm{D},\varepsilon} = \{H \in \mathrm{L}^2 : \operatorname{rot} H = 0,\ \operatorname{div} \varepsilon H = 0,\ \nu \times H|_{\Gamma_t} = 0,\ \nu \cdot \varepsilon H|_{\Gamma_n} = 0\}$
additional condition on Dirichlet/Neumann fields for uniqueness

$$\pi_{\mathrm{D}} E = K \in \mathcal{H}_{\mathrm{D},\varepsilon}$$

$$\{0\} \text{ or } \mathbb{R} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathrm{L}^2 \underset{-\operatorname{div}_{\Gamma_n} \varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}} \mathrm{L}^2_\varepsilon \underset{\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}} \mathrm{L}^2 \underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}} \mathrm{L}^2 \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathbb{R} \text{ or } \{0\}$$

$$\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_{-1} \underset{\mathrm{A}_{-1}^*}{\overset{\mathrm{A}_{-1}}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\mathrm{A}_2^*}{\overset{\mathrm{A}_2}{\rightleftarrows}} \mathrm{H}_3 \underset{\mathrm{A}_3^*}{\overset{\mathrm{A}_3}{\rightleftarrows}} \mathrm{H}_4 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$$

find $E \in \mathrm{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\mathrm{D}_{\Gamma_n}(\Omega)$ st (fos) find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ st

$$\begin{aligned} \operatorname{rot}_{\Gamma_t} E &= F \\ -\operatorname{div}_{\Gamma_n} \varepsilon E &= g \\ \pi_{\mathrm{D/N}} E &= K \end{aligned} \qquad \text{translation} \qquad \begin{aligned} \mathrm{A}_1 x &= f \\ \mathrm{A}_0^* x &= g \\ \pi_{K_1} x &= k \end{aligned}$$

classical de Rham complex in 3D (∇-rot-div-complex)

$c_{\mathrm{A}_0} = c_{\mathrm{fp}}$ (Friedrichs/Poincaré constant) and $c_{\mathrm{A}_1} = c_{\mathrm{m}}$ (Maxwell constant)

Lemma/Theorem $\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\to \mathrm{L}^2_\varepsilon(\Omega) \text{ compact}}$

(i) all Friedrichs-Poincaré type est hold

$$\forall\, \varphi \in D(\mathcal{A}_0) \quad |\varphi|_{\mathrm{H}_0} \le c_{\mathrm{A}_0} |\mathrm{A}_0 \varphi|_{\mathrm{H}_1} \quad \Leftrightarrow \quad \forall\, \varphi \in \mathrm{H}^1_{\Gamma_t} \quad |\varphi|_{\mathrm{L}^2} \le c_{\mathrm{fp}} |\nabla \varphi|_{\mathrm{L}^2_\varepsilon}$$

$$\forall\, \phi \in D(\mathcal{A}_0^*) \quad |\phi|_{\mathrm{H}_1} \le c_{\mathrm{A}_0} |\mathrm{A}_0^* \phi|_{\mathrm{H}_0} \quad \Leftrightarrow \quad \forall\, \Phi \in \varepsilon^{-1}\mathrm{D}_{\Gamma_n} \cap \nabla \mathrm{H}^1_{\Gamma_t} \quad |\Phi|_{\mathrm{L}^2_\varepsilon} \le c_{\mathrm{fp}} |\operatorname{div} \varepsilon \Phi|_{\mathrm{L}^2}$$

$$\forall\, \varphi \in D(\mathcal{A}_1) \quad |\varphi|_{\mathrm{H}_1} \le c_{\mathrm{A}_1} |\mathrm{A}_1 \varphi|_{\mathrm{H}_2} \quad \Leftrightarrow \quad \forall\, \Phi \in \mathrm{R}_{\Gamma_t} \cap \varepsilon^{-1} \operatorname{rot} \mathrm{R}_{\Gamma_n} \quad |\Phi|_{\mathrm{L}^2_\varepsilon} \le c_{\mathrm{m}} |\operatorname{rot} \Phi|_{\mathrm{L}^2}$$

$$\forall\, \psi \in D(\mathcal{A}_1^*) \quad |\psi|_{\mathrm{H}_2} \le c_{\mathrm{A}_1} |\mathrm{A}_1^* \psi|_{\mathrm{H}_1} \quad \Leftrightarrow \quad \forall\, \Psi \in \mathrm{R}_{\Gamma_n} \cap \operatorname{rot} \mathrm{R}_{\Gamma_t} \quad |\Psi|_{\mathrm{L}^2} \le c_{\mathrm{m}} |\operatorname{rot} \Psi|_{\mathrm{L}^2_\varepsilon}$$

(ii) all ranges $R(\mathrm{A}_0) = \nabla \mathrm{H}^1_{\Gamma_t}$, $R(\mathrm{A}_1) = \operatorname{rot} \mathrm{R}_{\Gamma_t}$, $R(\mathrm{A}_0^*) = \operatorname{div} \mathrm{D}_{\Gamma_n}$ are cl in L^2

(iii) the inverse ops $(\widetilde{\nabla}_{\Gamma_t})^{-1}$, $(\widetilde{\operatorname{div}_{\Gamma_n} \varepsilon})^{-1}$, $(\widetilde{\operatorname{rot}}_{\Gamma_t})^{-1}$, $(\widetilde{\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}})^{-1}$ are cont, even cpt

(iv) all Helmholtz decomposition hold, e.g.,

$$\mathrm{H}_1 = R(\mathrm{A}_0) \oplus K_1 \oplus R(\mathrm{A}_1^*) \quad \Leftrightarrow \quad \mathrm{L}^2_\varepsilon = \nabla \mathrm{H}^1_{\Gamma_t} \oplus_{\mathrm{L}^2_\varepsilon} \mathcal{H}_{\mathrm{D},\varepsilon} \oplus_{\mathrm{L}^2_\varepsilon} \varepsilon^{-1} \operatorname{rot} \mathrm{R}_{\Gamma_n}$$

(v) solution theory

(vi) variational formulations

(vii) functional a posteriori error estimates

(viii) div-curl-lemma

(ix) ...

classical de Rham complex in 3D (∇-rot-div-complex)

find $E \in \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n}$ s.t. / think of $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$

$$\begin{aligned} \operatorname{rot}_{\Gamma_t} E &= F \\ \operatorname{div}_{\Gamma_n} \varepsilon E &= g \\ \pi_{\mathcal{H}_{\mathrm{D},\varepsilon}} E &= K \end{aligned} \qquad / \qquad \text{think of} \qquad \begin{aligned} \mathrm{A}_1 x &= f \\ \mathrm{A}_0^* x &= g \\ \pi_{K_1} x &= k \end{aligned}$$

sol is simply $x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus K_1 = D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$

with $\boxed{x_f := \mathcal{A}_1^{-1} f} \in D(\mathcal{A}_1)$ and $\boxed{x_g := (\mathcal{A}_0^*)^{-1} g} \in D(\mathcal{A}_0^*)$

i.e., $E = E_F + E_g + K$, where

$$\boxed{E_F := (\widetilde{\operatorname{rot}}_{\Gamma_t})^{-1} F} \in D(\widetilde{\operatorname{rot}}_{\Gamma_t}) = \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1} \operatorname{rot} \mathsf{R}_{\Gamma_n} = \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1} \mathsf{D}_{\Gamma_n,0} \cap \mathcal{H}_{\mathrm{D},\varepsilon}^{\perp},$$

$$\boxed{E_g := (\widetilde{\operatorname{div}}_{\Gamma_n} \varepsilon)^{-1} g} \in D(\widetilde{\operatorname{div}}_{\Gamma_n} \varepsilon) = \varepsilon^{-1} \mathsf{D}_{\Gamma_n} \cap \nabla \mathsf{H}_{\Gamma_t}^1 = \varepsilon^{-1} \mathsf{D}_{\Gamma_n} \cap \mathsf{R}_{\Gamma_t,0} \cap \mathcal{H}_{\mathrm{D},\varepsilon}^{\perp}$$

classical de Rham complex in 3D (∇-rot-div-complex)

Theorem (sharp upper bounds)

Let $\boxed{\tilde{E} \in \mathsf{L}^2_\varepsilon}$ *(very non-conforming approximation of E!) and* $\boxed{e := E - \tilde{E}}$. *Then*

$$|e|^2_{\mathsf{L}^2_\varepsilon} = |\pi_{R(\nabla_{\Gamma_t})} e|^2_{\mathsf{L}^2_\varepsilon} + |\pi_{R(\varepsilon^{-1}\,\mathrm{rot}_{\Gamma_n})} e|^2_{\mathsf{L}^2_\varepsilon} + |\pi_{\mathcal{H}_{\mathrm{D},\varepsilon}} e|^2_{\mathsf{L}^2_\varepsilon}$$

$$= \min_{\Phi \in \varepsilon^{-1}\mathsf{D}_{\Gamma_n}} \big(c_{\mathrm{fp}} |\operatorname{div} \varepsilon\Phi + g|_{\mathsf{L}^2} + |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon}\big)^2$$

$\boxed{\text{reg } (-\nabla_{\Gamma_t} \operatorname{div}_{\Gamma_n} + 1)\text{-prbl in } \mathsf{D}_{\Gamma_n}}$

$$+ \min_{\Psi \in \mathsf{R}_{\Gamma_t}} \big(c_{\mathrm{m}} |\operatorname{rot} \Psi - F|_{\mathsf{L}^2} + |\Psi - \tilde{E}|_{\mathsf{L}^2_\varepsilon}\big)^2$$

$\boxed{\text{reg } (\mathrm{rot}_{\Gamma_n} \mathrm{rot}_{\Gamma_t} + 1)\text{-prbl in } \mathsf{R}_{\Gamma_t}}$

$$+ \min_{\theta \in \mathsf{H}^1_{\Gamma_t},\, \Theta \in \mathsf{R}_{\Gamma_n}} |\nabla\theta + \varepsilon^{-1} \operatorname{rot} \Theta + \tilde{E} - K|^2_{\mathsf{L}^2_\varepsilon}$$

$\boxed{\text{cpld } (-\operatorname{div}_{\Gamma_n} \nabla_{\Gamma_t})\text{-}(\mathrm{rot}_{\Gamma_t} \mathrm{rot}_{\Gamma_n})\text{-sys in } \mathsf{H}^1_{\Gamma_t}\text{-}\mathsf{R}_{\Gamma_n}}$

Remark

- $(\mathrm{rot}_{\Gamma_t} \mathrm{rot}_{\Gamma_n})$*-prbl needs saddle point formulation*
- Ω *top trv* $\Rightarrow$ $\pi_{\mathrm{D}} = 0$ *and* $\mathsf{R}_{\Gamma_t,0} = \nabla \mathsf{H}^1_{\Gamma_t}$ *and* $\mathsf{D}_{\Gamma_n,0} = \operatorname{rot} \mathsf{R}_{\Gamma_n}$
- $\boxed{\Omega \text{ convex and } \varepsilon = \mu = 1 \text{ and } \Gamma_t = \Gamma \text{ or } \Gamma_n = \Gamma \Rightarrow c_{\mathrm{f}} \le c_{\mathrm{m}} \le c_{\mathrm{p}} \le \frac{\mathrm{diam}_\Omega}{\pi}}$

functional a posteriori error estimates for BEM

(joint work with

Stefan Kurz, Dirk Praetorius, Sergey Repin, Daniel Sebastian)

problem: num approx with $\boxed{\text{BEM}}$

$$\Delta u = 0 \quad \text{in } \Omega, \qquad\qquad u|_\Gamma = g \quad \text{on } \Gamma.$$

functional a posteriori error estimates: num approx with $\boxed{\text{FEM}}$

$$\max_{\substack{E \in \mathsf{L}^2(\Omega) \\ \operatorname{div} E = 0}} \Big(2 \langle n \cdot E, g - \widetilde{u}|_\Gamma \rangle_{\mathsf{H}^{-1/2}(\Gamma)} - |E|^2_{\mathsf{L}^2(\Omega)} \Big) = |\nabla (u - \widetilde{u})|^2_{\mathsf{L}^2(\Omega)} = \min_{\substack{v \in \mathsf{H}^1(\Omega) \\ v|_\Gamma = g - \widetilde{u}|_\Gamma}} |\nabla v|^2_{\mathsf{L}^2(\Omega)}$$

natural energy norm ($\mathsf{H}^1(\Omega)$-volume norm)

$\boxed{\text{idea:}}$ compute upper and lower bounds in a thin boundary layer using $\boxed{\text{FEM}}$

functional a posteriori error estimates for BEM

$$\max_{\substack{E\in\mathrm{L}^2(\Omega)\\ \operatorname{div}E=0}}\Big(2\langle n\cdot E,g-\widetilde{u}|_\Gamma\rangle_{\mathrm{H}^{-1/2}(\Gamma)}-|E|^2_{\mathrm{L}^2(\Omega)}\Big)=|\nabla(u-\widetilde{u})|^2_{\mathrm{L}^2(\Omega)}=\min_{\substack{v\in\mathrm{H}^1(\Omega)\\ v|_\Gamma=g-\widetilde{u}|_\Gamma}}|\nabla v|^2_{\mathrm{L}^2(\Omega)}$$

minimiser of upper bound:

$$\Delta v=0\quad\text{in }\Omega,\qquad\qquad v|_\Gamma=g-\widetilde{u}|_\Gamma\quad\text{on }\Gamma.$$

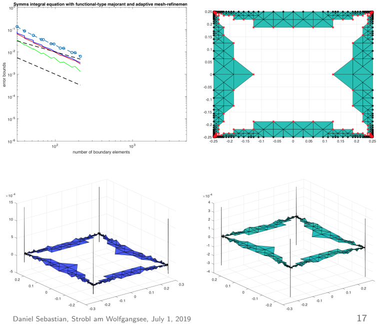
maximiser of lower bound: saddle point formulation (mixed/dual Laplacian)

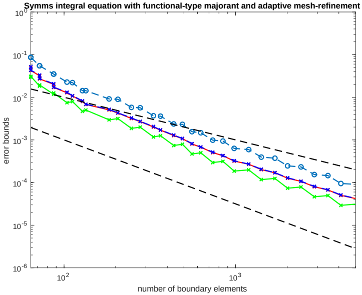
Find $(E,u)\in\mathrm{D}_0(\Omega)\times\mathrm{L}^2(\Omega)$ s.t. for all $(\Phi,\phi)\in\mathrm{D}_0(\Omega)\times\mathrm{L}^2(\Omega)$

$$\langle E,\Phi\rangle_{\mathrm{L}^2(\Omega)}+\langle\operatorname{div}\Phi,u\rangle_{\mathrm{L}^2(\Omega)}=\langle n\cdot\Phi,g-\widetilde{u}|_\Gamma\rangle_{\mathrm{H}^{-1/2}(\Gamma)},$$
$$\langle\operatorname{div}E,\phi\rangle_{\mathrm{L}^2(\Omega)}=0$$

note: $\mathrm{D}_0(\Omega)=\{\Phi\in\mathrm{L}^2(\Omega)\,:\,\operatorname{div}\Phi=0\}$

functional a posteriori error estimates for BEM - some pics





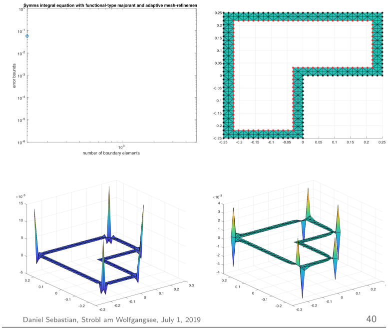
Ω: unit square, $g(x) = cosh(x_1)cos(x_2)$

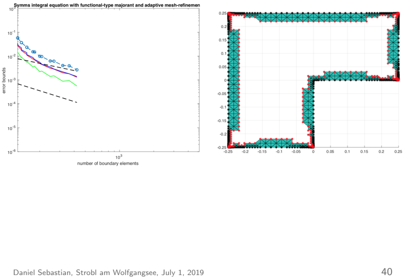
$\circ \circ \circ$ residual based estimator by Dirk Praetorius

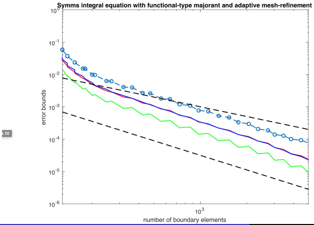
upper bound
exact error
lower bound

– – – order 3/2

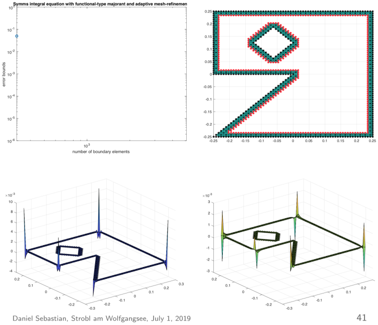
functional a posteriori error estimates for BEM - some pics

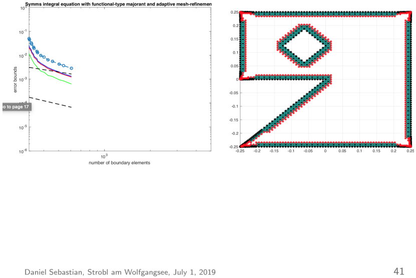


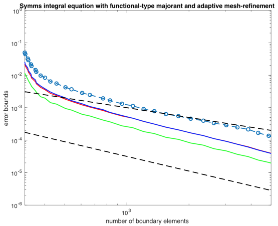




functional a posteriori error estimates for BEM - some pics







div-curl-lemma

Lemma (div-curl-lemma (global version))

Assumptions:

(i) (E_n) *bounded in* $\mathrm{L}^2(\Omega)$

(i') (H_n) *bounded in* $\mathrm{L}^2(\Omega)$

(ii) $(\operatorname{rot} E_n)$ *bounded in* $\mathrm{L}^2(\Omega)$

(ii') $(\operatorname{div} \varepsilon H_n)$ *bounded in* $\mathrm{L}^2(\Omega)$

(iii) $\nu \times E_n = 0$ *on* Γ_t, *i.e.,* $E_n \in \mathrm{R}_{\Gamma_t}(\Omega)$

(iii') $\nu \cdot \varepsilon H_n = 0$ *on* Γ_n, *i.e.,* $H_n \in \varepsilon^{-1}\mathrm{D}_{\Gamma_n}(\Omega)$

$\Rightarrow \exists\ E, H$ *and subsequences st*

$E_n \rightharpoonup E$, $\operatorname{rot} E_n \rightharpoonup \operatorname{rot} E$ *and* $H_n \rightharpoonup H$, $\operatorname{div} H_n \rightharpoonup \operatorname{div} H$ *in* $\mathrm{L}^2(\Omega)$ *and*

$$\langle E_n, H_n \rangle_{\mathrm{L}^2_\varepsilon(\Omega)} \to \langle E, H \rangle_{\mathrm{L}^2_\varepsilon(\Omega)}$$

$\Rightarrow$ classical local version

key tools to prove compact embeddings

crucial tool: compact embeddings

- localisation to top triv domains by partition of unity
- Helmholtz decompositions
- regular potentials
  (Here is the hard analysis: weak/strong Lipschitz domains, mixed bc, ...)
- Rellich's selection theorem

literature (complexes, applications to FEM, ...)

Arnold, Falk, Winther, Christiansen, Gopalakrishnan, Schöberl, Zulehner, ...

literature (fa-toolbox, complexes, a posteriori error estimates, ...)

some results of this talk:

- Py: *Solution Theory, Variational Formulations, and Functional a Posteriori Error Estimates for General First Order Systems with Applications to Electro-Magneto-Statics and More*,
  (NFAO) Numerical Functional Analysis and Optimization, 2019

literature (complexes, Friedrichs type constants, Maxwell constants)

some results of this talk:

- Py: *On Constants in Maxwell Inequalities for Bounded and Convex Domains*, Zapiski POMI/ (JMS)Journal of Mathematical Sciences (Springer New York), 2015
- Py: *On Maxwell's and Poincare's Constants*, (DCDS) Discrete and Continuous Dynamical Systems - Series S, 2015
- Py: *On the Maxwell Constants in 3D*, (M2AS) Mathematical Methods in the Applied Sciences, 2017
- Py: *On the Maxwell and Friedrichs/Poincaré Constants in ND*, (MZ) Mathematische Zeitschrift, 2019

- Py: *...some (so far) unpublished results*

literature (complexes, Friedrichs type constants, compact embeddings)

compact embeddings for Maywell:

- Weck: *Maxwell's boundary value problems on Riemannian manifolds with nonsmooth boundaries*,
  (JMA2) Journal of Mathematical Analysis and Applications, 1974 (1972)
- Picard: *An elementary proof for a compact imbedding result in generalized electromagnetic theory*,
  (MZ) Mathematische Zeitschrift, 1984
- Witsch: *A remark on a compactness result in electromagnetic theory*,
  (M2AS) Mathematical Methods in the Applied Sciences, 1993

(Weber '80, Costabel '90, Jochmann '97, Fernandes/Gilardi '97, Kuhn '99, Picard/Weck/Witsch '01, Py '96, '03, '06, '07, '08)

literature (complexes, Friedrichs type constants, compact embeddings)

some results of this talk:

- Bauer, Py, Schomburg: *The Maxwell Compactness Property in Bounded Weak Lipschitz Domains with Mixed Boundary Conditions*,
  (SIMA) SIAM Journal on Mathematical Analysis, 2016
- Py, Zulehner: *The divDiv-Complex and Applications to Biharmonic Equations*,
  (AA) Applicable Analysis, 2019
- Hiptmair, Pechstein, Py, Schomburg, Zulehner,: *Regular Potentials and Regular Decompositions for Bounded Strong Lipschitz Domains with Mixed Boundary Conditions in Arbitrary Dimensions*,
  (almost) submitted
- Py, Schomburg, Zulehner: *The Elasticity Complex*,
  (almost) submitted

literature (div-curl-lemma)

original papers (local div-curl-lemma):

- Murat: *Compacité par compensation*,
  Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 1978
- Tartar: *Compensated compactness and applications to partial differential equations*,
  Nonlinear analysis and mechanics, Heriot-Watt symposium, 1979

literature (div-curl-lemma)

recent papers (global div-curl-lemma, H^1-detour):

- Gloria, Neukamm, Otto: *Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics*, (IM) Invent. Math., 2015
- Kozono, Yanagisawa: *Global compensated compactness theorem for general differential operators of first order*, (ARMA) Arch. Ration. Mech. Anal., 2013
- Schweizer: *On Friedrichs inequality, Helmholtz decomposition, vector potentials, and the div-curl lemma*, accepted preprint, 2018

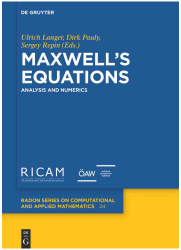
recent papers (global div-curl-lemma, general results/this talk):

- Waurick: *A Functional Analytic Perspective to the div-curl Lemma*, (JOP) J. Operator Theory, 2018
- Py: *A Global div-curl-Lemma for Mixed Boundary Conditions in Weak Lipschitz Domains and a Corresponding Generalized* A_0^*-A_1*-Lemma in Hilbert Spaces*, (ANA) Analysis (Munich), 2019

literature (full time-dependent Maxwell equations)

- Py, Picard: *A Note on the Justification of the Eddy Current Model in Electrodynamics*,
  (M2AS) Mathematical Methods in the Applied Sciences, 2017
- Py, Picard, Trostorff, Waurick: *On a Class of Degenerate Abstract Parabolic Problems and Applications to Some Eddy Current Models*,
  submitted, 2019

literature (Maxwell's equations and more...)



- Langer, Py, Repin (Eds): *Maxwell's equations. Analysis and numerics*, Radon Series on Applied Mathematics, De Gruyter, July 2019