FA-Toolbox: Part 1
The divDiv-Complex: Analytical Foundations

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Open-Minded ;-)

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classical de Rham complex in 3D (∇-rot-div-complex)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot\cup \Gamma_n}$

(electro-magnetics, Maxwell's equations)

$$\{0\} \quad \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{-\,\mathrm{div}}{\overset{\mathring{\nabla}}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{\mathrm{rot}}{\overset{\mathring{\mathrm{rot}}}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{-\nabla}{\overset{\mathring{\mathrm{div}}}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}} \quad \mathbb{R}$$

mixed boundary conditions and inhomogeneous and anisotropic media are possible

de Rham complex in ND or on Riemannian manifolds (d-complex)

$\Omega \subset \mathbb{R}^N$ bd w. Lip. dom. or Ω Riemannian manifold with cpt cl. and Lip. boundary Γ

(generalized Maxwell equations | the mother of all complexes |)

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^{2,0} \underset{-\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \mathsf{L}^{2,1} \underset{-\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \cdots \boxed{\mathsf{L}^{2,q} \underset{-\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \mathsf{L}^{2,q+1}} \cdots \mathsf{L}^{2,N-1} \underset{-\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \mathsf{L}^{2,N} \underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}} \mathbb{R}$$

elasticity complex in 3D (sym ∇-Rot $\mathrm{Rot}_{\mathbb{S}}^{\top}$-$\mathrm{Div}_{\mathbb{S}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\mathrm{Div}_{\mathbb{S}}}{\overset{\mathrm{sym}\,\mathring{\nabla}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top}}{\overset{\mathrm{Rot}\,\mathring{\mathrm{Rot}}_{\mathbb{S}}^{\top}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{-\mathrm{sym}\,\nabla}{\overset{\mathring{\mathrm{Div}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathsf{RM}}}{\overset{\pi_{\mathsf{RM}}}{\rightleftarrows}} \mathsf{RM}$$

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\operatorname{Rot}_{\mathbb{S}}$-$\operatorname{Div}_{\mathbb{T}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\nabla\nabla}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}} \underset{-\operatorname{dev}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathsf{RT}}}{\overset{\pi_{\mathsf{RT}}}{\rightleftarrows}} \mathsf{RT}$$

general complex

$$A_0 : D(A_0) \subset H_0 \to H_1, \qquad A_1 : D(A_1) \subset H_1 \to H_2 \qquad \text{(lddc)}$$

$$A_0^* : D(A_0^*) \subset H_1 \to H_0, \qquad A_1^* : D(A_1^*) \subset H_2 \to H_1$$

general complex property $\boxed{A_1 A_0 = 0}$,

i.e., $R(A_0) \subset N(A_1)$ and/or eq $R(A_1^*) \subset N(A_0^*)$

$$\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} H_0 \underset{A_0^*}{\overset{A_0}{\rightleftarrows}} H_1 \underset{A_1^*}{\overset{A_1}{\rightleftarrows}} H_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$$

general observations

$$Ax = f$$

general theory

- compact embeddings
  $\Downarrow$
- closed ranges
  $\Downarrow$
- solution theory
- Friedrichs/Poincaré estimates and constants
- Helmholtz/Hodge/Weyl decompositions
- continuous and compact inverse operators
- variational formulations
- functional a posteriori error estimates
- generalized div-curl-lemma
- ...

idea: solve problem with general and simple linear functional analysis

general observations

$$\mathrm{A}x = f$$

let's say $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ linear and $\mathrm{H}_0, \mathrm{H}_1$ Hilbert spaces

question: How to solve?

general observations

$$\mathrm{A}x = f$$

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ linear

solution theory in the sense of Hadamard

- existence $\Leftrightarrow$ $f \in R(\mathrm{A})$
- uniqueness $\Leftrightarrow$ A inj $\Leftrightarrow$ $N(\mathrm{A}) = \{0\}$ $\Leftrightarrow$ A^{-1} exists
- cont dep on f $\Leftrightarrow$ A^{-1} cont

$\Rightarrow$ $x = \mathrm{A}^{-1} f \in D(\mathrm{A})$ and cont estimate (Friedrichs/Poincaré type estimate)

$$|x|_{\mathrm{H}_0} = |\mathrm{A}^{-1} f|_{\mathrm{H}_0} \le c_{\mathrm{A}} |f|_{\mathrm{H}_1} = c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$$

$\Rightarrow$ best constant $c_{\mathrm{A}} = |\mathrm{A}^{-1}|_{R(\mathrm{A}),\mathrm{H}_0}$ $\quad |\mathrm{A}^{-1}|_{R(\mathrm{A}),D(\mathrm{A})} = (c_{\mathrm{A}}^2 + 1)^{1/2}$

general observations

$A : D(A) \subset H_0 \to H_1$

$A^* : D(A^*) \subset H_1 \to H_0$ Hilbert space adjoint

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$H_1 = \overline{R(A)} \oplus N(A^*) \qquad H_0 = N(A) \oplus \overline{R(A^*)}$$

$$Ax = f$$

solution theory in the sense of Hadamard

- existence $\Leftrightarrow$ $f \in R(A) = N(A^*)^\perp$
- uniqueness $\Leftrightarrow$ A inj $\Leftrightarrow$ $N(A) = \{0\}$ $\Leftrightarrow$ A^{-1} exists
- cont dep on f $\Leftrightarrow$ A^{-1} cont $\Leftrightarrow$ $R(A)$ cl (cl range theo)

fund range cond: $\boxed{R(A) = \overline{R(A)} \text{ closed}}$ (must hold $\rightsquigarrow$ right setting!)

kernel cond: $\boxed{N(A) = \{0\}}$ (fails in gen $\rightsquigarrow$ proj onto $N(A)^\perp = \overline{R(A^*)}$)

general observations

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$\mathrm{H}_1 = \overline{R(\mathrm{A})} \oplus N(\mathrm{A}^*) \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

remarkable observations

- time-dependent problems are simple
  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H} \to \mathrm{H}$, $\boxed{\mathrm{A} = \partial_t + \mathrm{T}}$ (gen T skew-sa, or alt lsast Re $\mathrm{T} \geq 0$)

  $$N(\mathrm{A}) = \{0\} \qquad N(\mathrm{A}^*) = \{0\} \qquad R(\mathrm{A}) \text{ (cl) } = N(\mathrm{A}^*)^\perp = \mathrm{H}$$

- time-harmonic problems are more complicated
  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H} \to \mathrm{H}$, $\boxed{\mathrm{A} = -\omega + \mathrm{T}}$

  $$N(\mathrm{A}),\ N(\mathrm{A}^*) \text{ (fin dim)} \qquad R(\mathrm{A}) \text{ (cl, fin co-dim) } = N(\mathrm{A}^*)^\perp$$

  (Fredholm alternative)

- stat problems are most complicated
  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$, $\boxed{\mathrm{A} = 0 + \mathrm{T}}$

  $$\dim N(\mathrm{A}) = \dim N(\mathrm{A}^*) = \infty \text{ (possibly)} \qquad R(\mathrm{A}) \text{ (cl, infin co-dim) } = N(\mathrm{A}^*)^\perp$$

1st fundamental observations

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ lddc, $\mathrm{A}^* : D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0$ Hilbert space adjoint

$(\mathrm{A}, \mathrm{A}^*)$ dual pair as $(\mathrm{A}^*)^* = \overline{\mathrm{A}} = \mathrm{A}$

A, A^* may not be inj

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$\mathrm{H}_1 = N(\mathrm{A}^*) \oplus \overline{R(\mathrm{A})} \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

reduced operators restr to $N(\mathrm{A})^\perp$ and $N(\mathrm{A}^*)^\perp$

$$\mathcal{A} := \mathrm{A}|_{N(\mathrm{A})^\perp} = \mathrm{A}|_{\overline{R(\mathrm{A}^*)}} \qquad \mathcal{A}^* := \mathrm{A}^*|_{N(\mathrm{A}^*)^\perp} = \mathrm{A}^*|_{\overline{R(\mathrm{A})}}$$

$\mathcal{A}$, $\mathcal{A}^*$ inj $\quad\Rightarrow\quad$ $\mathcal{A}^{-1}$, $(\mathcal{A}^*)^{-1}$ ex

1st fundamental observations

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1, \quad \mathrm{A}^* : D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0$ lddc $\qquad (\mathrm{A}, \mathrm{A}^*)$ dual pair

$$\mathrm{H}_1 = N(\mathrm{A}^*) \oplus \overline{R(\mathrm{A})} \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

more precisely

$$\mathcal{A} := \mathrm{A}|_{\overline{R(\mathrm{A}^*)}} : D(\mathcal{A}) \subset \overline{R(\mathrm{A}^*)} \to \overline{R(\mathrm{A})}, \qquad D(\mathcal{A}) := D(\mathrm{A}) \cap N(\mathrm{A})^\perp = D(\mathrm{A}) \cap \overline{R(\mathrm{A}^*)}$$

$$\mathcal{A}^* := \mathrm{A}^*|_{\overline{R(\mathrm{A})}} : D(\mathcal{A}^*) \subset \overline{R(\mathrm{A})} \to \overline{R(\mathrm{A}^*)}, \quad D(\mathcal{A}^*) := D(\mathrm{A}^*) \cap N(\mathrm{A}^*)^\perp = D(\mathrm{A}^*) \cap \overline{R(\mathrm{A})}$$

$(\mathcal{A}, \mathcal{A}^*)$ dual pair $\quad$ and $\quad \mathcal{A}, \mathcal{A}^*$ inj $\quad \Rightarrow$

inverse ops exist (and bij)

$$\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A}) \qquad (\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$$

refined decompositions

$$D(\mathrm{A}) = N(\mathrm{A}) \oplus D(\mathcal{A}) \qquad D(\mathrm{A}^*) = N(\mathrm{A}^*) \oplus D(\mathcal{A}^*)$$

$\Rightarrow$

$$R(\mathrm{A}) = R(\mathcal{A}) \qquad R(\mathrm{A}^*) = R(\mathcal{A}^*)$$

1st fundamental observations

closed range theorem & closed graph theorem $\Rightarrow$

Lemma (Friedrichs-Poincaré type est/cl range/cont inv)

The following assertions are equivalent:

- **(i)** $\exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathrm{H}_0} \le c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$
- **(i*)** $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathrm{H}_1} \le c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$
- **(ii)** $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_1.
- **(ii*)** $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_0.
- **(iii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is continuous and bijective.*
- **(iii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is continuous and bijective.*

In case that one of the latter assertions is true, e.g., (ii), $R(\mathrm{A})$ *is closed, we have*

$$\mathrm{H}_0 = N(\mathrm{A}) \oplus R(\mathrm{A}^*) \qquad \mathrm{H}_1 = N(\mathrm{A}^*) \oplus R(\mathrm{A})$$
$$D(\mathrm{A}) = N(\mathrm{A}) \oplus D(\mathcal{A}) \qquad D(\mathrm{A}^*) = N(\mathrm{A}^*) \oplus D(\mathcal{A}^*)$$
$$D(\mathcal{A}) = D(\mathrm{A}) \cap R(\mathrm{A}^*) \qquad D(\mathcal{A}^*) = D(\mathrm{A}^*) \cap R(\mathrm{A})$$

and $\quad \mathcal{A} : D(\mathcal{A}) \subset R(\mathrm{A}^*) \to R(\mathrm{A}), \quad \mathcal{A}^* : D(\mathcal{A}^*) \subset R(\mathrm{A}) \to R(\mathrm{A}^*).$

Note: trivial equivalence to inf-sup condition

1st fundamental observations

recall

$$\textbf{(i)}\quad \exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathrm{H}_0} \leq c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$$

$$\textbf{(i}^*\textbf{)}\quad \exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$$

'best' consts in **(i)** and **(i*)** equal norms of the inv ops and Rayleigh quotients

$$c_{\mathrm{A}} = |\mathcal{A}^{-1}|_{R(\mathrm{A}),R(\mathrm{A}^*)} \qquad c_{\mathrm{A}^*} = |(\mathcal{A}^*)^{-1}|_{R(\mathrm{A}^*),R(\mathrm{A})}$$

$$\frac{1}{c_{\mathrm{A}}} = \inf_{0\neq x\in D(\mathcal{A})} \frac{|\mathrm{A}x|_{\mathrm{H}_1}}{|x|_{\mathrm{H}_0}} \qquad \frac{1}{c_{\mathrm{A}^*}} = \inf_{0\neq y\in D(\mathcal{A}^*)} \frac{|\mathrm{A}^* y|_{\mathrm{H}_0}}{|y|_{\mathrm{H}_1}}$$

Lemma (Friedrichs-Poincaré type const)

$c_{\mathrm{A}} = c_{\mathrm{A}^*}$

1st fundamental observations

Lemma (cpt emb/cpt inv)

The following assertions are equivalent:

- **(i)** $D(\mathcal{A}) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_0$ *is compact.*
- **(i*)** $D(\mathcal{A}^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1$ *is compact.*
- **(ii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to R(\mathrm{A}^*)$ *is compact.*
- **(ii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to R(\mathrm{A})$ *is compact.*

Lemma (Friedrichs-Poincaré type est/cl range/cont inv)

$\Downarrow$ $\boxed{D(\mathcal{A}) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_0 \text{ compact}}$

- **(i)** $\exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathrm{H}_0} \le c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$
- **(i*)** $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathrm{H}_1} \le c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$
- **(ii)** $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_1.
- **(ii*)** $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_0.
- **(iii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is continuous and bijective.*
- **(iii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is continuous and bijective.*

(i)-(iii*) *equi* & *the resp Helm deco hold* & $|\mathcal{A}^{-1}| = c_{\mathrm{A}} = c_{\mathrm{A}^*} = |(\mathcal{A}^*)^{-1}|$

2nd fundamental observations

So far no complex...

$$\mathrm{A}_0 : D(\mathrm{A}_0) \subset \mathrm{H}_0 \to \mathrm{H}_1, \quad \mathrm{A}_1 : D(\mathrm{A}_1) \subset \mathrm{H}_1 \to \mathrm{H}_2 \text{ (lddc)}$$
$$\mathrm{A}_0^* : D(\mathrm{A}_0^*) \subset \mathrm{H}_1 \to \mathrm{H}_0, \quad \mathrm{A}_1^* : D(\mathrm{A}_1^*) \subset \mathrm{H}_2 \to \mathrm{H}_1 \text{ (lddc)}$$

general complex ($\mathrm{A}_1\mathrm{A}_0 = 0$, i.e., $R(\mathrm{A}_0) \subset N(\mathrm{A}_1)$ and $R(\mathrm{A}_1^*) \subset N(\mathrm{A}_0^*)$)

$$\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$$

recall Helmholtz deco

$$\begin{array}{ccc} \mathrm{H}_1 = \overline{R(\mathrm{A}_0)} & \oplus & N(\mathrm{A}_0^*) \\ \bigcap & & \bigcup \\ = N(\mathrm{A}_1) & \oplus & \overline{R(\mathrm{A}_1^*)} \end{array} \quad \Rightarrow \text{ (e.g.) } N(\mathrm{A}_1) = \overline{R(\mathrm{A}_0)} \oplus \Big(\underbrace{N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)}_{=:K_1} \Big)$$

$\Rightarrow$ refined Helmholtz deco

$$\mathrm{H}_1 = \overline{R(\mathrm{A}_0)} \oplus K_1 \oplus \overline{R(\mathrm{A}_1^*)}$$

2nd fundamental observations

recall

$$D(\mathrm{A}_1) = D(\mathcal{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \qquad R(\mathrm{A}_1) = R(\mathcal{A}_1) \qquad R(\mathrm{A}_1^*) = R(\mathcal{A}_1^*)$$

$$D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \cap \overline{R(\mathrm{A}_0)} \qquad R(\mathrm{A}_0^*) = R(\mathcal{A}_0^*) \qquad R(\mathrm{A}_0) = R(\mathcal{A}_0)$$

cohomology group $K_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)$

Lemma (Helmholtz deco I)

$$\mathrm{H}_1 = \overline{R(\mathrm{A}_0)} \oplus N(\mathrm{A}_0^*) \qquad \mathrm{H}_1 = \overline{R(\mathrm{A}_1^*)} \oplus N(\mathrm{A}_1)$$

$$D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus N(\mathrm{A}_0^*) \qquad D(\mathrm{A}_1) = D(\mathcal{A}_1) \oplus N(\mathrm{A}_1)$$

$$N(\mathrm{A}_1) = D(\mathcal{A}_0^*) \oplus K_1 \qquad N(\mathrm{A}_0^*) = D(\mathcal{A}_1) \oplus K_1$$

$$D(\mathrm{A}_1) = \overline{R(\mathrm{A}_0)} \oplus \big(D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)\big) \qquad D(\mathrm{A}_0^*) = \overline{R(\mathrm{A}_1^*)} \oplus \big(D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1)\big)$$

Lemma (Helmholtz deco II)

$$\mathrm{H}_1 = \overline{R(\mathrm{A}_0)} \oplus K_1 \oplus \overline{R(\mathrm{A}_1^*)}$$

$$D(\mathrm{A}_1) = \overline{R(\mathrm{A}_0)} \oplus K_1 \oplus D(\mathcal{A}_1)$$

$$D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus K_1 \oplus \overline{R(\mathrm{A}_1^*)}$$

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus K_1 \oplus D(\mathcal{A}_1)$$

2nd fundamental observations

$$K_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \qquad D(\mathrm{A}_1) = D(\mathcal{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \qquad D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \cap \overline{R(\mathrm{A}_0)}$$

Lemma (cpt emb II)

The following assertions are equivalent:

(i) $D(\mathcal{A}_0) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_0$, $D(\mathcal{A}_1) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1$, *and* $K_1 \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1$ *are compact.*

(ii) $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1$ *is compact.*

In this case $K_1 < \infty$.

Theorem (fa-toolbox I)

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1 \text{ compact}}$

(i) *all emb cpt, i.e.,* $D(\mathcal{A}_0) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_0$, $D(\mathcal{A}_1) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1$, $D(\mathcal{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1$, $D(\mathcal{A}_1^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_2$ *cpt*

(ii) *cohomology group* K_1 *finite dim*

(iii) *all ranges closed, i.e.,* $R(\mathrm{A}_0)$, $R(\mathrm{A}_0^*)$, $R(\mathrm{A}_1)$, $R(\mathrm{A}_1^*)$ *cl*

(iv) *all Friedrichs-Poincaré type est hold*

(v) *all Hodge-Helmholtz-Weyl type deco I & II hold with closed ranges*

2nd fundamental observations

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

Theorem (fa-toolbox I (Friedrichs-Poincaré type est))

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1 \text{ compact}}$ $\Rightarrow \quad \exists \quad |\mathcal{A}_i^{-1}| = c_{\mathrm{A}_i} = c_{\mathrm{A}_i^*} = |(\mathcal{A}_i^*)^{-1}| \in (0, \infty)$

(i) $\forall\, x \in D(\mathcal{A}_0)$ $\qquad |x|_{\mathrm{H}_0} \le c_{\mathrm{A}_0} |\mathrm{A}_0 x|_{\mathrm{H}_1}$

(i*) $\forall\, y \in D(\mathcal{A}_0^*)$ $\qquad |y|_{\mathrm{H}_1} \le c_{\mathrm{A}_0} |\mathrm{A}_0^* y|_{\mathrm{H}_0}$

(ii) $\forall\, y \in D(\mathcal{A}_1)$ $\qquad |y|_{\mathrm{H}_1} \le c_{\mathrm{A}_1} |\mathrm{A}_1 y|_{\mathrm{H}_2}$

(ii*) $\forall\, z \in D(\mathcal{A}_1^*)$ $\qquad |z|_{\mathrm{H}_2} \le c_{\mathrm{A}_1} |\mathrm{A}_1^* z|_{\mathrm{H}_1}$

(iii) $\forall\, y \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ $\qquad |(1 - \pi_{K_1}) y|_{\mathrm{H}_1} \le c_{\mathrm{A}_1} |\mathrm{A}_1 y|_{\mathrm{H}_2} + c_{\mathrm{A}_0} |\mathrm{A}_0^* y|_{\mathrm{H}_0}$

note $\pi_{K_1} y \in K_1$ and $(1 - \pi_{K_1}) y \in K_1^{\perp}$

Remark

enough $R(\mathrm{A}_0)$ *and* $R(\mathrm{A}_1)$ *cl*

2nd fundamental observations

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

Theorem (fa-toolbox I (Helmholtz deco))

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1 \textit{ compact}}$

$$\mathrm{H}_1 = R(\mathrm{A}_0) \oplus N(\mathrm{A}_0^*) \qquad \mathrm{H}_1 = R(\mathrm{A}_1^*) \oplus N(\mathrm{A}_1)$$

$$D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus N(\mathrm{A}_0^*) \qquad D(\mathrm{A}_1) = D(\mathcal{A}_1) \oplus N(\mathrm{A}_1)$$

$$N(\mathrm{A}_1) = D(\mathcal{A}_0^*) \oplus K_1 \qquad N(\mathrm{A}_0^*) = D(\mathcal{A}_1) \oplus K_1$$

$$D(\mathrm{A}_1) = R(\mathrm{A}_0) \oplus \big(D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)\big) \qquad D(\mathrm{A}_0^*) = R(\mathrm{A}_1^*) \oplus \big(D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1)\big)$$

$$\begin{aligned}
\mathrm{H}_1 &= R(\mathrm{A}_0) \oplus K_1 \oplus R(\mathrm{A}_1^*) \\
D(\mathrm{A}_1) &= R(\mathrm{A}_0) \oplus K_1 \oplus D(\mathcal{A}_1) \\
D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus K_1 \oplus R(\mathrm{A}_1^*) \\
D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus K_1 \oplus D(\mathcal{A}_1)
\end{aligned}$$

Remark

enough $R(\mathrm{A}_0)$ and $R(\mathrm{A}_1)$ cl

(stat) first order system - solution theory

complex $\cdots \overset{\cdots}{\underset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \overset{\mathrm{A}_0}{\underset{\mathrm{A}_0^*}{\rightleftarrows}} \mathrm{H}_1 \overset{\mathrm{A}_1}{\underset{\mathrm{A}_1^*}{\rightleftarrows}} \mathrm{H}_2 \overset{\cdots}{\underset{\cdots}{\rightleftarrows}} \cdots$

$\boxed{\mathrm{A}_1 x = f}$ $\qquad \dim N(\mathrm{A}_1) = \infty$

find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ such that the fos

$$\begin{aligned} \mathrm{A}_1 x &= f \\ \mathrm{A}_0^* x &= g \\ \pi_{K_1} x &= k \end{aligned} \quad \text{think of} \quad \begin{aligned} \mathring{\mathrm{rot}} E &= F \\ -\operatorname{div} E &= g \\ \pi_{\mathrm{D}} E &= K \end{aligned} \quad \text{or} \quad \begin{aligned} \mathring{\mathrm{Rot}}_{\mathbb{S}} S &= F \\ \operatorname{div} \operatorname{Div}_{\mathbb{S}} S &= G \\ \pi_{\mathrm{D}} S &= K \end{aligned} \quad \text{or} \quad \begin{aligned} \mathring{\mathrm{Div}}_{\mathbb{T}} T &= F \\ \operatorname{sym} \operatorname{Rot}_{\mathbb{T}} T &= G \\ \pi_{\mathrm{D}} T &= K \end{aligned}$$

kernel = cohomology group = $K_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)$

trivially necessary $\quad f \in R(\mathrm{A}_1) \quad g \in R(\mathrm{A}_0^*) \quad k \in K_1$

apply fa-toolbox

(stat) first order system - solution theory

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ st fos $\quad \mathrm{A}_1 x = f \quad \mathrm{A}_0^* x = g \quad \pi_{K_1} x = k$

Theorem (fa-toolbox II (solution theory))

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1 \textit{ compact}}$

fos is uniq sol $\Leftrightarrow$ $f \in R(\mathrm{A}_1) \quad g \in R(\mathrm{A}_0^*) \quad k \in K_1$

$$x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus K_1 = D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$$

$$\boxed{x_f := \mathcal{A}_1^{-1} f} \in D(\mathcal{A}_1)$$

$$\boxed{x_g := (\mathcal{A}_0^*)^{-1} g} \in D(\mathcal{A}_0^*)$$

dep cont on data $|x|_{\mathrm{H}_1} \leq |x_f|_{\mathrm{H}_1} + |x_g|_{\mathrm{H}_1} + |k|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |f|_{\mathrm{H}_2} + c_{\mathrm{A}_0} |g|_{\mathrm{H}_0} + |k|_{\mathrm{H}_1}$

moreover

$$\pi_{R(\mathrm{A}_1^*)} x = x_f \qquad \pi_{R(\mathrm{A}_0)} x = x_g \qquad \pi_{K_1} x = k \qquad |x|_{\mathrm{H}_1}^2 = |x_f|_{\mathrm{H}_1}^2 + |x_g|_{\mathrm{H}_1}^2 + |k|_{\mathrm{H}_1}^2$$

Remark

enough $R(\mathrm{A}_0)$ *and* $R(\mathrm{A}_1)$ *cl*

A_0^*-A_1-lemma (generalized global div-curl-lemma)

Lemma (A_0^*-A_1-lemma)

Let $D(A_1) \cap D(A_0^*) \hookrightarrow H_1$ be compact, and

(i) (x_n) *bounded in* $D(A_1)$,

(ii) (y_n) *bounded in* $D(A_0^*)$.

$\Rightarrow \exists\, x \in D(A_1)$, $y \in D(A_0^*)$ *and subsequences st*

$x_n \rightharpoonup x$ *in* $D(A_1)$ *and* $y_n \rightharpoonup y$ *in* $D(A_0^*)$ *as well as*

$$\langle x_n, y_n \rangle_{H_1} \to \langle x, y \rangle_{H_1}.$$

A_0^*-A_1-lemma (generalized global div-curl-lemma)

Lemma (generalized A_0^*-A_1-lemma)

Let $R(A_0)$ and $R(A_1$ be closed, and let K_1 be finite dimensional. Moreover, let $(x_n), (y_n) \subset H_1$ be bounded such that

(i) $\widetilde{A}_1 x_n$ *is relatively compact in* $D(A_1^*)'$,

(ii) $\widetilde{A}_0^* y_n$ *is relatively compact in* $D(A_0)'$.

$\Rightarrow \exists\ x, y \in H_1$ *and subsequences st* $x_n \rightharpoonup x$ *in* H_1 *and* $y_n \rightharpoonup y$ *in* H_1 *as well as*

$$\langle x_n, y_n \rangle_{H_1} \to \langle x, y \rangle_{H_1}.$$

proof uses key observation

Lemma

Let $R(A)$ be closed. For $(x_n) \subset H_0$ the following statements are equivalent:

(i) $\widetilde{A} x_n$ *is relatively compact in* $D(A^*)'$.

(ii) $\pi_{R(A^*)} x_n$ *is relatively compact in* $R(A^*)$ *resp.* H_1.

If $x_n \rightharpoonup x$ in H_1, then either of cond. (i) or (ii) implies $\pi_{R(A^*)} x_n \to \pi_{R(A^*)} x$ in H_1.

nice results (and joint work/communication with) Marcus Waurick

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\mathrm{Rot}_{\mathbb{S}}$-$\mathrm{Div}_{\mathbb{T}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \;\underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\nabla\nabla}}{\rightleftarrows}}\; \mathsf{L}^2_{\mathbb{S}} \;\underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}}\; \mathsf{L}^2_{\mathbb{T}} \;\underset{-\operatorname{dev}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota_{\mathsf{RT}}}{\overset{\pi_{\mathsf{RT}}}{\rightleftarrows}}\; \mathsf{RT}$$

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\mathrm{Rot}_{\mathbb{S}}$-$\mathrm{Div}_{\mathbb{T}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \;\underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\nabla}\nabla}{\rightleftarrows}}\; \mathsf{L}^2_{\mathbb{S}} \;\underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}}\; \mathsf{L}^2_{\mathbb{T}} \;\underset{-\operatorname{dev}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota_{\mathsf{RT}}}{\overset{\pi_{\mathsf{RT}}}{\rightleftarrows}}\; \mathsf{RT}$$

related fos ($\mathring{\nabla}\nabla$, $\operatorname{div}\operatorname{Div}_{\mathbb{S}}$ first order operators!)

$\mathring{\nabla}\nabla u = M$ in Ω	$\mathring{\operatorname{Rot}}_{\mathbb{S}} M = F$ in Ω	$\mathring{\operatorname{Div}}_{\mathbb{T}} N = g$ in Ω	$\pi v = r$ in Ω
$\pi u = 0$ in Ω	$\operatorname{div}\operatorname{Div}_{\mathbb{S}} M = f$ in Ω	$\operatorname{sym}\operatorname{Rot}_{\mathbb{T}} N = G$ in Ω	$-\operatorname{dev}\nabla v = T$ in Ω

related sos ($\operatorname{div}\operatorname{Div}_{\mathbb{S}}\mathring{\nabla}\nabla = \mathring{\Delta}^2$ second order operator!)

$\operatorname{div}\operatorname{Div}_{\mathbb{S}}\mathring{\nabla}\nabla u = \mathring{\Delta}^2 u = f$ in Ω	$\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}\mathring{\operatorname{Rot}}_{\mathbb{S}} M = G$ in Ω	$-\operatorname{dev}\nabla\mathring{\operatorname{Div}}_{\mathbb{T}} N = T$ in Ω
$\pi u = 0$ in Ω	$\operatorname{div}\operatorname{Div}_{\mathbb{S}} M = f$ in Ω	$\operatorname{sym}\operatorname{Rot}_{\mathbb{T}} N = G$ in Ω

corresponding compact embeddings:

$$D(\mathring{\nabla}\nabla) \cap D(\pi) = D(\mathring{\nabla}\nabla) = \mathring{\mathsf{H}}^2 \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

$$D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \cap D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \hookrightarrow \mathsf{L}^2_{\mathbb{S}} \qquad \text{(new selection theorem)}$$

$$D(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \hookrightarrow \mathsf{L}^2_{\mathbb{T}} \qquad \text{(new selection theorem)}$$

$$D(\pi) \cap D(\operatorname{dev}\nabla) = D(\operatorname{dev}\nabla) = D(\nabla) = \mathsf{H}^1 \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem and Korn type ineq.)}$$

two new selection theorems for strong Lip. dom. and Korn Type ineq.: Py/Zulehner ('16)

These do not simply follow by BGG!

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\operatorname{Rot}_{\mathbb{S}}$-$\operatorname{Div}_{\mathbb{T}}$-complex)

Lemma/Theorem $\Downarrow$ $\boxed{D(\mathrm{A}_1)\cap D(\mathrm{A}_0^*)\hookrightarrow\!\!\!\rightarrow \mathrm{H}_1,\quad D(\mathrm{A}_2)\cap D(\mathrm{A}_1^*)\hookrightarrow\!\!\!\rightarrow \mathrm{H}_2 \quad \text{cpt}}$

(i) all Friedrichs-Poincaré type est hold

$$\text{est for } \mathcal{A}_0 \quad\Leftrightarrow\quad \forall\,\varphi\in D(\mathring{\nabla\nabla})\cap R(\operatorname{div}\operatorname{Div}_{\mathbb{S}})=\mathring{\mathrm{H}}^2 \qquad |\varphi|_{\mathrm{L}^2}\le c_0|\nabla\nabla\varphi|_{\mathrm{L}^2}$$

$$\text{est for } \mathcal{A}_0^* \quad\Leftrightarrow\quad \forall\,\Phi\in D(\operatorname{div}\operatorname{Div}_{\mathbb{S}})\cap R(\mathring{\nabla\nabla}) \qquad |\Phi|_{\mathrm{L}^2}\le c_0|\operatorname{div}\operatorname{Div}\Phi|_{\mathrm{L}^2}$$

$$\text{est for } \mathcal{A}_1 \quad\Leftrightarrow\quad \forall\,\Phi\in D(\mathring{\operatorname{Rot}}_{\mathbb{S}})\cap R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \qquad |\Phi|_{\mathrm{L}^2}\le c_1|\operatorname{Rot}\Phi|_{\mathrm{L}^2}$$

$$\text{est for } \mathcal{A}_1^* \quad\Leftrightarrow\quad \forall\,\Phi\in D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}})\cap R(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \qquad |\Phi|_{\mathrm{L}^2}\le c_1|\operatorname{sym}\operatorname{Rot}\Phi|_{\mathrm{L}^2}$$

$$\text{est for } \mathcal{A}_2 \quad\Leftrightarrow\quad \forall\,\Phi\in D(\mathring{\operatorname{Div}}_{\mathbb{T}})\cap R(\operatorname{dev}\nabla) \qquad |\Phi|_{\mathrm{L}^2}\le c_2|\operatorname{Div}\Phi|_{\mathrm{L}^2}$$

$$\text{est for } \mathcal{A}_2^* \quad\Leftrightarrow\quad \forall\,\varphi\in D(\operatorname{dev}\nabla)\cap R(\mathring{\operatorname{Div}}_{\mathbb{T}})=\mathrm{H}^1\cap\mathrm{RT}^{\perp} \qquad |\varphi|_{\mathrm{L}^2}\le c_2|\operatorname{dev}\nabla\varphi|_{\mathrm{L}^2}$$

(ii) all ranges $R(\mathrm{A}_n)=R(\mathcal{A}_n)$, $R(\mathrm{A}_n^*)=R(\mathcal{A}_n^*)$ are cl in L^2

(iii) all inverse ops $\mathcal{A}_n^{-1}$, $(\mathcal{A}_n^*)^{-1}$ are cont, even cpt

(iv) all Helmholtz decomposition hold, e.g.,

$$\mathrm{H}_1=R(\mathrm{A}_0)\oplus K_1\oplus R(\mathrm{A}_1^*) \quad\Leftrightarrow\quad \mathrm{L}^2_{\mathbb{S}}=R(\mathring{\nabla\nabla})\oplus_{\mathrm{L}^2_{\mathbb{S}}}\mathcal{H}_{\mathrm{D},\mathbb{S}}\oplus_{\mathrm{L}^2_{\mathbb{S}}}R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}),$$

$$\mathrm{H}_2=R(\mathrm{A}_1)\oplus K_2\oplus R(\mathrm{A}_2^*) \quad\Leftrightarrow\quad \mathrm{L}^2_{\mathbb{T}}=R(\mathring{\operatorname{Rot}}_{\mathbb{S}})\oplus_{\mathrm{L}^2_{\mathbb{T}}}\mathcal{H}_{\mathrm{N},\mathbb{T}}\oplus_{\mathrm{L}^2_{\mathbb{T}}}R(\operatorname{dev}\nabla)$$

(v)-(ix) solution theories, variational formulations, functional a posteriori error estimates, div-curl-lemmas, ...

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\mathrm{Rot}_{\mathbb{S}}$-$\mathrm{Div}_{\mathbb{T}}$-complex)

find $S \in D(\mathring{\mathrm{Rot}}_{\mathbb{S}}) \cap D(\operatorname{div}\mathrm{Div}_{\mathbb{S}})$ s.t. / think of $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$

$$\mathring{\mathrm{Rot}}_{\mathbb{S}} S = F \qquad \mathrm{A}_1 x = f$$
$$\operatorname{div}\mathrm{Div}_{\mathbb{S}} S = G \qquad / \quad \text{think of} \qquad \mathrm{A}_0^* x = g$$
$$\pi_{\mathcal{H}_{\mathrm{D},\mathbb{S}}} S = K \qquad \pi_{K_1} x = k$$

sol is simply $x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus K_1 = D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$

with $\boxed{x_f := \mathcal{A}_1^{-1} f} \in D(\mathcal{A}_1)$ and $\boxed{x_g := (\mathcal{A}_0^*)^{-1} g} \in D(\mathcal{A}_0^*)$

i.e., $S = S_F + S_G + K$, where

$$\boxed{S_F := \mathring{\mathcal{R}ot}_{\mathbb{S}}^{-1} F} \in D(\mathring{\mathcal{R}ot}_{\mathbb{S}}) = D(\mathring{\mathrm{Rot}}_{\mathbb{S}}) \cap R(\operatorname{sym}\mathrm{Rot}_{\mathbb{T}}) = D(\mathring{\mathrm{Rot}}_{\mathbb{S}}) \cap N(\operatorname{div}\mathrm{Div}_{\mathbb{S}}) \cap \mathcal{H}_{\mathrm{D},\mathbb{S}}^{\perp},$$

$$\boxed{S_G := div\mathcal{D}iv_{\mathbb{S}}^{-1} G} \in D(div\mathcal{D}iv_{\mathbb{S}}) = D(\operatorname{div}\mathrm{Div}_{\mathbb{S}}) \cap R(\mathring{\nabla}\nabla) = D(\operatorname{div}\mathrm{Div}_{\mathbb{S}}) \cap N(\mathring{\mathrm{Rot}}_{\mathbb{S}}) \cap \mathcal{H}_{\mathrm{D},\mathbb{S}}^{\perp}$$

key tools to prove compact embeddings

crucial tool: compact embeddings

- localisation to top triv domains by partition of unity
  (2nd order operators = problems / sol: regular decompositions in H^{-1}, ...)
- Helmholtz decompositions
- regular potentials
  (Here is the hard analysis: weak/strong Lipschitz domains, mixed bc, ...)
- Rellich's selection theorem

regular potentials

Theorem (regular potentials)

Let (Ω, Γ_t) *be a bounded strong Lipschitz pair and* $k \geq 0$. *Then there exists a continuous linear operator*

$$\mathcal{S}^q_{\mathrm{d},k} : \mathring{\mathsf{H}}^{k,q}_{\Gamma_t}(\Omega) \cap \mathring{\mathsf{D}}^q_{\Gamma_t,0}(\Omega) \cap \mathcal{H}^q(\Omega)^\perp \longrightarrow \mathring{\mathsf{H}}^{k+1,q-1}_{\Gamma_t}(\Omega),$$

such that $\mathrm{d}\,\mathcal{S}^q_{\mathrm{d},k} = \mathrm{id}\,|_{\mathring{\mathsf{H}}^{k,q}_{\Gamma_t}(\Omega)\cap\mathring{\mathsf{D}}^q_{\Gamma_t,0}(\Omega)\cap\mathcal{H}^q(\Omega)^\perp}$. *In particular,*

$$\begin{aligned}\mathring{\mathsf{H}}^{k,q}_{\Gamma_t}(\Omega) \cap \mathring{\mathsf{D}}^q_{\Gamma_t,0}(\Omega) \cap \mathcal{H}^q(\Omega)^\perp &= \mathring{\mathsf{H}}^{k,q}_{\Gamma_t}(\Omega) \cap \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_t}(\Omega)\\ &= \mathrm{d}\,\mathring{\mathsf{H}}^{k+1,q-1}_{\Gamma_t}(\Omega)\\ &= \mathrm{d}\,\mathring{\mathsf{D}}^{k,q-1}_{\Gamma_t}(\Omega)\end{aligned}$$

and the regular $\mathring{\mathsf{H}}^{k+1,q-1}_{\Gamma_t}(\Omega)$*-potential depends continuously on the data. Especially, these spaces are closed subspaces of* $\mathring{\mathsf{H}}^{k,q}(\Omega)$ *and* $\mathcal{S}^q_{\mathrm{d},k}$ *is a right inverse to* d.

regular docompositions

Theorem (regular decompositions)

Let (Ω, Γ_t) be a bounded strong Lipschitz pair and $k \geq 0$. Then the regular decompositions

$$\mathring{\mathsf{D}}^{k,q}_{\Gamma_t}(\Omega) = \mathring{\mathsf{H}}^{k+1,q}_{\Gamma_t}(\Omega) + \mathrm{d}\,\mathring{\mathsf{H}}^{k+1,q-1}_{\Gamma_t}(\Omega)$$

$$\bigcap \qquad\qquad \bigcup$$

$$= \mathcal{S}^{q+1}_{\mathrm{d},k}\,\mathrm{d}\,\mathring{\mathsf{D}}^{k,q}_{\Gamma_t}(\Omega) \dot{+} \big(\mathring{\mathsf{H}}^{k,q}_{\Gamma_t}(\Omega) \cap \mathring{\mathsf{D}}^{q}_{\Gamma_t,0}(\Omega)\big)$$

hold with linear and continuous regular decomposition resp. potential operators, which can be defined explicitly by the orthonormal Helmholtz projectors and the operators $\mathcal{S}^{q}_{\mathrm{d},k}$.

joint work with
Ralf Hiptmair, Clemens Pechstein, Michael Schomburg, Walter Zulehner

generalises and improves work/results of Costabel and McIntosh

dual regular potentials and decompositions

Dual regular potentials and decompositions involving

$$\mathring{\mathrm{H}}^{-k,q}_{\Gamma_n}(\Omega) = \mathring{\mathrm{H}}^{k,q}_{\Gamma_t}(\Omega)'$$

can be proved by Banach space duality. E.g.:

- $\mathring{\mathrm{D}}^{k,q}_{\Gamma_t}(\Omega)' = \mathring{\Delta}^{-k-1,q}_{\Gamma_n}(\Omega) := \{E' \in \mathring{\mathrm{H}}^{-k-1,q}_{\Gamma_n}(\Omega) \, : \, \delta E' \in \mathring{\mathrm{H}}^{-k-1,q-1}_{\Gamma_n}(\Omega)\}$
- dual ranges are closed
- dual Friedrichs/Poincaré typ estimates, inf-sup condition, i.e.,

$$\delta^{-1} : \delta\, \mathring{\mathrm{H}}^{-k,q}_{\Gamma_n}(\Omega) \longrightarrow \big(\mathrm{d}\, \mathring{\mathrm{H}}^{k+1,q-1}_{\Gamma_t}(\Omega)\big)' \qquad \text{cont}$$

$$\Updownarrow$$

$$\forall\, H' \in \big(\mathrm{d}\, \mathring{\mathrm{H}}^{k+1,q-1}_{\Gamma_t}(\Omega)\big)' \qquad \frac{1}{c}|H'|_{\mathring{\mathrm{H}}^{-k,q}_{\Gamma_n}(\Omega)} \le |\,\delta\, H'|_{\mathring{\mathrm{H}}^{-k-1,q-1}_{\Gamma_n}(\Omega)} \le c|H'|_{\mathring{\mathrm{H}}^{-k,q}_{\Gamma_n}(\Omega)}$$

$$\Updownarrow$$

$$0 < \inf_{0 \ne H \in \mathrm{d}\, \mathring{\mathrm{H}}^{k+1,q-1}_{\Gamma_t}(\Omega)} \; \sup_{0 \ne E \in \mathring{\mathrm{H}}^{k+1,q-1}_{\Gamma_t}(\Omega)} \frac{\langle H, \mathrm{d}\, E\rangle_{\mathring{\mathrm{H}}^{k,q}_{\Gamma_t}(\Omega)}}{|H|_{\mathring{\mathrm{H}}^{k,q}_{\Gamma_t}(\Omega)} |E|_{\mathring{\mathrm{H}}^{k+1,q-1}_{\Gamma_t}(\Omega)}}$$

dual regular potentials and decompositions

Ω top triv $\quad\Rightarrow$

$$\mathrm{d}\,\mathring{\mathsf{H}}^{k+1,q-1}_{\Gamma_t}(\Omega) = \mathring{\mathsf{H}}^{k,q}_{\Gamma_t}(\Omega) \cap \mathring{\mathsf{D}}^{q}_{\Gamma_t,0}(\Omega)$$

$$\delta\,\mathring{\mathsf{H}}^{-k,q}_{\Gamma_n}(\Omega) = \mathring{\Delta}^{-k-1,q-1}_{\Gamma_n,0}(\Omega) = \{H' \in \mathring{\mathsf{H}}^{-k-1,q-1}_{\Gamma_n}(\Omega) \,:\, \delta\,H' = 0\}$$

literature (complexes, applications to FEM, ...)

Arnold, Falk, Winther, Christiansen, Gopalakrishnan, Schöberl, Zulehner, ...

literature (fa-toolbox, complexes, a posteriori error estimates, ...)

some results of this talk:

- Py: *Solution Theory, Variational Formulations, and Functional a Posteriori Error Estimates for General First Order Systems with Applications to Electro-Magneto-Statics and More*,
  (NFAO) Numerical Functional Analysis and Optimization, 2019

literature (complexes, Friedrichs type constants, Maxwell constants)

some results of this talk:

- Py: *On Constants in Maxwell Inequalities for Bounded and Convex Domains*, Zapiski POMI/ (JMS)Journal of Mathematical Sciences (Springer New York), 2015
- Py: *On Maxwell's and Poincare's Constants*, (DCDS) Discrete and Continuous Dynamical Systems - Series S, 2015
- Py: *On the Maxwell Constants in 3D*, (M2AS) Mathematical Methods in the Applied Sciences, 2017
- Py: *On the Maxwell and Friedrichs/Poincaré Constants in ND*, (MZ) Mathematische Zeitschrift, 2019

- Py: *. . . some (so far) unpublished results*

literature (complexes, Friedrichs type constants, compact embeddings)

compact embeddings for Maywell:

- Weck: *Maxwell's boundary value problems on Riemannian manifolds with nonsmooth boundaries*,
  (JMA2) Journal of Mathematical Analysis and Applications, 1974 (1972)
- Picard: *An elementary proof for a compact imbedding result in generalized electromagnetic theory*,
  (MZ) Mathematische Zeitschrift, 1984
- Witsch: *A remark on a compactness result in electromagnetic theory*,
  (M2AS) Mathematical Methods in the Applied Sciences, 1993

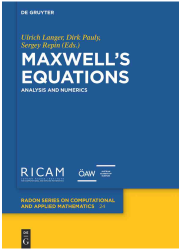
(Weber '80, Costabel '90, Jochmann '97, Fernandes/Gilardi '97, Kuhn '99, Picard/Weck/Witsch '01, Py '96, '03, '06, '07, '08)

literature (complexes, Friedrichs type constants, compact embeddings)

some results of this talk:

- Bauer, Py, Schomburg: *The Maxwell Compactness Property in Bounded Weak Lipschitz Domains with Mixed Boundary Conditions*,
  (SIMA) SIAM Journal on Mathematical Analysis, 2016
- Py, Zulehner: *The divDiv-Complex and Applications to Biharmonic Equations*,
  (AA) Applicable Analysis, 2019
- Hiptmair, Pechstein, Py, Schomburg, Zulehner,: *Regular Potentials and Regular Decompositions for Bounded Strong Lipschitz Domains with Mixed Boundary Conditions in Arbitrary Dimensions*,
  (almost) submitted
- Py, Schomburg, Zulehner: *The Elasticity Complex*,
  (almost) submitted

literature (Maxwell's equations and more...)



- Langer, Py, Repin (Eds): *Maxwell's equations. Analysis and numerics*, Radon Series on Applied Mathematics, De Gruyter, July 2019