On Grad-grad, div-Div, and Rot-Rot$^{\top}$ complexes for problems related to the biharmonic equation and elasticity

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Open-Minded ;-)

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Some Well-Known and Some Not so Well-Known Complexes

De-Rham complex (grad = ∇, rot = curl = ∇×, div = ∇·)

$$\xrightarrow{0} D(\mathring{\text{grad}}) \xrightarrow{\mathring{\text{grad}}} D(\mathring{\text{rot}}) \xrightarrow{\mathring{\text{rot}}} D(\mathring{\text{div}}) \xrightarrow{\mathring{\text{div}}} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathbb{R}}} \mathbb{R}$$

dual or adjoint complex

$$\xleftarrow{0} \mathsf{L}^2(\Omega) \xleftarrow{-\operatorname{div}} D(\operatorname{div}) \xleftarrow{\operatorname{rot}} D(\operatorname{rot}) \xleftarrow{-\operatorname{grad}} D(\operatorname{grad}) \xleftarrow{\iota_{\mathbb{R}}} \mathbb{R}$$

- typical fos electro statics ($\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain)

$$\mathring{\text{rot}}\, E = F, \qquad -\operatorname{div} E = \mathring{\text{grad}}^* E = G, \qquad E \perp \mathcal{H}_D = N(\mathring{\text{rot}}) \cap N(\operatorname{div})$$

- typical Helmholtz type decompositions

$$\mathsf{L}^2 = \underbrace{R(\mathring{\text{grad}}) \oplus \mathcal{H}_D}_{=N(\mathring{\text{rot}})} \oplus R(\operatorname{rot}), \qquad \overbrace{\mathcal{H}_D \oplus R(\operatorname{rot})}^{=N(\operatorname{div})}$$

- typical Friedrichs/Poincaré type estimates - Maxwell estimates

$$\forall\, u \in D(\mathring{\text{grad}}) \qquad |u|_{\mathsf{L}^2} \leq c_f |\mathring{\text{grad}}\, u|_{\mathsf{L}^2},$$

$$\forall\, E \in D(\mathring{\text{rot}}) \cap R(\operatorname{rot}) = D(\mathring{\text{rot}}) \cap N(\operatorname{div}) \cap \mathcal{H}_D^{\perp} \qquad |E|_{\mathsf{L}^2} \leq c_m |\mathring{\text{rot}}\, E|_{\mathsf{L}^2},$$

$$\forall\, E \in D(\mathring{\text{rot}}) \cap D(\operatorname{div}) \cap \mathcal{H}_D^{\perp} \qquad |E|_{\mathsf{L}^2} \leq \hat{c}\big(|\mathring{\text{rot}}\, E|_{\mathsf{L}^2} + |\operatorname{div} E|_{\mathsf{L}^2}\big)$$

Some Well-Known and Some Not so Well-Known Complexes

De-Rham complex (d exterior derivative, $\delta = \pm * \mathrm{d} *$ co-derivative)

$$\cdots \xrightarrow{\mathring{\mathrm{d}}_{q-1}} D(\mathring{\mathrm{d}}_q) \xrightarrow{\mathring{\mathrm{d}}_q} D(\mathring{\mathrm{d}}_{q+1}) \xrightarrow{\mathring{\mathrm{d}}_{q+1}} \cdots$$

dual or adjoint complex

$$\cdots \xleftarrow{\delta_q} D(\delta_q) \xleftarrow{\delta_{q+1}} D(\delta_{q+1}) \xleftarrow{\delta_{q+2}} \cdots$$

- typical fos gen. el./mag. stat. ($\Omega \subset \mathbb{R}^N$ bd, weak-Lip or Ω Riemannian manifold)

$$\mathring{\mathrm{d}}_q \omega = f, \qquad -\delta_q \omega = \mathring{\mathrm{d}}_{q-1}^* \omega = g, \qquad \omega \perp \mathcal{H}_D^q = N(\mathring{\mathrm{d}}_q) \cap N(\delta_q)$$

- typical Helmholtz type decompositions

$$\mathrm{L}^{2,q} = \underbrace{R(\mathring{\mathrm{d}}_{q-1}) \oplus \overbrace{\mathcal{H}_D^q}_{=N(\mathring{\mathrm{d}}_q)} \oplus R(\delta_{q+1})}^{=N(\delta_q)}$$

- typical Friedrichs/Poincaré type estimates - Maxwell estimates

$$\forall\, \omega \in D(\mathring{\mathrm{d}}_q) \cap R(\delta_{q+1}) = D(\mathring{\mathrm{d}}_q) \cap N(\delta_q) \cap \mathcal{H}_D^{q\,\perp} \qquad |\omega|_{\mathrm{L}^{2,q}} \leq c_q |\mathring{\mathrm{d}}_q \omega|_{\mathrm{L}^{2,q+1}},$$

$$\forall\, \zeta \in D(\delta_{q+1}) \cap R(\mathring{\mathrm{d}}_q) = D(\delta_{q+1}) \cap N(\mathring{\mathrm{d}}_{q+1}) \cap \mathcal{H}_D^{q+1\,\perp} \qquad |\zeta|_{\mathrm{L}^{2,q+1}} \leq c_q |\delta_{q+1} \zeta|_{\mathrm{L}^{2,q}},$$

$$\forall\, \omega \in D(\mathring{\mathrm{d}}_q) \cap D(\delta_q) \cap \mathcal{H}_D^{q\,\perp} \qquad |\omega|_{\mathrm{L}^{2,q}} \leq \hat{c}\big(|\mathring{\mathrm{d}}_q \omega|_{\mathrm{L}^{2,q+1}} + |\delta_q \omega|_{\mathrm{L}^{2,q-1}}\big)$$

Some Well-Known and Some Not so Well-Known Complexes

bi-harmonic Grad grad - div Div cmplx: $0 \to L^2(\Omega,\mathbb{R}) \to L^2(\Omega,\mathbb{R}^{3\times3}_{\mathrm{sym}}) \to L^2(\Omega,\mathbb{R}^{3\times3}_{\mathrm{dev}}) \to L^2(\Omega,\mathbb{R}^3) \to \mathrm{RT}_0$

$$\xrightarrow{\ 0\ } D(\mathring{\mathrm{Grad}}\,\mathrm{grad}) \xrightarrow{\mathring{\mathrm{Grad}}\,\mathrm{grad}} D(\mathring{\mathrm{Rot}}_{\mathbb{S}}) \xrightarrow{\mathring{\mathrm{Rot}}_{\mathbb{S}}} D(\mathring{\mathrm{Div}}_{\mathbb{T}}) \xrightarrow{\mathring{\mathrm{Div}}_{\mathbb{T}}} L^2(\Omega) \xrightarrow{\pi_{\mathrm{RT}_0}} \mathrm{RT}_0$$

dual or adjoint complex

$$\xleftarrow{\ 0\ } L^2(\Omega) \xleftarrow{\mathrm{div}\,\mathrm{Div}_{\mathbb{S}}} D(\mathrm{div}\,\mathrm{Div}_{\mathbb{S}}) \xleftarrow{\mathrm{sym}\,\mathrm{Rot}_{\mathbb{T}}} D(\mathrm{sym}\,\mathrm{Rot}_{\mathbb{T}}) \xleftarrow{-\,\mathrm{dev}\,\mathrm{Grad}} D(\mathrm{dev}\,\mathrm{Grad}) \xleftarrow{\iota_{\mathrm{RT}_0}} \mathrm{RT}_0$$

- typical fos bi-harmonic eq. ($\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain)

$$\mathring{\mathrm{Rot}}_{\mathbb{S}} S = F, \quad \mathrm{div}\,\mathrm{Div}_{\mathbb{S}} S = \mathring{\mathrm{Grad}}\,\mathrm{grad}^* S = G, \qquad S \perp \mathcal{H}^D_{\mathbb{S}} = N(\mathring{\mathrm{Rot}}_{\mathbb{S}}) \cap N(\mathrm{div}\,\mathrm{Div}_{\mathbb{S}})$$

or
$$\mathring{\mathrm{Div}}_{\mathbb{T}} T = F, \qquad \mathrm{sym}\,\mathrm{Rot}_{\mathbb{T}} S = \mathring{\mathrm{Rot}}^*_{\mathbb{S}} T = G, \quad T \perp \mathcal{H}^N_{\mathbb{T}} = N(\mathring{\mathrm{Div}}_{\mathbb{T}}) \cap N(\mathrm{sym}\,\mathrm{Rot}_{\mathbb{T}})$$

- typical Helmholtz type decompositions

$$L^2_{\mathbb{S}} = \underbrace{R(\mathring{\mathrm{Grad}}\,\mathrm{grad}) \oplus \overbrace{\mathcal{H}^D_{\mathbb{S}}}^{}}_{=N(\mathring{\mathrm{Rot}}_{\mathbb{S}})} \overbrace{\oplus R(\mathrm{sym}\,\mathrm{Rot}_{\mathbb{T}})}^{=N(\mathrm{div}\,\mathrm{Div}_{\mathbb{S}})}$$

- typical Friedrichs/Poincaré type estimates

$$\forall\, S \in D(\mathring{\mathrm{Rot}}_{\mathbb{S}}) \cap \underbrace{N(\mathrm{div}\,\mathrm{Div}_{\mathbb{S}}) \cap \mathcal{H}^{D\,\perp}_{\mathbb{S}}}_{=R(\mathrm{sym}\,\mathrm{Rot}_{\mathbb{T}})} \qquad |S|_{L^2_{\mathbb{S}}} \le c_R |\mathring{\mathrm{Rot}}_{\mathbb{S}} S|_{L^2_{\mathbb{T}}},$$

$$\forall\, T \in D(\mathring{\mathrm{Div}}_{\mathbb{T}}) \cap D(\mathrm{sym}\,\mathrm{Rot}_{\mathbb{T}}) \cap \mathcal{H}^{N\,\perp}_{\mathbb{T}} \qquad |T|_{L^2_{\mathbb{T}}} \le \hat{c}\big(|\mathring{\mathrm{Div}}_{\mathbb{T}} T|_{L^2} + |\mathrm{sym}\,\mathrm{Rot}_{\mathbb{T}} T|_{L^2_{\mathbb{S}}}\big)$$

Some Well-Known and Some Not so Well-Known Complexes

elasticity $\operatorname{Rot}\operatorname{Rot}^\top$ complex: $0 \to L^2(\Omega,\mathbb{R}^3) \to L^2(\Omega,\mathbb{R}^{3\times3}_{\mathrm{sym}}) \to L^2(\Omega,\mathbb{R}^{3\times3}_{\mathrm{sym}}) \to L^2(\Omega,\mathbb{R}^3) \to \mathrm{RM}$

$$\xrightarrow{\;0\;} D(\operatorname{sym}\mathring{\operatorname{Grad}}) \xrightarrow{\operatorname{sym}\mathring{\operatorname{Grad}}} D(\operatorname{Rot}\mathring{\operatorname{Rot}}^\top_{\mathbb{S}}) \xrightarrow{\operatorname{Rot}\mathring{\operatorname{Rot}}^\top_{\mathbb{S}}} D(\mathring{\operatorname{Div}}_{\mathbb{S}}) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{S}}} L^2(\Omega) \xrightarrow{\pi_{\mathrm{RM}}} \mathrm{RM}$$

dual or adjoint complex

$$\xleftarrow{\;0\;} L^2(\Omega) \xleftarrow{-\operatorname{Div}_{\mathbb{S}}} D(\operatorname{Div}_{\mathbb{S}}) \xleftarrow{\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S}}} D(\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S}}) \xleftarrow{-\operatorname{sym}\operatorname{Grad}} D(\operatorname{sym}\operatorname{Grad}) \xleftarrow{\iota_{\mathrm{RM}}} \mathrm{RM}$$

- typical fos elasticity ($\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain)

$$\operatorname{Rot}\mathring{\operatorname{Rot}}^\top_{\mathbb{S}} S = F, \quad -\operatorname{Div}_{\mathbb{S}} S = \operatorname{sym}\mathring{\operatorname{Grad}} S = G, \quad S \perp \mathcal{H}^D_{\mathbb{S}} = N(\operatorname{Rot}\mathring{\operatorname{Rot}}^\top_{\mathbb{S}}) \cap N(\operatorname{Div}_{\mathbb{S}})$$

- typical Helmholtz type decompositions

$$L^2_{\mathbb{S}} = \underbrace{R(\operatorname{sym}\mathring{\operatorname{Grad}}) \oplus \mathcal{H}^D_{\mathbb{S}}}_{=N(\operatorname{Rot}\mathring{\operatorname{Rot}}^\top_{\mathbb{S}})} \overbrace{\phantom{\mathcal{H}^D_{\mathbb{S}}}\oplus R(\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S}})}^{=N(\operatorname{Div}_{\mathbb{S}})}$$

- typical Friedrichs/Poincaré type estimates

$$\forall\, S \in D(\operatorname{Rot}\mathring{\operatorname{Rot}}^\top_{\mathbb{S}}) \cap \underbrace{N(\operatorname{Div}_{\mathbb{S}}) \cap \mathcal{H}^{D\,\perp}_{\mathbb{S}}}_{=R(\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S}})} \qquad |S|_{L^2_{\mathbb{S}}} \le c_{RR} |\operatorname{Rot}\mathring{\operatorname{Rot}}^\top_{\mathbb{S}} S|_{L^2_{\mathbb{S}}},$$

$$\forall\, S \in D(\mathring{\operatorname{Div}}_{\mathbb{S}}) \cap D(\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S}}) \cap \mathcal{H}^{N\,\perp}_{\mathbb{S}} \qquad |S|_{L^2_{\mathbb{S}}} \le \hat{c}\big(|\mathring{\operatorname{Div}}_{\mathbb{S}} S|_{L^2} + |\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S}} S|_{L^2_{\mathbb{S}}}\big)$$

General Complexes

H_i Hilbert spaces
densely defined, closed (unbounded), linear operators A_i with adjoints A_i^*

$$\mathrm{A}_i : D(\mathrm{A}_i) \subset \mathsf{H}_i \to \mathsf{H}_{i+1}, \quad \mathrm{A}_i^* : D(\mathrm{A}_i^*) \subset \mathsf{H}_{i+1} \to \mathsf{H}_i, \quad i \in \mathbb{Z}$$

Hilbert complex (sequence) with adjoint Hilbert complex:

$$\ldots \xrightarrow{\cdots} D(\mathrm{A}_{i-1}) \xrightarrow{\mathrm{A}_{i-1}} D(\mathrm{A}_i) \xrightarrow{\mathrm{A}_i} D(\mathrm{A}_{i+1}) \xrightarrow{\mathrm{A}_{i+1}} \ldots$$

$$\ldots \xleftarrow{\mathrm{A}_{i-1}^*} D(\mathrm{A}_{i-1}^*) \xleftarrow{\mathrm{A}_i^*} D(\mathrm{A}_i^*) \xleftarrow{\mathrm{A}_{i+1}^*} D(\mathrm{A}_{i+1}^*) \xleftarrow{\cdots} \ldots$$

complex: 'range $\subset$ kernel', i.e., $\mathrm{A}_i \mathrm{A}_{i-1} = 0$, $\mathrm{A}_{i-1}^* \mathrm{A}_i^* = 0$, i.e.,

$$R(\mathrm{A}_{i-1}) \subset N(\mathrm{A}_i), \quad R(\mathrm{A}_i^*) \subset N(\mathrm{A}_{i-1}^*)$$

<u>related problem:</u> find $x \in D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)$ s.t.

$$\boxed{\mathrm{A}_i x = f, \quad \mathrm{A}_{i-1}^* x = g, \quad \pi_i x = h,}$$

where $f \in R(\mathrm{A}_i)$, $g \in R(\mathrm{A}_{i-1}^*)$ and $h \in \mathcal{H}_i$
with kernel/cohomology group $\mathcal{H}_i := N(\mathrm{A}_i) \cap N(\mathrm{A}_{i-1}^*)$

General Complexes

densely defined, closed (unbounded), linear operators with adjoints

$$\mathrm{A}_i : D(\mathrm{A}_i) \subset \mathsf{H}_i \to \mathsf{H}_{i+1}, \quad \mathrm{A}_i^* : D(\mathrm{A}_i^*) \subset \mathsf{H}_{i+1} \to \mathsf{H}_i, \quad i \in \mathbb{Z}$$

Hilbert complex with adjoint Hilbert complex:

$$\ldots \xrightarrow{\;\cdots\;} D(\mathrm{A}_{i-1}) \xrightarrow{\mathrm{A}_{i-1}} D(\mathrm{A}_i) \xrightarrow{\mathrm{A}_i} D(\mathrm{A}_{i+1}) \xrightarrow{\mathrm{A}_{i+1}} \ldots$$

$$\ldots \xleftarrow{\mathrm{A}_{i-1}^*} D(\mathrm{A}_{i-1}^*) \xleftarrow{\mathrm{A}_i^*} D(\mathrm{A}_i^*) \xleftarrow{\mathrm{A}_{i+1}^*} D(\mathrm{A}_{i+1}^*) \xleftarrow{\;\cdots\;} \ldots$$

- complex is <u>closed</u>, iff all ranges are closed, i.e., $R(\mathrm{A}_i) = \overline{R(\mathrm{A}_i)}$
- complex is <u>exact</u>, iff ranges=kernels, i.e., $R(\mathrm{A}_{i-1}) = N(\mathrm{A}_i)$
- complex is <u>compact</u>, iff “all” embeddings are compact, i.e., $D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*) \hookrightarrow\!\!\!\!\rightarrow \mathsf{H}_i$

General Complexes

Hodge/Helmholtz/Weyl decomposition (projection theorem)

$$\mathrm{H}_i = N(\mathrm{A}_i) \oplus_{\mathrm{H}_i} \overline{R(\mathrm{A}_i^*)}, \qquad N(\mathrm{A}_i)^{\perp_{\mathrm{H}_i}} = \overline{R(\mathrm{A}_i^*)},$$
$$\mathrm{H}_i = \overline{R(\mathrm{A}_{i-1})} \oplus_{\mathrm{H}_i} N(\mathrm{A}_{i-1}^*), \qquad N(\mathrm{A}_{i-1}^*)^{\perp_{\mathrm{H}_i}} = \overline{R(\mathrm{A}_{i-1})}$$

reduced operators (<u>inj.</u>, dd, cl. (unbd.), lin. operators with adjoints)

$$\mathcal{A}_i : D(\mathcal{A}_i) \subset \overline{R(\mathrm{A}_i^*)} \to \overline{R(\mathrm{A}_i)}, \quad \mathcal{A}_i^* : D(\mathcal{A}_i^*) \subset \overline{R(\mathrm{A}_i)} \to \overline{R(\mathrm{A}_i^*)}, \quad R(\mathrm{A}_i) = R(\mathcal{A}_i),$$
$$D(\mathcal{A}_i) = D(\mathrm{A}_i) \cap \overline{R(\mathrm{A}_i^*)}, \quad D(\mathcal{A}_i^*) = D(\mathrm{A}_i^*) \cap \overline{R(\mathrm{A}_i)}, \quad R(\mathrm{A}_i^*) = R(\mathcal{A}_i^*)$$

- complex is <u>closed</u>

$\Leftrightarrow$ $R(\mathrm{A}_i) = \overline{R(\mathrm{A}_i)}$

$\Leftrightarrow$ $R(\mathrm{A}_i^*) = \overline{R(\mathrm{A}_i^*)}$

$\Leftrightarrow$ $\mathcal{A}_i^{-1}$ continuous, i.e., $\mathcal{A}_i^{-1} : R(\mathrm{A}_i) \to D(\mathcal{A}_i)$ continuous

$\Leftrightarrow$ $(\mathcal{A}_i^*)^{-1}$ continuous, i.e., $(\mathcal{A}_i^*)^{-1} : R(\mathrm{A}_i^*) \to D(\mathcal{A}_i^*)$ continuous

$\Leftrightarrow$ Friedrichs/Poincaré type estimates for A_i, i.e.,

$$\exists\, c_{\mathrm{A}_i} > 0 \quad \forall\, x \in D(\mathcal{A}_i) \qquad |x|_{\mathrm{H}_i} \le c_{\mathrm{A}_i} |\mathrm{A}_i x|_{\mathrm{H}_{i+1}}$$

$\Leftrightarrow$ Friedrichs/Poincaré type estimates for A_i, i.e., $(c_{\mathrm{A}_i} = c_{\mathrm{A}_i^*})$

$$\exists\, c_{\mathrm{A}_i^*} > 0 \quad \forall\, y \in D(\mathcal{A}_i^*) \qquad |y|_{\mathrm{H}_{i+1}} \le c_{\mathrm{A}_i^*} |\mathrm{A}_i^* y|_{\mathrm{H}_i}$$

General Complexes

Hodge/Helmholtz/Weyl decomposition (projection theorem)

$$\mathrm{H}_i = N(\mathrm{A}_i) \oplus_{\mathrm{H}_i} \overline{R(\mathrm{A}_i^*)}, \qquad \Rightarrow \qquad \mathrm{H}_i = \overline{R(\mathrm{A}_{i-1})} \oplus_{\mathrm{H}_i} \mathcal{H}_i \oplus_{\mathrm{H}_i} \overline{R(\mathrm{A}_i^*)},$$

$$\cup \qquad\qquad \cap$$

$$\mathrm{H}_i = \overline{R(\mathrm{A}_{i-1})} \oplus_{\mathrm{H}_i} N(\mathrm{A}_{i-1}^*)$$

reduced operators (inj., dd, cl. (unbd.), lin. operators with adjoints)

$$\mathcal{A}_i : D(\mathcal{A}_i) \subset \overline{R(\mathrm{A}_i^*)} \to \overline{R(\mathrm{A}_i)}, \quad \mathcal{A}_i^* : D(\mathcal{A}_i^*) \subset \overline{R(\mathrm{A}_i)} \to \overline{R(\mathrm{A}_i^*)}, \quad R(\mathrm{A}_i) = R(\mathcal{A}_i),$$

$$D(\mathcal{A}_i) = D(\mathrm{A}_i) \cap \overline{R(\mathrm{A}_i^*)}, \qquad D(\mathcal{A}_i^*) = D(\mathrm{A}_i^*) \cap \overline{R(\mathrm{A}_i)}, \qquad R(\mathrm{A}_i^*) = R(\mathcal{A}_i^*)$$

- complex is exact

$$\Leftrightarrow \quad R(\mathrm{A}_{i-1}) = N(\mathrm{A}_i) \quad \Leftrightarrow \quad \mathcal{H}_i = \underbrace{N(\mathrm{A}_i)}_{=R(\mathrm{A}_{i-1})} \cap N(\mathrm{A}_{i-1}^*) = \{0\} \quad \wedge \quad R(\mathrm{A}_{i-1}) \text{ closed}$$

note: $\overline{R(\mathrm{A}_{i-1})} = N(\mathrm{A}_i) \quad \Leftrightarrow \quad \mathcal{H}_i = \{0\}$

General Complexes

Hodge/Helmholtz/Weyl decomposition (projection theorem)

$$D(\mathrm{A}_i) = N(\mathrm{A}_i) \oplus_{\mathrm{H}_i} D(\mathcal{A}_i), \qquad \Rightarrow \qquad D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*) = D(\mathcal{A}_{i-1}^*) \oplus_{\mathrm{H}_i} \mathcal{H}_i \oplus_{\mathrm{H}_i} D(\mathcal{A}_i),$$

$$\cup \qquad\qquad \cap$$

$$D(\mathrm{A}_{i-1}^*) = D(\mathcal{A}_{i-1}^*) \oplus_{\mathrm{H}_i} N(\mathrm{A}_{i-1}^*)$$

reduced operators (inj., dd, cl. (unbd.), lin. operators with adjoints)

$$\mathcal{A}_i : D(\mathcal{A}_i) \subset \overline{R(\mathrm{A}_i^*)} \to \overline{R(\mathrm{A}_i)}, \quad \mathcal{A}_i^* : D(\mathcal{A}_i^*) \subset \overline{R(\mathrm{A}_i)} \to \overline{R(\mathrm{A}_i^*)}, \quad R(\mathrm{A}_i) = R(\mathcal{A}_i),$$

$$D(\mathcal{A}_i) = D(\mathrm{A}_i) \cap \overline{R(\mathrm{A}_i^*)}, \quad D(\mathcal{A}_i^*) = D(\mathrm{A}_i^*) \cap \overline{R(\mathrm{A}_i)}, \quad R(\mathrm{A}_i^*) = R(\mathcal{A}_i^*)$$

- complex is compact

$$\Leftrightarrow \quad D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_i \quad \Leftrightarrow \quad D(\mathcal{A}_i) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_i \ \wedge\ D(\mathcal{A}_{i-1}^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_i \ \wedge\ \mathcal{H}_i \hookrightarrow\!\!\!\rightarrow \mathrm{H}_i$$

$$\Leftrightarrow \quad D(\mathcal{A}_i) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_i \ \wedge\ D(\mathcal{A}_{i-1}) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_i \ \wedge\ \dim \mathcal{H}_i < \infty$$

$\Rightarrow$ complex is closed and $\dim \mathcal{H}_i < \infty$

note: $D(\mathcal{A}_i) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_i \quad \Leftrightarrow \quad \mathcal{A}_i^{-1}$ cpt $\quad \Leftrightarrow \quad \sigma(\mathcal{A}_i^* \mathcal{A}_i)$ disc. $\wedge$ (and exp. theo.)

$\Leftrightarrow \quad D(\mathcal{A}_i^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_{i+1} \quad \Leftrightarrow \quad (\mathcal{A}_i^*)^{-1}$ cpt $\quad \Leftrightarrow \quad \sigma(\mathcal{A}_i \mathcal{A}_i^*)$ disc. $\wedge$ (and exp. theo.)

De-Rham Complexes: grad-rot-div Complexes

complex (grad = ∇, rot = curl = $\nabla\times$, div = $\nabla\cdot$)

$$\xrightarrow{\;0\;} D(\mathring{\text{grad}}) \xrightarrow{\;\mathring{\text{grad}}\;} D(\mathring{\text{rot}}) \xrightarrow{\;\mathring{\text{rot}}\;} D(\mathring{\text{div}}) \xrightarrow{\;\mathring{\text{div}}\;} \mathsf{L}^2(\Omega) \xrightarrow{\;\pi_{\mathbb{R}}\;} \mathbb{R}$$

dual or adjoint complex

$$\xleftarrow{\;0\;} \mathsf{L}^2(\Omega) \xleftarrow{\;-\text{div}\;} D(\text{div}) \xleftarrow{\;\text{rot}\;} D(\text{rot}) \xleftarrow{\;-\text{grad}\;} D(\text{grad}) \xleftarrow{\;\iota_{\mathbb{R}}\;} \mathbb{R}$$

complex property: "range $\subset$ kernel" (rot grad = 0, div rot = 0)

$$R(0) = \{0\} = N(\mathring{\text{grad}}), \quad R(\mathring{\text{grad}}) \subset N(\mathring{\text{rot}}), \quad R(\mathring{\text{rot}}) \subset N(\mathring{\text{div}}), \quad R(\mathring{\text{div}}) = N(\pi_{\mathbb{R}}) = \mathbb{R}^{\perp},$$
$$R(\iota_{\mathbb{R}}) = \mathbb{R} = N(\text{grad}), \quad R(-\text{grad}) \subset N(\text{rot}), \quad R(\text{rot}) \subset N(-\text{div}), \quad R(-\text{div}) = N(0) = \mathsf{L}^2(\Omega)$$

complex closed $\Leftrightarrow$ all ranges are closed
complex exact $\Leftrightarrow$ complex closed and all cohomology groups trivial, i.e.,

$$\mathcal{H}_D := N(\mathring{\text{rot}}) \ominus R(\mathring{\text{grad}}) = N(\text{div}) \ominus R(\text{rot}) = N(\mathring{\text{rot}}) \cap N(\text{div}) = \{0\} \qquad \text{(Dirichlet fields)},$$
$$\mathcal{H}_N := N(\mathring{\text{div}}) \ominus R(\mathring{\text{rot}}) = N(\text{rot}) \ominus R(\text{grad}) = N(\mathring{\text{div}}) \cap N(\text{rot}) = \{0\} \qquad \text{(Neumann fields)}$$

dimension depends only on topology, Betti numbers

De-Rham Complexes: grad-rot-div Complexes

complex:

$$\xrightarrow{0} D(\mathring{\operatorname{grad}}) \xrightarrow{\mathring{\operatorname{grad}}} D(\mathring{\operatorname{rot}}) \xrightarrow{\mathring{\operatorname{rot}}} D(\mathring{\operatorname{div}}) \xrightarrow{\mathring{\operatorname{div}}} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathbb{R}}} \mathbb{R}$$

dual or adjoint complex

$$\xleftarrow{0} \mathsf{L}^2(\Omega) \xleftarrow{-\operatorname{div}} D(\operatorname{div}) \xleftarrow{\operatorname{rot}} D(\operatorname{rot}) \xleftarrow{-\operatorname{grad}} D(\operatorname{grad}) \xleftarrow{\iota_{\mathbb{R}}} \mathbb{R}$$

"crucial"/"best" compact embeddings (for, e.g., Ω bounded weak Lipschitz)

$D(\mathring{\operatorname{grad}}) \subset D(\operatorname{grad}) \hookrightarrow\!\!\!\rightarrow \mathsf{L}^2$ (Rellich's selection theorem),

$D(\mathring{\operatorname{rot}}) \cap D(\operatorname{div}) \hookrightarrow\!\!\!\rightarrow \mathsf{L}^2$ (Weck's selection theorem: Weck '74, Weber' 80, Picard '84, Jochmann' 97, Picard-Weck-Witsch '01, Bauer-P.-Schomburg '16, '17),

$D(\operatorname{rot}) \cap D(\mathring{\operatorname{div}}) \hookrightarrow\!\!\!\rightarrow \mathsf{L}^2$ (Weck's selection theorem)

$\Rightarrow$ closed complexes and finite cohomology groups
$\Rightarrow$ Helmholtz decompositions
$\Rightarrow$ Friedrichs/Poincaré type estimates
$\Rightarrow$ continuous and compact inverses of reduced operators
$\Rightarrow$ ...all from general fa-toolbox $\surd$

De-Rham Complexes: grad-rot-div Complexes

complex:

$$\xrightarrow{0} D(\mathring{\text{grad}}) \xrightarrow{\mathring{\text{grad}}} D(\mathring{\text{rot}}) \xrightarrow{\mathring{\text{rot}}} D(\mathring{\text{div}}) \xrightarrow{\mathring{\text{div}}} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathbb{R}}} \mathbb{R}$$

dual or adjoint complex

$$\xleftarrow{0} \mathsf{L}^2(\Omega) \xleftarrow{-\operatorname{div}} D(\operatorname{div}) \xleftarrow{\operatorname{rot}} D(\operatorname{rot}) \xleftarrow{-\operatorname{grad}} D(\operatorname{grad}) \xleftarrow{\iota_{\mathbb{R}}} \mathbb{R}$$

e.g.: Helmholtz decompositions

$$\mathsf{L}^2 = R(\mathring{\text{grad}}) \oplus \mathcal{H}_D \oplus R(\operatorname{rot}) = R(\operatorname{grad}) \oplus \mathcal{H}_N \oplus R(\mathring{\text{rot}})$$

e.g.: Friedrichs/Poincaré type estimates

$$\begin{aligned}
&\forall\, u \in D(\mathring{\text{grad}}) && |u|_{\mathsf{L}^2} \leq c_f |\operatorname{grad} u|_{\mathsf{L}^2},\\
&\forall\, E \in D(\operatorname{div}) \cap R(\mathring{\text{grad}}) = D(\operatorname{div}) \cap N(\mathring{\text{rot}}) \cap \mathcal{H}_D^{\perp} && |E|_{\mathsf{L}^2} \leq c_f |\operatorname{div} E|_{\mathsf{L}^2},\\
&\forall\, E \in D(\mathring{\text{rot}}) \cap R(\operatorname{rot}) = D(\mathring{\text{rot}}) \cap N(\operatorname{div}) \cap \mathcal{H}_D^{\perp} && |E|_{\mathsf{L}^2} \leq c_m |\operatorname{rot} E|_{\mathsf{L}^2},\\
&\forall\, E \in D(\operatorname{rot}) \cap R(\mathring{\text{rot}}) = D(\operatorname{rot}) \cap N(\mathring{\text{div}}) \cap \mathcal{H}_N^{\perp} && |E|_{\mathsf{L}^2} \leq c_m |\operatorname{rot} E|_{\mathsf{L}^2},\\
&\forall\, E \in D(\mathring{\text{div}}) \cap R(\operatorname{grad}) = D(\mathring{\text{div}}) \cap N(\operatorname{rot}) \cap \mathcal{H}_N^{\perp} && |E|_{\mathsf{L}^2} \leq c_p |\operatorname{div} E|_{\mathsf{L}^2},\\
&\forall\, u \in D(\operatorname{grad}) \cap R(\mathring{\text{div}}) = D(\operatorname{grad}) \cap \mathbb{R}^{\perp} && |u|_{\mathsf{L}^2} \leq c_p |\operatorname{grad} u|_{\mathsf{L}^2}
\end{aligned}$$

De-Rham Complexes: d Complexes

complex (d exterior derivative, $\delta = \pm * \mathrm{d} *$ co-derivative)

$$\cdots \xrightarrow{\mathring{\mathrm{d}}_{q-1}} D(\mathring{\mathrm{d}}_q) \xrightarrow{\mathring{\mathrm{d}}_q} D(\mathring{\mathrm{d}}_{q+1}) \xrightarrow{\mathring{\mathrm{d}}_{q+1}} \cdots$$

dual or adjoint complex

$$\cdots \xleftarrow{\delta_q} D(\delta_q) \xleftarrow{\delta_{q+1}} D(\delta_{q+1}) \xleftarrow{\delta_{q+2}} \cdots$$

complex property: "range ⊂ kernel" ($\mathrm{d}_q \, \mathrm{d}_{q-1} = 0$, $\delta_q \, \delta_{q+1} = 0$)

$$R(\mathring{\mathrm{d}}_{q-1}) \subset N(\mathring{\mathrm{d}}_q), \qquad R(\delta_{q+1}) \subset N(\delta_q)$$

complex closed ⇔ all ranges are closed
complex exact ⇔ complex closed and all cohomology groups trivial, i.e.,

$$\mathcal{H}^q_D := N(\mathring{\mathrm{d}}_q) \cap N(\delta_q) = \{0\} \qquad \text{(Dirichlet forms)},$$
$$\mathcal{H}^q_N := N(\mathrm{d}_q) \cap N(\mathring{\delta}_q) = \{0\} \qquad \text{(Neumann forms)}$$

dimension depends only on topology, Betti numbers

De-Rham Complexes: d Complexes

complex (d exterior derivative, $\delta = \pm * \mathrm{d} *$ co-derivative)

$$\cdots \xrightarrow{\mathring{\mathrm{d}}_{q-1}} D(\mathring{\mathrm{d}}_q) \xrightarrow{\mathring{\mathrm{d}}_q} D(\mathring{\mathrm{d}}_{q+1}) \xrightarrow{\mathring{\mathrm{d}}_{q+1}} \cdots$$

dual or adjoint complex

$$\cdots \xleftarrow{\delta_q} D(\delta_q) \xleftarrow{\delta_{q+1}} D(\delta_{q+1}) \xleftarrow{\delta_{q+2}} \cdots$$

"crucial" / "best" compact embeddings (for, e.g., Ω bounded weak Lipschitz domain or manifold)

$D(\mathring{\mathrm{d}}_q) \cap D(\delta_q) \hookrightarrow\!\!\!\!\!\rightarrow \mathrm{L}^{2,q}$ (Weck's selection theorem: Weck '74, Picard '84, Bauer-P.-Schomburg '17),

$D(\mathrm{d}_q) \cap D(\mathring{\delta}_q) \hookrightarrow\!\!\!\!\!\rightarrow \mathrm{L}^{2,q}$ (Weck's selection theorem)

$\Rightarrow$ closed complexes and finite cohomology groups
$\Rightarrow$ Helmholtz decompositions
$\Rightarrow$ Friedrichs/Poincaré type estimates
$\Rightarrow$ continuous and compact inverses of reduced operators
$\Rightarrow$ ...all from general fa-toolbox $\surd$

De-Rham Complexes: d Complexes

complex (d exterior derivative, $\delta = \pm * \mathrm{d} *$ co-derivative)

$$\cdots \xrightarrow{\mathring{\mathrm{d}}_{q-1}} D(\mathring{\mathrm{d}}_q) \xrightarrow{\mathring{\mathrm{d}}_q} D(\mathring{\mathrm{d}}_{q+1}) \xrightarrow{\mathring{\mathrm{d}}_{q+1}} \cdots$$

dual or adjoint complex

$$\cdots \xleftarrow{\delta_q} D(\delta_q) \xleftarrow{\delta_{q+1}} D(\delta_{q+1}) \xleftarrow{\delta_{q+2}} \cdots$$

e.g.: Helmholtz decompositions

$$\mathrm{L}^{2,q} = R(\mathring{\mathrm{d}}_{q-1}) \oplus \mathcal{H}^q_D \oplus R(\delta_{q+1}) = R(\mathrm{d}_{q-1}) \oplus \mathcal{H}^q_N \oplus R(\mathring{\delta}_{q+1})$$

e.g.: Friedrichs/Poincaré type estimates

$$\forall\, \omega \in D(\mathring{\mathrm{d}}_q) \cap R(\delta_{q+1}) = D(\mathring{\mathrm{d}}_q) \cap N(\delta_q) \cap \mathcal{H}^{q\,\perp}_D \qquad |\omega|_{\mathrm{L}^{2,q}} \leq c_q |\mathrm{d}\,\omega|_{\mathrm{L}^{2,q+1}},$$

$$\forall\, \zeta \in D(\delta_{q+1}) \cap R(\mathring{\mathrm{d}}_q) = D(\delta_{q+1}) \cap N(\mathring{\mathrm{d}}_{q+1}) \cap \mathcal{H}^{q+1\,\perp}_D \qquad |\zeta|_{\mathrm{L}^{2,q+1}} \leq c_q |\delta\,\zeta|_{\mathrm{L}^{2,q}}$$

(same for the other boundary condition on $\mathring{\delta}_{q+1}$,
note: by Hodge $*$-operator $\tilde{c}_{N-q} = c_q$, same constant for 4 estimates)

Grad grad - div Div Complexes (Bi-Harmonic Equation)

complex: $0 \to L^2(\Omega,\mathbb{R}) \to L^2(\Omega,\mathbb{R}^{3\times 3}_{\mathrm{sym}}) \to L^2(\Omega,\mathbb{R}^{3\times 3}_{\mathrm{dev}}) \to L^2(\Omega,\mathbb{R}^3) \to \mathrm{RT}_0$

$$\xrightarrow{\ 0\ } D(\mathring{\mathrm{Grad}}\,\mathrm{grad}) \xrightarrow{\mathring{\mathrm{Grad}}\,\mathrm{grad}} D(\mathring{\mathrm{Rot}}_{\mathbb{S}}) \xrightarrow{\mathring{\mathrm{Rot}}_{\mathbb{S}}} D(\mathring{\mathrm{Div}}_{\mathbb{T}}) \xrightarrow{\mathring{\mathrm{Div}}_{\mathbb{T}}} L^2(\Omega) \xrightarrow{\pi_{\mathrm{RT}_0}} \mathrm{RT}_0$$

dual or adjoint complex

$$\xleftarrow{\ 0\ } L^2(\Omega) \xleftarrow{\mathrm{div}\,\mathrm{Div}_{\mathbb{S}}} D(\mathrm{div}\,\mathrm{Div}_{\mathbb{S}}) \xleftarrow{\mathrm{sym}\,\mathrm{Rot}_{\mathbb{T}}} D(\mathrm{sym}\,\mathrm{Rot}_{\mathbb{T}}) \xleftarrow{-\mathrm{dev}\,\mathrm{Grad}} D(\mathrm{dev}\,\mathrm{Grad}) \xleftarrow{\iota_{\mathrm{RT}_0}} \mathrm{RT}_0$$

complex property: "range ⊂ kernel"

$$\mathring{\mathrm{Rot}}_{\mathbb{S}}\mathring{\mathrm{Grad}}\,\mathrm{grad} = 0, \qquad \mathring{\mathrm{Div}}_{\mathbb{T}}\mathring{\mathrm{Rot}}_{\mathbb{S}} = 0, \qquad \pi_{\mathrm{RT}_0}\mathring{\mathrm{Div}}_{\mathbb{T}} = 0,$$
$$\mathrm{div}\,\mathrm{Div}_{\mathbb{S}}\,\mathrm{sym}\,\mathrm{Rot}_{\mathbb{T}} = 0, \qquad \mathrm{sym}\,\mathrm{Rot}_{\mathbb{T}}\,\mathrm{dev}\,\mathrm{Grad} = 0, \qquad \mathrm{dev}\,\mathrm{Grad}\,\iota_{\mathrm{RT}_0} = 0$$

complex closed ⇔ all ranges are closed
complex exact ⇔ complex closed and all cohomology groups trivial, i.e.,

$$\mathcal{H}^D_{\mathbb{S}} := N(\mathring{\mathrm{Rot}}_{\mathbb{S}}) \cap N(\mathrm{div}\,\mathrm{Div}_{\mathbb{S}}) \qquad \text{(symmetric Dirichlet fields)},$$
$$\mathcal{H}^N_{\mathbb{T}} := N(\mathring{\mathrm{Div}}_{\mathbb{T}}) \cap N(\mathrm{sym}\,\mathrm{Rot}_{\mathbb{T}}) \qquad \text{(trace-free Neumann fields)}$$

dimension depends only on topology, Betti numbers

Grad grad - div Div Complexes (Bi-Harmonic Equation)

complex: $0 \to L^2(\Omega,\mathbb{R}) \to L^2(\Omega,\mathbb{R}^{3\times3}_{\text{sym}}) \to L^2(\Omega,\mathbb{R}^{3\times3}_{\text{dev}}) \to L^2(\Omega,\mathbb{R}^3) \to RT_0$

$$\xrightarrow{0} D(\mathring{\text{Grad}}\,\text{grad}) \xrightarrow{\mathring{\text{Grad}}\,\text{grad}} D(\mathring{\text{Rot}}_{\mathbb{S}}) \xrightarrow{\mathring{\text{Rot}}_{\mathbb{S}}} D(\mathring{\text{Div}}_{\mathbb{T}}) \xrightarrow{\mathring{\text{Div}}_{\mathbb{T}}} L^2(\Omega) \xrightarrow{\pi_{RT_0}} RT_0$$

dual or adjoint complex

$$\xleftarrow{0} L^2(\Omega) \xleftarrow{\text{div}\,\text{Div}_{\mathbb{S}}} D(\text{div}\,\text{Div}_{\mathbb{S}}) \xleftarrow{\text{sym}\,\text{Rot}_{\mathbb{T}}} D(\text{sym}\,\text{Rot}_{\mathbb{T}}) \xleftarrow{-\text{dev}\,\text{Grad}} D(\text{dev}\,\text{Grad}) \xleftarrow{\iota_{RT_0}} RT_0$$

"crucial" / "best" compact embeddings (for, e.g., Ω bounded strong Lipschitz)

$$D(\mathring{\text{Grad}}\,\text{grad}) = \mathring{H}^2 \hookrightarrow\!\!\!\!\rightarrow L^2 \qquad \text{(Rellich's selection theorem)},$$
$$D(\mathring{\text{Rot}}_{\mathbb{S}}) \cap D(\text{div}\,\text{Div}_{\mathbb{S}}) \hookrightarrow\!\!\!\!\rightarrow L^2_{\mathbb{S}} \qquad \text{(P.-Zulehner '16)},$$
$$D(\mathring{\text{Div}}_{\mathbb{T}}) \cap D(\text{sym}\,\text{Rot}_{\mathbb{T}}) \hookrightarrow\!\!\!\!\rightarrow L^2_{\mathbb{T}} \qquad \text{(P.-Zulehner '16)},$$
$$D(\text{dev}\,\text{Grad}) = H^1 \hookrightarrow\!\!\!\!\rightarrow L^2 \qquad \text{(P.-Zulehner '16, Rellich's selection theorem)}$$

$\Rightarrow$ closed complexes and finite cohomology groups
$\Rightarrow$ Helmholtz decompositions
$\Rightarrow$ Friedrichs/Poincaré type estimates
$\Rightarrow$ continuous and compact inverses of reduced operators
$\Rightarrow$ ... all from general fa-toolbox $\surd$

Grad grad - div Div Complexes (Bi-Harmonic Equation)

complex: $0 \to \mathsf{L}^2(\Omega, \mathbb{R}) \to \mathsf{L}^2(\Omega, \mathbb{R}^{3\times3}_{\mathrm{sym}}) \to \mathsf{L}^2(\Omega, \mathbb{R}^{3\times3}_{\mathrm{dev}}) \to \mathsf{L}^2(\Omega, \mathbb{R}^3) \to \mathsf{RT}_0$

$$\xrightarrow{\;0\;} D(\mathring{\operatorname{Grad}}\operatorname{grad}) \xrightarrow{\mathring{\operatorname{Grad}}\operatorname{grad}} D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \xrightarrow{\mathring{\operatorname{Rot}}_{\mathbb{S}}} D(\mathring{\operatorname{Div}}_{\mathbb{T}}) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathsf{RT}_0}} \mathsf{RT}_0$$

dual or adjoint complex

$$\xleftarrow{\;0\;} \mathsf{L}^2(\Omega) \xleftarrow{\operatorname{div}\operatorname{Div}_{\mathbb{S}}} D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \xleftarrow{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}} D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \xleftarrow{-\operatorname{dev}\operatorname{Grad}} D(\operatorname{dev}\operatorname{Grad}) \xleftarrow{\iota_{\mathsf{RT}_0}} \mathsf{RT}_0$$

e.g.: Helmholtz decompositions: $\mathsf{L}^2(\Omega) = R(\mathring{\operatorname{Div}}_{\mathbb{T}}) \oplus \mathsf{RT}_0,$

$$\mathsf{L}^2_{\mathbb{S}} = R(\mathring{\operatorname{Grad}}\operatorname{grad}) \oplus \mathcal{H}^D_{\mathbb{S}} \oplus R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}), \qquad \mathsf{L}^2_{\mathbb{T}} = R(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \oplus \mathcal{H}^N_{\mathbb{T}} \oplus R(\operatorname{dev}\operatorname{Grad})$$

e.g.: Friedrichs/Poincaré type estimates

$\forall\, u \in D(\mathring{\operatorname{Grad}}\operatorname{grad})$ $\qquad |u|_{\mathsf{L}^2} \leq c_{\mathrm{Gg}} |\operatorname{Grad}\operatorname{grad} u|_{\mathsf{L}}$

$\forall\, S \in D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \cap R(\mathring{\operatorname{Grad}}\operatorname{grad}) = D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \cap N(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \cap \mathcal{H}^{D\,\perp}_{\mathbb{S}}$ $\qquad |S|_{\mathsf{L}^2_{\mathbb{S}}} \leq c_{\mathrm{Gg}} |\operatorname{div}\operatorname{Div} S|_{\mathsf{L}^2},$

$\forall\, S \in D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \cap R(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) = D(\mathring{\operatorname{Rot}}_{\mathbb{S}}) \cap N(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \cap \mathcal{H}^{D\,\perp}_{\mathbb{S}}$ $\qquad |S|_{\mathsf{L}^2_{\mathbb{S}}} \leq c_{\mathrm{R}} |\operatorname{Rot} S|_{\mathsf{L}^2_{\mathbb{T}}},$

$\forall\, T \in D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \cap R(\mathring{\operatorname{Rot}}_{\mathbb{S}}) = D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \cap N(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap \mathcal{H}^{N\,\perp}_{\mathbb{T}}$ $\qquad |T|_{\mathsf{L}^2_{\mathbb{T}}} \leq c_{\mathrm{R}} |\operatorname{sym}\operatorname{Rot} T|_{\mathsf{L}^2_{\mathbb{S}}},$

$\forall\, T \in D(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap R(\operatorname{dev}\operatorname{Grad}) = D(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap N(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \cap \mathcal{H}^{N\,\perp}_{\mathbb{T}}$ $\qquad |T|_{\mathsf{L}^2_{\mathbb{T}}} \leq c_{\mathrm{D}} |\operatorname{Div} T|_{\mathsf{L}^2},$

$\forall\, E \in D(\operatorname{dev}\operatorname{Grad}) \cap R(\mathring{\operatorname{Div}}_{\mathbb{T}}) = D(\operatorname{dev}\operatorname{Grad}) \cap \mathsf{RT}_0^{\perp}$ $\qquad |E|_{\mathsf{L}^2} \leq c_{\mathrm{D}} |\operatorname{dev}\operatorname{Grad} E|_{\mathsf{L}^2_{\mathbb{T}}}$

Rot Rot$^\top$ Complexes (Elasticity)

complex: $0 \to \mathsf{L}^2(\Omega,\mathbb{R}^3) \to \mathsf{L}^2(\Omega,\mathbb{R}^{3\times3}_{\mathrm{sym}}) \to \mathsf{L}^2(\Omega,\mathbb{R}^{3\times3}_{\mathrm{sym}}) \to \mathsf{L}^2(\Omega,\mathbb{R}^3) \to \mathsf{RM}$

$$\xrightarrow{0} D(\operatorname{sym}\mathring{\operatorname{Grad}}) \xrightarrow{\operatorname{sym}\mathring{\operatorname{Grad}}} D(\operatorname{Rot}\mathring{\operatorname{Rot}}^\top_{\mathbb{S}}) \xrightarrow{\operatorname{Rot}\mathring{\operatorname{Rot}}^\top_{\mathbb{S}}} D(\mathring{\operatorname{Div}}_{\mathbb{S}}) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{S}}} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathsf{RM}}} \mathsf{RM}$$

dual or adjoint complex

$$\xleftarrow{0} \mathsf{L}^2(\Omega) \xleftarrow{-\operatorname{Div}_{\mathbb{S}}} D(\operatorname{Div}_{\mathbb{S}}) \xleftarrow{\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S}}} D(\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S}}) \xleftarrow{-\operatorname{sym}\operatorname{Grad}} D(\operatorname{sym}\operatorname{Grad}) \xleftarrow{\iota_{\mathsf{RM}}} \mathsf{RM}$$

complex property: "range ⊂ kernel"

$$\operatorname{Rot}\mathring{\operatorname{Rot}}^\top_{\mathbb{S}}\operatorname{sym}\mathring{\operatorname{Grad}} = 0, \qquad \mathring{\operatorname{Div}}_{\mathbb{S}}\operatorname{Rot}\mathring{\operatorname{Rot}}^\top_{\mathbb{S}} = 0, \qquad \pi_{\mathsf{RM}}\mathring{\operatorname{Div}}_{\mathbb{S}} = 0,$$
$$\operatorname{Div}_{\mathbb{S}}\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S}} = 0, \qquad \operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S}}\operatorname{sym}\operatorname{Grad} = 0, \qquad \operatorname{sym}\operatorname{Grad}\iota_{\mathsf{RM}} = 0$$

complex closed ⇔ all ranges are closed
complex exact ⇔ complex closed and all cohomology groups trivial, i.e.,

$$\mathcal{H}^D_{\mathbb{S}} := N(\operatorname{Rot}\mathring{\operatorname{Rot}}^\top_{\mathbb{S}}) \cap N(\operatorname{Div}_{\mathbb{S}}) \qquad \text{(symmetric Dirichlet fields II)},$$
$$\mathcal{H}^N_{\mathbb{S}} := N(\mathring{\operatorname{Div}}_{\mathbb{S}}) \cap N(\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S}}) \qquad \text{(symmetric Neumann fields)}$$

dimension depends only on topology, Betti numbers

$\operatorname{Rot}\operatorname{Rot}^\top$ Complexes (Elasticity)

complex: $0 \to \mathsf{L}^2(\Omega,\mathbb{R}^3) \to \mathsf{L}^2(\Omega,\mathbb{R}^{3\times 3}_{\mathrm{sym}}) \to \mathsf{L}^2(\Omega,\mathbb{R}^{3\times 3}_{\mathrm{sym}}) \to \mathsf{L}^2(\Omega,\mathbb{R}^3) \to \mathrm{RM}$

$$\xrightarrow{\;0\;} D(\operatorname{sym}\mathring{\operatorname{Grad}}) \xrightarrow{\operatorname{sym}\mathring{\operatorname{Grad}}} D(\operatorname{Rot}\mathring{\operatorname{Rot}}^\top_{\mathbb{S}}) \xrightarrow{\operatorname{Rot}\mathring{\operatorname{Rot}}^\top_{\mathbb{S}}} D(\mathring{\operatorname{Div}}_{\mathbb{S}}) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{S}}} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathrm{RM}}} \mathrm{RM}$$

dual or adjoint complex

$$\xleftarrow{\;0\;} \mathsf{L}^2(\Omega) \xleftarrow{-\operatorname{Div}_{\mathbb{S}}} D(\operatorname{Div}_{\mathbb{S}}) \xleftarrow{\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S}}} D(\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S}}) \xleftarrow{-\operatorname{sym}\operatorname{Grad}} D(\operatorname{sym}\operatorname{Grad}) \xleftarrow{\iota_{\mathrm{RM}}} \mathrm{RM}$$

"crucial" / "best" compact embeddings (for, e.g., Ω bounded strong Lipschitz)

$$\begin{aligned}
D(\operatorname{sym}\mathring{\operatorname{Grad}}) = \mathring{\mathsf{H}}^1 &\hookrightarrow\!\!\!\!\to \mathsf{L}^2 && \text{(Rellich's selection theorem)},\\
D(\operatorname{Rot}\mathring{\operatorname{Rot}}^\top_{\mathbb{S}}) \cap D(\operatorname{Div}_{\mathbb{S}}) &\hookrightarrow\!\!\!\!\to \mathsf{L}^2_{\mathbb{S}} && \text{(P.-Zulehner '17)},\\
D(\mathring{\operatorname{Div}}_{\mathbb{S}}) \cap D(\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S}}) &\hookrightarrow\!\!\!\!\to \mathsf{L}^2_{\mathbb{S}} && \text{(P.-Zulehner '17)},\\
D(\operatorname{sym}\operatorname{Grad}) = \mathsf{H}^1 &\hookrightarrow\!\!\!\!\to \mathsf{L}^2 && \text{(Rellich's selection theorem)}
\end{aligned}$$

$\Rightarrow$ closed complexes and finite cohomology groups
$\Rightarrow$ Helmholtz decompositions
$\Rightarrow$ Friedrichs/Poincaré type estimates
$\Rightarrow$ continuous and compact inverses of reduced operators
$\Rightarrow$ ...all from general fa-toolbox $\surd$

Rot Rot$^\top$ Complexes (Elasticity)

complex: $0 \to \mathsf{L}^2(\Omega,\mathbb{R}^3) \to \mathsf{L}^2(\Omega,\mathbb{R}^{3\times3}_{\mathrm{sym}}) \to \mathsf{L}^2(\Omega,\mathbb{R}^{3\times3}_{\mathrm{sym}}) \to \mathsf{L}^2(\Omega,\mathbb{R}^3) \to \mathsf{RM}$

$$\xrightarrow{\;0\;} D(\operatorname{sym}\mathring{\operatorname{Grad}}) \xrightarrow{\operatorname{sym}\mathring{\operatorname{Grad}}} D(\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^\top) \xrightarrow{\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^\top} D(\mathring{\operatorname{Div}}_{\mathbb{S}}) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{S}}} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathsf{RM}}} \mathsf{RM}$$

dual or adjoint complex

$$\xleftarrow{\;0\;} \mathsf{L}^2(\Omega) \xleftarrow{-\operatorname{Div}_{\mathbb{S}}} D(\operatorname{Div}_{\mathbb{S}}) \xleftarrow{\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^\top} D(\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^\top) \xleftarrow{-\operatorname{sym}\operatorname{Grad}} D(\operatorname{sym}\operatorname{Grad}) \xleftarrow{\iota_{\mathsf{RM}}} \mathsf{RM}$$

e.g.: Helmholtz decompositions: $\mathsf{L}^2(\Omega) = R(\mathring{\operatorname{Div}}_{\mathbb{S}}) \oplus \mathsf{RM}$,

$$\mathsf{L}^2_{\mathbb{S}} = R(\operatorname{sym}\mathring{\operatorname{Grad}}) \oplus \mathcal{H}^D_{\mathbb{S}} \oplus R(\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^\top), \qquad \mathsf{L}^2_{\mathbb{S}} = R(\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^\top) \oplus \mathcal{H}^N_{\mathbb{S}} \oplus R(\operatorname{sym}\operatorname{Grad})$$

e.g.: Friedrichs/Poincaré type estimates

$\forall\, E \in D(\operatorname{sym}\mathring{\operatorname{Grad}})$ $\quad |E|_{\mathsf{L}^2} \le c_{\mathsf{sG}} |\operatorname{sym}\operatorname{Grad} E|_{\mathsf{L}^2_{\mathbb{S}}}$

$\forall\, S \in D(\operatorname{Div}_{\mathbb{S}}) \cap R(\operatorname{sym}\mathring{\operatorname{Grad}}) = D(\operatorname{Div}_{\mathbb{S}}) \cap N(\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^\top) \cap \mathcal{H}^{D\,\perp}_{\mathbb{S}}$ $\quad |S|_{\mathsf{L}^2_{\mathbb{S}}} \le c_{\mathsf{sG}} |\operatorname{Div} S|_{\mathsf{L}^2}$,

$\forall\, S \in D(\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^\top) \cap R(\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^\top) = D(\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^\top) \cap N(\operatorname{Div}_{\mathbb{S}}) \cap \mathcal{H}^{D\,\perp}_{\mathbb{S}}$ $\quad |S|_{\mathsf{L}^2_{\mathbb{S}}} \le c_{\mathsf{RR}} |\operatorname{Rot}\operatorname{Rot}^\top S|_{\mathsf{L}^2_{\mathbb{S}}}$

$\forall\, S \in D(\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^\top) \cap R(\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^\top) = D(\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^\top) \cap N(\mathring{\operatorname{Div}}_{\mathbb{S}}) \cap \mathcal{H}^{N\,\perp}_{\mathbb{S}}$ $\quad |S|_{\mathsf{L}^2_{\mathbb{S}}} \le c_{\mathsf{RR}} |\operatorname{Rot}\operatorname{Rot}^\top S|_{\mathsf{L}^2_{\mathbb{S}}}$

$\forall\, S \in D(\mathring{\operatorname{Div}}_{\mathbb{S}}) \cap R(\operatorname{sym}\operatorname{Grad}) = D(\mathring{\operatorname{Div}}_{\mathbb{S}}) \cap N(\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^\top) \cap \mathcal{H}^{N\,\perp}_{\mathbb{S}}$ $\quad |S|_{\mathsf{L}^2_{\mathbb{S}}} \le c_{\mathsf{D}} |\operatorname{Div} S|_{\mathsf{L}^2}$,

$\forall\, E \in D(\operatorname{sym}\operatorname{Grad}) \cap R(\mathring{\operatorname{Div}}_{\mathbb{S}}) = D(\operatorname{sym}\operatorname{Grad}) \cap \mathsf{RM}^\perp$ $\quad |E|_{\mathsf{L}^2} \le c_{\mathsf{D}} |\operatorname{sym}\operatorname{Grad} E|_{\mathsf{L}^2_{\mathbb{S}}}$