

Traces for Hilbert Complexes

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Traces for Hilbert Complexes

ToC

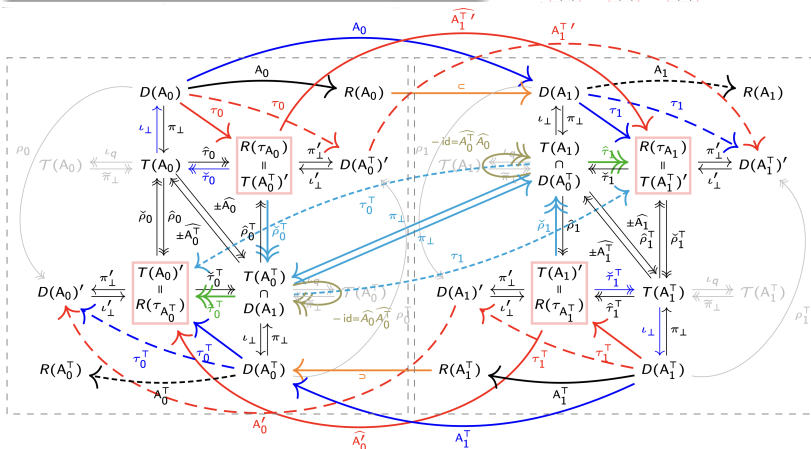
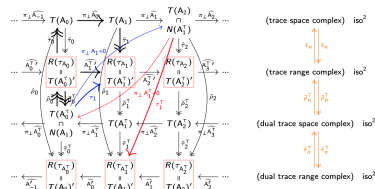
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Traces for Hilbert Complexes

Question: Why are traces so complicated?

Question: What is $\tilde{H}_{\parallel/\perp/0(0)}^{\pm 1/2}(\text{div}_\Gamma / \text{rot}_\Gamma, \Gamma)$?

... some answers below ...



Traces for Hilbert Complexes

OVERVIEW and BASIC IDEAS

paper in JFA 2023:

R. Hiptmair, D. Pauly, and E. Schulz: *Traces for Hilbert Complexes*

? Traces ?

Traces without any regularity of the domain?

Is this even possible?

even better:

? Traces ?

Traces without domains (or boundaries)?

Traces

$$A : D(A) \subset H_0 \rightarrow H_1$$

lddc: lin, dedef, cl

Traces for $D(A)$?

$\Omega \subset \mathbb{R}^N$ Lipschitz:

very classical

$$D(A) = H^1 \text{ or } W^{1,p},$$

scalar trace $u_s = u|_\Gamma$

classical (we stay in Hilbert spaces)

$$D(A) = H(\text{rot}) \text{ or } H(\text{div}), \quad \text{tan or nor traces } v_t = (\nu \times v \times \nu)|_\Gamma, \quad v_n = (\nu \cdot v)|_\Gamma$$

more recent (BGG, zoo of complexes)

$$D(A) = H(\text{Rot}^\top \text{Rot}_\mathbb{S}), H(\text{div Div}_\mathbb{S}), H(\text{sym Rot}_\mathbb{T}), \dots$$

$$\dots H(\text{Rot Rot Rot}), H(\text{rot Div}), H(\text{Grad rot}) \dots$$

traces?

$$A : D(A) \subset H_0 \rightarrow H_1 \quad \text{lddc}$$

Traces for $D(A)$?

$\Omega \subset \mathbb{R}^N$ Lipschitz

What if less regularity? What if

- Ω just open / no regularity and $D(A) = H^1(\Omega), H(\text{rot}, \Omega), H(\text{div}, \Omega), \dots$?
- no Ω at all, just $D(A)$?

Traces & Literature

Trace for $D(A)$ where $A : D(A) \subset H_0 \rightarrow H_1$

Iddc

What is known?

well known/basic idea:

$$D(\tau_A)/N(\tau_A) = D(A)/D(\mathring{A}) = \mathcal{T}(A) \cong T(A) = D(\mathring{A})^{\perp D(A)}$$

not so well known:

$$T(A)' \cong_{\hat{\rho}_A} \boxed{T(A) = N(A^\top A + 1)} \cong_{\hat{\tau}_A} T(A^\top)' = R(\tau_A) = D(A^*)^\circ$$

- Friedrichs (Friedrichs systems): 1958 (S skew-sym, i.e., $S + S^\top = 0$)
- Ern, Guermont (Aubin): 2006, 2007 (S alm skew-sym, i.e., $|(S + S^\top)x|_H \leq c|x|_H$)
- Picard, Trostorff: 2024 (S skew-sym)

theory: bc $\Rightarrow$ well-posedness / skew-selfadjoint extensions / accretive extensions

connection to our setting: $S = \begin{bmatrix} 0 & -A^\top \\ A & 0 \end{bmatrix}$ but no complexes, $S^2 = \begin{bmatrix} -A^\top A & 0 \\ 0 & -A A^\top \end{bmatrix}$

maybe not known (at all?):

connection to Hilbert complexes

connection to trace complexes

NOT!

$$\boxed{\tau_{A_1} A_0 = A_0^\top{}' \tau_{A_0}}$$

BUT!
index shift

$$\boxed{\tau_{A_1} A_0 = -A_1^\top{}' \tau_{A_0}}$$

surface differential operators

compact embeddings

... dual representations by regular potentials and continuous extensions

Traces

$$A : D(A) \subset H_0 \rightarrow H_1$$

lddc

$$A^* : D(A^*) \subset H_1 \rightarrow H_0$$

lddc, Hilbert space adjoint

Traces for $D(A)$?

basic idea: integration by parts / extension of adjoints

$$\forall x \in D(A) \quad \forall y \in D(A^*)$$

$$\langle y, Ax \rangle_{H_1} - \langle A^* y, x \rangle_{H_0} = 0$$

think of $A = \mathring{\text{grad}} : D(A) = \mathring{H}^1 \subset L^2 \rightarrow L^2$

and $A^* = -\text{div} : D(A^*) = H(\text{div}) \subset L^2 \rightarrow L^2$

$$\langle y, \mathring{\text{grad}} x \rangle_{L^2} + \langle \text{div } y, x \rangle_{L^2} = 0$$

Traces

$$\mathring{A} \subset A$$

Iddc

$$A^* \subset A^\top := \mathring{A}^* \quad (A^\top \text{ formal transpose of } A)$$

Iddc, Hilbert space adjoints

Traces for $D(A)$?

basic idea and setting: integration by parts / extension of adjoints

$$\exists x \in D(A) \quad \exists y \in D(A^\top)$$

$$\langle y, Ax \rangle_{H_1} - \langle A^\top y, x \rangle_{H_0} \neq 0$$

think of $\mathring{\text{grad}} = \mathring{A} \subset A = \text{grad}$

$$D(\mathring{\text{grad}}) = \mathring{H}^1 \subset D(\text{grad}) = H^1$$

and $-\mathring{\text{div}} = \text{grad}^* = A^* \subset A^\top = \mathring{A}^* = \mathring{\text{grad}}^* = -\mathring{\text{div}}$

$$D(\mathring{\text{div}}) = \mathring{H}(\text{div}) \subset D(\text{div}) = H(\text{div})$$

$$\langle y, \text{grad } x \rangle_{L^2(\Omega)} + \langle \mathring{\text{div}} y, x \rangle_{L^2(\Omega)} = \underbrace{\langle y_n, x_s \rangle_{L^2(\Gamma)}}_{\neq 0}$$

$$= \left\langle \int_\Gamma y_n x_s \right\rangle = \langle \langle y_n, x_s \rangle \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}$$

for some $x \in H^1$, $y \in H(\text{div})$

For simplicity of this talk: real Hilbert spaces

Traces

$$\mathring{A} \subset A$$

$$A^* \subset A^\top = \mathring{A}^*$$

$(\mathring{A}, A^*)$ pair “with” boundary conditions

$(A, A^\top)$ pair “without” boundary conditions

lddc

lddc, Hilbert space adjoints

$$(A, A^*), (\mathring{A}, A^\top = \mathring{A}^*)$$

dual/adjoint pairs

Traces for $D(A)$?

basic idea and setting: integration by parts / extension of adjoints

bd trace

$$\tau_A : D(A) \rightarrow D(A^\top)', \quad \tau_A x(y) := \langle y, Ax \rangle_{H_1} - \langle A^\top y, x \rangle_{H_0}$$

$$x \mapsto \tau_A x$$

$$x \in D(A), y \in D(A^\top)$$

bd dual trace

$$\tau_{A^\top} : D(A^\top) \rightarrow D(A)', \quad \tau_{A^\top} y(x) := \langle x, A^\top y \rangle_{H_0} - \langle Ax, y \rangle_{H_1}$$

$$y \mapsto \tau_{A^\top} y$$

note

$$\tau_{A^\top} y(x) = -\tau_A x(y)$$

$$\text{skew-symmetric structure } S = \begin{bmatrix} 0 & -A^\top \\ A & 0 \end{bmatrix}$$

equivalently **bilinear form** on $D(A) \times D(A^\top)$ resp. $D(A^\top) \times D(A)$

$$\langle\langle x, y \rangle\rangle := \tau_A x(y) = -\tau_{A^\top} y(x) = \langle y, Ax \rangle_{H_1} - \langle A^\top y, x \rangle_{H_0} = \left\langle S \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle_{H_0 \times H_1}$$

Hilbert Complexes

Traces for Hilbert Complexes

- We give Traces for Hilbert Complexes.
- On the other hand Hilbert Complexes are necessary for Traces.

Traces for Single Operators

Traces for Single Operators

Traces for Single Operators (and Adjoints)

$$\mathring{A} \subset A \quad \text{and} \quad A^* \subset A^\top = \mathring{A}^* \text{ lddc} \quad (\text{Hilbert space adjoints})$$

Traces for $D(A)$ and $D(A^\top)$ — traces come always in pairs

$$\tau_A x(y) = \langle y, Ax \rangle_{H_1} - \langle A^\top y, x \rangle_{H_0}$$

$$\text{primal / dual traces} \quad \tau_A : D(A) \rightarrow D(A^\top)', \quad \tau_{A^\top} : D(A^\top) \rightarrow D(A)'$$

$$\text{primal / dual adjoint traces} \quad \tau'_A : D(A^\top)'' \rightarrow D(A)', \quad \tau'_{A^\top} : D(A)'' \rightarrow D(A^\top)'$$

note: Hilbert spaces $H \stackrel{\text{here}}{=} D(A) \vee D(A^\top)$ are self-dual (Riesz) and reflexive
 $\Rightarrow$ isometric isomorphisms $\rho_H : H \rightarrow H'$ and $\iota_d : H \rightarrow H''$

Theorem (kernels, boundedness, and adjoints)

- $N(\tau_A) = D(\mathring{A})$ and $N(\tau_{A^\top}) = D(A^*)$ and $\|\tau_A\|, \|\tau_{A^\top}\| \leq 1$
- $\tau'_A \iota_d = -\tau_{A^\top}$ and $\tau'_{A^\top} \iota_d = -\tau_A$

Remark

$$\overline{R(\tau_{A^\top})} = \overline{R(\tau'_A)} = N(\tau_A)^\circ = D(\mathring{A})^\circ \text{ and } \overline{R(\tau_A)} = D(A^*)^\circ$$

Traces for Single Operators (Riesz Isometric Isometries)

$$\mathring{A} \subset A \quad \text{and} \quad A^* \subset A^\top = \mathring{A}^* \text{ lddc} \quad (\text{Hilbert space adjoints})$$

Let $x \in D(A)$. What is / solves

$$\check{y} := -\rho_{D(A^\top)}^{-1} \tau_A x \in D(A^\top) \quad \text{and} \quad \check{x} := A^\top \check{y} ?$$

Lemma (extension / right inverse)

$$(\check{x}, \check{y}) \in N(A^\top A + 1) \times N(AA^\top + 1) \quad \text{and} \quad \check{x} - x \in D(\mathring{A}) = N(\tau_A)$$

$$\Rightarrow \boxed{\tau_A A^\top \check{y} = \tau_A \check{x} = \tau_A x}$$

$$\Rightarrow \boxed{-\tau_A A^\top \rho_{D(A^\top)}^{-1} = \text{id}_{R(\tau_A)}} \quad \Rightarrow \quad -A^\top \rho_{D(A^\top)}^{-1} \text{ right inverse of } \tau_A \text{ on } R(\tau_A)$$

$$\text{note} \quad (S+1) \begin{bmatrix} \check{x} \\ \check{y} \end{bmatrix} = 0 \quad \text{with} \quad S = \begin{bmatrix} 0 & -A^\top \\ A & 0 \end{bmatrix} \quad (\text{formally skew-symmetric})$$

Traces for Single Operators (Riesz Isometric Isometries)

$$\mathring{A} \subset A \quad \text{and} \quad A^* \subset A^\top = \mathring{A}^* \text{ lddc}$$

(Hilbert space adjoints)

Lemma (extensions / right inverses)

$$-\tau_A A^\top \rho_{D(A^\top)}^{-1} = \text{id}_{R(\tau_A)} \quad \text{and} \quad \check{\tau}_A := -A^\top \rho_{D(A^\top)}^{-1} \text{ right inverse of } \tau_A \text{ on } R(\tau_A)$$

$$-\tau_{A^\top} A \rho_{D(A)}^{-1} = \text{id}_{R(\tau_{A^\top})} \quad \text{and} \quad \check{\tau}_{A^\top} := -A \rho_{D(A)}^{-1} \text{ right inverse of } \tau_{A^\top} \text{ on } R(\tau_{A^\top})$$

Definition (extensions / right inverses)

Let $\phi \in R(\tau_A)$ and $\psi \in R(\tau_{A^\top})$. We call:

- $\check{\phi} = -\rho_{D(A^\top)}^{-1} \phi \in N(AA^\top + 1)$ harm Neumann ext of ϕ since $\tau_A A^\top \check{\phi} = \phi$
- $\check{\check{\phi}} = A^\top \check{\phi} = -A^\top \rho_{D(A^\top)}^{-1} \phi \in N(A^\top A + 1)$ harm Dirichlet ext of ϕ since $\tau_A \check{\check{\phi}} = \phi$
- $\check{\psi} = -\rho_{D(A)}^{-1} \psi \in N(A^\top A + 1)$ harm Neumann ext of ψ since $\tau_{A^\top} A \check{\psi} = \psi$
- $\check{\check{\psi}} = A \check{\psi} = -A \rho_{D(A)}^{-1} \psi \in N(AA^\top + 1)$ harm Dirichlet ext of ψ since $\tau_{A^\top} \check{\check{\psi}} = \psi$

Traces for Single Operators

$$\mathring{A} \subset A \quad \text{and} \quad A^* \subset A^\top = \mathring{A}^* \quad (\text{Iddc})$$

Theorem (kernels, ranges = annihilators)

- $N(\tau_A) = D(\mathring{A})$
- $R(\tau_A) = D(A^*)^\circ = \{\Phi \in D(A^\top)' : D(A^*) \subset N(\Phi)\}$
- $N(\tau_{A^\top}) = D(A^*)$
- $R(\tau_{A^\top}) = D(\mathring{A})^\circ = \{\Phi \in D(A)' : D(\mathring{A}) \subset N(\Phi)\}$

In particular, the kernels and ranges are closed.

Definition and Lemma (trace spaces)

- $T(A) := D(\mathring{A})^{\perp D(A)} = N(A^\top A + 1) \cong D(\tau_A)/N(\tau_A) = D(A)/D(\mathring{A}) =: \mathcal{T}(A)$
- $T(A^\top) := D(A^*)^{\perp D(A^\top)} = N(AA^\top + 1) \cong D(\tau_{A^\top})/N(\tau_{A^\top}) = D(A^\top)/D(A^*) =: \mathcal{T}(A^\top)$

$\Rightarrow$ red traces $\hat{\tau}_A := \tau_A|_{T(A)} : T(A) \rightarrow R(\tau_A)$

$\hat{\tau}_{A^\top} := \tau_{A^\top}|_{T(A^\top)} : T(A^\top) \rightarrow R(\tau_{A^\top})$

Theorem (ranges and trace isometries)

$$R(\hat{\tau}_A) = R(\tau_A) = D(A^*)^\circ = \rho_{D(A^\top)} T(A^\top) = T(A^\top)'$$

$$R(\hat{\tau}_{A^\top}) = R(\tau_{A^\top}) = D(\mathring{A})^\circ = \rho_{D(A)} T(A) = T(A)'$$

The reduced traces are isometric isomorphisms.

Traces for Single Operators

$$\check{A} \subset A \quad \text{and} \quad A^* \subset A^\top = \check{A}^* \quad (\text{Iddc})$$

Remark (trace / Riesz isometric isomorphisms $\rightarrow$)

$$\begin{aligned} \tau_A : D(A) &\rightarrow R(\tau_A) \subset D(A^\top)', & \rho_A &:= \rho_{D(A)} : D(A) \rightarrow D(A)' \\ \tau_{A^\top} : D(A^\top) &\rightarrow R(\tau_{A^\top}) \subset D(A)', & \rho_{A^\top} &:= \rho_{D(A^\top)} : D(A^\top) \rightarrow D(A^\top)' \\ \hat{\tau}_A = \tau_A|_{T(A)} : T(A) &\twoheadrightarrow R(\tau_A) = T(A^\top)', & \hat{\rho}_A &:= \rho_A|_{T(A)} : T(A) \rightarrow T(A)' \\ \hat{\tau}_{A^\top} = \tau_{A^\top}|_{T(A^\top)} : T(A^\top) &\twoheadrightarrow R(\tau_{A^\top}) = T(A)', & \hat{\rho}_{A^\top} &:= \rho_{A^\top}|_{T(A^\top)} : T(A^\top) \rightarrow T(A^\top)' \end{aligned}$$

Lemma (trace / Riesz isometric isomorphisms $\rightarrow$)

$$\begin{aligned} R(\tau_A) = R(\hat{\tau}_A) = R(\hat{\rho}_{A^\top}) &= T(A^\top)', & \hat{\tau}_A : T(A) &\twoheadrightarrow T(A^\top)', & \hat{\rho}_{A^\top} : T(A^\top) &\twoheadrightarrow T(A^\top)' \\ R(\tau_{A^\top}) = R(\hat{\tau}_{A^\top}) = R(\hat{\rho}_A) &= T(A)', & \hat{\tau}_{A^\top} : T(A^\top) &\twoheadrightarrow T(A)', & \hat{\rho}_A : T(A) &\twoheadrightarrow T(A)' \end{aligned}$$

Definition (inverses of trace / Riesz isometric isomorphisms $\rightarrow$)

$$\begin{aligned} \check{\tau}_A &:= \hat{\tau}_A^{-1} : T(A^\top)' \twoheadrightarrow T(A), & \check{\rho}_{A^\top} &:= \hat{\rho}_{A^\top}^{-1} : T(A^\top)' \twoheadrightarrow T(A^\top) \\ \check{\tau}_{A^\top} &:= \hat{\tau}_{A^\top}^{-1} : T(A)' \twoheadrightarrow T(A^\top), & \check{\rho}_A &:= \hat{\rho}_A^{-1} : T(A)' \twoheadrightarrow T(A) \end{aligned}$$

Remark

Continuity of traces and extensions for free!

(no ass on $R(A)$ or domains Ω)



EVEN ISOMETRIES



Traces for Single Operators

$$\mathring{A} \subset A \quad \text{and} \quad A^* \subset A^\top = \mathring{A}^* \quad (\text{Iddc})$$

Remark (trace / Riesz isometric isomorphisms $\rightarrow$)

Continuity/isometry of traces and extensions for free! (no ass on $R(A)$ or domains Ω)



NO compact embeddings or Friedrichs/Poincaré type estimates NEEDED



We do NOT need:

e.g.: $\Omega \subset \mathbb{R}^3$ bd, weak Lip

(compact embeddings)

(Weck 1972/'74, Weber 1980, Picard 1984)

$$H^1(\Omega) \hookleftrightarrow L^2(\Omega) \quad \text{or} \quad \mathring{H}(\text{rot}, \Omega) \cap H(\text{div}, \Omega) \hookleftrightarrow L^2(\Omega)$$

or even weaker

(Friedrichs/Poincaré type estimate $\Leftrightarrow$ closed range)

Ω (weak Lip, bd in **ONE** direction) $\Rightarrow$ $R(\mathring{\text{grad}})$ and $R(\text{div})$ closed

Ω (weak Lip, bd in **TWO** directions) $\Rightarrow$ $R(\mathring{\text{rot}})$ and $R(\text{rot})$ closed

Ω (weak Lip, bd in **THREE** directions) $\Rightarrow$ $R(\mathring{\text{div}})$ and $R(\text{grad})$ closed

Traces for Single Operators

$$\mathring{A} \subset A \quad \text{and} \quad A^* \subset A^\top = \mathring{A}^* \quad (\text{Iddc})$$

Theorem (trace /Riesz isometric isomorphisms $\rightarrow$)

$$T(A)' \cong_{\hat{\rho}_A} T(A) \cong_{\hat{\tau}_A} T(A^\top)', \quad T(A^\top)' \cong_{\hat{\rho}_{A^\top}} T(A^\top) \cong_{\hat{\tau}_{A^\top}} T(A)'$$

bilinear (sesquilinear) forms on $T(A) \times T(A^\top)$ or $D(A) \times D(A^\top)$

$$\begin{aligned} \langle\langle x, y \rangle\rangle &:= \langle\langle x, y \rangle\rangle_\tau := \tau_A x(y) = -\tau_{A^\top} y(x) = \langle Ax, y \rangle_{H_1} - \langle x, A^\top y \rangle_{H_0}, \\ \langle\langle x, y \rangle\rangle_\rho &:= \rho_A x(y) = \langle x, y \rangle_{D(A)} = \langle x, y \rangle_{H_0} + \langle Ax, Ay \rangle_{H_1} \end{aligned}$$

Corollary (“integration by parts”)

$$\langle Ax, y \rangle_{H_1} = \langle x, A^\top y \rangle_{H_0} + \langle\langle x, y \rangle\rangle$$

Traces for Single Operators

$$\mathring{A} \subset A \quad \text{and} \quad A^* \subset A^\top = \mathring{A}^* \quad (\text{Iddc})$$

Isometric Isomorphisms ($\rightarrow$)

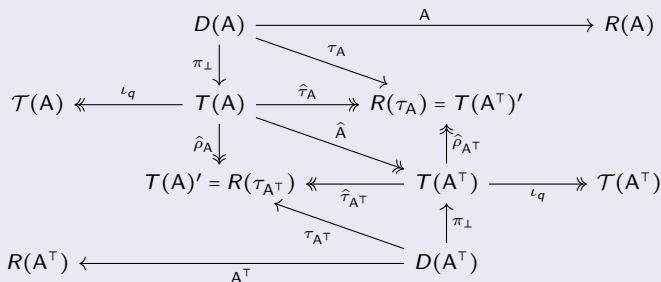
$$\begin{array}{ccccc}
 & D(A) & & & \\
 & \downarrow \pi_\perp & & & \\
 \mathcal{T}(A) & \xleftarrow{\iota_q} T(A) & \xrightarrow{\hat{\tau}_A} & R(\tau_A) = T(A^\top)' & \\
 & \downarrow \hat{\rho}_A & & \uparrow \hat{\rho}_{A^\top} & \\
 & T(A)' = R(\tau_{A^\top}) & \xleftarrow{\hat{\tau}_{A^\top}} & T(A^\top) & \xrightarrow{\iota_q} \mathcal{T}(A^\top) \\
 & & & \uparrow \pi_\perp & \\
 & & & D(A^\top) &
 \end{array}$$

$$D(A) = D(\mathring{A}) \oplus_{D(A)} T(A)$$

$$[x_\perp] = [x] \quad \text{and} \quad \tau_A x_\perp = \tau_A x = \tau_A [x]$$

Traces for Single Operators

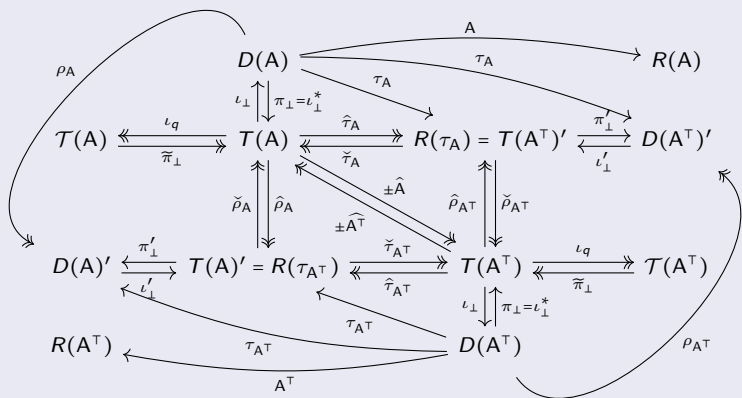
$$\mathring{A} \subset A \quad \text{and} \quad A^* \subset A^\top = \mathring{A}^* \quad (\text{Iddc})$$

Isometric Isomorphisms ($\rightarrow$)

$$\hat{A} := A|_{T(A)}$$

Traces for Single Operators

$$\mathring{A} \subset A \quad \text{and} \quad A^* \subset A^\top = \mathring{A}^* \quad (\text{Iddc})$$

Isometric Isomorphisms ($\rightarrow$)

“on $T(A) = N(A^\top A + 1)$ and $T(A^\top) = N(AA^\top + 1)$ ”:

$$\begin{aligned} \check{\tau}_A &= -\hat{A}^\top \check{\rho}_{A^\top} & \hat{\tau}_A &= \hat{\rho}_{A^\top} \hat{A} & \hat{A}^{-1} &= -\hat{A}^\top & \hat{A} &= A|_{T(A)} \\ \check{\tau}_{A^\top} &= -\hat{A} \check{\rho}_A & \hat{\tau}_{A^\top} &= \hat{\rho}_A \hat{A}^\top & (\hat{A}^\top)^{-1} &= -\hat{A} & \hat{A}^\top &= A^\top|_{T(A^\top)} \end{aligned}$$

Traces for Single Operators

$$\mathring{A} \subset A \quad \text{and} \quad A^* \subset A^\top = \mathring{A}^* \quad (\text{Iddc})$$

Theorem (kernels and ranges of traces / isometric isomorphisms)

- $N(\tau_A) = D(\mathring{A})$
- $N(\tau_{A^\top}) = D(A^*)$
- $R(\tau_A) = R(\hat{\tau}_A) = D(A^*)^\circ = R(\hat{\rho}_{A^\top}) = T(A^\top)'$
- $R(\tau_{A^\top}) = R(\hat{\tau}_{A^\top}) = D(\mathring{A})^\circ = R(\hat{\rho}_A) = T(A)'$
- $T(A) = D(\mathring{A})^{\perp D(A)} = N(A^\top A + 1)$
- $T(A^\top) = D(A^*)^{\perp D(A^\top)} = N(AA^\top + 1)$
- $T(A) \cong \mathcal{T}(A) = D(A)/D(\mathring{A})$
- $T(A^\top) \cong \mathcal{T}(A^\top) = D(A^\top)/D(A^*)$

Remark (summary)

- trace ranges are annihilators of trace kernels
- trace ranges are duals of reduced transpose trace spaces
- trace spaces are kernels $N(A^\top A + 1)$ and $N(AA^\top + 1)$ (“harmonic fields”)
- trace spaces are orthogonal complements of trace kernels (—)
- trace spaces are minimal norm extensions (—)
- trace spaces are quotient spaces of trace kernels

note:

- elements of the trace spaces are “smooth”
- regularity is never a problem (regularity not a good term)
- integrability is always the problem

Traces for Hilbert Complexes

so far NO (Hilbert) complexes

Traces for Hilbert Complexes

Traces for Hilbert Complexes

Traces for Hilbert Complexes

Different Hilbert Complexes

$$\cdots \begin{array}{c} \xrightarrow{\cdots} \\ \xleftarrow{\cdots} \end{array} H_0 \begin{array}{c} \xrightarrow{\check{A}_0} \\ \xleftarrow{A_0^\top = \check{A}_0^*} \end{array} H_1 \begin{array}{c} \xrightarrow{\check{A}_1} \\ \xleftarrow{A_1^\top = \check{A}_1^*} \end{array} H_2 \begin{array}{c} \xrightarrow{\cdots} \\ \xleftarrow{\cdots} \end{array} \cdots \quad (\text{unbd prim/dual HilComs})$$

$$\cdots \begin{array}{c} \xrightarrow{\cdots} \\ \xleftarrow{\cdots} \end{array} H_0 \begin{array}{c} \xrightarrow{A_0} \\ \xleftarrow{A_0^*} \end{array} H_1 \begin{array}{c} \xrightarrow{A_1} \\ \xleftarrow{A_1^*} \end{array} H_2 \begin{array}{c} \xrightarrow{\cdots} \\ \xleftarrow{\cdots} \end{array} \cdots \quad (\text{unbd prim/dual HilComs})$$

$$\cdots \xrightarrow{\cdots} D(A_0) \xrightarrow{A_0} D(A_1) \xrightarrow{A_1} D(A_2) \xrightarrow{\cdots} \cdots \quad (\text{bd DomCom})$$

$$\cdots \xleftarrow{\cdots} D(A_0)' \xleftarrow{A_0'} D(A_1)' \xleftarrow{A_1'} D(A_2)' \xleftarrow{\cdots} \cdots \quad (\text{bd adjoint DomCom})$$

$$\cdots \xleftarrow{\cdots} D(A_{-1}^\top) \xleftarrow{A_0^\top} D(A_0^\top) \xleftarrow{A_1^\top} D(A_1^\top) \xleftarrow{\cdots} \cdots \quad (\text{bd DomCom})$$

$$\cdots \xrightarrow{\cdots} D(A_{-1}^\top)' \xrightarrow{A_0^{\top'}} D(A_0^{\top'}) \xrightarrow{A_1^{\top'}} D(A_1^{\top'}) \xrightarrow{\cdots} \cdots \quad (\text{bd adjoint DomCom})$$

setting

$$\bullet \check{A}_\ell \in A_\ell \quad \text{and} \quad A_\ell^* \subset A_\ell^\top = \check{A}_\ell^* \quad (\text{lddc})$$

$$\bullet R(\check{A}_0) \subset N(\check{A}_1), \quad R(A_0) \subset N(A_1) \quad (\text{prim HilComs})$$

$$\bullet R(A_1^*) \subset N(A_0^*), \quad R(A_1^\top) \subset N(A_0^\top) \quad (\text{dual HilComs})$$

$$\bullet R(A_1') \subset N(A_0'), \quad R(A_0^{\top'}) \subset N(A_1^{\top'}) \quad (\text{adjoint HilComs})$$

note

$$\begin{aligned} & A_n^{\top'} : D(A_{n-1}^\top)' \rightarrow D(A_n^\top)' \\ \Rightarrow & A_n^{\top'} \not\subseteq A_n : D(A_n) \rightarrow D(A_{n+1}) \quad \text{but} \quad A_n^{\top'} \cong A_{n-1} : D(A_{n-1}) \rightarrow D(A_n) \quad (\text{index shift}) \end{aligned}$$

Traces for Hilbert Complexes

$$\mathring{A}_\ell \subset A_\ell \quad \text{and} \quad R(\dots) \subset N(\dots)$$

$$A_0 : D(A_0) \rightarrow D(A_1)$$

$$\tau_{A_0} : D(A_0) \rightarrow D(A_0^\top)'$$

$$A_1^\top{}' : D(A_0^\top)'\rightarrow D(A_1^\top)'$$

$$A_1^\top : D(A_1^\top) \rightarrow D(A_0^\top)$$

$$\tau_{A_1^\top} : D(A_1^\top) \rightarrow D(A_1)'$$

$$A_0' : D(A_1)'\rightarrow D(A_0)'$$

(vol diff ops)

(trace ops)

(surf diff ops)

Theorem (surface differential operators / commutators with traces)

$$\tau_{A_1} A_0 = -A_1^\top{}' \tau_{A_0} \quad \text{and} \quad \tau_{A_0^\top} A_1^\top = -A_0' \tau_{A_1^\top}$$

Theorem (integration by parts ...)

$$\bullet \dots \text{on domains: } x \in D(A), y \in D(A^\top) \text{ or } x \in T(A), y \in T(A^\top) \Rightarrow \langle Ax, y \rangle_{H_1} = \langle x, A^\top y \rangle_{H_0} + \langle \langle x, y \rangle \rangle$$

$$\bullet \dots \text{on trace domains} \quad \tau_{A_1} A_0 = -A_1^\top{}' \tau_{A_0} \quad x \in D(A_0), z \in D(A_1^\top) \Rightarrow \langle \langle A_0 x, z \rangle \rangle_1 = \tau_{A_1} (A_0 x)(z) = -\tau_{A_0} (x)(A_1^\top z) = -\langle \langle x, A_1^\top z \rangle \rangle_0$$

$$\bullet \dots \text{on trace spaces} \quad \hat{\tau}_{A_1} \pi_\perp \widehat{A_0} = -\widehat{A_1^\top}{}' \pi_\perp{}' \hat{\tau}_{A_0} \quad x \in T(A_0), z \in T(A_1^\top) \Rightarrow \langle \langle \pi_\perp \widehat{A_0} x, z \rangle \rangle_1 = \hat{\tau}_{A_1} (\pi_\perp \widehat{A_0} x)(z) = -\hat{\tau}_{A_0} (x)(\pi_\perp \widehat{A_1^\top} z) = -\langle \langle x, \pi_\perp \widehat{A_1^\top} z \rangle \rangle_0$$

$$\bullet \dots \text{on trace ranges} \quad \widehat{A_1^\top}{}' = -\hat{\tau}_{A_1} \iota_d^{-1} (\widehat{A_0'})' \iota_d \check{\tau}_{A_0} \quad \varphi \in R(\tau_{A_0}), \psi \in R(\tau_{A_1^\top}) \Rightarrow \langle \langle \widehat{A_1^\top}{}' \varphi, \psi \rangle \rangle_1 = \langle \langle \check{\tau}_{A_1} \widehat{A_1^\top}{}' \varphi, \check{\tau}_{A_1^\top} \psi \rangle \rangle_1 = -\langle \langle \check{\tau}_{A_0} \varphi, \check{\tau}_{A_0^\top} \widehat{A_0'} \psi \rangle \rangle_0 = -\langle \langle \varphi, \widehat{A_0'} \psi \rangle \rangle_0$$

Traces for Hilbert Complexes

$$\dot{A}_\ell \subset A_\ell \quad \text{and} \quad R(\dots) \subset N(\dots)$$

$$A_0 : D(A_0) \rightarrow D(A_1)$$

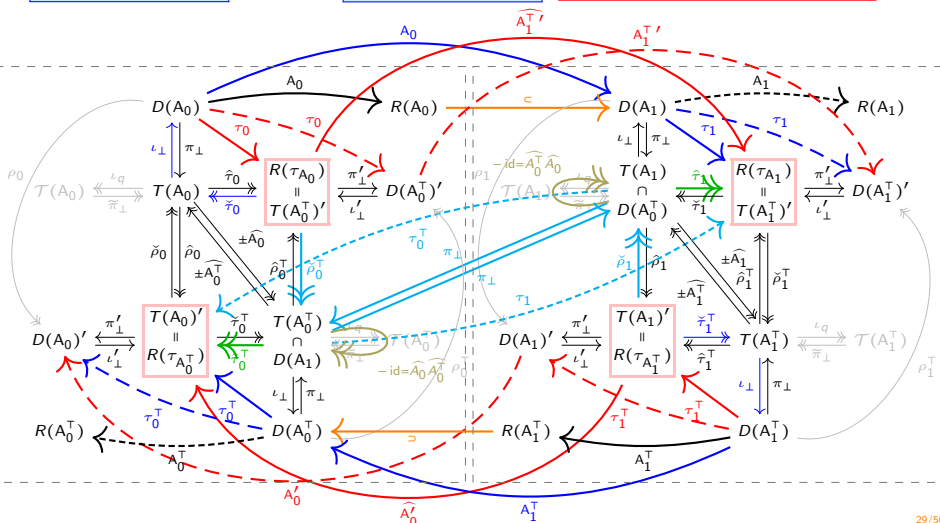
(vol diff ops)

$$A_1^\top : D(A_1^\top) \rightarrow D(A_0^\top)$$

(surf diff ops)

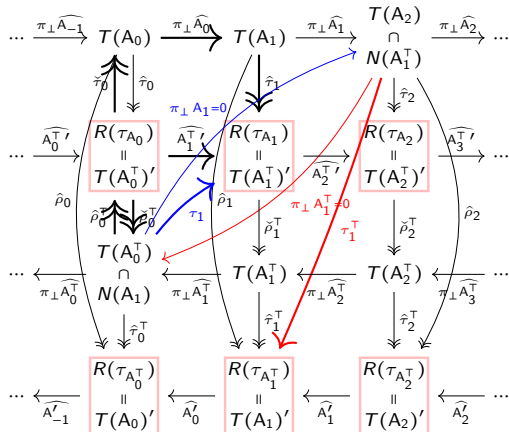
$$A_1^{\top'} : D(A_0^\top)' \rightarrow D(A_1^\top)'$$

$$A_0' : D(A_1)' \rightarrow D(A_0)'$$

**CRAZY**simple idea $\Rightarrow$ amazing complexity

Traces for Hilbert Complexes

$$\mathring{A}_\ell \subset A_\ell \quad \text{and} \quad R(\dots) \subset N(\dots)$$

(trace space complex) iso^2

$$\begin{array}{c} \uparrow \hat{\tau}_n \\ \downarrow \check{\tau}_n \end{array}$$

(trace range complex) iso^2

$$\begin{array}{c} \uparrow \hat{\rho}_n^T \\ \downarrow \check{\rho}_n^T \end{array}$$

(dual trace space complex) iso^2

$$\begin{array}{c} \uparrow \hat{\tau}_n^T \\ \downarrow \check{\tau}_n^T \end{array}$$

(dual trace range complex) iso^2

... Examples of Traces...

Examples

Example 1

De Rham Complex – ∇ / div

$$\mathring{A} \subset A$$

lddc

$$A^* \subset A^\top = \mathring{A}^*$$

 $(\mathring{A}, A^*)$ pair “with” and $(A, A^\top)$ pair “without” bc

$$\text{bd trace } \tau_A : D(A) \rightarrow D(A^\top)'$$

$$\text{with } \langle\langle x, y \rangle\rangle = \tau_A x(y) = \langle y, Ax \rangle_{H_1} - \langle A^\top y, x \rangle_{H_0} = -\tau_{A^\top} y(x)$$

classical ibp

$$\langle\langle u, E \rangle\rangle = \tau_\nabla u(E) = \langle \nabla u, E \rangle_{L^2(\Omega)} - \langle u, -\text{div } E \rangle_{L^2(\Omega)} = \langle u|_\Gamma, \nu \cdot E|_\Gamma \rangle_{L^2(\Gamma)} = -\tau_{-\text{div}} E(u)$$

$$\mathring{\nabla} : D(\mathring{\nabla}) = \mathring{H}^1(\Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega) \quad (\text{minimal gradient}) \quad \mathring{\nabla}^* = -\text{div}$$

$$\nabla : D(\nabla) = H^1(\Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega) \quad (\text{maximal gradient}) \quad \nabla^* = -\mathring{\text{div}}$$

$$\mathring{\text{div}} : D(\mathring{\text{div}}) = \mathring{H}(\text{div}, \Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega) \quad (\text{minimal divergence}) \quad \mathring{\text{div}}^* = -\nabla$$

$$\text{div} : D(\text{div}) = H(\text{div}, \Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega) \quad (\text{maximal divergence}) \quad \text{div}^* = -\mathring{\nabla}$$

$$\mathring{\nabla} = \mathring{A} \subset A = \nabla$$

$$-\mathring{\text{div}} = \nabla^* = A^* \subset A^\top = \mathring{A}^* = \mathring{\nabla}^* = -\text{div}$$

Example 1

De Rham Complex – ∇ / div

$$\mathring{A} \subset A$$

$$\text{lddc}$$

$$A^* \subset A^\top = \mathring{A}^*$$

$$(\mathring{A}, A^*) \text{ pair "with" and } (A, A^\top) \text{ pair "without" bc}$$

$$\text{bd trace } \tau_A : D(A) \rightarrow D(A^\top)'$$

$$\text{with } \langle\langle x, y \rangle\rangle = \tau_A x(y) = \langle y, Ax \rangle_{H_1} - \langle A^\top y, x \rangle_{H_0} = -\tau_{A^\top} y(x)$$

classical ibp

$$\langle\langle u, E \rangle\rangle = \tau_\nabla u(E) = \langle \nabla u, E \rangle_{L^2(\Omega)} - \langle u, -\text{div } E \rangle_{L^2(\Omega)} = \langle u|_\Gamma, \nu \cdot E|_\Gamma \rangle_{L^2(\Gamma)} = -\tau_{-\text{div}} E(u)$$

$$\mathring{\nabla} = \mathring{A} \subset A = \nabla$$

$$-\text{div} = A^* \subset A^\top = -\text{div}$$

$$\tau_\nabla : H^1(\Omega) \rightarrow H(\text{div}, \Omega)',$$

$$\tau_\nabla u(E) \stackrel{\text{class}}{=} \langle u|_\Gamma, \nu \cdot E|_\Gamma \rangle_{L^2(\Gamma)}$$

$$\tau_{-\text{div}} : H(\text{div}, \Omega) \rightarrow H^1(\Omega)',$$

$$\tau_{-\text{div}} E(u) \stackrel{\text{class}}{=} -\langle \nu \cdot E|_\Gamma, u|_\Gamma \rangle_{L^2(\Gamma)}$$

$$H(\text{div}, \Omega)' \ni \tau_\nabla u(\cdot) = \langle\langle u, \cdot \rangle\rangle \stackrel{\text{class}}{=} \langle u|_\Gamma, \nu \cdot (\cdot)|_\Gamma \rangle_{L^2(\Gamma)}$$

$$\tau_\nabla u \rightsquigarrow u|_\Gamma \text{ class scalar trace}$$

$$H^1(\Omega)' \ni \tau_{-\text{div}} E(\cdot) = -\langle\langle \cdot, E \rangle\rangle \stackrel{\text{class}}{=} -\langle \nu \cdot E|_\Gamma, \cdot|_\Gamma \rangle_{L^2(\Gamma)}$$

$$\text{class normal trace}$$

$$N(-\nabla \text{div} + 1)' = \mathring{H}(\text{div}, \Omega)^\circ = R(\tau_\nabla) \ni \tau_\nabla u \stackrel{\text{class}}{\in} H^{-1/2}(\Gamma)' \stackrel{\iota_d}{=} H^{1/2}(\Gamma)$$

$$N(-\text{div} \nabla + 1)' = \mathring{H}^1(\Omega)^\circ = R(\tau_{\text{div}}) \ni \tau_{\text{div}} E \stackrel{\text{class}}{\in} H^{1/2}(\Gamma)' = H^{-1/2}(\Gamma)$$

Example 2

De Rham Complex – rot

$$\mathring{A} \subset A$$

lddc

$$A^* \subset A^\top = \mathring{A}^*$$

 $(\mathring{A}, A^*)$ pair “with” and $(A, A^\top)$ pair “without” bc

$$\text{bd trace } \tau_A : D(A) \rightarrow D(A^\top)'$$

$$\text{with } \langle\langle x, y \rangle\rangle = \tau_A x(y) = \langle y, Ax \rangle_{H_1} - \langle A^\top y, x \rangle_{H_0} = -\tau_{A^\top} y(x)$$

$$\text{classical ibp } \langle\langle E, H \rangle\rangle = \tau_{\text{rot}} E(H) = \langle \text{rot } E, H \rangle_{L^2(\Omega)} - \langle E, \text{rot } H \rangle_{L^2(\Omega)} = \langle \nu \times \nu \times E|_\Gamma, \nu \times H|_\Gamma \rangle_{L^2(\Gamma)} = -\tau_{\text{rot}} H(E)$$

$$\text{rot} : D(\text{rot}) = \mathring{H}(\text{rot}, \Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega) \quad (\text{minimal rotation}) \quad \text{rot}^* = \text{rot}$$

$$\text{rot} : D(\text{rot}) = H(\text{rot}, \Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega) \quad (\text{maximal rotation}) \quad \text{rot}^* = \text{rot}$$

$$\text{rot} = \mathring{A} \subset A = \text{rot}$$

$$\text{rot} = \text{rot}^* = A^* \subset A^\top = \mathring{A}^* = \text{rot}^* = \text{rot}$$

Example 2

De Rham Complex – rot

$$\mathring{A} \subset A$$

$$\text{lddc}$$

$$A^* \subset A^\top = \mathring{A}^*$$

$$(\mathring{A}, A^*) \text{ pair "with" and } (A, A^\top) \text{ pair "without" bc}$$

$$\text{bd trace } \tau_A : D(A) \rightarrow D(A^\top)'$$

$$\text{with } \langle\langle x, y \rangle\rangle = \tau_A x(y) = \langle y, A x \rangle_{H_1} - \langle A^\top y, x \rangle_{H_0} = -\tau_{A^\top} y(x)$$

$$\text{classical ibp } \langle\langle E, H \rangle\rangle = \tau_{\text{rot}} E(H) = \langle \text{rot } E, H \rangle_{L^2(\Omega)} - \langle E, \text{rot } H \rangle_{L^2(\Omega)} = \langle \nu \times \nu \times E|_\Gamma, \nu \times H|_\Gamma \rangle_{L^2(\Gamma)} = -\tau_{\text{rot}} H(E)$$

$$\text{rot} = \mathring{A} \subset A = \text{rot}$$

$$\text{rot} = A^* \subset A^\top = \text{rot}$$

$$\tau_{\text{rot}} : H(\text{rot}, \Omega) \rightarrow H(\text{rot}, \Omega)', \quad \tau_{\text{rot}} E(H) \stackrel{\text{class}}{=} \langle \nu \times \nu \times E|_\Gamma, \nu \times H|_\Gamma \rangle_{L^2(\Gamma)} \stackrel{\text{class}}{=} -\tau_{\text{rot}} H(E)$$

$$H(\text{rot}, \Omega)' \ni \tau_{\text{rot}} E(\cdot) = \langle\langle E, \cdot \rangle\rangle \stackrel{\text{class}}{=} \langle \nu \times \nu \times E|_\Gamma, \nu \times (\cdot)|_\Gamma \rangle_{L^2(\Gamma)} \quad \text{class tangential trace}$$

$$H(\text{rot}, \Omega)' \ni \tau_{\text{rot}} H(\cdot) = -\langle\langle \cdot, H \rangle\rangle \stackrel{\text{class}}{=} \langle \nu \times H|_\Gamma, \nu \times (\cdot)|_\Gamma \rangle_{L^2(\Gamma)} \quad \text{cl twisted tan trace}$$

$$N(\text{rot rot} + 1)' = \mathring{H}(\text{rot}, \Omega)^\circ = R(\tau_{\text{rot}}) \ni \tau_{\text{rot}} E \stackrel{\text{class}}{\in} H_x^{-1/2}(\text{rot}_\Gamma, \Gamma) \stackrel{\iota_d}{=} H_x^{-1/2}(\text{div}_\Gamma, \Gamma)'$$

$$N(\text{rot rot} + 1)' = \mathring{H}(\text{rot}, \Omega)^\circ = R(\tau_{\text{rot}}) \ni \tau_{\text{rot}} H \stackrel{\text{class}}{\in} H_x^{-1/2}(\text{div}_\Gamma, \Gamma) = H^{-1/2}(\text{rot}_\Gamma, \Gamma)'$$

Example 3

De Rham Complex – mixed bc

$$\mathring{A} \subset A$$

Iddc

$$A^* \subset A^\top = \mathring{A}^*$$

 $(\mathring{A}, A^*)$ pair “with” and $(A, A^\top)$ pair “without” bc

$$\text{bd trace } \tau_A : D(A) \rightarrow D(A^\top)'$$

$$\text{with } \langle\langle x, y \rangle\rangle = \tau_A x(y) = \langle y, Ax \rangle_{H_1} - \langle A^\top y, x \rangle_{H_0} = -\tau_{A^\top} y(x)$$

classical ibp

$$\langle\langle u, E \rangle\rangle = \tau_\nabla u(E) = \langle \nabla u, E \rangle_{L^2(\Omega)} - \langle u, -\text{div } E \rangle_{L^2(\Omega)} = \langle u|_\Gamma, \nu \cdot E|_\Gamma \rangle_{L^2(\Gamma)} = -\tau_{-\text{div}} E(u)$$

$$\langle\langle E, H \rangle\rangle = \tau_{\text{rot}} E(H) = \langle \text{rot } E, H \rangle_{L^2(\Omega)} - \langle E, \text{rot } H \rangle_{L^2(\Omega)} = \langle \nu \times \nu \times E|_\Gamma, \nu \times H|_\Gamma \rangle_{L^2(\Gamma)} = -\tau_{\text{rot}} H(E)$$

$$\Gamma = \Gamma_0 \cup \Gamma_1$$

$$\nabla_{\Gamma_0} : D(\nabla_{\Gamma_0}) = H_{\Gamma_0}^1(\Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega) \quad (\text{gradient}) \quad \nabla_{\Gamma_0}^* = -\text{div}_{\Gamma_1}$$

$$\text{div}_{\Gamma_0} : D(\text{div}_{\Gamma_0}) = H_{\Gamma_0}(\text{div}, \Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega) \quad (\text{divergence}) \quad \text{div}_{\Gamma_0}^* = -\nabla_{\Gamma_1}$$

$$\text{rot}_{\Gamma_0} : D(\text{rot}_{\Gamma_0}) = H_{\Gamma_0}(\text{rot}, \Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega) \quad (\text{rotation}) \quad \text{rot}_{\Gamma_0}^* = \text{rot}_{\Gamma_1}$$

$$\nabla_{\Gamma_0} = \mathring{A} \subset A = \nabla$$

$$-\text{div} = \nabla^* = A^* \subset A^\top = \mathring{A}^* = \nabla_{\Gamma_0}^* = -\text{div}_{\Gamma_1}$$

$$\tau_\nabla \hat{=} (\cdot)|_{\Gamma_0} : \text{partial scalar trace on } \Gamma_0$$

$$\mathring{\nabla} = \mathring{A} \subset A = \nabla_{\Gamma_0}$$

$$-\text{div}_{\Gamma_1} = \nabla_{\Gamma_0}^* = A^* \subset A^\top = \mathring{A}^* = \mathring{\nabla}^* = -\text{div}$$

$$\tau_{\text{div}} \hat{=} \nu \cdot (\cdot)|_{\Gamma_1} : \text{partial normal trace on } \Gamma_1$$

$$\text{rot}_{\Gamma_0} = \mathring{A} \subset A = \text{rot}$$

$$\text{rot} = \text{rot}^* = A^* \subset A^\top = \mathring{A}^* = \text{rot}_{\Gamma_0}^* = \text{rot}_{\Gamma_1}$$

$$\tau_{\text{rot}} \hat{=} \nu \times (\cdot)|_{\Gamma_0} : \text{partial tangential trace on } \Gamma_0$$

Example 4

De Rham Complex – Laplacians

$$\mathring{\Delta} \subset A$$

$$A^* \subset A^T = \mathring{\Delta}^*$$

$$\text{bd trace } \tau_A : D(A) \rightarrow D(A^T)'$$

lddc

 $(\mathring{\Delta}, A^*)$ pair “with” and (A, A^T) pair “without” bc

$$\text{with } \langle\langle x, y \rangle\rangle = \tau_A x(y) = \langle y, Ax \rangle_{H_1} - \langle A^T y, x \rangle_{H_0} = -\tau_{A^T} y(x)$$

$$\text{cl ibp } \langle\langle u, v \rangle\rangle = \tau_\Delta u(v) = \langle \Delta u, v \rangle_{L^2(\Omega)} - \langle u, \Delta v \rangle_{L^2(\Omega)} = \langle \partial_\nu u|_\Gamma, v|_\Gamma \rangle_{L^2(\Gamma)} - \langle u|_\Gamma, \partial_\nu v|_\Gamma \rangle_{L^2(\Gamma)} = -\tau_\Delta v(u)$$

$$\mathring{\Delta} : D(\mathring{\Delta}) = \mathring{H}^2(\Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega) \quad (\text{minimal Laplacian}) \quad \mathring{\Delta}^* = \Delta$$

$$\mathring{\Delta}_D = \text{div } \mathring{\nabla} : D(\mathring{\Delta}_D) \subset L^2(\Omega) \rightarrow L^2(\Omega) \quad (\text{Dirichlet Laplacian, s.a.}) \quad \mathring{\Delta}_D^* = \mathring{\Delta}_D$$

$$\mathring{\Delta}_N = \text{div } \nabla : D(\mathring{\Delta}_N) \subset L^2(\Omega) \rightarrow L^2(\Omega) \quad (\text{Neumann Laplacian, s.a.}) \quad \mathring{\Delta}_N^* = \mathring{\Delta}_N$$

$$\Delta : D(\Delta) = H(\Delta, \Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega) \quad (\text{maximal Laplacian}) \quad \Delta^* = \mathring{\Delta}$$

$$\mathring{\Delta} = \mathring{\Delta} \subset A = \Delta$$

$$\mathring{\Delta} = \Delta^* = A^* \subset A^T = \mathring{\Delta}^* = \mathring{\Delta}^* = \Delta$$

$$\mathring{\Delta}_D = \mathring{\Delta} \subset A = \Delta$$

$$\mathring{\Delta} = \Delta^* = A^* \subset A^T = \mathring{\Delta}^* = \mathring{\Delta}_D^* = \mathring{\Delta}_D$$

$$\mathring{\Delta} = \mathring{\Delta} \subset A = \mathring{\Delta}_{D/N}$$

$$\mathring{\Delta}_{D/N} = \mathring{\Delta}_{D/N}^* = A^* \subset A^T = \mathring{\Delta}^* = \mathring{\Delta}^* = \Delta$$

$$\mathring{\Delta}_N = \mathring{\Delta} \subset A = \Delta$$

$$\mathring{\Delta} = \Delta^* = A^* \subset A^T = \mathring{\Delta}^* = \mathring{\Delta}_N^* = \mathring{\Delta}_N$$

Example 4

De Rham Complex – Laplacians

$$\mathring{A} \subset A$$

$$A^* \subset A^\top = \mathring{A}^*$$

$$\text{bd trace } \tau_A : D(A) \rightarrow D(A^\top)'$$

lddc

 $(\mathring{A}, A^*)$ pair “with” and $(A, A^\top)$ pair “without” bc

$$\text{with } \langle\langle x, y \rangle\rangle = \tau_A x(y) = \langle y, Ax \rangle_{H_1} - \langle A^\top y, x \rangle_{H_0} = -\tau_{A^\top} y(x)$$

$$\text{cl ibp } \langle\langle u, v \rangle\rangle = \tau_\Delta u(v) = \langle \Delta u, v \rangle_{L^2(\Omega)} - \langle u, \Delta v \rangle_{L^2(\Omega)} = \langle \partial_\nu u|_\Gamma, v|_\Gamma \rangle_{L^2(\Gamma)} - \langle u|_\Gamma, \partial_\nu v|_\Gamma \rangle_{L^2(\Gamma)} = -\tau_\Delta v(u)$$

$$\mathring{\Delta} = \mathring{A} \subset A = \Delta$$

$$\mathring{\Delta} = A^* \subset A^\top = \Delta$$

$$\mathring{\Delta}_D = \mathring{A} \subset A = \Delta$$

$$\mathring{\Delta} = A^* \subset A^\top = \mathring{\Delta}_D$$

$$\mathring{\Delta}_N = \mathring{A} \subset A = \Delta$$

$$\mathring{\Delta} = A^* \subset A^\top = \mathring{\Delta}_N$$

$$\tau_\Delta : H(\Delta, \Omega) \rightarrow H(\Delta, \Omega)', \quad \tau_\Delta u(v) \stackrel{\text{class}}{=} \langle \partial_\nu u|_\Gamma, v|_\Gamma \rangle_{L^2(\Gamma)} - \langle u|_\Gamma, \partial_\nu v|_\Gamma \rangle_{L^2(\Gamma)}$$

$$\tau_\Delta : H(\Delta, \Omega) \rightarrow D(\mathring{\Delta}_D)', \quad \tau_\Delta u(v) \stackrel{\text{class}}{=} -\langle u|_\Gamma, \partial_\nu v|_\Gamma \rangle_{L^2(\Gamma)}, \quad v \in \mathring{H}^1(\Omega)$$

$$\tau_\Delta : H(\Delta, \Omega) \rightarrow D(\mathring{\Delta}_N)', \quad \tau_\Delta u(v) \stackrel{\text{class}}{=} \langle \partial_\nu u|_\Gamma, v|_\Gamma \rangle_{L^2(\Gamma)}, \quad \nabla v \in \mathring{H}(\text{div}, \Omega)$$

$$\tau_{\mathring{\Delta}_D} : D(\mathring{\Delta}_D) \rightarrow H(\Delta, \Omega)', \quad \tau_{\mathring{\Delta}_D} v(u) \stackrel{\text{class}}{=} \langle u|_\Gamma, \partial_\nu v|_\Gamma \rangle_{L^2(\Gamma)}, \quad v \in \mathring{H}^1(\Omega)$$

$$\tau_{\mathring{\Delta}_N} : D(\mathring{\Delta}_N) \rightarrow H(\Delta, \Omega)', \quad \tau_{\mathring{\Delta}_N} v(u) \stackrel{\text{class}}{=} -\langle \partial_\nu u|_\Gamma, v|_\Gamma \rangle_{L^2(\Gamma)}, \quad \nabla v \in \mathring{H}(\text{div}, \Omega)$$

Example 4

De Rham Complex – Laplacians

$$\tilde{A} \subset A$$

$$A^* \subset A^T = \tilde{A}^*$$

$$\text{bd trace } \tau_A : D(A) \rightarrow D(A^T)'$$

lddc

$(\tilde{A}, A^*)$ pair “with” and (A, A^T) pair “without” bc

$$\text{with } \langle\langle x, y \rangle\rangle = \tau_A x(y) = \langle y, Ax \rangle_{H_1} - \langle A^T y, x \rangle_{H_0} = -\tau_{A^T} y(x)$$

$$\text{cl ibp } \langle\langle u, v \rangle\rangle = \tau_\Delta u(v) = \langle \Delta u, v \rangle_{L^2(\Omega)} - \langle u, \Delta v \rangle_{L^2(\Omega)} = \langle \partial_\nu u|_\Gamma, v|_\Gamma \rangle_{L^2(\Gamma)} - \langle u|_\Gamma, \partial_\nu v|_\Gamma \rangle_{L^2(\Gamma)} = -\tau_\Delta v(u)$$

$$\tilde{\Delta} = \tilde{A} \subset A = \Delta$$

$$\tilde{\Delta} = A^* \subset A^T = \Delta$$

$$\tilde{\Delta}_D = \tilde{A} \subset A = \Delta$$

$$\tilde{\Delta} = A^* \subset A^T = \tilde{\Delta}_D$$

$$\tilde{\Delta}_N = \tilde{A} \subset A = \Delta$$

$$\tilde{\Delta} = A^* \subset A^T = \tilde{\Delta}_N$$

$$\tau_\Delta : H(\Delta, \Omega) \rightarrow H(\Delta, \Omega)', \quad \tau_\Delta u(v) \stackrel{\text{class}}{=} \langle \partial_\nu u|_\Gamma, v|_\Gamma \rangle_{L^2(\Gamma)} - \langle u|_\Gamma, \partial_\nu v|_\Gamma \rangle_{L^2(\Gamma)}$$

$$\tau_\Delta : H(\Delta, \Omega) \rightarrow D(\tilde{\Delta}_D)', \quad \tau_\Delta u(v) \stackrel{\text{class}}{=} -\langle u|_\Gamma, \partial_\nu v|_\Gamma \rangle_{L^2(\Gamma)}, \quad v \in \dot{H}^1(\Omega)$$

$$\tau_\Delta : H(\Delta, \Omega) \rightarrow D(\tilde{\Delta}_N)', \quad \tau_\Delta u(v) \stackrel{\text{class}}{=} \langle \partial_\nu u|_\Gamma, v|_\Gamma \rangle_{L^2(\Gamma)}, \quad \nabla v \in \dot{H}(\text{div}, \Omega)$$

$$H(\Delta, \Omega)' \ni \tau_\Delta u(\cdot) = \langle\langle u, \cdot \rangle\rangle \stackrel{\text{class}}{=} \langle \partial_\nu u|_\Gamma, \cdot|_\Gamma \rangle_{L^2(\Gamma)} - \langle u|_\Gamma, (\partial_\nu \cdot)|_\Gamma \rangle_{L^2(\Gamma)}$$

Dir and Neu trace as sum

$$D(\tilde{\Delta}_D)' \ni \tau_\Delta u(\cdot) = \langle\langle u, \cdot \rangle\rangle \stackrel{\text{class}}{=} -\langle u|_\Gamma, (\partial_\nu \cdot)|_\Gamma \rangle_{L^2(\Gamma)}$$

non-standard Dirichlet trace

$$D(\tilde{\Delta}_N)' \ni \tau_\Delta u(\cdot) = \langle\langle u, \cdot \rangle\rangle \stackrel{\text{class}}{=} \langle \partial_\nu u|_\Gamma, \cdot|_\Gamma \rangle_{L^2(\Gamma)}$$

non-standard Neumann trace

Example 4

De Rham Complex – Laplacians

$$\tilde{A} \subset A$$

lddc

$$A^* \subset A^T = \tilde{A}^*$$

$(\tilde{A}, A^*)$ pair “with” and (A, A^T) pair “without” bc

$$\text{bd trace } \tau_A : D(A) \rightarrow D(A^T)'$$

$$\text{with } \langle\langle x, y \rangle\rangle = \tau_A x(y) = \langle y, Ax \rangle_{H_1} - \langle A^T y, x \rangle_{H_0} = -\tau_{A^T} y(x)$$

$$\text{cl ibp } \langle\langle u, v \rangle\rangle = \tau_\Delta u(v) = \langle \Delta u, v \rangle_{L^2(\Omega)} - \langle u, \Delta v \rangle_{L^2(\Omega)} = \langle \partial_\nu u|_\Gamma, v|_\Gamma \rangle_{L^2(\Gamma)} - \langle u|_\Gamma, \partial_\nu v|_\Gamma \rangle_{L^2(\Gamma)} = -\tau_\Delta v(u)$$

$$\tau_\Delta : H(\Delta, \Omega) \rightarrow H(\Delta, \Omega)', \quad \tau_\Delta u(v) \stackrel{\text{class}}{=} \langle \partial_\nu u|_\Gamma, v|_\Gamma \rangle_{L^2(\Gamma)} - \langle u|_\Gamma, \partial_\nu v|_\Gamma \rangle_{L^2(\Gamma)}$$

$$\tau_\Delta : H(\Delta, \Omega) \rightarrow D(\tilde{\Delta}_D)', \quad \tau_\Delta u(v) \stackrel{\text{class}}{=} -\langle u|_\Gamma, \partial_\nu v|_\Gamma \rangle_{L^2(\Gamma)}, \quad v \in \dot{H}^1(\Omega)$$

$$\tau_\Delta : H(\Delta, \Omega) \rightarrow D(\tilde{\Delta}_N)', \quad \tau_\Delta u(v) \stackrel{\text{class}}{=} \langle \partial_\nu u|_\Gamma, v|_\Gamma \rangle_{L^2(\Gamma)}, \quad \nabla v \in \dot{H}(\text{div}, \Omega)$$

$$H(\Delta, \Omega)' \ni \tau_\Delta u(\cdot) = \langle\langle u, \cdot \rangle\rangle \stackrel{\text{class}}{=} \langle \partial_\nu u|_\Gamma, \cdot|_\Gamma \rangle_{L^2(\Gamma)} - \langle u|_\Gamma, (\partial_\nu \cdot)|_\Gamma \rangle_{L^2(\Gamma)}$$

Dir and Neu trace as sum

$$D(\tilde{\Delta}_D)' \ni \tau_\Delta u(\cdot) = \langle\langle u, \cdot \rangle\rangle \stackrel{\text{class}}{=} -\langle u|_\Gamma, (\partial_\nu \cdot)|_\Gamma \rangle_{L^2(\Gamma)}$$

non-standard Dirichlet trace

$$D(\tilde{\Delta}_N)' \ni \tau_\Delta u(\cdot) = \langle\langle u, \cdot \rangle\rangle \stackrel{\text{class}}{=} \langle \partial_\nu u|_\Gamma, \cdot|_\Gamma \rangle_{L^2(\Gamma)}$$

non-standard Neumann trace

$$N(\Delta\Delta + 1)' = \dot{H}^2(\Omega)^\circ = R(\tau_\Delta) \ni \tau_\Delta u \stackrel{\text{class}}{\in} ?$$

$$N(\Delta\Delta + 1)' = \dot{H}^2(\Omega)^\circ = R(\tau_\Delta) \ni \tau_\Delta u \stackrel{\text{class}}{\in} H^{-1/2}(\Gamma)' \stackrel{L^2}{\not=} H^{1/2}(\Gamma)$$

$$N(\Delta\Delta + 1)' = \dot{H}^2(\Omega)^\circ = R(\tau_\Delta) \ni \tau_\Delta u \stackrel{\text{class}}{\in} H^{1/2}(\Gamma)' = H^{-1/2}(\Gamma)$$

Grisvard trace (Lions/Magenes)

... to be continued ...

Appendix

Regular Subspaces and Duals

“Regular Subspaces” and Duals

Regular Subspaces and Duals

$$A'_0 : D(A_1)' \rightarrow D(A_0)'$$

- $H_1^+ \subset D(A_1) \subset H_1$ (bd dense embs of reg subspes)
- $D(A_1) = H_1^+ + A_0 H_0^+$ (bd reg deco ops)
- $H_0^+(A_0) = \{x \in H_0^+ : A_0 x \in H_1^+\} \subset H_0^+ \subset D(A_0) \subset H_0$ (bd dense embs)
- $\dot{H}_0^- = H_0^{+'}$ (duals)
- $H_1^+ \subset D(A_1) \cap D(A_0^T) \subset H_1$ (bd dense embs of reg subspes)

note: $H_0^+(A_0) \subset H_0^+ \subset D(A_0) \subset H_0$ and $H_0' \subset D(A_0)' \subset \dot{H}_0^- \subset H_0^+(A_0)'$

$\Rightarrow$ extend A'_0 to $A'_0 : \dot{H}_1^- \rightarrow H_0^+(A_0)'$ by

$$\forall x \in H_0^+(A_0) \quad A'_0 \psi(x) := \psi(A_0 x)$$

$$\bullet H_1' \subset D(A_1)' \stackrel{!}{=} \left[\dot{H}_1^-(A'_0) := \{ \psi \in \dot{H}_1^- : A'_0 \psi \in \dot{H}_0^- \} \right] \subset \dot{H}_1^- = H_1^{+'} \subset H_1^+(A_1)'$$

$$\text{c: } H_1^+ \subset D(A_1) \Rightarrow \psi \in D(A_1)' \subset \dot{H}_1^- \quad \text{and} \quad A'_0 \psi \in D(A_0)' \subset \dot{H}_0^-$$

$$\text{d: } \psi \in \dot{H}_1^-(A'_0) \quad \text{and} \quad D(A_1) \ni y = y_1 + A_0 y_0 \in H_1^+ + A_0 H_0^+ \quad (\text{reg deco})$$

$$\Rightarrow \psi y := \psi y_1 + (A'_0 \psi) y_0 \quad \text{and} \quad |\psi y| \leq c |\psi|_{\dot{H}_1^-(A'_0)} |y|_{D(A_1)} \Rightarrow \psi \in D(A_1)'$$

Characterisation of Dual Spaces by Regular Subspaces

Characterisation of Dual Spaces by “Regular Subspaces”

Characterisation of Dual Spaces by Regular Subspaces

Theorem (Characterisation of Dual Spaces by Regular Subspaces)

$$D(A_1)' = \mathring{H}_1^-(A_0') = \{\psi \in \mathring{H}_1^- : A_0' \psi \in \mathring{H}_0^-\}$$

$$D(A_0^\top)' = \mathring{H}_1^-(A_1^\top)' = \{\psi \in \mathring{H}_1^- : A_1^\top' \psi \in \mathring{H}_2^-\}$$

with equivalent norms.

Theorem (Characterisation of Dual Spaces by Regular Subspaces)

$$D(\mathring{A}_1)' = H_1^-(\mathring{A}_0') = \{\psi \in H_1^- : \mathring{A}_0' \psi \in H_0^-\}$$

$$D(A_0^*)' = H_1^-(A_1^*)' = \{\psi \in H_1^- : A_1^* \psi \in H_2^-\}$$

with equivalent norms.

Characterisation of Trace Ranges by Regular Subspaces

Characterisation of Trace Ranges by “Regular Subspaces”

Characterisation of Trace Ranges by Regular Subspaces

recall traces: $\tau_{A_0} : D(A_0) \rightarrow D(A_0^\top)'$, $\tau_{A_1^\top} : D(A_1^\top) \rightarrow D(A_1)'$

- $N(\tau_{A_1^\top}) = D(A_1^*)$
- $R(\tau_{A_1^\top}) = D(\mathring{A}_1)^\circ = \{\psi \in D(A_1)' : \psi|_{D(\mathring{A}_1)} = 0\}$
- $N(\tau_{A_0}) = D(\mathring{A}_0)$
- $R(\tau_{A_0}) = D(A_0^*)^\circ = \{\psi \in D(A_0^\top)' : \psi|_{D(A_0^*)} = 0\}$

density of $\mathring{H}_1^+ \subset D(\mathring{A}_1)$ and $\mathring{H}_1^* \subset D(A_0^*) \Rightarrow$

- $R(\tau_{A_1^\top}) = \mathring{H}_1^{+\circ}$ as closed subspace of $D(A_1)'$
 - $R(\tau_{A_0}) = \mathring{H}_1^{+\circ}$ as closed subspace of $D(A_0^\top)'$
- $\Rightarrow$ more detailed

Theorem (Characterisation of Trace Ranges by Regular Subspaces)

$$R(\tau_{A_1^\top}) = D(A_1)' \cap D(\mathring{A}_1)^\circ = \mathring{H}_1^-(A_0') \cap \mathring{H}_1^{+\circ} = \{\psi \in \mathring{H}_1^- : A_0' \psi \in \mathring{H}_0^- \wedge \psi|_{\mathring{H}_1^+} = 0\}$$

$$R(\tau_{A_0}) = D(A_0^\top)' \cap D(A_0^*)^\circ = \mathring{H}_1^-(A_1^\top)' \cap \mathring{H}_1^{+\circ} = \{\psi \in \mathring{H}_1^- : A_1^\top' \psi \in \mathring{H}_2^- \wedge \psi|_{\mathring{H}_1^+}^* = 0\}$$

with equivalent norms.

Trace Hilbert Complexes

Hilbert Complexes of Traces and Trace Spaces

Trace Hilbert Complexes

Hilbert Complexes of Traces and Trace Spaces

... to be continued ...

- different unbounded versions of “surface differential operators”

$$\begin{aligned} \cdots &\xrightarrow{\cdots} D(A_0^\top)' \xrightarrow{A_1^{\top\prime}} D(A_1^\top)' \xrightarrow{A_2^{\top\prime}} D(A_2^\top)' \xrightarrow{\cdots} \cdots \\ \cdots &\xleftarrow{\cdots} D(A_1)' \xleftarrow{A_1'} D(A_2)' \xleftarrow{A_2'} D(A_3)' \xleftarrow{\cdots} \cdots \end{aligned}$$

$$\begin{aligned} \cdots &\xrightarrow{\cdots} \mathring{H}_1^{+\circ} \xrightarrow{A_1^{\top\prime}} \mathring{H}_2^{+\circ} \xrightarrow{A_2^{\top\prime}} \mathring{H}_3^{+\circ} \xrightarrow{\cdots} \cdots \\ \cdots &\xleftarrow{\cdots} \mathring{H}_1^{+\circ} \xleftarrow{A_1'} \mathring{H}_2^{+\circ} \xleftarrow{A_2'} \mathring{H}_3^{+\circ} \xleftarrow{\cdots} \cdots \end{aligned}$$

$$\cdots \xleftarrow{\cdots} \mathring{H}_1^- \xleftarrow{A_1^{\top\prime}} \mathring{H}_2^- \xleftarrow{A_2^{\top\prime}} \mathring{H}_3^- \xleftarrow{\cdots} \cdots$$

- compact embeddings for trace Hilbert complexes
- boundary value problems on trace Hilbert complexes

three interpretations

- $D(A_{n+1}^\top)' = R(\tau_{A_n}) \subset D(A_n^\top)'$
- $D(A_n') = R(\tau_{A_{n+1}^\top}) \subset D(A_{n+1})'$
- $R(\tau_{A_0}) = \mathring{H}_1^{+\circ}$ cl subsp of both $D(A_0^\top)' \subset \mathring{H}_1^-$
- $A_1^{\top\prime} : R(\tau_{A_0}) \rightarrow R(\tau_{A_1})$
- $R(\tau_{A_n}) \subset \mathring{H}_{n+1}^{+\circ} \subset \mathring{H}_{n+1}^-$
- $R(\tau_{A_0}) = D(A_0^\top)' \cap D(A_0^*)^\circ$

$$= \mathring{H}_1^- (A_1^{\top\prime}) \cap \mathring{H}_1^{+\circ}$$

$$= \{ \psi \in \mathring{H}_1^- : A_1^{\top\prime} \psi \in \mathring{H}_2^- \wedge \psi|_{\mathring{H}_1^+} = 0 \}$$

$$= \underbrace{\{ \psi \in \mathring{H}_1^{+\circ} : A_1^{\top\prime} \psi \in \mathring{H}_2^{+\circ} \}}_{=: \mathring{H}_1^{+\circ} (A_1^{\top\prime})}$$

...to be continued ...

...to be continued ...