# Functional A Posteriori Error Estimates for Electro-Magneto Static Problems 

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University of Leicester
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## universitat



## Introduction: Static Maxwell Problem

■ $\Omega \subset \mathbb{R}^{3}$ bounded domain with Lipschitz boundary $\Gamma=\partial \Omega$
■ $\varepsilon, \mu: \Omega \rightarrow \mathbb{R}^{3 \times 3}$ medium properties:

- $F$ given right hand side (current),
- E electric field or vector potential for
- $\tau$ tangential trace, i.e., $\tau E=\nu \times\left. E\right|_{r}$
- $\perp$ orthogonality w.r.t. $L^{2}(\Omega)$-scalar product $\langle E, H\rangle_{\Omega}:=\int_{\Omega} E \cdot H$
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magneto static problem with (electric) vector potential

- goal: NON-CONFORMING error estimates for $e:=E-\tilde{E}$ and $h:=\mu H-\tilde{H}$, where $\tilde{E}, \tilde{H}$ approx. of $E, \mu H$, i.e., $\tilde{E}, \tilde{H} \in \mathrm{~L}^{2}(\Omega)$ just
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## Introduction: Sobolev Spaces

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\begin{aligned}
\mathrm{H}(\text { curl } \Omega) & :=\left\{E \in \mathrm{~L}^{2}(\Omega): \text { curl } E \in \mathrm{~L}^{2}(\Omega)\right\} \\
\mathrm{H}\left(\text { curl }_{0} ; \Omega\right) & :=\{E \in \mathrm{H}(\text { curl } ; \Omega): \text { curl } E=0\} \\
\stackrel{\circ}{\mathrm{H}}(\text { curl } \Omega) & :=\left\{E \in \mathrm{H}(\text { curl } ; \Omega)^{\circ}: \tau E=0\right\}=\stackrel{\circ}{\mathrm{C}} \infty(\Omega) \\
\mathrm{H}\left(\text { curl }_{0} ; \Omega\right) & :=\stackrel{\circ}{\mathrm{H}}(\text { curl } ; \Omega) \cap \mathrm{H}\left(\text { curl }_{0} ; \Omega\right)
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analogously:

```
    \(H(\operatorname{div} ; \Omega):=\left\{E \in L^{2}(\Omega): \operatorname{div} E \in L^{2}(\Omega)\right\}\)
    \(H\left(\operatorname{div}_{0} ; \Omega\right):=\{E \in H(\operatorname{div} ; \Omega): \operatorname{div} E=0\}\)
\(E \in \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \Leftrightarrow \varepsilon E \in \mathrm{H}\left(\operatorname{div}_{n} ; \Omega\right)\)
and:
```

```
\(\mathcal{H}_{\varepsilon}(\Omega):=\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)\)
    \(=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E=0, \operatorname{div} \varepsilon E=0, \tau E=0\right\}\)
```


## Introduction: Sobolev Spaces

## spaces:

$\mathrm{H}($ curl $; \Omega):=\left\{E \in \mathrm{~L}^{2}(\Omega):\right.$ curl $\left.E \in \mathrm{~L}^{2}(\Omega)\right\}$
$H($ curlo $; \Omega):=\{E \in H($ curl $; \Omega):$ curl $E=0\}$
$\stackrel{\circ}{\mathrm{H}}($ curl $; \Omega):=\{E \in \mathrm{H}($ curl $; \Omega): \tau E=0\}=\bar{\circ}{ }^{\circ}(\Omega)$ $H($ curlo: $\Omega):-H($ curl $: \Omega) \cap \mathrm{H}($ curlo: $\Omega)$
analogously:
$H(\operatorname{div} ; \Omega):=\left\{E \in L^{2}(\Omega): \operatorname{div} E \in L^{2}(\Omega)\right\}$
$\mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right):=\{E \in \mathrm{H}(\operatorname{div} ; \Omega): \operatorname{div} E=0\}$
$E \in \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{n} ; \Omega\right) \Leftrightarrow \varepsilon E \in \mathrm{H}\left(\operatorname{div}_{n} ; \Omega\right)$
and:
$\mathcal{H}_{\varepsilon}(\Omega):=\dot{H}\left(\operatorname{curl}_{0} ; \Omega\right) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)$ (finite dimension)

$$
=\left\{E \in L^{2}(\Omega): \operatorname{curl} E=0, \operatorname{div} \varepsilon E=0, \tau E=0\right\}
$$

## Introduction: Sobolev Spaces

spaces:

$$
\begin{aligned}
\mathrm{H}(\operatorname{curl} ; \Omega) & :=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E \in \mathrm{~L}^{2}(\Omega)\right\} \\
\mathrm{H}\left(\text { curl }_{0} ; \Omega\right) & :=\{E \in \mathrm{H}(\text { curl } ; \Omega): \operatorname{curl} E=0\}
\end{aligned}
$$

$\stackrel{\circ}{\mathrm{H}}(\mathrm{curl} ; \Omega):=\{E \in \mathrm{H}($ curl $; \Omega): \tau E=0\}={ }_{\circ}^{\circ} \infty(\Omega)$
$\mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right):=\mathrm{H}($ curl $; \Omega) \cap \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
analogously:
$\mathrm{H}(\operatorname{div} ; \Omega):=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{div} E \in \mathrm{~L}^{2}(\Omega)\right\}$
$\mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right):=\{E \in \mathrm{H}(\operatorname{div} ; \Omega): \operatorname{div} E=0\}$
$E \in \varepsilon^{-1} H\left(\operatorname{div}_{0} ; \Omega\right) \Leftrightarrow \varepsilon E \in H\left(\operatorname{div}_{0} ; \Omega\right)$
and:
$\mathcal{H}_{\varepsilon}(\Omega):=\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)$

## Introduction: Sobolev Spaces

spaces:

$$
\begin{aligned}
\mathrm{H}(\operatorname{curl} ; \Omega) & :=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E \in \mathrm{~L}^{2}(\Omega)\right\} \\
\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) & :=\{E \in \mathrm{H}(\operatorname{curl} ; \Omega): \operatorname{curl} E=0\} \\
\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega) & :=\{E \in \mathrm{H}(\operatorname{curl} ; \Omega): \tau E=0\}=\overline{{ }^{\circ} \infty(\Omega)} \mathrm{H}(\operatorname{curl} ; \Omega)
\end{aligned}
$$

## Introduction: Sobolev Spaces

spaces:

$$
\begin{aligned}
\mathrm{H}(\text { curl } ; \Omega) & :=\left\{E \in \mathrm{~L}^{2}(\Omega): \text { curl } E \in \mathrm{~L}^{2}(\Omega)\right\} \\
\mathrm{H}\left(\text { curl }_{0} ; \Omega\right) & :=\{E \in \mathrm{H}(\text { curl } ; \Omega): \text { curl } E=0\} \\
\mathrm{H}(\text { curl } ; \Omega) & :=\{E \in \mathrm{H}(\text { curl } ; \Omega): \tau E=0\}=\overline{\mathrm{C}^{\circ} \infty(\Omega)} \mathrm{H}(\text { curl } ; \Omega)
\end{aligned}
$$

(Gauß' theorem)
analogously:
$\mathrm{H}(\operatorname{div} ; \Omega):=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{div} E \in \mathrm{~L}^{2}(\Omega)\right\}$ $\mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right):=\{E \in \mathrm{H}(\operatorname{div} ; \Omega): \operatorname{div} E=0\}$ $E \in \varepsilon^{-1} H\left(\operatorname{div}_{0} ; \Omega\right) \Leftrightarrow \varepsilon E \in H\left(\operatorname{div}_{0} ; \Omega\right)$
and
$\mathcal{H}_{\varepsilon}(\Omega):=\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)$ $=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E=0, \operatorname{div} \varepsilon E=0, \tau E=0\right\}$

## Introduction: Sobolev Spaces

spaces:

$$
\begin{aligned}
& \mathrm{H}(\operatorname{curl} ; \Omega):=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E \in \mathrm{~L}^{2}(\Omega)\right\} \\
& \mathrm{H}\left(\text { curl }_{0} ; \Omega\right):=\{E \in \mathrm{H}(\text { curl } ; \Omega): \operatorname{curl} E=0\} \\
& \mathrm{O}(\operatorname{curl} ; \Omega):=\{E \in \mathrm{H}(\operatorname{curl} ; \Omega): \tau E=0\}=\stackrel{\circ}{\mathrm{C}} \infty(\Omega) \\
& \mathrm{H}\left(\text { curl }^{\prime} \Omega\right) \\
& \mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right):=\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega) \cap \mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right)
\end{aligned}
$$

analogously:

$$
\begin{aligned}
\mathrm{H}(\operatorname{div} ; \Omega) & :=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{div} E \in \mathrm{~L}^{2}(\Omega)\right\} \\
\mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) & :=\{E \in \mathrm{H}(\operatorname{div} ; \Omega): \operatorname{div} E=0\}
\end{aligned}
$$

$E \in \varepsilon^{-1} H\left(\operatorname{div}_{0} ; \Omega\right) \Leftrightarrow \varepsilon E \in H\left(\operatorname{div}_{0} ; \Omega\right)$
and:

# $\mathcal{H}_{\varepsilon}(\Omega):=\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)$ 

## Introduction: Sobolev Spaces

spaces:

$$
\begin{aligned}
& \mathrm{H}(\operatorname{curl} ; \Omega):=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E \in \mathrm{~L}^{2}(\Omega)\right\} \\
& \mathrm{H}\left(\text { curl }_{0} ; \Omega\right):=\{E \in \mathrm{H}(\text { curl } ; \Omega): \operatorname{curl} E=0\} \\
& \mathrm{O}(\operatorname{curl} ; \Omega):=\{E \in \mathrm{H}(\operatorname{curl} ; \Omega): \tau E=0\}=\stackrel{\circ}{\mathrm{C}} \infty(\Omega) \\
& \mathrm{H}\left(\text { curl }^{\prime} \Omega\right) \\
& \mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right):=\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega) \cap \mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right)
\end{aligned}
$$

analogously:

$$
\begin{aligned}
\mathrm{H}(\operatorname{div} ; \Omega) & :=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{div} E \in \mathrm{~L}^{2}(\Omega)\right\} \\
\mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) & :=\{E \in \mathrm{H}(\operatorname{div} ; \Omega): \operatorname{div} E=0\} \\
E \in \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) & \Leftrightarrow \varepsilon E \in \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)
\end{aligned}
$$

and:

$$
\mathcal{H}_{\varepsilon}(\Omega):=\stackrel{\circ}{\mathrm{H}}\left(\operatorname{curl}_{0} ; \Omega\right) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)
$$

## Introduction: Sobolev Spaces

spaces:

$$
\begin{aligned}
\mathrm{H}(\operatorname{curl} ; \Omega) & :=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E \in \mathrm{~L}^{2}(\Omega)\right\} \\
\mathrm{H}\left(\text { curl }_{0} ; \Omega\right) & :=\{E \in \mathrm{H}(\text { curl } ; \Omega): \operatorname{curl} E=0\} \\
\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega) & :=\{E \in \mathrm{H}(\operatorname{curl} ; \Omega): \tau E=0\}=\overline{\mathrm{C}^{\circ}(\Omega)} \mathrm{H}(\text { curl } ; \Omega) \\
\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) & :=\stackrel{\circ}{\mathrm{H}}(\text { curl } ; \Omega) \cap \mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right)
\end{aligned}
$$

(Gauß' theorem)
analogously:

$$
\begin{aligned}
& \mathrm{H}(\operatorname{div} \Omega):=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{div} E \in \mathrm{~L}^{2}(\Omega)\right\} \\
& \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right):=\{E \in \mathrm{H}(\operatorname{div} ; \Omega): \operatorname{div} E=0\} \\
& E \in \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \Leftrightarrow \varepsilon E \in \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \\
& \text { and: }
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{H}_{\varepsilon}(\Omega) & :=\stackrel{\circ}{\mathrm{H}}\left(\operatorname{curl}_{0} ; \Omega\right) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \\
& =\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E=0, \operatorname{div} \varepsilon E=0, \tau E=0\right\}
\end{aligned}
$$

(finite dimension)

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## Variational Formulation

testing curl $\mu^{-1}$ curl $E=F$ with $\Phi \in \stackrel{\circ}{\mathrm{H}}(\mathrm{curl} ; \Omega)$
unfortunately: $\mathrm{H}($ curl $; \Omega$ ) is not the proper Hilbert space! (kernel of curl)
Poincaré Friedrichs incquality: $\exists$ apr $>0 \quad \forall E \in H($ curl $\cdot \Omega) \cap \varepsilon^{-1} \mathrm{H}($ div: $\Omega)$

special case: $\forall E \in \mathbb{H}:=\mathrm{H}\left(\operatorname{curl}_{1} ; \Omega\right) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{\varepsilon}(\Omega)^{\perp_{\varepsilon}}$

$\Rightarrow b$ bilinear, continuous and coercive over $\mathbb{H}, \varphi$ linear and continuous over $\mathbb{H}$
Lax-Milgram $\Rightarrow$ unique solution $E \in \mathbb{T}+\varkappa G$ with proper tang. ext. operator $\asymp$
key tool: compact embedding of $\mathrm{H}($ curl $; \Omega) \cap \mathrm{H}(\operatorname{div} ; \Omega)$ into $\mathrm{L}^{2}(\Omega)$

## Variational Formulation

testing curl $\mu^{-1}$ curl $E=F$ with $\Phi \in \stackrel{\circ}{\mathrm{H}}($ curl $; \Omega)$

$$
\varphi(\Phi):=\langle F, \Phi\rangle_{\Omega}=\left\langle\mu^{-1} \operatorname{curl} E, \operatorname{curl} \Phi\right\rangle_{\Omega}=: b(E, H)
$$

unfortunately: $\mathrm{H}($ curl $; \Omega)$ is not the proper Hilbert space! (kernel of curl)
Doincaré Friedrichs incquality: $\exists$ apr $>0 \quad \forall E \in H($ curl $; \Omega) \cap c^{-1} H($ div: $\Omega)$

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$\qquad$
$\Rightarrow b$ bilinear, continuous and coercive over $\mathbb{H}, \varphi$ linear and continuous over $\mathbb{H}$
Lax-Milgram $\Rightarrow$ unique solution $E \in \mathbb{H}+\mathscr{F}$ with proper tang. ext. operator $\neq$
key tool: compact embedding of $\mathrm{H}($ curl $; \Omega) \cap \mathrm{H}(\operatorname{div} ; \Omega)$ into $\mathrm{L}^{2}(\Omega)$

## Variational Formulation

testing curl $\mu^{-1}$ curl $E=F$ with $\Phi \in \stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega)$

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\varphi(\Phi):=\langle F, \Phi\rangle_{\Omega}=\left\langle\mu^{-1} \operatorname{curl} E, \operatorname{curl} \Phi\right\rangle_{\Omega}=: b(E, H)
$$

unfortunately: $\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega)$ is not the proper Hilbert space! (kernel of curl)
Poincaré-Friedrichs inequality: $\exists C_{\mathrm{PF}}>0 \quad \forall E \in H(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} H(\operatorname{div} ; \Omega)$

special case: $\forall E \in \mathbb{H}:=\mathrm{H}(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{\varepsilon}(\Omega)^{\perp \varepsilon}$
$\square$
$\Rightarrow b$ bilinear, continuous and coercive over $\mathbb{H}, \varphi$ linear and continuous over $\mathbb{H}$
Lax-Milgram $\Rightarrow$ unique solution $E \in \mathbb{H}+千 G$ with proper tang. ext. operator $\not \subset$
key tool: compact embedding of $\mathrm{H}(\operatorname{curl} ; \Omega) \cap \mathrm{H}(\operatorname{div} ; \Omega)$ into $\mathrm{L}^{2}(\Omega)$

## Variational Formulation

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special case: $\forall E \in \mathbb{H}:=\mathrm{H}(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{\varepsilon}(\Omega)^{\perp \varepsilon}$ $\|E\|_{L^{2}(\Omega)} \leq C_{P F}\|c u r \mid E\|_{L^{2}(\Omega)}$
$\Rightarrow b$ bilinear, continuous and coercive over $\mathbb{H}, \varphi$ linear and continuous over $\mathbb{H}$
Lax-Milgram $\Rightarrow$ unique solution $E \in \mathbb{H}+千 G$ with proper tang. ext. operator $\not \subset$
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## Variational Formulation

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$$
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Poincaré-Friedrichs inequality: $\exists c_{\mathrm{PF}}>0 \quad \forall E \in \mathrm{H}(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}(\operatorname{div} ; \Omega)$

$$
c_{\mathrm{PF}}^{-1}\|E\|_{\mathrm{L}^{2}(\Omega)} \leq\|\operatorname{curl} E\|_{\mathrm{L}^{2}(\Omega)}+\|\operatorname{div} \varepsilon E\|_{\mathrm{L}^{2}(\Omega)}+\|\tau E\|_{\text {trace }}+\sum_{\ell \text { finite }}\left|\left\langle\varepsilon E, E_{\ell}\right\rangle_{\Omega}\right|
$$

special case: $\forall E \in \mathbb{H}:=\mathrm{H}(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{\varepsilon}(\Omega)^{\perp_{\varepsilon}}$

## Variational Formulation

testing curl $\mu^{-1}$ curl $E=F$ with $\Phi \in \stackrel{\circ}{\mathrm{H}}($ curl $; \Omega)$

$$
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$$
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$$
\|E\|_{L^{2}(\Omega)} \leq c_{\mathrm{PF}}\|\operatorname{curl} E\|_{\mathrm{L}^{2}(\Omega)}
$$

$\Rightarrow b$ bilinear, continuous and coercive over $\mathbb{H}, \varphi$ linear and continuous over $\mathbb{H}$
Lax-Milgram $\Rightarrow$ unique solution $E \in \mathbb{H}+\check{\tau} G$ with proper tang. ext. operator $\check{\tau}$
key tool: compact embedding of $\mathrm{H}(\operatorname{curl} ; \Omega) \cap \mathrm{H}(\operatorname{div} ; \Omega)$ into $\mathrm{L}^{2}(\Omega)$

## Variational Formulation

testing curl $\mu^{-1}$ curl $E=F$ with $\Phi \in \stackrel{\circ}{\mathrm{H}}($ curl $; \Omega)$

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$$
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$$

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## Variational Formulation

testing curl $\mu^{-1}$ curl $E=F$ with $\Phi \in \stackrel{\circ}{\mathrm{H}}($ curl $; \Omega)$

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$$
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$$
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## Variational Formulation

testing curl $\mu^{-1}$ curl $E=F$ with $\Phi \in \stackrel{\circ}{\mathrm{H}}($ curl $; \Omega)$

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$$
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$$

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$$
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## Upper and Lower Bounds for Non-Conforming Approximations

$\tilde{H} \in \mathrm{~L}^{2}(\Omega)$ approximation of curl $E=\mu H$ (first, only approximation of magnetic field)
Theorem 1 For all $H \in L^{2}(\Omega)$ the estimates


## hold. Here,



## Upper and Lower Bounds for Non-Conforming Approximations

$\tilde{H} \in \mathrm{~L}^{2}(\Omega)$ approximation of curl $E=\mu H$ (first, only approximation of magnetic field)
Theorem 1 For all $\tilde{H} \in \mathrm{~L}^{2}(\Omega)$ the estimates

$$
\begin{aligned}
& \|\mu H-\tilde{H}\|_{L^{2}(\Omega)} \leq \inf _{Y \in \mathrm{H}(\operatorname{curl} ; \Omega)} M_{+}(\tilde{H} ; Y, \operatorname{curl} Y)+\inf _{\substack{X \in \mathrm{H}(\operatorname{curl} ; \Omega) \\
\tau X=G}} m_{+}(\tilde{H} ; \operatorname{curl} X), \\
& \|\mu H-\tilde{H}\|_{L^{2}(\Omega)}^{2} \geq \sup _{\substack{\circ \\
X \in \mathrm{H}(\operatorname{curl} ; \Omega)}} M_{-}(\tilde{H} ; X, \operatorname{curl} X)+\sup _{\substack{\mu^{-1} Y, Z \in \mathrm{H}(\operatorname{curl} ; \Omega) \\
\operatorname{curl} \mu^{-1} Y=0 \\
\tau Z=G}} m_{-}(\tilde{H} ; Y, \operatorname{curl} Z)
\end{aligned}
$$

hold.
Here,
$\square$

## Upper and Lower Bounds for Non-Conforming Approximations

$\tilde{H} \in \mathrm{~L}^{2}(\Omega)$ approximation of curl $E=\mu H$ (first, only approximation of magnetic field)
Theorem 1 For all $\tilde{H} \in \mathrm{~L}^{2}(\Omega)$ the estimates

$$
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& \|\mu H-\tilde{H}\|_{L^{2}(\Omega)} \leq \inf _{Y \in H(\operatorname{curl} ; \Omega)} M_{+}(\tilde{H} ; Y, \operatorname{curl} Y)+\inf _{\substack{x \in \mathrm{H}(\text { curl } ; \Omega) \\
\tau X=G}} m_{+}(\tilde{H} ; \text { curl } X), \\
& \|\mu H-\tilde{H}\|_{L^{2}(\Omega)}^{2} \geq \sup _{\substack{\dot{\circ}(\operatorname{cur} \mid \Omega)}} M_{-}(\tilde{H} ; X, \operatorname{curl} X)+\sup _{\substack{\mu^{-1} Y, Z \in H(\operatorname{cur} ; \Omega) \\
\operatorname{cur} \mid{ }^{-1} Y=0 \\
\tau Z=G}} m_{-}(\tilde{H} ; Y, \text { curl } Z)
\end{aligned}
$$

hold. Here,

$$
\begin{aligned}
M_{+}(\tilde{H} ; Y, \operatorname{curl} Y) & :=c_{\mu, 1}\left(c_{\mathrm{PF}}\|F-\operatorname{curl} Y\|_{\mathrm{L}^{2}(\Omega)}+\left\|\mu^{-1} \tilde{H}-Y\right\|_{\mathrm{L}^{2}(\Omega)}\right), \\
\quad m_{+}(\tilde{H} ; \operatorname{curl} X) & :=c_{\mu, 2}\|\operatorname{curl} X-\tilde{H}\|_{\mathrm{L}^{2}(\Omega)}, \\
M_{-}(\tilde{H} ; X, \operatorname{curl} X) & :=2\langle F, X\rangle_{\mathrm{L}^{2}(\Omega)}-\left\langle\mu^{-1}(\operatorname{curl} X+2 \tilde{H}), \operatorname{curl} X\right\rangle_{\mathrm{L}^{2}(\Omega)}, \\
m_{-}(\tilde{H} ; Y, \operatorname{curl} Z) & :=2\left\langle\mu^{-1}(\operatorname{curl} Z-\tilde{H}), Y\right\rangle_{\mathrm{L}^{2}(\Omega)}-\|Y\|_{\mathrm{L}^{2}(\Omega)}^{2} .
\end{aligned}
$$

## Upper and Lower Bounds for (Very) Conforming Approximations

$\tilde{E} \in \mathrm{H}($ curl $; \Omega)$ approx. of $E$ and $\tilde{H}:=\operatorname{curl} \tilde{E} \in \mathrm{~L}^{2}(\Omega)$ approx. of curl $E=\mu H$
Corollary $\mathbf{1}$ For all $\tilde{E} \in H(c u r l ; \Omega)$


Corollary 2 For all $\tilde{E} \in \mathrm{H}(\operatorname{curl} ; \Omega)$ with $\tau \tilde{E}=G$, i.e., $E-\tilde{E} \in \mathrm{H}($ curl $; \Omega)$
sup
$X \in H($ curl $; \Omega)$

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## Upper and Lower Bounds for (Very) Conforming Approximations

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Corollary 1 For all $\tilde{E} \in \mathrm{H}($ curl $; \Omega)$

$$
\begin{aligned}
\|\operatorname{curl}(E-\tilde{E})\|_{L^{2}(\Omega)} \leq & \inf _{Y \in \mathrm{H}(\operatorname{curl} ; \Omega)} M_{+}(\tilde{H} ; Y, \operatorname{curl} Y)+c_{\mu, 2} c_{\tau}\|G-\tau \tilde{E}\|_{\text {trace }} \\
\|\operatorname{curl}(E-\tilde{E})\|_{L^{2}(\Omega)}^{2} \geq & \sup _{X \in \mathrm{H}(\operatorname{curl} ; \Omega)} M_{-}(\tilde{H} ; X, \operatorname{curl} X) \\
& \quad+\sup _{Y \in \mu \mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right)}\left(2\left\langle G-\tau \tilde{E}, \mu^{-1} Y\right\rangle_{\text {trace }}-\|Y\|_{L^{2}(\Omega)}^{2}\right) .
\end{aligned}
$$

Corollary 2 For all $\tilde{E} \in \mathrm{H}(\operatorname{curl} ; \Omega)$ with $\tau \tilde{E}=G$, i.e., $E-\tilde{E} \in \mathrm{H}(\operatorname{curl} ; \Omega)$

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\end{aligned}
$$

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$$
\sup _{X \in \mathrm{H}(\operatorname{curl} ; \Omega)} M_{-}(\tilde{H} ; X, \operatorname{curl} X) \leq\|\operatorname{curl}(E-\tilde{E})\|_{L^{2}(\Omega)}^{2} \leq \inf _{Y \in H(\operatorname{curl} ; \Omega)} M_{+}^{2}(\tilde{H} ; Y, \operatorname{curl} Y) .
$$

## Norm Estimates

- norm estimates for $h=\mu H-\tilde{H}$
- $E \mapsto \|$ curl $E \|_{L^{2}(\Omega)}$ semi-norm but not norm on $H($ curl $; \Omega)$ or $H(c u r l ; \Omega)$ (not controlling $\|\operatorname{div} E\|_{L^{2}(\Omega)}$ and projection on Dirichlet fields)
- for conforming anproximations $E \in \mathrm{H}($ curl $; \Omega) \cap \varepsilon^{-1} \mathrm{H}($ div $; \Omega)$ the semi-norm

is well defined and a norm on $\mathrm{H}($ curl; $\Omega) \cap \varepsilon^{-1} \mathrm{H}(\operatorname{div} ; \Omega)$
- $\Rightarrow$ norm estimates for $e=E-\tilde{E}$


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## Norm Estimates

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$$
E \mapsto\|E\|:=\|\operatorname{curl} E\|_{L^{2}(\Omega)}+\|\operatorname{div} \varepsilon E\|_{L^{2}(\Omega)}+\sum_{\ell=1}^{d}\left|\left\langle\varepsilon E, E_{\ell}\right\rangle_{L^{2}(\Omega)}\right|
$$

is well defined and a norm on $\stackrel{\circ}{\mathrm{H}}(\mathrm{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}(\operatorname{div} ; \Omega)$

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- $\Rightarrow$ norm estimates for $e=E-\tilde{E}$

Corollary 3 For all $\tilde{E} \in H(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}(\operatorname{div} ; \Omega)$ with $\tau \tilde{E}=G$ and $\operatorname{div} \varepsilon \tilde{E}=0$ and $\varepsilon \tilde{E} \perp \mathcal{H}_{\varepsilon}(\Omega)$, i.e., $E-\tilde{E} \in \mathbb{H}$,
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## Norm Estimates

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$$

is well defined and a norm on $\stackrel{\circ}{\mathrm{H}}(\mathrm{curl} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}(\operatorname{div} ; \Omega)$
■ $\Rightarrow$ norm estimates for $e=E-\tilde{E}$, e.g.,
Corollary 3 For all $\tilde{E} \in \mathrm{H}(\tilde{\operatorname{curl}} ; \Omega) \cap \varepsilon^{-1} \mathrm{H}(\operatorname{div} ; \Omega)$ with $\tau \tilde{E}=G$ and $\operatorname{div} \varepsilon \tilde{E}=0$ and $\varepsilon \tilde{E} \perp \mathcal{H}_{\varepsilon}(\Omega)$, i.e., $E-\tilde{E} \in \mathbb{H}$,

$$
\sup _{X \in \stackrel{H}{(c u r l} ; \Omega)} M_{-}(\tilde{H} ; X, \operatorname{curl} X) \leq\|E-\tilde{E}\|^{2} \leq \inf _{Y \in \mathrm{H}(\text { curl } ; \Omega)} M_{+}^{2}(\tilde{H} ; Y, \text { curl } Y) .
$$

## Constants and Sharpness

typical features of functional a posteriori error estimates

- estimates for errors: basic (integral) relations, constants for embedding inequalities $C_{\mathrm{PF}}, c_{\mu, i}, c_{\tau}$
- recall e.g.: For $\tilde{E} \in H(\operatorname{curl} ; \Omega) \cap \varepsilon^{-1} H\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{\varepsilon}(\Omega)^{\perp \varepsilon}$ with $\tau \tilde{E}=G$ $\|E-\tilde{E}\| \leq \inf _{Y \in H(\operatorname{curl} ; \Omega)} M_{+}(\tilde{H} ; Y, \operatorname{curl} Y)$



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$$
\begin{aligned}
\|E-\tilde{E}\| & \leq \inf _{Y \in \mathrm{H}(\operatorname{curl} ; \Omega)} M_{+}(\tilde{H} ; Y, \operatorname{curl} Y) \\
& =\inf _{Y \in \mathrm{H}(\operatorname{curl} ; \Omega)} c_{\mu, 1}\left(\operatorname{cpF}\|F-\operatorname{curl} Y\|_{L^{2}(\Omega)}+\left\|\mu^{-1} \tilde{H}-Y\right\|_{L^{2}(\Omega)}\right) .
\end{aligned}
$$

$\Rightarrow \quad M_{+}(\tilde{H} ; Y, \operatorname{curl} Y)=0 \quad \Leftrightarrow \quad \mu Y=H=\tilde{H}=\operatorname{curl} \tilde{E} \wedge \tilde{E}=E \quad$ sharp

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$$
\begin{aligned}
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& \\
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\end{aligned}
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## Proofs (Helmholtz Decomposition)

main tools: standard techniques and Helmholtz decomposition
$\tilde{H}$ approximation of $\mu H=\operatorname{curl} E, h:=\mu H-\tilde{H}$ error simplicity $\varepsilon=\mu=\mathrm{id}$

$$
\begin{aligned}
\mathrm{L}^{2}(\Omega) \ni h & =\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \\
\text { curl } E_{\mathrm{c}} & \in \operatorname{curl} \stackrel{\circ}{(\operatorname{curl} ; \Omega)} \\
H_{\mathrm{d}} & \in \mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right)
\end{aligned}
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\end{aligned}
$$

recall $\mathbb{H}=H($ curl $; \Omega) \cap H\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}(\Omega)^{\perp}$ curl $E_{c} \quad$ regular/conforming error $H_{d}$ non-conforming error (bounclary error) orthogonality $\Rightarrow\|h\|_{\mathrm{L}^{2}(\Omega)}^{2}=\|$ curl $E_{\mathrm{c}}\left\|_{\mathrm{L}^{2}(\Omega)}^{2}+\right\| H_{d} \|_{\mathrm{L}^{2}(\Omega)}^{2}$

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\text { recall } \mathbb{H} & =\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega) \cap \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}(\Omega)^{\perp}
\end{aligned}
$$

curl $E_{c} \quad$ regular/conforming error
$H_{\mathrm{d}}$ non-conforming error (boundary error)
orthogonality $\Rightarrow\left\|h_{L^{2}(\Omega)}^{\|^{2}}=\right\| \operatorname{curl} E_{C}\left\|_{L^{2}(\Omega)}^{2}+\right\| \boldsymbol{H}_{\mathrm{d}} \|_{L^{2}(\Omega)}^{2}$

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\text { curl } E_{\mathrm{c}} \quad & \text { regular/conforming error }
\end{aligned}
$$

orthogonality $\Rightarrow\|h\|_{\mathrm{L}^{2}(\Omega)}^{2}=\|$ curl $E_{\mathrm{c}}\left\|_{\mathrm{L}^{2}(\Omega)}^{2}+\right\| H_{\mathrm{d}} \|_{\mathrm{L}^{2}(\Omega)}^{2}$

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\text { recall } \mathbb{H} & =\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega) \cap \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}(\Omega)^{\perp} \\
\operatorname{curl} E_{\mathrm{c}} & \text { regular/conforming error } \\
H_{\mathrm{d}} & \text { non-conforming error (boundary error) } \\
\text { orthogonality } \Rightarrow h_{L^{2}(\Omega)}^{2} & =\| \text { curl } E_{\mathrm{c}}\left\|_{L^{2}(\Omega)}^{2}+\right\| H_{\mathrm{d}} \|_{L^{2}(\Omega)}^{2}
\end{aligned}
\end{gathered}
$$

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\end{aligned}
\end{gathered}
$$

## Proofs (Upper Bounds)

recall $h=H-\tilde{H}=$ curl $E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
standard argument for curl $E_{c}$ : for all $\Phi \in \mathbb{H}$

since $\langle\text { curl } Y, \Phi\rangle_{L^{2}(\Omega)}=\langle Y, \operatorname{curl} \Phi\rangle_{L^{2}(\Omega)}$ for all $Y \in \mathrm{H}($ curl; $\Omega) ;$ note $\mathbb{H} \subset \mathrm{H}($ curl $; \Omega)$.
Cauchy-Schwarz and Poincaré-Friedrichs and $\Phi:=E_{c} \in \mathbb{H} \quad \Rightarrow$


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standard argument for curl $E_{\mathrm{c}}$ :
$\left\langle\operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} \Phi\right\rangle_{\mathrm{L}^{2}(\Omega)}=\langle h, \operatorname{curl} \mid\rangle_{\mathrm{L}^{2}(\Omega)} \quad\left(H_{d} \perp \operatorname{curl} \Phi\right)$ $=\langle F, \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\tilde{H}, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)}$ $=\langle F-\operatorname{curl} Y, \Phi\rangle_{L^{2}(\Omega)}-\langle\tilde{H}-Y, \operatorname{curl} \phi\rangle_{L^{2}(\Omega)}$
since $\langle\text { curl } Y, \Phi\rangle_{L^{2}(\Omega)}=\langle Y, \operatorname{curl} \Phi\rangle_{L^{2}(\Omega)}$ for all $Y \in \mathrm{H}($ curl; $\Omega)$; note $\mathbb{H} \subset \mathrm{H}($ curl $; \Omega)$.
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$$
\left\langle\operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} \Phi\right\rangle_{\mathrm{L}^{2}(\Omega)}=\langle h, \operatorname{curl} \Phi\rangle_{L^{2}(\Omega)} \quad\left(H_{d} \perp \operatorname{curl} \Phi\right)
$$


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$$
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$$


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$$
\begin{aligned}
\left\langle\operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} \Phi\right\rangle_{\mathrm{L}^{2}(\Omega)} & =\langle h, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)} \quad\left(H_{\mathrm{d}} \perp \operatorname{curl} \Phi\right) \\
& =\langle F, \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\tilde{H}, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)}
\end{aligned}
$$


since $\langle\text { curl } Y, \Phi\rangle_{L^{2}(\Omega)}=\langle Y, \operatorname{curl} \Phi\rangle_{L^{2}(\Omega)}$ for all $Y \in \mathrm{H}($ curl; $\Omega)$; note $\mathbb{H} \subset \mathrm{H}($ curl $; \Omega)$.
Cauchy-Schwarz and Poincaré-Friedrichs and $\Phi:=E_{c} \in \mathbb{H} \quad \Rightarrow$
$\left\|\operatorname{curl} E_{C}\right\|_{L^{2}(\Omega)} \leq C_{P F}\|F-\operatorname{curl} Y\|_{L^{2}(\Omega)}+\|\tilde{H}-Y\|_{L^{2}(\Omega)}=M_{+}(\tilde{H} ; Y, \operatorname{curl} Y)$

## Proofs (Upper Bounds)

recall $h=H-\tilde{H}=$ curl $E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
standard argument for curl $E_{\mathrm{c}}$ : for all $\Phi \in \mathbb{H}$

$$
\begin{aligned}
\left\langle\operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} \Phi\right\rangle_{\mathrm{L}^{2}(\Omega)} & =\langle h, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)} \quad\left(H_{\mathrm{d}} \perp \operatorname{curl} \Phi\right) \\
& =\langle F, \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\tilde{H}, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)} \\
& =\langle F-\operatorname{curl} Y, \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\tilde{H}-Y, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)}
\end{aligned}
$$

since $\langle\text { curl } Y, \Phi\rangle_{L^{2}(\Omega)}=\langle Y, \operatorname{curl} \Phi\rangle_{L^{2}(\Omega)}$ for all $Y \in \mathrm{H}($ curl $; \Omega) ;$ note $\mathbb{H} \subset \stackrel{\circ}{\mathrm{H}}($ curl $; \Omega)$.
Cauchy-Schwarz and Poincaré-Friedrichs and $\Phi:=E_{c} \in \mathbb{H} \quad \Rightarrow$

$$
\| \text { curl } E_{\mathrm{c}}\left\|_{L^{2}(\Omega)} \leq c_{\mathrm{PF}}\right\| F-\operatorname{curl} Y\left\|_{L^{2}(\Omega)}+\right\| \tilde{H}-Y \|_{L^{2}(\Omega)}=M_{+}(\tilde{H} ; Y, \text { curl } Y)
$$

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## Proofs (Upper Bounds)

recall $h=H-\tilde{H}=$ curl $E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
standard argument for curl $E_{\mathrm{c}}$ : for all $\Phi \in \mathbb{H}$

$$
\begin{aligned}
\left\langle\operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} \Phi\right\rangle_{\mathrm{L}^{2}(\Omega)} & =\langle h, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)} \quad\left(H_{\mathrm{d}} \perp \operatorname{curl} \Phi\right) \\
& =\langle F, \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\tilde{H}, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)} \\
& =\langle F-\operatorname{curl} Y, \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\tilde{H}-Y, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)}
\end{aligned}
$$

since $\langle\text { curl } Y, \Phi\rangle_{L^{2}(\Omega)}=\langle Y, \operatorname{curl} \Phi\rangle_{L^{2}(\Omega)}$ for all $Y \in \mathrm{H}($ curl $; \Omega)$; note $\mathbb{H} \subset \stackrel{\circ}{\mathrm{H}}($ curl $; \Omega)$.
Cauchy-Schwarz and Poincaré-Friedrichs and $\Phi:=E_{c} \in \mathbb{H} \quad \Rightarrow$

$$
\left\|\operatorname{curl} E_{\mathrm{c}}\right\|_{\mathrm{L}^{2}(\Omega)} \leq c_{\mathrm{PF}}\|F-\operatorname{curl} Y\|_{\mathrm{L}^{2}(\Omega)}+\|\tilde{H}-Y\|_{\mathrm{L}^{2}(\Omega)}=M_{+}(\tilde{H} ; Y, \operatorname{curl} Y)
$$

## Proofs (Upper Bounds Continued)

recall $h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
argument for $H_{d}$ : for all $\psi \in H($ curlo; $\Omega)$

$\in H($ curl $; \Omega)$
for all $X \in \mathrm{H}($ curl $; \Omega)$ with $\tau X=G$.
Cauchy-Schwarz and $\psi:=H_{d} \Rightarrow$

$$
\left\|H_{\mathrm{d}}\right\|_{\mathrm{L}^{2}(\Omega)} \leq\|\operatorname{curl} X-\tilde{H}\|_{\mathrm{L}^{2}(\Omega)}=m_{+}(\tilde{H} ; \operatorname{curl} X)
$$

$\Rightarrow$ finally

$$
\left\|h_{L^{2}(\Omega)}^{2}=\right\| \operatorname{cur} \mid E_{c}\left\|_{L^{2}(\Omega)}^{2}+\right\| H_{d} \|_{L^{2}(\Omega)}^{2} \leq M_{+}^{2}(\tilde{H} ; Y, \operatorname{curl} Y)+m^{2}(\tilde{H} ; \operatorname{curl} X)
$$

for all $Y \in \mathrm{H}(\operatorname{curl} ; \Omega)$ and all $X \in \mathrm{H}($ curl $; \Omega)$ with $\tau X=G$.
$\Rightarrow \quad m_{+}\left(\tilde{H} ; \operatorname{curl}^{\prime} X\right)=\left\|\operatorname{cur}{ }^{\prime}(X-\tilde{E})\right\|_{L^{2}(\Omega)}=\left\|\operatorname{cur} \prime^{\prime}(G-\tau \tilde{E})^{\prime}\right\|_{L^{2}(\Omega)} \leq c_{T}\|G-\tau \tilde{E}\|_{\text {trace }}$
if $\tilde{H}=\operatorname{curl} \tilde{E}$ with $\tilde{E} \in \mathrm{H}(\operatorname{curl} ; \Omega)$
and we choose $X:=\tilde{E}-\check{\tau} \tau \tilde{E}+\check{\tau} G \in H($ curl $; \Omega) \quad$ 正
(then $\tau X=G$ )

## Proofs (Upper Bounds Continued)

recall $h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
argument for $H_{d}$ : for all $\psi \in H($ curlo; $\Omega)$

for all $X \in \mathrm{H}($ curl $; \Omega)$ with $\tau X=G$.
Cauchy-Schwarz and $\psi:=H_{d} \Rightarrow$

$$
\left\|H_{\mathrm{d}}\right\|_{\mathrm{L}^{2}(\Omega)} \leq\|\operatorname{curl} X-\tilde{H}\|_{\mathrm{L}^{2}(\Omega)}=m_{+}(\tilde{H} ; \operatorname{curl} X)
$$

$\Rightarrow$ finally

$$
\|h\|_{L^{2}(\Omega)}^{2}=\left\|\operatorname{curl} E_{c}\right\|_{L^{2}(\Omega)}^{2}+\left\|H_{d}\right\|_{L^{2}(\Omega)}^{2} \leq M_{+}^{2}(\tilde{H} ; Y, \operatorname{curl} Y)+m^{2}(\tilde{H} ; \operatorname{curl} X)
$$

for all $Y \in \mathrm{H}(\operatorname{curl} ; \Omega)$ and all $X \in \mathrm{H}(\operatorname{curl} ; \Omega)$ with $\tau X=G$.

if $\tilde{H}=\operatorname{curl} \tilde{E}$ with $\tilde{E} \in \mathrm{H}(\operatorname{curl} ; \Omega)$

(then $\tau X=G$ )

## Proofs (Upper Bounds Continued)

recall $h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
argument for $H_{d}$ : for all $\Psi \in \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
$\left\langle H_{d}, \Psi\right\rangle_{L^{2}(\Omega)}$

for all $X \in \mathrm{H}($ curl $; \Omega)$ with $\tau X=G$.
Cauchy-Schmarz and $\Psi:=H_{a} \quad \Rightarrow$

$$
\left\|H_{d}\right\|_{L^{2}(\Omega)} \leq\|\operatorname{curl} X-\tilde{H}\|_{L^{2}(\Omega)}=m_{+}(\tilde{H} ; \operatorname{curl} X)
$$

$\Rightarrow$ finally

$$
\|h\|_{L^{2}(\Omega)}^{2}=\| \text { curl } E_{d}\left\|_{L^{2}(\Omega)}^{2}+\right\| H_{d} \|_{L^{2}(\Omega)}^{2} \leq M_{+}^{2}(\tilde{H} ; Y, \text { curl } Y)+m^{2}(\tilde{H} ; \text { curl } X)
$$

for all $Y \in H(c u r l ; \Omega)$ and all $X \in H(c u r l ; \Omega)$ with $\tau X=G$.

if $\tilde{H}=\operatorname{curl} \tilde{E}$ with $\tilde{E} \in \mathrm{H}(\operatorname{curl} ; \Omega)$
and we choose $X:=\tilde{E}-\check{\tau} \tau \tilde{E}+\check{\tau} G \in H(c u r l ; \Omega)$
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(then $\tau X=G$ )


## Proofs (Upper Bounds Continued)

recall $h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
argument for $H_{d}$ : for all $\Psi \in \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$

$$
\left\langle H_{\mathrm{d}}, \Psi\right\rangle_{\mathrm{L}^{2}(\Omega)}=\langle\operatorname{curl} X-\tilde{H}, \Psi\rangle_{\mathrm{L}^{2}(\Omega)} \quad(H_{\mathrm{d}}+\tilde{H}-\operatorname{curl} X=\operatorname{curl}(\underbrace{E-X-E_{\mathrm{c}}}_{\in \mathrm{H}(\mathrm{curl} ; \Omega)}) \perp \Psi)
$$

for all $X \in \mathrm{H}($ curl $; \Omega)$ with $\tau X=G$.
Cauchy-Schwarz and $\psi:=H_{d}$

$$
\left\|H_{d}\right\|_{L^{2}(\Omega)} \leq\|\operatorname{curl} X-\tilde{H}\|_{L^{2}(\Omega)}=m_{+}(\tilde{H} ; \operatorname{curl} X)
$$

$\Rightarrow$ finally

$$
\|h\|_{L^{2}(\Omega)}^{2}=\| \text { curl } E_{d}\left\|_{L^{2}(\Omega)}^{2}+\right\| H_{d} \|_{L^{2}(\Omega)}^{2} \leq M_{+}^{2}(\tilde{H} ; Y, \text { curl } Y)+m^{2}(\tilde{H} ; \text { curl } X)
$$

for all $Y \in H(c u r l ; \Omega)$ and all $X \in H(c u r l ; \Omega)$ with $\tau X=G$.
$\square$ $m_{+}(\tilde{H} ; \operatorname{curl} X)=\|\operatorname{curl}(X-\tilde{E})\|_{L^{2}(\Omega)}=\|\operatorname{curl} \check{\tau}(G-\tau \tilde{E})\|_{L^{2}(\Omega)} \leq c_{\tau}\|G-\tau \tilde{E}\|$ trace
if $\tilde{H}=\operatorname{curl} \tilde{E}$ with $\tilde{E} \in \mathrm{H}(\operatorname{curl} ; \Omega)$
and we choose $X:=\tilde{E}-\check{\tau} \tau \tilde{E}+\check{\tau} G \in H(c u r l ; \Omega)$
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(then $\tau X=G$ )

## Proofs (Upper Bounds Continued)

recall $h=H-\tilde{H}=\operatorname{curl} E_{c} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
argument for $H_{d}$ : for all $\Psi \in \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$

$$
\left\langle H_{\mathrm{d}}, \Psi\right\rangle_{\mathrm{L}^{2}(\Omega)}=\langle\operatorname{curl} X-\tilde{H}, \Psi\rangle_{\mathrm{L}^{2}(\Omega)} \quad(H_{\mathrm{d}}+\tilde{H}-\operatorname{curl} X=\operatorname{curl}(\underbrace{E-X-E_{\mathrm{c}}}_{\in \mathrm{H}(\mathrm{curl} ; \Omega)}) \perp \Psi)
$$

for all $X \in \mathrm{H}($ curl $; \Omega)$ with $\tau X=G$.
Cauchy-Schwarz and $\Psi:=H_{d} \quad \Rightarrow$

$$
\left\|H_{d}\right\|_{L^{2}(\Omega)} \leq\|\operatorname{curl} X-\tilde{H}\|_{L^{2}(\Omega)}=m_{+}(\tilde{H} ; \operatorname{curl} X)
$$

$\Rightarrow$ finally

$$
\|h\|_{L^{2}(\Omega)}^{2}=\| \text { curl } E_{d}\left\|_{L^{2}(\Omega)}^{2}+\right\| H_{d} \|_{L^{2}(\Omega)}^{2} \leq M_{+}^{2}(\tilde{H} ; Y, \text { curl } Y)+m^{2}(\tilde{H} ; \text { curl } X)
$$

for all $Y \in H(c u r l ; \Omega)$ and all $X \in H(c u r l ; \Omega)$ with $\tau X=G$
$\square$ $m_{+}(\tilde{H} ; \operatorname{curl} X)=\|\operatorname{curl}(X-\tilde{E})\|_{L^{2}(\Omega)}=\|\operatorname{curl} \check{\tau}(G-\tau \tilde{E})\|_{L^{2}(\Omega)} \leq c_{\tau}\|G-\tau \tilde{E}\|$ trace
if $\tilde{H}=\operatorname{curl} \tilde{E}$ with $\tilde{E} \in \mathrm{H}($ curl $: \Omega)$
and we choose $X:=\tilde{E}-\check{\tau} \tau \tilde{E}+\check{\tau} G \in H(c u r l ; \Omega)$
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(then $\tau X=G$ )

## Proofs (Upper Bounds Continued)

recall $h=H-\tilde{H}=\operatorname{curl} E_{c} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
argument for $H_{d}$ : for all $\Psi \in \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$

$$
\left\langle H_{\mathrm{d}}, \Psi\right\rangle_{\mathrm{L}^{2}(\Omega)}=\langle\operatorname{curl} X-\tilde{H}, \Psi\rangle_{\mathrm{L}^{2}(\Omega)} \quad(H_{\mathrm{d}}+\tilde{H}-\operatorname{curl} X=\operatorname{curl}(\underbrace{E-X-E_{\mathrm{c}}}_{\in \dot{\mathrm{H}(\mathrm{curl} ; \Omega)}}) \perp \Psi)
$$

for all $X \in \mathrm{H}($ curl $; \Omega)$ with $\tau X=G$.
Cauchy-Schwarz and $\Psi:=H_{d} \quad \Rightarrow$

$$
\left\|H_{d}\right\|_{L^{2}(\Omega)} \leq\|\operatorname{curl} X-\tilde{H}\|_{L^{2}(\Omega)}=m_{+}(\tilde{H} ; \operatorname{curl} X)
$$

$\Rightarrow$ finally

$$
\|h\|_{L^{2}(\Omega)}^{2}=\left\|\operatorname{curl} E_{\mathrm{c}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2}+\left\|H_{d}\right\|_{\mathrm{L}^{2}(\Omega)}^{2} \leq M_{+}^{2}(\tilde{H} ; Y, \operatorname{curl} Y)+m_{+}^{2}(\tilde{H} ; \operatorname{curl} X)
$$

for all $Y \in \mathrm{H}(\operatorname{curl} ; \Omega)$ and all $X \in \mathrm{H}(\operatorname{curl} ; \Omega)$ with $\tau X=G$.

## Proofs (Upper Bounds Continued)

recall $h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
argument for $H_{d}$ : for all $\Psi \in \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$

$$
\left\langle H_{\mathrm{d}}, \Psi\right\rangle_{\mathrm{L}^{2}(\Omega)}=\langle\operatorname{curl} X-\tilde{H}, \Psi\rangle_{\mathrm{L}^{2}(\Omega)} \quad(H_{\mathrm{d}}+\tilde{H}-\operatorname{curl} X=\operatorname{curl}(\underbrace{E-X-E_{\mathrm{c}}}_{\in \dot{\mathrm{H}}(\mathrm{curl} ; \Omega)}) \perp \Psi)
$$

for all $X \in \mathrm{H}($ curl $; \Omega)$ with $\tau X=G$.
Cauchy-Schwarz and $\Psi:=H_{d} \quad \Rightarrow$

$$
\left\|H_{d}\right\|_{L^{2}(\Omega)} \leq\|\operatorname{curl} X-\tilde{H}\|_{L^{2}(\Omega)}=m_{+}(\tilde{H} ; \operatorname{curl} X)
$$

$\Rightarrow$ finally

$$
\|h\|_{L^{2}(\Omega)}^{2}=\left\|\operatorname{curl} E_{\mathrm{c}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2}+\left\|H_{d}\right\|_{\mathrm{L}^{2}(\Omega)}^{2} \leq M_{+}^{2}(\tilde{H} ; Y, \operatorname{curl} Y)+m_{+}^{2}(\tilde{H} ; \operatorname{curl} X)
$$

for all $Y \in \mathrm{H}(\operatorname{curl} ; \Omega)$ and all $X \in \mathrm{H}(\operatorname{curl} ; \Omega)$ with $\tau X=G$.
$\Rightarrow \quad m_{+}(\tilde{H} ; \operatorname{curl} X)=\|\operatorname{curl}(X-\tilde{E})\|_{L^{2}(\Omega)}=\|\operatorname{curl} \check{\tau}(G-\tau \tilde{E})\|_{L^{2}(\Omega)} \leq c_{\tau}\|G-\tau \tilde{E}\|_{\text {trace }}$
if $\tilde{H}=\operatorname{curl} \tilde{E}$ with $\tilde{E} \in \mathrm{H}($ curl $; \Omega)$
and we choose $X:=\tilde{E}-\check{\tau} \tau \tilde{E}+\check{\tau} G \in \mathrm{H}($ curl $; \Omega)$
(then $\tau X=G$ )

## Proofs (Lower Bounds)

recall $h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
standard argument for curl $E_{c}$ : for all $X \in \mathrm{H}($ curl $; \Omega)$
$\left\|\operatorname{cur}\left|E_{C}\left\|_{L^{2}(\Omega)}^{2} \geq 2\langle\operatorname{cur}| E_{C}, \operatorname{cur}|X\rangle_{L^{2}(\Omega)}-\right\| \operatorname{cur}\right| X\right\|_{L^{2}(\Omega)}^{2} \quad\left(H_{d} \perp \operatorname{cur} \mid X\right)$ $=\langle h, \operatorname{curl} X\rangle_{L^{2}(\Omega)}$
$=2\langle F, \Phi\rangle_{L^{2}(\Omega)}-\langle\operatorname{curl} X+2 \tilde{H}, \operatorname{curl} X\rangle_{L^{2}(\Omega)}=M_{-}(\tilde{H} ; X, \operatorname{curl} X)$
similar for $H_{d}$ : for all $Y \in H\left(\right.$ curl $\left._{0} ; \Omega\right)$ and for all $Z \in H(\operatorname{curl} ; \Omega)$ with $\tau Z=G$

$=m_{-}(\tilde{H} ; Y, \operatorname{curl} Z)$

## Proofs (Lower Bounds)

recall $h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
standard argument for curl $E_{\mathrm{c}}$ : for all $X \in H($ curl $; \Omega)$
$\left\|\operatorname{curl} E_{\mathrm{C}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2} \geq 2\left\langle\operatorname{curl} E_{\mathrm{C}}, \operatorname{curl} X\right\rangle_{\mathrm{L}^{2}(\Omega)}-\|\operatorname{curl} X\|_{L^{2}(\Omega)}^{2} \quad\left(H_{\mathrm{d}} \perp \operatorname{curl} X\right)$

similar for $H_{d}:$ for all $Y \in H\left(\right.$ curl $\left._{0} ; \Omega\right)$ and for all $Z \in H(\operatorname{curl} ; \Omega)$ with $\tau Z=G$

$=m_{-}(\tilde{H} ; Y, \operatorname{curl} Z)$
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## Proofs (Lower Bounds)

recall $h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
standard argument for curl $E_{\mathrm{c}}$ : for all $X \in H($ curl $; \Omega)$
$\|$ curl $E_{c} \|_{L^{2}(\Omega)}^{2} \geq 2\left\langle\text { curl } E_{\mathrm{c}}, \text { curl } X\right\rangle_{L^{2}(\Omega)}-\|$ curl $X \|_{L^{2}(\Omega)}^{2} \quad\left(H_{d} \perp\right.$ curl $\left.X\right)$

similar for $H_{d}$ : for all $Y \in H\left(\right.$ curl $\left._{0} ; \Omega\right)$ and for all $Z \in H($ curl $; \Omega)$ with $\tau Z=G$

$=m_{-}(\tilde{H} ; Y, \operatorname{curl} Z)$

## Proofs (Lower Bounds)

recall $h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
standard argument for curl $E_{\mathrm{c}}$ : for all $X \in \stackrel{\circ}{\mathrm{H}}($ curl $; \Omega)$

$$
\left\|\operatorname{curl} E_{\mathrm{c}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2} \geq 2 \underbrace{\left\langle\operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} X\right\rangle_{\mathrm{L}^{2}(\Omega)}}_{=\langle h, \operatorname{curl} X\rangle_{\mathrm{L}^{2}(\Omega)}}-\|\operatorname{curl} X\|_{\mathrm{L}^{2}(\Omega)}^{2} \quad\left(H_{\mathrm{d}} \perp \operatorname{curl} X\right)
$$


similar for $H_{d}$ : for all $Y \in H\left(\right.$ curl $\left._{0} ; \Omega\right)$ and for all $Z \in H(\operatorname{curl} ; \Omega)$ with $\tau Z=G$

$=m_{-}(\tilde{H} ; Y, \operatorname{curl} Z)$
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## Proofs (Lower Bounds)

recall $h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
standard argument for curl $E_{\mathrm{c}}$ : for all $X \in \stackrel{\circ}{\mathrm{H}}($ curl $; \Omega)$

$$
\begin{aligned}
\left\|\operatorname{curl} E_{\mathrm{c}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2} & \geq 2 \underbrace{\left\langle\operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} X\right\rangle_{\mathrm{L}^{2}(\Omega)}}_{=\langle h, \operatorname{curl} X\rangle_{\mathrm{L}^{2}(\Omega)}}-\|\operatorname{curl} X\|_{\mathrm{L}^{2}(\Omega)}^{2} \quad\left(H_{\mathrm{d}} \perp \operatorname{curl} X\right) \\
& =2\langle F, \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\operatorname{curl} X+2 \tilde{H}, \operatorname{curl} X\rangle_{\mathrm{L}^{2}(\Omega)}=M_{-}(\tilde{H} ; X, \operatorname{curl} X)
\end{aligned}
$$

similar for $H_{d}$ : for all $Y \in H($ curlo; $\Omega)$ and for all $Z \in H(c u r l ; \Omega)$ with $\tau Z=G$


## Proofs (Lower Bounds)

recall $h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
standard argument for curl $E_{\mathrm{c}}$ : for all $X \in \stackrel{\circ}{\mathrm{H}}($ curl $; \Omega)$

$$
\begin{aligned}
\left\|\operatorname{curl} E_{\mathrm{c}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2} & \geq 2 \underbrace{}_{=\langle h, \operatorname{curl} X\rangle_{\mathrm{L}^{2}(\Omega)}\left\langle\operatorname{curl}^{c} E_{\mathrm{c}}, \operatorname{curl} X\right\rangle_{\mathrm{L}^{2}(\Omega)}}-\|\operatorname{curl} X\|_{\mathrm{L}^{2}(\Omega)}^{2} \quad\left(H_{\mathrm{d}} \perp \operatorname{curl} X\right) \\
& =2\langle F, \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\operatorname{curl} X+2 \tilde{H}, \operatorname{curl} X\rangle_{\mathrm{L}^{2}(\Omega)}=M_{-}(\tilde{H} ; X, \operatorname{curl} X)
\end{aligned}
$$

similar for $H_{d}$ : for all $Y \in H($ curlo; $\Omega)$ and for all $Z \in H($ curl $; \Omega)$ with $\tau Z=G$

$=m_{-}(\tilde{H} ; Y, \operatorname{curl} Z)$

## Proofs (Lower Bounds)

recall $h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
standard argument for curl $E_{\mathrm{c}}$ : for all $X \in \stackrel{\circ}{\mathrm{H}}($ curl $; \Omega)$

$$
\begin{aligned}
\left\|\operatorname{curl} E_{\mathrm{c}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2} & \geq 2 \underbrace{}_{=\langle h, \operatorname{curl} X\rangle_{\mathrm{L}^{2}(\Omega)}\left\langle\operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} X\right\rangle_{\mathrm{L}^{2}(\Omega)}}-\|\operatorname{curl} X\|_{\mathrm{L}^{2}(\Omega)}^{2} \quad\left(H_{\mathrm{d}} \perp \operatorname{curl} X\right) \\
& =2\langle F, \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\operatorname{curl} X+2 \tilde{H}, \operatorname{curl} X\rangle_{\mathrm{L}^{2}(\Omega)}=M_{-}(\tilde{H} ; X, \operatorname{curl} X)
\end{aligned}
$$

similar for $H_{d}$ : for all $Y \in H($ curlo; $\Omega)$ and for all $Z \in H($ curl $; \Omega)$ with $\tau Z=G$
$\left\|H_{d}\right\|_{L^{2}(\Omega)}^{2}$

(curl $E_{c} \perp Y$ )


## Proofs (Lower Bounds)

recall $h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
standard argument for curl $E_{\mathrm{c}}$ : for all $X \in \stackrel{\circ}{\mathrm{H}}($ curl $; \Omega)$

$$
\begin{aligned}
\left\|\operatorname{curl} E_{\mathrm{c}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2} & \geq 2 \underbrace{}_{=\langle h, \operatorname{curl} X\rangle_{\mathrm{L}^{2}(\Omega)}\left\langle\operatorname{curl}^{c} E_{\mathrm{c}}, \operatorname{curl} X\right\rangle_{\mathrm{L}^{2}(\Omega)}}-\|\operatorname{curl} X\|_{\mathrm{L}^{2}(\Omega)}^{2} \quad\left(H_{\mathrm{d}} \perp \operatorname{curl} X\right) \\
& =2\langle F, \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\operatorname{curl} X+2 \tilde{H}, \operatorname{curl} X\rangle_{\mathrm{L}^{2}(\Omega)}=M_{-}(\tilde{H} ; X, \operatorname{curl} X)
\end{aligned}
$$

similar for $H_{d}$ : for all $Y \in \mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right)$ and for all $Z \in \mathrm{H}($ curl $; \Omega)$ with $\tau Z=G$

$$
\left\|H_{d}\right\|_{L^{2}(\Omega)}^{2} \geq 2 \underbrace{\left\langle H_{d}, Y\right\rangle_{L^{2}(\Omega)}}_{=\langle h, Y\rangle_{L^{2}(\Omega)}}-\|Y\|_{L^{2}(\Omega)}^{2}
$$

$$
\text { (curl } E_{c} \perp Y \text { ) }
$$



## Proofs (Lower Bounds)

recall $h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
standard argument for curl $E_{\mathrm{c}}$ : for all $X \in \stackrel{\circ}{\mathrm{H}}($ curl $; \Omega)$

$$
\begin{aligned}
\left\|\operatorname{curl} E_{\mathrm{c}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2} & \geq 2 \underbrace{}_{=\langle h, \operatorname{curl} X\rangle_{\mathrm{L}^{2}(\Omega)}\left\langle\operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} X\right\rangle_{\mathrm{L}^{2}(\Omega)}}-\|\operatorname{curl} X\|_{\mathrm{L}^{2}(\Omega)}^{2} \quad\left(H_{\mathrm{d}} \perp \operatorname{curl} X\right) \\
& =2\langle F, \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\operatorname{curl} X+2 \tilde{H}, \operatorname{curl} X\rangle_{\mathrm{L}^{2}(\Omega)}=M_{-}(\tilde{H} ; X, \operatorname{curl} X)
\end{aligned}
$$

similar for $H_{d}$ : for all $Y \in \mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right)$ and for all $Z \in \mathrm{H}($ curl $; \Omega)$ with $\tau Z=G$

$$
\begin{aligned}
&\left\|H_{d}\right\|_{L^{2}(\Omega)}^{2} \geq 2 \underbrace{\left\langle H_{d}, Y\right\rangle_{L^{2}(\Omega)}}_{=\langle h, Y\rangle_{L^{2}(\Omega)}}-\|Y\|_{L^{2}(\Omega)}^{2} \\
&=2\langle\operatorname{curl} Z-\tilde{H}, Y\rangle_{L^{2}(\Omega)}-\|Y\|_{L^{2}(\Omega)}^{2} \quad\left(\operatorname{curl} E_{c} \perp Y\right) \\
&\text { curl } \underbrace{(E-Z)}_{\in \dot{\mathrm{H}}(\text { curl } ; \Omega)} \perp Y)
\end{aligned}
$$

## Proofs (Lower Bounds)

recall $h=H-\tilde{H}=\operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$
standard argument for curl $E_{\mathrm{c}}$ : for all $X \in \stackrel{\circ}{\mathrm{H}}($ curl $; \Omega)$

$$
\begin{aligned}
\left\|\operatorname{curl} E_{\mathrm{c}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2} & \geq 2 \underbrace{}_{=\langle h, \operatorname{curl} X\rangle_{\mathrm{L}^{2}(\Omega)}\left\langle\operatorname{curl} E_{\mathrm{c}}, \operatorname{curl} X\right\rangle_{\mathrm{L}^{2}(\Omega)}}-\|\operatorname{curl} X\|_{\mathrm{L}^{2}(\Omega)}^{2} \quad\left(H_{\mathrm{d}} \perp \operatorname{curl} X\right) \\
& =2\langle F, \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\operatorname{curl} X+2 \tilde{H}, \operatorname{curl} X\rangle_{\mathrm{L}^{2}(\Omega)}=M_{-}(\tilde{H} ; X, \operatorname{curl} X)
\end{aligned}
$$

similar for $H_{d}$ : for all $Y \in \mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right)$ and for all $Z \in \mathrm{H}($ curl $; \Omega)$ with $\tau Z=G$

$$
\begin{aligned}
\left\|H_{d}\right\|_{L^{2}(\Omega)}^{2} & \geq 2 \underbrace{\left\langle H_{d}, Y\right\rangle_{L^{2}(\Omega)}}_{=\langle h, Y\rangle_{L^{2}(\Omega)}}-\|Y\|_{L^{2}(\Omega)}^{2} \\
& =2\langle\operatorname{curl} Z-\tilde{H}, Y\rangle_{L^{2}(\Omega)}-\|Y\|_{L^{2}(\Omega)}^{2} \\
& =m_{-}(\tilde{H} ; Y, \operatorname{curl} Z)
\end{aligned}
$$

$$
(\operatorname{curl} \underbrace{(E-Z)}_{\in \stackrel{\circ}{\mathrm{H}}(\mathrm{curl} ; \Omega)} \perp Y)
$$

## Proofs (Lower Bounds Continued)

$\Rightarrow$ finally

$$
\|h\|_{\mathrm{L}^{2}(\Omega)}^{2}=\left\|\operatorname{curl} E_{\mathrm{c}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2}+\left\|H_{d}\right\|_{\mathrm{L}^{2}(\Omega)}^{2} \geq M_{-}(\tilde{H} ; X, \operatorname{curl} X)+m_{-}(\tilde{H} ; Y, \operatorname{curl} Z)
$$

for all $Y \in \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$ and all $Z \in \mathrm{H}(\operatorname{curl} ; \Omega)$ with $\tau Z=G$.
if $\tilde{H}=\operatorname{curl} \tilde{E}$ with $\tilde{E} \in H(c u r l ; \Omega) \Rightarrow$


## Proofs (Lower Bounds Continued)

$\Rightarrow$ finally

$$
\|h\|_{\mathrm{L}^{2}(\Omega)}^{2}=\left\|\operatorname{curl} E_{\mathrm{c}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2}+\left\|H_{d}\right\|_{\mathrm{L}^{2}(\Omega)}^{2} \geq M_{-}(\tilde{H} ; X, \operatorname{curl} X)+m_{-}(\tilde{H} ; Y, \operatorname{curl} Z)
$$

for all $Y \in \mathrm{H}\left(\right.$ curl $\left._{0} ; \Omega\right)$ and all $Z \in \mathrm{H}($ curl $; \Omega)$ with $\tau Z=G$.
if $\tilde{H}=\operatorname{curl} \tilde{E}$ with $\tilde{E} \in \mathrm{H}($ curl $; \Omega) \Rightarrow$

$$
\begin{aligned}
m_{-}(\tilde{H} ; Y, \operatorname{curl} Z)= & 2\langle\operatorname{curl}(Z-\tilde{E}), Y\rangle_{\mathrm{L}^{2}(\Omega)}-\|Y\|_{\mathrm{L}^{2}(\Omega)}^{2} \\
= & 2 \underbrace{\left\langle G-\tau \tilde{E}, \tau_{n} Y\right\rangle_{\text {trace }}}-\|Y\|_{\mathrm{L}^{2}(\Omega)}^{2} \\
& =\int_{\Gamma}(G-\nu \times \tilde{E}) Y "
\end{aligned}
$$

## Last Slide!

## Thank You!

