Functional A Posteriori Error Estimates for Electro-Magneto Static Problems

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Introduction: Static Maxwell Problem

- $\Omega \subset \mathbb{R}^3$ bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$
- $\varepsilon, \mu : \Omega \to \mathbb{R}^{3\times3}$ medium properties: bd., sym., unif. pos. def. matrices
- F given right hand side (current), G given boundary data (boundary current)
- E electric field or vector potential for $H := \mu^{-1} \operatorname{curl} E$ magnetic field
- τ tangential trace, i.e., $\tau E = \nu \times E|_\Gamma$
- $\perp$ orthogonality w.r.t. $\mathsf{L}^2(\Omega)$-scalar product $\langle E, H\rangle_\Omega := \int_\Omega E \cdot H$
- $\mathcal{H}_\varepsilon(\Omega)$ Dirichlet fields; $H \in \mathcal{H}_\varepsilon(\Omega)$, iff $\operatorname{curl} H = 0$ and $\operatorname{div} \varepsilon H = 0$ and $\tau H = 0$

magneto static problem with (electric) vector potential

$$\operatorname{curl} \mu^{-1} \operatorname{curl} E = F \quad \text{in } \Omega$$
$$\operatorname{div} \varepsilon E = 0 \quad \text{in } \Omega$$
$$\tau E = G \quad \text{on } \Gamma$$
$$\varepsilon E \perp \mathcal{H}_\varepsilon(\Omega)$$

- goal: NON-CONFORMING error estimates for $e := E - \tilde{E}$ and $h := \mu H - \tilde{H}$, where $\tilde{E}, \tilde{H}$ approx. of $E, \mu H$, i.e., $\tilde{E}, \tilde{H} \in \mathsf{L}^2(\Omega)$ just
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$$\begin{aligned} \operatorname{curl} \mu^{-1} \operatorname{curl} E &= F && \text{in } \Omega \\ \operatorname{div} \varepsilon E &= 0 && \text{in } \Omega \\ \tau E &= G && \text{on } \Gamma \\ \varepsilon E &\perp \mathcal{H}_\varepsilon(\Omega) \end{aligned}$$

- goal: NON-CONFORMING error estimates for $e := E - \tilde{E}$ and $h := \mu H - \tilde{H}$, where $\tilde{E}, \tilde{H}$ approx. of $E, \mu H$, i.e., $\tilde{E}, \tilde{H} \in \mathsf{L}^2(\Omega)$ just
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Introduction: Static Maxwell Problem

- $\Omega \subset \mathbb{R}^3$ bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$
- $\varepsilon, \mu : \Omega \to \mathbb{R}^{3\times 3}$ medium properties: bd., sym., unif. pos. def. matrices
- F given right hand side (current), G given boundary data (boundary current)
- E electric field or vector potential for $H := \mu^{-1} \operatorname{curl} E$ magnetic field
- τ tangential trace, i.e., $\tau E = \nu \times E|_\Gamma$
- $\perp$ orthogonality w.r.t. $\mathsf{L}^2(\Omega)$-scalar product $\langle E, H\rangle_\Omega := \int_\Omega E \cdot H$
- $\mathcal{H}_\varepsilon(\Omega)$ Dirichlet fields; $H \in \mathcal{H}_\varepsilon(\Omega)$, iff $\operatorname{curl} H = 0$ and $\operatorname{div} \varepsilon H = 0$ and $\tau H = 0$

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spaces:

$$\mathsf{H}(\mathrm{curl};\Omega) := \{E \in \mathsf{L}^2(\Omega) \,:\, \mathrm{curl}\, E \in \mathsf{L}^2(\Omega)\}$$

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testing $\operatorname{curl} \mu^{-1} \operatorname{curl} E = F$ with $\Phi \in \overset{\circ}{\mathrm{H}}(\operatorname{curl}; \Omega)$

$$\varphi(\Phi) := \langle F, \Phi \rangle_\Omega = \langle \mu^{-1} \operatorname{curl} E, \operatorname{curl} \Phi \rangle_\Omega =: b(E, H)$$

unfortunately: $\overset{\circ}{\mathrm{H}}(\operatorname{curl}; \Omega)$ is not the proper Hilbert space! (kernel of curl)

Poincaré-Friedrichs inequality: $\exists\, c_{\mathrm{PF}} > 0 \quad \forall\, E \in \mathrm{H}(\operatorname{curl}; \Omega) \cap \varepsilon^{-1}\mathrm{H}(\operatorname{div}; \Omega)$

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special case: $\forall\, E \in \mathbb{H} := \overset{\circ}{\mathrm{H}}(\operatorname{curl}; \Omega) \cap \varepsilon^{-1}\mathrm{H}(\operatorname{div}_0; \Omega) \cap \mathcal{H}_\varepsilon(\Omega)^{\perp_\varepsilon}$

$$\|E\|_{\mathrm{L}^2(\Omega)} \le c_{\mathrm{PF}} \, \|\operatorname{curl} E\|_{\mathrm{L}^2(\Omega)}$$

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Upper and Lower Bounds for Non-Conforming Approximations

$\tilde{H} \in \mathrm{L}^2(\Omega)$ approximation of $\operatorname{curl} E = \mu H$ (first, only approximation of magnetic field)

Theorem 1 *For all $\tilde{H} \in \mathrm{L}^2(\Omega)$ the estimates*

$$\left\| \mu H - \tilde{H} \right\|_{\mathrm{L}^2(\Omega)} \leq \inf_{Y \in \mathrm{H}(\operatorname{curl};\Omega)} M_+(\tilde{H}; Y, \operatorname{curl} Y) + \inf_{\substack{X \in \mathrm{H}(\operatorname{curl};\Omega) \\ \tau X = G}} m_+(\tilde{H}; \operatorname{curl} X),$$

$$\left\| \mu H - \tilde{H} \right\|^2_{\mathrm{L}^2(\Omega)} \geq \sup_{X \in \overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega)} M_-(\tilde{H}; X, \operatorname{curl} X) + \sup_{\substack{\mu^{-1} Y, Z \in \mathrm{H}(\operatorname{curl};\Omega) \\ \operatorname{curl} \mu^{-1} Y = 0 \\ \tau Z = G}} m_-(\tilde{H}; Y, \operatorname{curl} Z)$$

hold. Here,

$$M_+(\tilde{H}; Y, \operatorname{curl} Y) := c_{\mu,1} \left(c_{\mathrm{PF}} \left\| F - \operatorname{curl} Y \right\|_{\mathrm{L}^2(\Omega)} + \left\| \mu^{-1} \tilde{H} - Y \right\|_{\mathrm{L}^2(\Omega)} \right),$$

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$$M_-(\tilde{H}; X, \operatorname{curl} X) := 2 \langle F, X \rangle_{\mathrm{L}^2(\Omega)} - \left\langle \mu^{-1}(\operatorname{curl} X + 2\tilde{H}), \operatorname{curl} X \right\rangle_{\mathrm{L}^2(\Omega)},$$

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Upper and Lower Bounds for (Very) Conforming Approximations

$\tilde{E} \in \mathrm{H}(\mathrm{curl};\Omega)$ approx. of E and $\tilde{H} := \mathrm{curl}\,\tilde{E} \in \mathrm{L}^2(\Omega)$ approx. of $\mathrm{curl}\,E = \mu H$

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Norm Estimates

- norm estimates for $h = \mu H - \tilde{H}$ $\quad\surd$
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typical features of functional a posteriori error estimates

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Proofs (Helmholtz Decomposition)

main tools: standard techniques and Helmholtz decomposition

$\tilde{H}$ approximation of $\mu H = \operatorname{curl} E$, $h := \mu H - \tilde{H}$ error
simplicity $\varepsilon = \mu = \mathrm{id}$

$$\mathrm{L}^2(\Omega) \ni h = \operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}}$$

$$\operatorname{curl} E_{\mathrm{c}} \in \operatorname{curl} \overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega) = \operatorname{curl} \mathbb{H}$$

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recall $\mathbb{H} = \overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega) \cap \mathrm{H}(\operatorname{div}_0;\Omega) \cap \mathcal{H}(\Omega)^{\perp}$

$\operatorname{curl} E_{\mathrm{c}}$ regular/conforming error

H_{d} non-conforming error (boundary error)

orthogonality $\Rightarrow$ $\|h\|^2_{\mathrm{L}^2(\Omega)} = \|\operatorname{curl} E_{\mathrm{c}}\|^2_{\mathrm{L}^2(\Omega)} + \|H_{\mathrm{d}}\|^2_{\mathrm{L}^2(\Omega)}$

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Proofs (Upper Bounds)

recall $h = H - \tilde{H} = \operatorname{curl} E_c \oplus H_d \in \operatorname{curl} \mathbb{H} \oplus \mathsf{H}(\operatorname{curl}_0; \Omega)$

standard argument for $\operatorname{curl} E_c$: for all $\Phi \in \mathbb{H}$

$$\begin{aligned}\langle \operatorname{curl} E_c, \operatorname{curl} \Phi \rangle_{\mathsf{L}^2(\Omega)} &= \langle h, \operatorname{curl} \Phi \rangle_{\mathsf{L}^2(\Omega)} \quad (H_d \perp \operatorname{curl} \Phi)\\ &= \langle F, \Phi \rangle_{\mathsf{L}^2(\Omega)} - \left\langle \tilde{H}, \operatorname{curl} \Phi \right\rangle_{\mathsf{L}^2(\Omega)}\\ &= \langle F - \operatorname{curl} Y, \Phi \rangle_{\mathsf{L}^2(\Omega)} - \left\langle \tilde{H} - Y, \operatorname{curl} \Phi \right\rangle_{\mathsf{L}^2(\Omega)}\end{aligned}$$

since $\langle \operatorname{curl} Y, \Phi \rangle_{\mathsf{L}^2(\Omega)} = \langle Y, \operatorname{curl} \Phi \rangle_{\mathsf{L}^2(\Omega)}$ for all $Y \in \mathsf{H}(\operatorname{curl}; \Omega)$; note $\mathbb{H} \subset \mathring{\mathsf{H}}(\operatorname{curl}; \Omega)$.

Cauchy-Schwarz and Poincaré-Friedrichs and $\Phi := E_c \in \mathbb{H} \quad \Rightarrow$

$$\|\operatorname{curl} E_c\|_{\mathsf{L}^2(\Omega)} \leq c_{PF} \|F - \operatorname{curl} Y\|_{\mathsf{L}^2(\Omega)} + \left\|\tilde{H} - Y\right\|_{\mathsf{L}^2(\Omega)} = M_+(\tilde{H}; Y, \operatorname{curl} Y)$$

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Proofs (Upper Bounds Continued)

recall $h = H - \tilde{H} = \operatorname{curl} E_{\mathrm{c}} \oplus H_{\mathrm{d}} \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}(\operatorname{curl}_0; \Omega)$

argument for H_{d}: for all $\Psi \in \mathrm{H}(\operatorname{curl}_0; \Omega)$

$$\langle H_{\mathrm{d}}, \Psi \rangle_{\mathrm{L}^2(\Omega)} = \left\langle \operatorname{curl} X - \tilde{H}, \Psi \right\rangle_{\mathrm{L}^2(\Omega)} \quad (H_{\mathrm{d}} + \tilde{H} - \operatorname{curl} X = \operatorname{curl}(\underbrace{E - X - E_{\mathrm{c}}}_{\in \overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega)}) \perp \Psi)$$

for all $X \in \mathrm{H}(\operatorname{curl}; \Omega)$ with $\tau X = G$.

Cauchy-Schwarz and $\Psi := H_{\mathrm{d}}$ $\Rightarrow$

$$\|H_{\mathrm{d}}\|_{\mathrm{L}^2(\Omega)} \leq \left\| \operatorname{curl} X - \tilde{H} \right\|_{\mathrm{L}^2(\Omega)} = m_+(\tilde{H}; \operatorname{curl} X)$$

$\Rightarrow$ finally

$$\|h\|^2_{\mathrm{L}^2(\Omega)} = \|\operatorname{curl} E_{\mathrm{c}}\|^2_{\mathrm{L}^2(\Omega)} + \|H_{\mathrm{d}}\|^2_{\mathrm{L}^2(\Omega)} \leq M_+^2(\tilde{H}; Y, \operatorname{curl} Y) + m_+^2(\tilde{H}; \operatorname{curl} X)$$

for all $Y \in \mathrm{H}(\operatorname{curl}; \Omega)$ and all $X \in \mathrm{H}(\operatorname{curl}; \Omega)$ with $\tau X = G$.

$$\Rightarrow \quad m_+(\tilde{H}; \operatorname{curl} X) = \left\| \operatorname{curl}(X - \tilde{E}) \right\|_{\mathrm{L}^2(\Omega)} = \left\| \operatorname{curl} \check{\tau}(G - \tau \tilde{E}) \right\|_{\mathrm{L}^2(\Omega)} \leq c_\tau \left\| G - \tau \tilde{E} \right\|_{\mathrm{trace}}$$

if $\tilde{H} = \operatorname{curl} \tilde{E}$ with $\tilde{E} \in \mathrm{H}(\operatorname{curl}; \Omega)$
and we choose $X := \tilde{E} - \check{\tau}\tau\tilde{E} + \check{\tau}G \in \mathrm{H}(\operatorname{curl}; \Omega)$
(then $\tau X = G$)

Proofs (Upper Bounds Continued)

recall $h = H - \tilde{H} = \operatorname{curl} E_c \oplus H_d \in \operatorname{curl} \mathbb{H} \oplus \mathsf{H}(\operatorname{curl}_0; \Omega)$

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for all $X \in \mathsf{H}(\operatorname{curl}; \Omega)$ with $\tau X = G$.

Cauchy-Schwarz and $\Psi := H_d \quad \Rightarrow$

$$\|H_d\|_{\mathsf{L}^2(\Omega)} \leq \left\| \operatorname{curl} X - \tilde{H} \right\|_{\mathsf{L}^2(\Omega)} = m_+(\tilde{H}; \operatorname{curl} X)$$

$\Rightarrow$ finally

$$\|h\|^2_{\mathsf{L}^2(\Omega)} = \|\operatorname{curl} E_c\|^2_{\mathsf{L}^2(\Omega)} + \|H_d\|^2_{\mathsf{L}^2(\Omega)} \leq M^2_+(\tilde{H}; Y, \operatorname{curl} Y) + m^2_+(\tilde{H}; \operatorname{curl} X)$$

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Proofs (Upper Bounds Continued)

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$$\|h\|^2_{\mathsf{L}^2(\Omega)} = \|\operatorname{curl} E_{\mathrm{c}}\|^2_{\mathsf{L}^2(\Omega)} + \|H_{\mathrm{d}}\|^2_{\mathsf{L}^2(\Omega)} \leq M_+^2(\tilde{H}; Y, \operatorname{curl} Y) + m_+^2(\tilde{H}; \operatorname{curl} X)$$

for all $Y \in \mathsf{H}(\operatorname{curl}; \Omega)$ and all $X \in \mathsf{H}(\operatorname{curl}; \Omega)$ with $\tau X = G$.

$$\Rightarrow \quad m_+(\tilde{H}; \operatorname{curl} X) = \left\| \operatorname{curl}(X - \tilde{E}) \right\|_{\mathsf{L}^2(\Omega)} = \left\| \operatorname{curl} \check{\tau}(G - \tau \tilde{E}) \right\|_{\mathsf{L}^2(\Omega)} \leq c_\tau \left\| G - \tau \tilde{E} \right\|_{\mathrm{trace}}$$

if $\tilde{H} = \operatorname{curl} \tilde{E}$ with $\tilde{E} \in \mathsf{H}(\operatorname{curl}; \Omega)$
and we choose $X := \tilde{E} - \check{\tau}\tau\tilde{E} + \check{\tau}G \in \mathsf{H}(\operatorname{curl}; \Omega)$
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Proofs (Upper Bounds Continued)

recall $h = H - \tilde{H} = \operatorname{curl} E_c \oplus H_d \in \operatorname{curl} \mathbb{H} \oplus \mathrm{H}(\operatorname{curl}_0;\Omega)$

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Proofs (Lower Bounds)

recall $h = H - \tilde{H} = \operatorname{curl} E_c \oplus H_d \in \operatorname{curl} \mathbb{H} \oplus \mathsf{H}(\operatorname{curl}_0; \Omega)$

standard argument for curl E_c: for all $X \in \overset{\circ}{\mathsf{H}}(\operatorname{curl}; \Omega)$

$$\|\operatorname{curl} E_c\|^2_{\mathsf{L}^2(\Omega)} \geq 2 \underbrace{\langle \operatorname{curl} E_c, \operatorname{curl} X\rangle_{\mathsf{L}^2(\Omega)}}_{=\langle h, \operatorname{curl} X\rangle_{\mathsf{L}^2(\Omega)}} - \|\operatorname{curl} X\|^2_{\mathsf{L}^2(\Omega)} \qquad (H_d \perp \operatorname{curl} X)$$

$$= 2\langle F, \Phi\rangle_{\mathsf{L}^2(\Omega)} - \left\langle \operatorname{curl} X + 2\tilde{H}, \operatorname{curl} X\right\rangle_{\mathsf{L}^2(\Omega)} = M_-(\tilde{H}; X, \operatorname{curl} X)$$

similar for H_d: for all $Y \in \mathsf{H}(\operatorname{curl}_0; \Omega)$ and for all $Z \in \mathsf{H}(\operatorname{curl}; \Omega)$ with $\tau Z = G$

$$\|H_d\|^2_{\mathsf{L}^2(\Omega)} \geq 2 \underbrace{\langle H_d, Y\rangle_{\mathsf{L}^2(\Omega)}}_{=\langle h, Y\rangle_{\mathsf{L}^2(\Omega)}} - \|Y\|^2_{\mathsf{L}^2(\Omega)} \qquad (\operatorname{curl} E_c \perp Y)$$

$$= 2\left\langle \operatorname{curl} Z - \tilde{H}, Y\right\rangle_{\mathsf{L}^2(\Omega)} - \|Y\|^2_{\mathsf{L}^2(\Omega)} \qquad (\operatorname{curl} \underbrace{(E - Z)}_{\in \overset{\circ}{\mathsf{H}}(\operatorname{curl}; \Omega)} \perp Y)$$

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similar for H_{d}: for all $Y \in \mathsf{H}(\operatorname{curl}_0; \Omega)$ and for all $Z \in \mathsf{H}(\operatorname{curl}; \Omega)$ with $\tau Z = G$

$$\|H_{\mathrm{d}}\|^2_{\mathsf{L}^2(\Omega)} \geq 2 \underbrace{\langle H_{\mathrm{d}}, Y\rangle_{\mathsf{L}^2(\Omega)}}_{=\langle h, Y\rangle_{\mathsf{L}^2(\Omega)}} - \|Y\|^2_{\mathsf{L}^2(\Omega)} \quad (\operatorname{curl} E_{\mathrm{c}} \perp Y)$$

$$= 2 \left\langle \operatorname{curl} Z - \tilde{H}, Y \right\rangle_{\mathsf{L}^2(\Omega)} - \|Y\|^2_{\mathsf{L}^2(\Omega)} \quad (\operatorname{curl} \underbrace{(E - Z)}_{\in \overset{\circ}{\mathsf{H}}(\operatorname{curl}; \Omega)} \perp Y)$$

$$= m_-(\tilde{H}; Y, \operatorname{curl} Z)$$

Proofs (Lower Bounds Continued)

$\Rightarrow$ finally

$$\|h\|^2_{\mathrm{L}^2(\Omega)} = \|\operatorname{curl} E_{\mathrm{c}}\|^2_{\mathrm{L}^2(\Omega)} + \|H_{\mathrm{d}}\|^2_{\mathrm{L}^2(\Omega)} \geq M_-(\tilde{H}; X, \operatorname{curl} X) + m_-(\tilde{H}; Y, \operatorname{curl} Z)$$

for all $Y \in \mathrm{H}(\operatorname{curl}_0; \Omega)$ and all $Z \in \mathrm{H}(\operatorname{curl}; \Omega)$ with $\tau Z = G$.

if $\tilde{H} = \operatorname{curl} \tilde{E}$ with $\tilde{E} \in \mathrm{H}(\operatorname{curl}; \Omega) \Rightarrow$

$$m_-(\tilde{H}; Y, \operatorname{curl} Z) = 2 \left\langle \operatorname{curl}(Z - \tilde{E}), Y \right\rangle_{\mathrm{L}^2(\Omega)} - \|Y\|^2_{\mathrm{L}^2(\Omega)}$$

$$= 2 \underbrace{\left\langle G - \tau\tilde{E}, \tau_n Y \right\rangle_{\mathtt{trace}}}_{\text{" } = \int_\Gamma (G - \nu \times \tilde{E}) Y \text{ "}} - \|Y\|^2_{\mathrm{L}^2(\Omega)}$$

Proofs (Lower Bounds Continued)

$\Rightarrow$ finally

$$\|h\|^2_{\mathsf{L}^2(\Omega)} = \|\operatorname{curl} E_{\mathsf{c}}\|^2_{\mathsf{L}^2(\Omega)} + \|H_{\mathsf{d}}\|^2_{\mathsf{L}^2(\Omega)} \geq M_-(\tilde{H}; X, \operatorname{curl} X) + m_-(\tilde{H}; Y, \operatorname{curl} Z)$$

for all $Y \in \mathsf{H}(\operatorname{curl}_0;\Omega)$ and all $Z \in \mathsf{H}(\operatorname{curl};\Omega)$ with $\tau Z = G$.
if $\tilde{H} = \operatorname{curl} \tilde{E}$ with $\tilde{E} \in \mathsf{H}(\operatorname{curl};\Omega) \Rightarrow$

$$\begin{aligned} m_-(\tilde{H}; Y, \operatorname{curl} Z) &= 2 \left\langle \operatorname{curl}(Z - \tilde{E}), Y \right\rangle_{\mathsf{L}^2(\Omega)} - \|Y\|^2_{\mathsf{L}^2(\Omega)} \\ &= 2 \underbrace{\left\langle G - \tau\tilde{E}, \tau_n Y \right\rangle_{\texttt{trace}}}_{\text{''} = \int_\Gamma (G - \nu \times \tilde{E}) Y \text{''}} - \|Y\|^2_{\mathsf{L}^2(\Omega)} \end{aligned}$$

Last Slide!

Thank You!