Hilbert Complexes and PDEs

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Hilbert Complexes and PDEs

Some Hilbert Complexes

PDEs: de Rham complex 3D

grad-complex

$$\{0\} \underset{\pi_0}{\overset{\iota_0}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{div}}{\overset{\mathring{\operatorname{grad}}}{\rightleftarrows}} \vec{\mathsf{L}}^2 \underset{\operatorname{rot}}{\overset{\mathring{\operatorname{rot}}}{\rightleftarrows}} \vec{\mathsf{L}}^2 \underset{-\operatorname{grad}}{\overset{\mathring{\operatorname{div}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}} \mathbb{R}$$

PDEs

$$\partial_t^n - \underbrace{\operatorname{div}\mathring{\operatorname{grad}}}_{=\Delta_{\mathsf{D}}}, \qquad \partial_t^n - \underbrace{\mathring{\operatorname{div}}\operatorname{grad}}_{=\Delta_{\mathsf{N}}}, \qquad \partial_t^n + \underbrace{\operatorname{rot}\mathring{\operatorname{rot}}}_{=\vec{\square}_{\mathsf{t}}}, \qquad \partial_t^n + \underbrace{\operatorname{rot}\mathring{\operatorname{rot}} - \mathring{\operatorname{grad}}\operatorname{div}}_{=-\vec{\Delta}_{\mathsf{t}}}$$

elliptic $(n = 0)$ / parabolic $(n = 1)$ / hyperbolic $(n = 2)$

or skew-selfadjoint FOSs

$$\partial_t^m - \underbrace{\begin{bmatrix} 0 & \operatorname{div} \\ \mathring{\operatorname{grad}} & 0 \end{bmatrix}}_{=\text{Maxwell}_{\mathsf{D},\text{acoustic}}}, \quad \partial_t^m - \underbrace{\begin{bmatrix} 0 & -\operatorname{rot} \\ \mathring{\operatorname{rot}} & 0 \end{bmatrix}}_{=\text{Maxwell}_{\mathsf{t},\text{electromagnetic}}}, \quad \partial_t^m - \underbrace{\begin{bmatrix} 0 & \operatorname{div} & 0 & 0 \\ \mathring{\operatorname{grad}} & 0 & -\operatorname{rot} & 0 \\ 0 & \mathring{\operatorname{rot}} & 0 & \operatorname{grad} \\ 0 & 0 & \mathring{\operatorname{div}} & 0 \end{bmatrix}}_{=\text{Picard's extended Maxwell}_{\mathsf{t},\text{electromagnetic}}}$$

PDEs: de Rham complex 3D

Picard's extended Maxwell $\Rightarrow$ factorisation of the Laplacian

$$\begin{bmatrix} 0 & \operatorname{div} & 0 & 0 \\ \mathring{\operatorname{grad}} & 0 & -\operatorname{rot} & 0 \\ 0 & \mathring{\operatorname{rot}} & 0 & \operatorname{grad} \\ 0 & 0 & \mathring{\operatorname{div}} & 0 \end{bmatrix}^2 = \begin{bmatrix} \operatorname{div}\mathring{\operatorname{grad}} & 0 & 0 & 0 \\ 0 & \mathring{\operatorname{grad}}\operatorname{div} - \operatorname{rot}\mathring{\operatorname{rot}} & 0 & 0 \\ 0 & 0 & -\mathring{\operatorname{rot}}\operatorname{rot} + \operatorname{grad}\mathring{\operatorname{div}} & 0 \\ 0 & 0 & 0 & \mathring{\operatorname{div}}\operatorname{grad} \end{bmatrix} = \begin{bmatrix} \Delta_{\mathrm{D}} & 0 & 0 & 0 \\ 0 & \vec{\Delta}_{\mathrm{t}} & 0 & 0 \\ 0 & 0 & \vec{\Delta}_{\mathrm{n}} & 0 \\ 0 & 0 & 0 & \Delta_{\mathrm{N}} \end{bmatrix}$$

solving Picard's extended Maxwell $\Rightarrow$ Helmholtz/Weyl decompositions

$$\begin{bmatrix} 0 & \operatorname{div} & 0 & 0 \\ \mathring{\operatorname{grad}} & 0 & -\operatorname{rot} & 0 \\ 0 & \mathring{\operatorname{rot}} & 0 & \operatorname{grad} \\ 0 & 0 & \mathring{\operatorname{div}} & 0 \end{bmatrix} \Rightarrow \begin{aligned} \mathsf{L}^2 &= R(\operatorname{div}) \\ \vec{\mathsf{L}}^2 &= R(\mathring{\operatorname{grad}}) \oplus R(\operatorname{rot}) \oplus \text{1st cohomology group} \\ \vec{\mathsf{L}}^2 &= R(\operatorname{grad}) \oplus R(\mathring{\operatorname{rot}}) \oplus \text{2nd cohomology group} \\ \mathsf{L}^2 &= R(\mathring{\operatorname{div}}) \oplus \mathbb{R} \end{aligned}$$

PDEs: de Rham complex ND / manifolds

d-complex (mother of all complexes)

$$\{0\} \underset{\pi_0}{\overset{\iota_0}{\rightleftarrows}} \cdots \mathsf{L}^{2,q-1} \underset{-\delta_q}{\overset{\mathring{\mathrm{d}}_{q-1}}{\rightleftarrows}} \mathsf{L}^{2,q} \underset{-\delta_{q+1}}{\overset{\mathring{\mathrm{d}}_q}{\rightleftarrows}} \mathsf{L}^{2,q+1} \cdots \underset{\iota_{*\mathbb{R}}}{\overset{\pi_{*\mathbb{R}}}{\rightleftarrows}} *\,\mathbb{R}$$

PDEs

$$\partial_t^n - \delta\,\mathring{\mathrm{d}}, \qquad \partial_t^n - \mathring{\delta}\,\mathrm{d}, \qquad \partial_t^n - \delta\,\mathring{\mathrm{d}}, \qquad \partial_t^n \underbrace{-\delta\,\mathring{\mathrm{d}} - \mathring{\mathrm{d}}\,\delta}_{=-\mathring{\Delta}_{q,\mathrm{t}}}$$

elliptic ($n=0$) / parabolic ($n=1$) / hyperbolic ($n=2$)

or skew-selfadjoint FOSs $\partial_t^m - \underbrace{\begin{bmatrix} 0 & \delta \\ \mathring{\mathrm{d}} & 0 \end{bmatrix}}_{=\mathrm{Maxwell}_{\mathrm{t}}}$, $\partial_t^m - \underbrace{\begin{bmatrix} 0 & \delta & 0 & 0 \\ \mathring{\mathrm{d}} & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \delta \\ 0 & 0 & \mathring{\mathrm{d}} & 0 \end{bmatrix}}_{=\mathrm{Picard's\ extended\ Maxwell}_{\mathrm{t}}}$

PDEs: de Rham complex ND / manifolds

Picard's extended Maxwell $\Rightarrow$ factorisation of the Hodge-Laplacian

$$\begin{bmatrix} 0 & \delta & 0 & 0 \\ \mathring{\mathrm{d}} & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \delta \\ 0 & 0 & \mathring{\mathrm{d}} & 0 \end{bmatrix}^2 = \begin{bmatrix} \delta\,\mathring{\mathrm{d}} & 0 & 0 & 0 & 0 \\ 0 & \mathring{\mathrm{d}}\,\delta + \delta\,\mathring{\mathrm{d}} & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \mathring{\mathrm{d}}\,\delta + \delta\,\mathring{\mathrm{d}} & 0 \\ 0 & 0 & 0 & 0 & \mathring{\mathrm{d}}\,\delta \end{bmatrix} = \begin{bmatrix} \mathring{\Delta}_{0,\mathrm{t}} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathring{\Delta}_{N,\mathrm{t}} \end{bmatrix}$$

solving Picard's extended Maxwell $\Rightarrow$ Hodge/Helmholtz/Weyl decompositions

$$\begin{bmatrix} 0 & \delta & 0 & 0 \\ \mathring{\mathrm{d}} & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \delta \\ 0 & 0 & \mathring{\mathrm{d}} & 0 \end{bmatrix} \Rightarrow \begin{aligned} \mathrm{L}^{2,0} &= R(\delta) \\ &\vdots \\ \mathrm{L}^{2,q} &= R(\mathring{\mathrm{d}}) \oplus R(\delta) \oplus \text{(q-1)th cohomology group} \\ &\vdots \\ \mathrm{L}^{2,N} &= R(\mathring{\mathrm{d}}) \oplus *\mathbb{R} \end{aligned}$$

PDEs: elasticity complex 3D

symGrad-complex

$$\{0\} \underset{\pi_0}{\overset{\iota_0}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{Div}_{\mathbb{S}}}{\overset{\operatorname{sym}\mathring{\operatorname{Grad}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{RotRot}_{\mathbb{S}}^\top}{\overset{\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^\top}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{-\operatorname{symGrad}}{\overset{\mathring{\operatorname{Div}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathbb{RM}}}{\overset{\pi_{\mathbb{RM}}}{\rightleftarrows}} \mathbb{RM}$$

PDEs

$$\partial_t^n - \underbrace{\operatorname{Div}_{\mathbb{S}} \operatorname{sym}\mathring{\operatorname{Grad}}}_{=\vec{\Delta}_{\mathsf{D}}}, \qquad \partial_t^n - \underbrace{\mathring{\operatorname{Div}}_{\mathbb{S}} \operatorname{symGrad}}_{=\vec{\Delta}_{\mathsf{N}}}, \qquad \partial_t^n + \operatorname{RotRot}_{\mathbb{S}}^\top \operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^\top,$$

$$\partial_t^n + \underbrace{\operatorname{RotRot}_{\mathbb{S}}^\top \operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^\top}_{\text{4th order}} - \underbrace{\operatorname{sym}\mathring{\operatorname{Grad}} \operatorname{Div}_{\mathbb{S}}}_{\text{2nd order}}$$

(apparently mixed order type, but NOT compact perturbation!)

elliptic ($n = 0$) / parabolic ($n = 1$) / hyperbolic ($n = 2$)

PDEs: elasticity complex 3D

extended elasticity $\Rightarrow$ factorisation of the gen Laplacian

$$\begin{bmatrix} 0 & \operatorname{Div}_{\mathbb{S}} & 0 & 0 \\ \operatorname{sym}\mathring{\operatorname{Grad}} & 0 & -\operatorname{RotRot}_{\mathbb{S}}^{\top} & 0 \\ 0 & \operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^{\top} & 0 & \operatorname{symGrad} \\ 0 & 0 & \mathring{\operatorname{Div}}_{\mathbb{S}} & 0 \end{bmatrix}^2 = \begin{bmatrix} \vec{\Delta}_{\mathrm{D}} & 0 & 0 & 0 \\ 0 & \mathring{\Delta}_{\mathbb{S},t} & 0 & 0 \\ 0 & 0 & \mathring{\Delta}_{\mathbb{S},n} & 0 \\ 0 & 0 & 0 & \vec{\Delta}_{\mathrm{N}} \end{bmatrix}$$

$$= \begin{bmatrix} \operatorname{Div}_{\mathbb{S}} \operatorname{sym}\mathring{\operatorname{Grad}} & 0 & 0 & 0 \\ 0 & \operatorname{sym}\mathring{\operatorname{Grad}} \operatorname{Div}_{\mathbb{S}} - \operatorname{RotRot}_{\mathbb{S}}^{\top} \operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^{\top} & 0 & 0 \\ 0 & 0 & -\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^{\top} \operatorname{RotRot}_{\mathbb{S}}^{\top} + \operatorname{symGrad} \mathring{\operatorname{Div}}_{\mathbb{S}} & 0 \\ 0 & 0 & 0 & \mathring{\operatorname{Div}}_{\mathbb{S}} \operatorname{symGrad} \end{bmatrix}$$

solving extended elasticity $\Rightarrow$ gen Hodge/Helmholtz/Weyl decompositions

$$\begin{bmatrix} 0 & \operatorname{Div}_{\mathbb{S}} & 0 & 0 \\ \operatorname{sym}\mathring{\operatorname{Grad}} & 0 & -\operatorname{RotRot}_{\mathbb{S}}^{\top} & 0 \\ 0 & \operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^{\top} & 0 & \operatorname{symGrad} \\ 0 & 0 & \mathring{\operatorname{Div}}_{\mathbb{S}} & 0 \end{bmatrix} \Rightarrow \begin{aligned} \vec{\mathrm{L}}^2 &= R(\operatorname{Div}_{\mathbb{S}}) \\ \mathrm{L}^2_{\mathbb{S}} &= R(\operatorname{sym}\mathring{\operatorname{Grad}}) \oplus R(\operatorname{RotRot}_{\mathbb{S}}^{\top}) \oplus \text{1st cohomology group} \\ \mathrm{L}^2_{\mathbb{S}} &= R(\operatorname{symGrad}) \oplus R(\operatorname{Rot}\mathring{\operatorname{Rot}}_{\mathbb{S}}^{\top}) \oplus \text{2nd cohomology group} \\ \vec{\mathrm{L}}^2 &= R(\mathring{\operatorname{Div}}_{\mathbb{S}}) \oplus \mathbb{RM} \end{aligned}$$

PDEs: 1st and 2nd biharmonic / general relativity complexes 3D

Gradgrad-complex

$$\{0\} \underset{\pi_0}{\overset{\iota_0}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{divDiv}_{\mathbb{S}}}{\overset{\mathring{\operatorname{Gradgrad}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{symRot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}} \underset{-\operatorname{devGrad}}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathbb{RT}}}{\overset{\pi_{\mathbb{RT}}}{\rightleftarrows}} \mathbb{RT}$$

devGrad-complex

$$\{0\} \underset{\pi_0}{\overset{\iota_0}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{Div}_{\mathbb{T}}}{\overset{\mathring{\operatorname{devGrad}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}} \underset{\operatorname{Rot}_{\mathbb{S}}}{\overset{\mathring{\operatorname{symRot}}_{\mathbb{T}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{Gradgrad}}{\overset{\mathring{\operatorname{divDiv}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathbb{P}_1}}{\overset{\pi_{\mathbb{P}_1}}{\rightleftarrows}} \mathbb{P}_1$$

PDEs

$$\partial_t^n + \underbrace{\operatorname{divDiv}_{\mathbb{S}} \mathring{\operatorname{Gradgrad}}}_{=\Delta_{\mathsf{D}}^2}, \qquad \partial_t^n - \underbrace{\mathring{\operatorname{Div}}_{\mathbb{T}} \operatorname{devGrad}}_{=\vec{\Delta}_{\mathsf{N}}}, \qquad \partial_t^n + \operatorname{Rot}_{\mathbb{S}} \mathring{\operatorname{symRot}}_{\mathbb{T}},$$

$$\partial_t^n + \underbrace{\operatorname{symRot}_{\mathbb{T}} \mathring{\operatorname{Rot}}_{\mathbb{S}}}_{\text{2nd order}} + \underbrace{\mathring{\operatorname{Gradgrad}} \operatorname{divDiv}_{\mathbb{S}}}_{\text{4th order}}$$

(apparently mixed order type, but NOT compact perturbation!)

elliptic $(n = 0)$ / parabolic $(n = 1)$ / hyperbolic $(n = 2)$

BGG

whole zoo more ...

BGG resolution

I.N. Bernstein, I.M. Gelfand, S.I. Gelfand ('70s)

M. Eastwood

D. Arnold, R. Falk, R. Winther, ...

also important for FEM, FEEC, DEC, ...
(construction and analysis)

General Complex / $\rightsquigarrow$ FA-ToolBox

densely defined and closed (unbounded) linear operators

$$A_0 : D(A_0) \subset H_0 \to H_1, \qquad A_1 : D(A_1) \subset H_1 \to H_2$$
$$A_0^* : D(A_0^*) \subset H_1 \to H_0, \qquad A_1^* : D(A_1^*) \subset H_2 \to H_1$$

general complex property $\boxed{A_1 A_0 = 0}$, i.e., $R(A_0) \subset N(A_1)$ $\quad (\Leftrightarrow \ R(A_1^*) \subset N(A_0^*))$

Hilbert complex

$$N(A_0) \underset{\pi_{N(A_0)}}{\overset{\iota_{N(A_0)}}{\rightleftarrows}} \cdots \ H_0 \underset{A_0^*}{\overset{A_0}{\rightleftarrows}} H_1 \underset{A_1^*}{\overset{A_1}{\rightleftarrows}} H_2 \ \cdots \underset{\iota_{N(A_n^*)}}{\overset{\pi_{N(A_n^*)}}{\rightleftarrows}} N(A_n^*)$$

General Complex $\rightsquigarrow$ FA-ToolBox

Hilbert complex

$$N(\mathrm{A}_0) \underset{\pi_{N(\mathrm{A}_0)}}{\overset{\iota_{N(\mathrm{A}_0)}}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \cdots \mathrm{H}_{n-1} \underset{\mathrm{A}_{n-1}^*}{\overset{\mathrm{A}_{n-1}}{\rightleftarrows}} \mathrm{H}_n \underset{\mathrm{A}_n^*}{\overset{\mathrm{A}_n}{\rightleftarrows}} \mathrm{H}_{n+1} \cdots \mathrm{H}_N \underset{\mathrm{A}_N^*}{\overset{\mathrm{A}_N}{\rightleftarrows}} \mathrm{H}_{N+1} \underset{\iota_{N(\mathrm{A}_N^*)}}{\overset{\pi_{N(\mathrm{A}_N^*)}}{\rightleftarrows}} N(\mathrm{A}_N^*)$$

some equations

$$\partial_t^n + \mathrm{A}_0^*\mathrm{A}_0, \qquad \partial_t^n + \mathrm{A}_1\mathrm{A}_1^*, \qquad \partial_t^n + \mathrm{A}_1^*\mathrm{A}_1 + \mathrm{A}_0\mathrm{A}_0^*, \qquad \partial_t^m - \begin{bmatrix} 0 & -\mathrm{A}_1^* \\ \mathrm{A}_1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} \partial_t^m & \mathrm{A}_1^* \\ -\mathrm{A}_1 & \partial_t^\ell \end{bmatrix}$$

elliptic ($n = 0$) / parabolic ($n = 1$) / hyperbolic ($n = 2$) $\qquad m, \ell \in \{0, 1\}$

ell ($m = \ell = 0$) / para ($m = 1, \ell = 0$ or $m = 0, \ell = 1$) / hyper ($m = \ell = 1$)

$$\partial_t^m - \underbrace{\begin{bmatrix} 0 & -\mathrm{A}_0^* & 0 & 0 \\ \mathrm{A}_0 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -\mathrm{A}_N^* \\ 0 & 0 & \mathrm{A}_N & 0 \end{bmatrix}}_{=\text{extended complex operator}}$$

General Complex $\rightsquigarrow$ FA-ToolBox

extended complex operator $\Rightarrow$ factorisation of gen Hodge-Laplacian

$$\begin{bmatrix} 0 & -A_0^* & 0 & 0 \\ A_0 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -A_N^* \\ 0 & 0 & A_N & 0 \end{bmatrix}^2 = -\begin{bmatrix} A_0^*A_0 & 0 & 0 & 0 & 0 \\ 0 & A_0A_0^* \oplus A_1^*A_1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & A_{N-1}A_{N-1}^* \oplus A_N^*A_N & 0 \\ 0 & 0 & 0 & 0 & A_NA_N^* \end{bmatrix}$$

$$= -\begin{bmatrix} \Delta_{A,0} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \Delta_{A,N+1} \end{bmatrix}$$

solving extended complex operator $\Rightarrow$ gen Hodge/Helmholtz/Weyl decompositions

$$\begin{bmatrix} 0 & -A_0^* & 0 & 0 \\ A_0 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -A_N^* \\ 0 & 0 & A_N & 0 \end{bmatrix} \Rightarrow \begin{aligned} H_0 &= R(A_0^*) \oplus N(A_0) \\ &\vdots \\ H_n &= R(A_{n-1}) \oplus R(A_n^*) \oplus \big(N(A_n) \cap N(A_{n-1}^*)\big) \\ &\vdots \\ H_{N+1} &= R(A_N) \oplus N(A_N^*) \end{aligned}$$

General Complex $\rightsquigarrow$ FA-ToolBox

extended complex operator

$$\begin{bmatrix} 0 & -A_0^* & 0 & 0 \\ A_0 & 0 & -A_1^* & 0 \\ 0 & A_1 & 0 & -A_2^* \\ 0 & 0 & A_2 & 0 \end{bmatrix}$$

another nice operator is

$$\begin{bmatrix} A_2 & 0 \\ A_1^* & A_0 \end{bmatrix}$$

solving the gen div-curl system (g st $y = 0$)

and giving the gen Hodge/Helmholtz/Weyl decomposition (eg $f = 0$)

$$A_2 x = f \in R(A_2) \qquad (\perp \text{ kernel of } A_2^*)$$

$$A_1^* x + A_0 y = g \in R(A_1^*) \oplus R(A_0) \qquad (\perp \text{ cohomology group})$$

think of

$$\begin{bmatrix} A_2 & 0 \\ A_1^* & A_0 \end{bmatrix} = \begin{bmatrix} \mathring{\operatorname{div}} & 0 \\ \operatorname{rot} & \mathring{\operatorname{grad}} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathring{\operatorname{div}}\, x & = f \\ \operatorname{rot} x + \mathring{\operatorname{grad}}\, y & = g \end{bmatrix}$$

$\mathring{\operatorname{grad}}\, y$ sometimes Lagrange parameter

Hilbert Complexes and PDEs

OVERVIEW

(I) general theory

FA-ToolBox (Hilbert complexes, tailor-made and simple functional analysis)

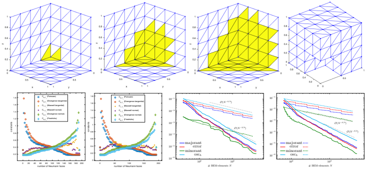
$$\boxed{Ax = f}$$

- solution theories
- closed ranges and continuous inverse operators
- Friedrichs/Poincaré estimates
- inf sup lemmas
- results about constants/eigenvalues
- Helmholtz/Hodge/Weyl decompositions
- compact embeddings
- compact inverse operators
- (compact) trace Hilbert complexes
- variational formulations
- regular potentials and regular decompositions
- generalized div-curl-lemmas
- index theorems
- dimensions and bases of cohomology groups
- functional a posteriori error estimates
- optimal control problems
- finding new FEs
- ...

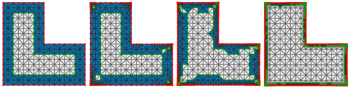
(II) applications to pdes (complexes)

- $\cdots \; L^2 \; \underset{-\operatorname{div}}{\overset{\mathring{\operatorname{grad}}}{\rightleftarrows}} \; L^2 \; \underset{\operatorname{rot}}{\overset{\mathring{\operatorname{rot}}}{\rightleftarrows}} \; L^2 \; \underset{-\operatorname{grad}}{\overset{\mathring{\operatorname{div}}}{\rightleftarrows}} \; L^2 \; \cdots$ (de Rham)
- $\cdots \; L^2 \; \underset{-\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \; L^2 \; \underset{-\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \; L^2 \; \cdots$ (de Rham)
- $\cdots \; L^2 \; \underset{-\operatorname{Div}_{\mathbb{S}}}{\overset{\operatorname{sym}\mathring{\operatorname{Grad}}}{\rightleftarrows}} \; L^2_{\mathbb{S}} \; \underset{\operatorname{Rot}\operatorname{Rot}^{\top}_{\mathbb{S}}}{\overset{\operatorname{Rot}\mathring{\operatorname{Rot}}^{\top}_{\mathbb{S}}}{\rightleftarrows}} \; L^2_{\mathbb{S}} \; \underset{-\operatorname{sym}\operatorname{Grad}}{\overset{\mathring{\operatorname{Div}}_{\mathbb{S}}}{\rightleftarrows}} \; L^2 \; \cdots$ (elasticity)
- $\cdots \; L^2 \; \underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\operatorname{Grad}\mathring{\operatorname{grad}}}{\rightleftarrows}} \; L^2_{\mathbb{S}} \; \underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \; L^2_{\mathbb{T}} \; \underset{-\operatorname{dev}\operatorname{Grad}}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}} \; L^2 \; \cdots$ (biharmonic/general relativity)
- $\cdots$ (... much more complexes)
- $\cdots \; H_0 \; \underset{A_0^*}{\overset{A_0}{\rightleftarrows}} \; H_1 \; \underset{A_1^*}{\overset{A_1}{\rightleftarrows}} \; H_2 \; \cdots$ (general)

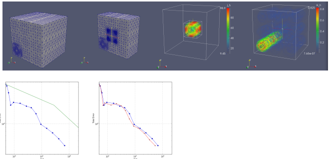
(III) numerical applications to pdes



- Friedrichs/Poincaré/Maxwell constants
  monotonicity?
  analytical results and FEM computations
  (with colleagues from Praha, Wien)



- functional a posteriori error estimates
  for BEM
  computations with BEM and FEM
  (with colleagues from Darmstadt, München, Wien, St. Petersburg, Bosch GmbH)



- functional a posteriori error estimates
  electro-magneto static optimal control problems
  computations with FEM
  (with colleagues from Darmstadt, Essen)

- DEC-FA-ToolBox
  DEC: Discrete Exterior Calculus as discrete version of FA-ToolBox
  (with colleagues from Jyväskylä)
  ⇝ `https://sites.google.com/jyu.fi/gfd/method/online-time-integrator`

Solving PDEs with Hilbert Complexes

Introduction and Motivation

Solving PDEs with Hilbert Complexes

FA-ToolBox

general observations

$$Ax = f$$

general observations

$$\boxed{\mathrm{A}x = f}$$

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ (lin, dd, cl, H_0, H_1 Hilbert spaces)

$$\boxed{? \qquad x = \mathrm{A}^{-1} f \qquad ?}$$

general observations

$$\boxed{\mathrm{A}x = f}$$

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ (lin, dd, cl)

solution theory in the sense of Hadamard

- existence $\Leftrightarrow$ $f \in R(\mathrm{A})$
- uniqueness $\Leftrightarrow$ A inj $\Leftrightarrow$ $N(\mathrm{A}) = \{0\}$ $\Leftrightarrow$ A^{-1} exists
- cont dep on f $\Leftrightarrow$ A^{-1} cont

aim: $x = \mathrm{A}^{-1} f \in D(\mathrm{A})$ and cont estimate (Friedrichs/Poincaré type estimate)

$$|x|_{\mathrm{H}_0} = |\mathrm{A}^{-1} f|_{\mathrm{H}_0} \leq c_{\mathrm{A}} |f|_{\mathrm{H}_1} = c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$$

note: best constant $c_{\mathrm{A}} = |\mathrm{A}^{-1}|_{R(\mathrm{A}),\mathrm{H}_0} = \frac{1}{\lambda_0}$,

λ_0^2 smallest eigenvalue of $\mathrm{A}^*\mathrm{A}$ (and $\mathrm{A}\mathrm{A}^*$)

general observations

$A : D(A) \subset H_0 \to H_1$

$A^* : D(A^*) \subset H_1 \to H_0$ Hilbert space adjoint

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$H_1 = \overline{R(A)} \oplus N(A^*), \qquad H_0 = N(A) \oplus \overline{R(A^*)}$$

$$\boxed{Ax = f}$$

solution theory in the sense of Hadamard

- existence $\Leftrightarrow$ $f \in R(A) = N(A^*)^\perp$ (Fredholm alt, if $R(A)$ cl)
- uniqueness $\Leftrightarrow$ A inj $\Leftrightarrow$ $N(A) = \{0\}$ $\Leftrightarrow$ A^{-1} exists
- cont dep on f $\Leftrightarrow$ A^{-1} cont $\Leftrightarrow$ $R(A)$ cl (cl graph theo)

fund range cond: $\boxed{R(A) = \overline{R(A)} \text{ closed}}$ (must hold $\rightsquigarrow$ right setting!)

kernel cond: $\boxed{N(A) = \{0\}}$ (fails in gen $\rightsquigarrow$ proj onto $N(A)^\perp = \overline{R(A^*)} = R(A^*)$)

general observations

observations (from this perspective)

- time-dependent problems are simple

  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H} \to \mathrm{H}$, $\boxed{\mathrm{A} = \partial_t + \mathrm{T}}$ (gen T skw-sa, or at least Re $\mathrm{T} \geq 0$)

  $$N(\mathrm{A}) = \{0\} \qquad N(\mathrm{A}^*) = \{0\} \qquad R(\mathrm{A}) \text{ (cl) } = N(\mathrm{A}^*)^{\perp} = \mathrm{H}$$

- time-harmonic problems are more complicated

  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H} \to \mathrm{H}$, $\boxed{\mathrm{A} = -\omega + \mathrm{T}}$

  $$N(\mathrm{A}),\ N(\mathrm{A}^*) \text{ (fin dim)} \qquad R(\mathrm{A}) \text{ (cl, fin co-dim) } = N(\mathrm{A}^*)^{\perp}$$

  (Fredholm alternative)

- static problems are most complicated

  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$, $\boxed{\mathrm{A} = 0 + \mathrm{T}}$

  $$\dim N(\mathrm{A}) = \dim N(\mathrm{A}^*) = \infty \text{ (possible/standard)} \qquad R(\mathrm{A}) \text{ (cl, infin co-dim) } = N(\mathrm{A}^*)^{\perp}$$

compactness $\Rightarrow$ cl (closed range)

general key observations I

$$A : D(A) \subset H_0 \to H_1 \qquad \text{(lddc, } H_0, H_1 \text{ Hilbert spaces)}$$
$$A^* : D(A^*) \subset H_1 \to H_0 \qquad \text{(Hilbert space adjoint, } (A, A^*) \text{ dual pair)}$$

A, A^* may not be inj $\Rightarrow$ Helmholtz/Hodge/Weyl decos (proj theorem)

$$H_1 = N(A^*) \oplus \overline{R(A)} \qquad H_0 = N(A) \oplus \overline{R(A^*)}$$

$$\mathcal{A} := A|_{N(A)^\perp} = A|_{\overline{R(A^*)}} : D(A) \cap N(A)^\perp \subset \overline{R(A^*)} \to \overline{R(A)} \qquad \text{(reduced operators)}$$
$$\mathcal{A}^* := A^*|_{N(A^*)^\perp} = A|_{\overline{R(A)}} : D(A^*) \cap N(A^*)^\perp \subset \overline{R(A)} \to \overline{R(A^*)}$$

$\mathcal{A}$, $\mathcal{A}^*$ inj $\Rightarrow$ $\mathcal{A}^{-1} : R(A) \to D(\mathcal{A})$, $(\mathcal{A}^*)^{-1} : R(A^*) \to D(\mathcal{A}^*)$ ex, ? bounded ?

general key observations I

FA-ToolBox

first simple lemmas

general key observations I

Lemma (FA-ToolBox Lemma 1)

The following assertions are equivalent:

- $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *bd*
- $\forall\, x \in D(\mathcal{A}) \quad |x|_{\mathrm{H}_0} \leq c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$
- $R(\mathrm{A}) = R(\mathcal{A})$ *cl*
- $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *bd*
- $\forall\, y \in D(\mathcal{A}^*) \quad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}} |\mathrm{A}^* y|_{\mathrm{H}_0}$
- $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *cl*

note: best const $c_{\mathrm{A}} = |\mathrm{A}^{-1}|_{R(\mathrm{A}),\mathrm{H}_0} = |(\mathrm{A}^*)^{-1}|_{R(\mathrm{A}^*),\mathrm{H}_1} = \frac{1}{\lambda_0}$,

λ_0^2 smallest pos ev of $\mathrm{A}^*\mathrm{A}$ and $\mathrm{A}\mathrm{A}^*$

Lemma (FA-ToolBox Lemma 2)

The following assertions are equivalent:

- $D(\mathcal{A}) \hookrightarrow \mathrm{H}_0$ *cpt*
- $\mathcal{A}^{-1} : R(\mathrm{A}) \to \mathrm{H}_0$ *cpt*
- $D(\mathcal{A}^*) \hookrightarrow \mathrm{H}_1$ *cpt*
- $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to \mathrm{H}_1$ *cpt*

Lemma (FA-ToolBox Lemma 3)

$D(\mathcal{A}) \hookrightarrow \mathrm{H}_0$ *cpt* $\quad \Rightarrow \quad$ *assertions of Lemma 1 hold.*

general key observations II

FA-ToolBox

so far no complex

time-dependent problems: OK
time-harmonic problems: OK
static problems: NOT OK

general key observations II

$$A_0 : D(A_0) \subset H_0 \to H_1, \qquad A_0^* : D(A_0^*) \subset H_1 \to H_0 \qquad \text{(lddc)}$$
$$A_1 : D(A_1) \subset H_1 \to H_2, \qquad A_1^* : D(A_1^*) \subset H_2 \to H_1$$

gen cmplx $\boxed{A_1 A_0 \subset 0}$ $\quad \big(\Leftrightarrow \ R(A_0) \subset N(A_1) \ \Leftrightarrow \ R(A_1^*) \subset N(A_0^*) \ \Leftrightarrow \ A_0^* A_1^* \subset 0\big)$

$$\boxed{\cdots \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad H_0 \quad \underset{A_0^*}{\overset{A_0}{\rightleftarrows}} \quad H_1 \quad \underset{A_1^*}{\overset{A_1}{\rightleftarrows}} \quad H_2 \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad \cdots}$$

recall Helmholtz deco

$$\begin{array}{rcl} H_1 = \overline{R(A_0)} & \oplus & N(A_0^*) \\ \bigcap & & \bigcup \\ = N(A_1) & \oplus & \overline{R(A_1^*)} \end{array} \quad \Rightarrow \ \text{(e.g.)} \quad N(A_1) = \overline{R(A_0)} \oplus \underbrace{\big(N(A_1) \cap N(A_0^*)\big)}_{=: N_{0,1} \text{ cohom gr}}$$

$\Rightarrow$ refined Helmholtz deco $\boxed{H_1 = \overline{R(A_0)} \oplus N_{0,1} \oplus \overline{R(A_1^*)}}$

general key observations II

refined Helmholtz deco

$$\mathrm{H}_1 = \overbrace{\overline{R(\mathrm{A}_0)}}^{=N(\mathrm{A}_0^*)^\perp} \oplus N_{0,1} \oplus \overbrace{\overline{R(\mathrm{A}_1^*)}}^{=N(\mathrm{A}_1)^\perp}$$

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus N_{0,1} \oplus D(\mathcal{A}_1)$$

Lemma (FA-ToolBox Lemma 4)

$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathrm{H}_1$ *compact* $\Rightarrow$ *Lemma 1 holds*

Lemma (FA-ToolBox Lemma 5)

The following assertions are equivalent:

- $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathrm{H}_1$ *compact*
- $D(\mathcal{A}_0) \hookrightarrow \mathrm{H}_0$ *and* $D(\mathcal{A}_1) \hookrightarrow \mathrm{H}_1$ *and* $N_{0,1} \hookrightarrow \mathrm{H}_1$ *compact*

In this case $\dim N_{0,1} < \infty$

Remark (FA-ToolBox Remark 1)

cohomology group: $N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) = N(\mathrm{A}_1) \cap R(\mathrm{A}_0)^\perp \cong N(\mathrm{A}_1)/R(\mathrm{A}_0)$

general key observations II

Remark (FA-ToolBox Remark 2)

$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathrm{H}_1$ compact $\quad\Rightarrow\quad$ everything holds!

Remark

Question: How to prove $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathrm{H}_1 \text{ compact}}$?

Answer: $\boxed{\text{FA-ToolBox}}$ + $\boxed{\text{regular decompositions and regular potentials}}$

Hilbert Complexes and PDEs

Some Selected Results

Assumption (on the domain where the PDEs are posed)

Let $\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain with bounded weak Lipschitz interface (mixed boundary conditions).

FA-ToolBox: Basics

Theorem (compact complexes)

All the latter Hilbert complexes are compact, i.e., for all n

$$D(\mathrm{A}_n) \cap D(\mathrm{A}_{n-1}^*) \hookrightarrow \mathrm{H}_n \text{ compact.}$$

best and gen assumption

Corollary (closed complexes)

All the latter Hilbert complexes are closed, i.e., for all n $R(\mathrm{A}_n)$, $R(\mathrm{A}_n^*)$ *closed.*

Corollary (Helmholtz/Hodge/Weyl decompositions)

For all n, e.g., $\mathrm{H}_n = R(\mathrm{A}_{n-1}) \oplus N_{n-1,n} \oplus R(\mathrm{A}_n^*)$

Corollary (Friedrichs/Poincaré estimates)

All the latter operators admit Friedrichs/Poincaré type estimates, i.e., for all n

$$\forall\, x \in D(\mathcal{A}_n) \quad |x|_{\mathrm{H}_n} \leq c_{\mathrm{A}_n} |\mathrm{A}_n x|_{\mathrm{H}_{n+1}}.$$

FA-ToolBox: Applications I

application: solution theories

e.g.: FOS (gen electro-magneto statics / saddle point problems)

$$\begin{aligned} \mathrm{A}_1 x &= f \\ \mathrm{A}_0^* x &= g \\ \pi_{N_{0,1}} x &= k \end{aligned}$$

Theorem (static solution theory)

FOS uniquely solvable $\Leftrightarrow$ $f \in R(\mathrm{A}_1)$, $g \in R(\mathrm{A}_0^*)$, $k \in N_{0,1}$
Moreover

$$x = x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus N_{0,1} = D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$$

with

$$\boxed{x_f := \mathcal{A}_1^{-1} f} \in D(\mathcal{A}_1), \qquad \boxed{x_g := (\mathcal{A}_0^*)^{-1} g} \in D(\mathcal{A}_0^*)$$

and dep cont on data $|x|_{\mathrm{H}_1}^2 = |x_f|_{\mathrm{H}_1}^2 + |x_g|_{\mathrm{H}_1}^2 + |k|_{\mathrm{H}_1}^2 \leq c_{\mathrm{A}_1}^2 |f|_{\mathrm{H}_2}^2 + c_{\mathrm{A}_0}^2 |g|_{\mathrm{H}_0}^2 + |k|_{\mathrm{H}_1}^2$

note: $x = \mathcal{A}_1^{-1} f + (\mathcal{A}_0^*)^{-1} g + k$

FA-ToolBox: Applications I

application: solution theories

same for other systems such as

$$\begin{aligned} \mathrm{A}_1^*\mathrm{A}_1 x &= h \\ \mathrm{A}_0^* x &= g \\ \pi_{N_{0,1}} x &= k \\ x = \mathcal{A}_1^{-1}(\mathcal{A}_1^*)^{-1}h &+ (\mathcal{A}_0^*)^{-1}g + k \end{aligned}$$

$$\begin{aligned} \mathrm{A}_1 x &= f \\ \mathrm{A}_0\mathrm{A}_0^* x &= j \\ \pi_{N_{0,1}} x &= k \\ x = \mathcal{A}_1^{-1}f &+ (\mathcal{A}_0^*)^{-1}\mathcal{A}_0^{-1}j + k \end{aligned}$$

$$\begin{aligned} \mathrm{A}_1^*\mathrm{A}_1 x &= h \\ \mathrm{A}_0\mathrm{A}_0^* x &= j \\ \pi_{N_{0,1}} x &= k \\ x = \mathcal{A}_1^{-1}(\mathcal{A}_1^*)^{-1}h &+ (\mathcal{A}_0^*)^{-1}\mathcal{A}_0^{-1}j + k \end{aligned}$$

$$\begin{aligned} (\mathrm{A}_1^*\mathrm{A}_1 + \mathrm{A}_0\mathrm{A}_0^*) x &= l \\ \pi_{N_{0,1}} x &= k \\ x = \mathcal{A}_1^{-1}(\mathcal{A}_1^*)^{-1}\pi_{R(\mathrm{A}_1^*)}l &+ k \\ &+ (\mathcal{A}_0^*)^{-1}\mathcal{A}_0^{-1}\pi_{R(\mathrm{A}_0)}l \end{aligned}$$

FA-ToolBox: Applications I

application: a posteriori error estimates for FOS/SOS

problem: find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ st $\mathrm{A}_1 x = f$ $\mathrm{A}_0^* x = g$ $\pi_{N_1} x = k$

'very' non-conforming 'approximation' of x: $\tilde{x} \in \mathrm{H}_1$

def., dcmp. err. $e = x - \tilde{x} = \pi_{R(\mathrm{A}_0)} e + \pi_{N_1} e + \pi_{R(\mathrm{A}_1^*)} e \in \mathrm{H}_1 = R(\mathrm{A}_0) \oplus N_1 \oplus R(\mathrm{A}_1^*)$

Theorem (sharp upper bounds)

Let $\tilde{x} \in \mathrm{H}_1$ *and* $e = x - \tilde{x}$. *Then*

$$|e|_{\mathrm{H}_1}^2 = |\pi_{R(\mathrm{A}_0)} e|_{\mathrm{H}_1}^2 + |\pi_{N_1} e|_{\mathrm{H}_1}^2 + |\pi_{R(\mathrm{A}_1^*)} e|_{\mathrm{H}_1}^2$$

$$|\pi_{R(\mathrm{A}_0)} e|_{\mathrm{H}_1} = \min_{\phi \in D(\mathrm{A}_0^*)} \big(c_{\mathrm{A}_0} |\mathrm{A}_0^* \phi - g|_{\mathrm{H}_0} + |\phi - \tilde{x}|_{\mathrm{H}_1} \big)$$

reg $(\mathrm{A}_0 \mathrm{A}_0^* + 1)$*-prbl in* $D(\mathrm{A}_0^*)$

$$|\pi_{R(\mathrm{A}_1^*)} e|_{\mathrm{H}_1} = \min_{\varphi \in D(\mathrm{A}_1)} \big(c_{\mathrm{A}_1} |\mathrm{A}_1 \varphi - f|_{\mathrm{H}_2} + |\varphi - \tilde{x}|_{\mathrm{H}_1} \big)$$

reg $(\mathrm{A}_1^* \mathrm{A}_1 + 1)$*-prbl in* $D(\mathrm{A}_1)$

$$|\pi_{N_1} e|_{\mathrm{H}_1} = |\pi_{N_1} \tilde{x} - k|_{\mathrm{H}_1} = \min_{\substack{\xi \in D(\mathrm{A}_0) \\ \zeta \in D(\mathrm{A}_1^*)}} |\mathrm{A}_0 \xi + \mathrm{A}_1^* \zeta + \tilde{x} - k|_{\mathrm{H}_1}$$

cpld $(\mathrm{A}_0^* \mathrm{A}_0)$-$(\mathrm{A}_1 \mathrm{A}_1^*)$*-sys in* $D(\mathcal{A}_0)$-$D(\mathcal{A}_1^*)$

Remark

also share lower bounds

FA-ToolBox: Applications I

application: div-curl lemmas

Theorem (div-curl lemma / A_0^*-A_1 lemma)

(x_n) bounded in $D(A_1)$ and (y_n) bounded in $D(A_0^*)$

$\Rightarrow \exists\, x \in D(A_1)$, $y \in D(A_0^*)$ and subseq st $x_n \rightharpoonup x$ in $D(A_1)$ and $y_n \rightharpoonup y$ in $D(A_0^*)$ and

$$\langle x_n, y_n \rangle_{H_1} \to \langle x, y \rangle_{H_1}$$

FA-ToolBox: Applications I

application: div-curl lemmas

Theorem (gen div-curl lemma / gen A_0^*-A_1 lemma)

enough: $R(A_0)$, $R(A_1$ *closed and* $\dim N_{0,1} < \infty$. *Then:* $(x_n), (y_n) \subset H_1$ *bounded and*

(i) $(\widetilde{A}_1 x_n)$ *is relatively compact in* $D(A_1^*)'$

(ii) $(\widetilde{A}_0^* y_n)$ *is relatively compact in* $D(A_0)'$

$\Rightarrow \exists\, x, y \in H_1$ *and subseq st* $x_n \rightharpoonup x$ *in* H_1 *and* $y_n \rightharpoonup y$ *in* H_1 *and*

$$\langle x_n, y_n \rangle_{H_1} \to \langle x, y \rangle_{H_1}.$$

proof uses key observation by Marcus Waurick

Lemma

Let $R(A)$ *be closed. For* $(x_n) \subset H_0$ *the following statements are equivalent:*

(i) $(\widetilde{A} x_n)$ *is relatively compact in* $D(A^*)'$.

(ii) $(\pi_{R(A^*)} x_n)$ *is relatively compact in* $R(A^*)$ *resp.* H_1.

If $x_n \rightharpoonup x$ *in* H_1, *then either of cond. (i) or (ii) implies* $\pi_{R(A^*)} x_n \to \pi_{R(A^*)} x$ *in* H_1.

FA-ToolBox: Continued

Corollary (compact resolvents)

All the latter corresponding inverses (of the reduced operators) are compact, i.e., for all n

$$\mathcal{A}_n^{-1},\ (\mathcal{A}_n^*)^{-1} \text{ compact.}$$

Corollary (spectra)

All the latter operators have discrete point spectra with finite eigenspaces, i.e., for all n

$$\sigma\Big(\begin{bmatrix} 0 & \mathrm{A}_n^* \\ \mathrm{A}_n & 0 \end{bmatrix}\Big) \setminus \{0\} = \pm\sqrt{\sigma(\mathrm{A}_n^*\mathrm{A}_n)} \setminus \{0\} = \pm\sqrt{\sigma(\mathrm{A}_n\mathrm{A}_n^*)} \setminus \{0\}$$

and $\sigma(\mathrm{A}_n^*\mathrm{A}_n) \setminus \{0\} = \{0 < \lambda_1^2 < \lambda_2^2 < \cdots < \lambda_\ell^2 \to \infty\}$ *and* λ_ℓ^2 *finite multiplicity.*

FA-ToolBox: Continued

Corollary (spectral theorems)

All the latter operators admit a spectral representation, i.e., for all n there exist orthonormal bases (ξ_n) and (ζ_n) with, e.g.,

$$x = \sum_\ell x_\ell \xi_\ell, \qquad A_n x = \sum_\ell \lambda_\ell x_\ell \zeta_\ell, \qquad A_n^* A_n x = \sum_\ell \lambda_\ell^2 x_\ell \xi_\ell,$$
$$y = \sum_\ell y_\ell \zeta_\ell, \qquad A_n^* y = \sum_\ell \lambda_\ell y_\ell \xi_\ell, \qquad A_n A_n^* y = \sum_\ell \lambda_\ell^2 y_\ell \zeta_\ell.$$

Corollary (Friedrichs/Poincaré estimates)

Friedrichs/Poincaré estimates type for higher eigenspaces.

FA-ToolBox: Continued

key ingredients

Lemma (bounded regular decompositions)

All the latter Hilbert complexes admit bounded regular decompositions, i.e., for all n

$$D(\mathrm{A}_n) = \mathrm{H}_n^+ + \mathrm{A}_{n-1}\mathrm{H}_{n-1}^+.$$

Corollary (bounded regular potentials)

All the latter Hilbert complexes admit bounded regular potentials, i.e., for all n

$$R(\mathrm{A}_n) = \mathrm{A}_n\mathrm{H}_n^+.$$

FA-ToolBox: Applications II

application: characterisation of duals

Theorem (characterisation of dual spaces)

Define regular dual spaces $\mathsf{H}_n^- := (\mathsf{H}_n^+)'$. *Then:*

regular decomposition	*dual space*
$D(\mathsf{A}_1) = \mathsf{H}_1^+ + \mathsf{A}_0\mathsf{H}_0^+$	$D(\mathsf{A}_1)' = \{\Phi \in \mathsf{H}_1^- : \mathsf{A}_0'\Phi \in \mathsf{H}_0^-\}$
$D(\mathsf{A}_0^*) = \mathsf{H}_1^+ + \mathsf{A}_1^*\mathsf{H}_2^+$	$D(\mathsf{A}_1^*)' = \{\Phi \in \mathsf{H}_1^- : (\mathsf{A}_1^*)'\Phi \in \mathsf{H}_2^-\}$

think of

regular decomposition	characterisation of dual space
$D(\mathring{\mathrm{rot}}) = \mathring{\mathsf{H}}^1 + \mathring{\mathrm{grad}}\,\mathring{\mathsf{H}}^1$	$D(\mathring{\mathrm{rot}})' = \{\Phi \in \mathsf{H}^{-1} : \mathrm{div}\,\Phi \in \mathsf{H}^{-1}\}$
$D(\mathrm{div}) = \mathsf{H}^1 + \mathrm{rot}\,\mathsf{H}^1$	$D(\mathrm{div})' = \{\Phi \in \mathring{\mathsf{H}}^{-1} : \mathrm{rot}\,\Phi \in \mathring{\mathsf{H}}^{-1}\}$

with $\mathring{\mathsf{H}}^1 := D(\mathring{\mathrm{grad}})$ and $\mathsf{H}^1 := D(\mathrm{grad})$ and $\mathsf{H}^{-1} := (\mathring{\mathsf{H}}^1)'$ and $\mathring{\mathsf{H}}^{-1} := (\mathsf{H}^1)'$
(everything works with mixed boundary conditions $\mathsf{H}_{\gamma_n}^{-1} := (\mathsf{H}_{\gamma_t}^1)'$)

FA-ToolBox: Applications II

application: cohomology groups

Theorem (Dirichlet-Neumann fields / cohomology groups)

The dimensions of the cohomology groups (generalised Dirichlet-Neumann fields) are independent of metric *and* Sobolev order *and admit a* duality*. In particular, there exists a finite C^∞-smooth per-bases (bases after projection).*

number of Dirichlet-Neumann fields

$$\mathcal{H}_{\epsilon,\gamma_t,\gamma_n} := N(\text{rot}_{\gamma_t}) \cap N(\text{div}_{\gamma_n} \epsilon) = N(\text{rot}_{\gamma_t}) \cap R(\text{grad}_{\gamma_t})^{\perp_{L^2_\epsilon}} \cong N(\text{rot}_{\gamma_t})/R(\text{grad}_{\gamma_t})$$

is independent of metric and Sobolev order, i.e.,

Lemma (Dirichlet-Neumann fields / cohomology groups)

For all k (Sobolev order)

$$\dim N(\text{rot}^k_{\gamma_t})/R(\text{grad}^k_{\gamma_t}) = \dim N(\text{div}^k_{\gamma_n})/R(\text{rot}^k_{\gamma_n}) = \dim \mathcal{H}_{\epsilon,\gamma_t,\gamma_n} < \infty.$$

FA-ToolBox: Applications II

application: cohomology groups

More precisely:

Lemma (Dirichlet-Neumann fields / cohomology groups)

There exist smooth pre-bases of Dirichlet-Neumann fields

$$\mathcal{B}_{\mathrm{rot},\gamma_t} \subset N(\mathrm{rot}^{\infty}_{\gamma_t}) = \mathsf{C}^{\infty}_{\gamma_t}(\overline{\Omega}) \cap N(\mathrm{rot}), \qquad \textit{(finite set)}$$

$$\mathcal{B}_{\mathrm{div},\gamma_n} \subset N(\mathrm{div}^{\infty}_{\gamma_n}) = \mathsf{C}^{\infty}_{\gamma_n}(\overline{\Omega}) \cap N(\mathrm{div}), \qquad \textit{(finite set)}$$

such that

$$\mathcal{H}_{\epsilon,\gamma_t,\gamma_n} = \mathrm{lin}\, \pi_{N(\mathrm{div}_{\gamma_n}\, \epsilon)} \mathcal{B}_{\mathrm{rot},\gamma_t} = \mathrm{lin}\, \pi_{N(\mathrm{rot}_{\gamma_t})} \mathcal{B}_{\mathrm{div},\gamma_n}. \qquad \textit{(bases)}$$

Corollary (Dirichlet-Neumann fields / cohomology groups)

For all Sobolev order k

$$N(\mathrm{rot}^{k}_{\gamma_t})/R(\mathrm{grad}^{k}_{\gamma_t}) \cong \mathrm{lin}\, \mathcal{B}_{\mathrm{rot},\gamma_t} \cong \mathcal{H}_{\epsilon,\gamma_t,\gamma_n} \cong \mathrm{lin}\, \mathcal{B}_{\mathrm{div},\gamma_n} \cong N(\mathrm{div}^{k}_{\gamma_n})/R(\mathrm{rot}^{k}_{\gamma_n})$$

FA-ToolBox: Applications II

application: biharmonic split

biharmonic equation $\Leftrightarrow$ to 3 elliptic 2nd order problems

$$\Delta_{\mathrm{D}}^2 u = f \quad \Leftrightarrow \quad \operatorname{div}\operatorname{Div}_{\mathbb{S}} \operatorname{Gra\mathring{d}grad} u = f$$

$$\Leftrightarrow \quad \begin{aligned} p &= \Delta_{\mathrm{D}}^{-1} f, \\ E &= (\mathring{\operatorname{Rot}}_{\mathbb{S}} \operatorname{symRot}_{\mathbb{T}})^{-1}_{\operatorname{Div}_{\mathbb{T}}=0} \operatorname{spn} \operatorname{grad} p, \\ u &= \Delta_{\mathrm{D}}^{-1}(3p + \operatorname{tr} \operatorname{symRot}_{\mathbb{T}} E) \end{aligned}$$

FEs for $\operatorname{symRot}_{\mathbb{T}}$ needed!

FA-ToolBox: some literature

some related literature

- Py: Discrete and Continuous Dynamical Systems 2015
On Maxwell's and Poincare's Constants

- Py: Mathematical Methods in the Applied Sciences 2017
On the Maxwell Constants in 3D

- Py and Irwin Yousept: Mathematical Modelling and Numerical Analysis 2017
A Posteriori Error Analysis for the Optimal Control of Magneto-Static Fields

- Py and Immanuel Anjam: Computational Methods in Applied Mathematics 2019
An Elementary Method of Deriving A Posteriori Error Equalities and Estimates for Linear Partial Differential Equations

- Py: Mathematische Zeitschrift 2019
On the Maxwell and Friedrichs/Poincaré Constants in ND

- Py: Analysis 2019
A Global div-curl-Lemma for Mixed Boundary Conditions in Weak Lipschitz Domains and a Corresponding Generalized A_0^**-*A_1*-Lemma in Hilbert Spaces*

- Py: Numerical Functional Analysis and Optimization 2020
Solution Theory, Variational Formulations, and Functional a Posteriori Error Estimates for General First Order Systems with Applications to Electro-Magneto-Statics and More

- Py and Jan Valdman: Computers and Mathematics with Applications 2020
Poincar-Friedrichs Type Constants for Operators Involving grad, curl, and div: Theory and Numerical Experiments

- Py and Walter Zulehner: Applicable Analysis 2020
The divDiv-Complex and Applications to Biharmonic Equations

FA-ToolBox: some literature

some recent related literature

• Stefan Kurz and Py and Dirk Praetorius and Sergey Repin and Daniel Sebastian: Numerische Mathematik 2021
Functional A Posteriori Error Estimates for Boundary Element Methods

• Py and Rainer Picard and Sascha Trostorff and Marcus Waurick: Journal of Functional Analysis 2021
On a Class of Degenerate Abstract Parabolic Problems and Applications to Some Eddy Current Models

• Py and Walter Zulehner: Applicable Analysis 2022
The Elasticity Complex: Compact Embeddings and Regular Decompositions

• Py and Michael Schomburg: Mathematical Methods in the Applied Sciences 2022
Hilbert Complexes with Mixed Boundary Conditions - Part 1: De Rham Complex

• Py and Michael Schomburg: Mathematical Methods in the Applied Sciences 2022
Hilbert Complexes with Mixed Boundary Conditions - Part 2: Elasticity Complex

• Py and Michael Schomburg: Mathematical Methods in the Applied Sciences 2023
Hilbert Complexes with Mixed Boundary Conditions - Part 3: Biharmonic Complex

• Py and Marcus Waurick: Mathematische Zeitschrift 2022
The Index of some Mixed Order Dirac-Type Operators
and Generalised Dirichlet-Neumann Tensor Fields

• Py and Nathanel Skrepek: Annali dell' Universita di Ferrara 2022
A Compactness Result for the div-curl System with Inhomogeneous Mixed Boundary Conditions for Bounded Lipschitz Domains and Some Applications

• Ralf Hiptmair and Py and Erick Schulz: Journal of Functional Analysis 2023
Traces for Hilbert Complexes

FA-ToolBox: This is Finnish but not the end.

Thank you