# Functional A Posteriori Error Estimates for Static and Eddy Current Maxwell Type Problems 

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## Introduction：General Electro－Magneto Static Maxwell Type Problem

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|}\Omega\mathrm{ smooth N-dim. Riemann. manifold. with comp.cl. and Lip. bound. }
|
    i.e., }\mp@subsup{\Gamma}{t}{}\subset\Gamma,\mp@subsup{\Gamma}{n}{}:=\Gamma\\overline{\mp@subsup{\Gamma}{t}{}}\mathrm{ , where }\gamma:=\overline{\mp@subsup{\Gamma}{t}{}}\cap\overline{\mp@subsup{\Gamma}{n}{}}\mathrm{ Lipschitz
|}\varepsilon\mathrm{ given medium property: bd., sym., unif. pos. def., lin. mapping on q-forms
⿴囗
m F,G,f,g}\mathrm{ given right hand side data: diff. forms on }\Omega\mathrm{ resp. Г, Г 
|}\tau\mathrm{ tangential trace, i.e., }\tauE=\mp@subsup{\iota}{\mp@subsup{\Gamma}{t}{}}{*}E\mathrm{ with }\mp@subsup{\iota}{\mp@subsup{\Gamma}{t}{}}{}:\mp@subsup{\Gamma}{t}{}\hookrightarrow\Gamma\hookrightarrow\hookrightarrow\Omega, canon. em
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|, *, Hodge's stars on \Omega resp. \Gamma resp. \Gamma \Gamma//n
|}\textrm{d},\delta=\mp@subsup{\textrm{d}}{}{\prime}=\pm*\textrm{d}*\mathrm{ exterior derivative and co-derivative
|
m H}\mp@subsup{\mathcal{L}}{\varepsilon}{q}(\Omega)\textrm{Di}./Neu.-forms: H\in\mp@subsup{\mathcal{H}}{\varepsilon}{q}(\Omega)\Leftrightarrow\textrm{d}H=0,\delta\varepsilonH=0 and \tauH=0,\nu\varepsilonH=
    d E=F, \delta\varepsilonE=G in \Omega
    \tauE=f in \Gamma 位 (static Maxwell type problem)
    \nu\varepsilonE=g in 「n
    \varepsilonE\perp \mathcal{H}}\mp@subsup{\varepsilon}{q}{q}(\Omega
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- GOAL: NON-CONFORMING estimates for error e $:=E-\tilde{E}$,

where $\tilde{E}$ (just) in $L^{2, q}(\Omega)$ approximation of $E$


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■ $\Omega$ smooth $N$－dim．Riemann．manifold．with comp．cl．and Lip．bound．$\Gamma=\partial \Omega$
－「 is decomposed in tangential and normal parts divided by a Lipschitz interface， －$\varepsilon$ given medium property：bd．，sym．，unif．pos．def．，lin．mapping on $q$－forms －$E$＇electric field＇：differential form（ $q$－form）on $\Omega$
－F $, ~ G, f, g$ given right hand side data：diff．forms on $\Omega$ resp．$\Gamma, \Gamma_{t / n}$ －$\tau$ tangential trace，i．e．，$\tau E=\iota_{\Gamma_{t}}^{*} E$ with $\iota_{t}: \Gamma_{t} \hookrightarrow \Gamma \hookrightarrow \Omega$ ，canon．emb

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■ \nu normal trace, i.e., }\nuE=\circledast\mp@subsup{\iota}{r}{*}*E\mathrm{ with }\mp@subsup{\iota}{\Gamma}{}:\mp@subsup{\Gamma}{n}{}\hookrightarrow\Gamma\hookrightarrow
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\＃＊，$*$ Hodge＇s stars on $\Omega$ resp．「 resp．「 ${ }_{t / n}$
－ $\mathrm{d}, \delta=\mathrm{d}^{\prime}= \pm * \mathrm{~d} *$ exterior derivative and co－derivative
－ 1 orthogonality w．r．t．$L^{2, q}(\Omega)$－scalar product $\langle E, H\rangle_{\Omega}:=\int_{\Omega} E \wedge * H$
－ $\mathcal{H}_{\varepsilon}^{q}(\Omega)$ Di．／Neu．－forms：$H \in \mathcal{H}_{\varepsilon}^{q}(\Omega) \Leftrightarrow \mathrm{d} H=0, \delta \varepsilon H=0$ and $\tau H=0, \nu \varepsilon H=0$

$$
\begin{array}{rlrl}
\mathrm{d} E & =F, \quad \delta \varepsilon E=G & & \text { in } \Omega \\
\\
\tau E & =f & & \text { in } \Gamma_{t}
\end{array} \quad \text { (static Maxwell type problem) }
$$

## Introduction: General Electro-Magneto Static Maxwell Type Problem

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1 $E$ 'electric field': differential form ( $q$-form) on $\Omega$
- $F, G, f, g$ given right hand side data: diff. forms on $\Omega$ resp. $\Gamma, \Gamma_{t / n}$

■ $\tau$ tangential trace, i.e., $\tau E=\iota_{\Gamma_{t}}^{*} E$ with $\iota_{t}: \Gamma_{t} \hookrightarrow \Gamma \hookrightarrow \Omega$, canon. emb

- II $\nu$ normal trace, i.e., $\nu E=* \iota_{\Gamma_{n}}^{*} * E$ with $\iota_{\Gamma_{n}}: \Gamma_{n} \hookrightarrow \Gamma \hookrightarrow \bar{\Omega}$
- ${ }^{\text {. }}$, Hodge's stars on $\Omega$ resp. 「 resp. $\Gamma_{t / n}$
- $\mathrm{d}, \delta=\mathrm{d}^{\prime}= \pm * \mathrm{~d} *$ exterior derivative and co-derivative
$\pm$ orthogonality w.r.t. $L^{2, q}(\Omega)$-scalar product $\langle E, H\rangle_{\Omega}:=\int_{\Omega} E \wedge * H$
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- v normal trace ie $\quad \nu F=\omega_{\imath}^{*} * E$ with $\quad$ : $\Gamma_{n} \hookrightarrow \Gamma \hookrightarrow \bar{\Omega}$

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- GOAL: NON-CONFORMING estimates for error $e:=E-\tilde{E}$, where $\tilde{E}$ (just) in $\mathrm{L}^{2, q}(\Omega)$ approximation of $E$


## Introduction：Electro－Magneto Static Maxwell Problem（special case $q=1$ ）

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| \Omega\subset\mp@subsup{\mathbb{R}}{}{3}\mathrm{ bounded domain with Lipschitz boundary }\Gamma=\partial\Omega\mathrm{ ,}
    \Gamma ~ \ ~ i s ~ d e c o m p o s e d ~ i n ~ t a n g e n t i a l ~ a n d ~ n o r m a l ~ p a r t s ~ d i v i d e d ~ b y ~ a ~ L i p s c h i t z ~ i n t e r f a c e ,
    i.e., }\mp@subsup{\Gamma}{t}{}\subset\mp@subsup{\Gamma}{,}{}\mp@subsup{\Gamma}{n}{}:=\Gamma\\overline{\mp@subsup{\Gamma}{t}{}}\mathrm{ , where }\gamma:=\overline{\mp@subsup{\Gamma}{t}{}}\cap\mp@subsup{\Gamma}{n}{}\mathrm{ Lipschitz
|=\varepsilon:\Omega->\mp@subsup{\mathbb{R}}{}{3\times3}}\mathrm{ given medium property: bd., sym., unif. pos. def. matrix field
⿴囗 Electric or magnetic (H) field
⿴囗 F,G,f,g given right hand side data
| \tau,\nu restr. tang. resp. norm. trace, i.e., \tauE =n\timesE | |rt resp. }\nuH=n=H\mp@subsup{|}{\mp@subsup{r}{n}{}}{
m orthogonality w.r.t. L L
|. H\in\mp@subsup{\mathcal{H}}{\varepsilon}{}(\Omega)\mathrm{ Dirich./Neum.-fields }\Leftrightarrow\mathrm{ curl H=0, div }\varepsilonH=0\mathrm{ and }\tauH=0,\nu\varepsilonH=0
    curl E=F in \Omega
    div}\varepsilonE=G\quad\mathrm{ in }
        \tauE=f in \Gamma 左 (electro-magneto static Maxwell problem)
    \nu\varepsilonE=g in \Gamman
        \varepsilonE\perp \mathcal{H}
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- note: $\Gamma_{t}=\Gamma \Rightarrow$ electro static (elec. bc); $\Gamma_{n}=\Gamma \Rightarrow$ magneto static (magn. bc)
GOAL: NON-CONFORMING estimates for error $e:=E-F \quad$.
where $\tilde{E}$ (just) in $L^{2}(\Omega)$ approximation of $E$


## Introduction: Electro-Magneto Static Maxwell Problem (special case $q=1$ )

$■ \Omega \subset \mathbb{R}^{3}$ bounded domain with Lipschitz boundary $\Gamma=\partial \Omega$,
$\Gamma$ is decomposed in tangential and normal parts divided by a Lipschitz interface, i.e., $\Gamma_{t} \subset \Gamma, \Gamma_{n}:=\Gamma \backslash \overline{\Gamma_{t}}$, where $\gamma:=\overline{\Gamma_{t}} \cap \overline{\Gamma_{n}}$ Lipschitz

- $\varepsilon: \Omega \rightarrow \mathbb{R}^{3 \times 3}$ given medium property: bd., sym., unif. pos. def. matrix field

E electric or magnetic $(H)$ field

- $F$, $G, f, g$ given right hand side data

■ $\tau, \nu$ restr. tang. resp. norm. trace, i.e., $\tau E=n \times\left. E\right|_{\Gamma_{t}}$ resp. $\nu H=\left.n \cdot H\right|_{\Gamma_{n}}$
■ orthogonality w.r.t. $L^{2}(\Omega)$-scalar product $\langle E, H\rangle_{\Omega}:=\int_{\Omega} E \cdot H$ - $H \in \mathcal{H}_{\varepsilon}(\Omega)$ Dirich./Neum.-fields $\Leftrightarrow \operatorname{curl} H=0, \operatorname{div} \varepsilon H=0$ and $\tau H=0, \nu \varepsilon H=0$

$$
\begin{aligned}
\operatorname{curl} E & =F & & \text { in } \Omega \\
\operatorname{div} \varepsilon E & =G & & \text { in } \Omega \\
\tau E & =f & & \text { in } \Gamma_{t} \\
\nu \varepsilon E & =g & & \text { in } \Gamma_{n} \\
\varepsilon E & \perp \mathcal{H}_{\varepsilon}(\Omega) & &
\end{aligned}
$$

$$
\tau E=f \quad \text { in } \Gamma_{t} \quad \text { (electro-magneto static Maxwell problem) }
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■ $\varepsilon: \Omega \rightarrow \mathbb{R}^{3 \times 3}$ given medium property：bd．，sym．，unif．pos．def．matrix field
－E electric or magnetic（H）field
－$F$ ，$G, f, g$ given right hand side data
■ $\tau, \nu$ restr．tang．resp．norm．trace，i．e．，$\tau E=n \times\left. E\right|_{\Gamma_{t}}$ resp．$\nu H=\left.n \cdot H\right|_{\Gamma_{n}}$
－$\perp$ orthogonality w．r．t．$L^{2}(\Omega)$－scalar product $\langle E, H\rangle_{\Omega}:=\int_{\Omega} E \cdot H$
－$H \in \mathcal{H}_{\varepsilon}(\Omega)$ Dirich．／Neum．－fields $\Leftrightarrow \operatorname{curl} H=0, \operatorname{div} \varepsilon H=0$ and $\tau H=0, \nu \varepsilon H=0$

$$
\begin{aligned}
\operatorname{curl} E & =F & & \text { in } \Omega \\
\operatorname{div} \varepsilon E & =G & & \text { in } \Omega \\
\tau E & =f & & \text { in } \Gamma_{t} \quad \text { (electro-magneto static Maxwell problem) } \\
\nu \varepsilon E & =g & & \text { in } \Gamma_{n} \\
\varepsilon E & \perp \mathcal{H}_{\varepsilon}(\Omega) & &
\end{aligned}
$$

note：$\Gamma_{t}=\Gamma \Rightarrow$ electro static（elec．bc）；$\Gamma_{n}=\Gamma \Rightarrow$ magneto static（magn．bc）
－GOAL：NON－CONFORMING estimates for error $e:=E-E$ ，
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where $\tilde{E}$（just）in $L^{2}(\Omega)$ approximation of $E$

## Introduction: Electro-Magneto Static Maxwell Problem (special case $q=1$ )

$■ \Omega \subset \mathbb{R}^{3}$ bounded domain with Lipschitz boundary $\Gamma=\partial \Omega$,
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■ $\tau, \nu$ restr. tang. resp. norm. trace, i.e., $\tau E=n \times\left. E\right|_{\Gamma_{t}}$ resp. $\nu H=\left.n \cdot H\right|_{\Gamma_{n}}$

- 1 orthogonality w.r.t. $L^{2}(\Omega)$-scalar product $\langle E, M\rangle_{\Omega}:=\int_{\Omega} E \cdot H$ - $H \in \mathcal{H}_{\varepsilon}(\Omega)$ Dirich./Neum.-fields $\Leftrightarrow \operatorname{curl} H=0, \operatorname{div} \varepsilon H=0$ and $\tau H=0, \nu \varepsilon H=0$

$$
\begin{array}{rlrl}
\operatorname{curl} E & =F & & \text { in } \Omega \\
& & \\
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& & \\
\tau E & =f & & \text { in } \Gamma_{t} \\
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& & \\
\varepsilon E & \perp \mathcal{H}_{\varepsilon}(\Omega) & &
\end{array}
$$

- note: $\Gamma_{t}=\Gamma \Rightarrow$ electro static (elec. bc); $\Gamma_{n}=\Gamma \Rightarrow$ magneto static (magn. bc)
- GOAL: NON-CONFORMING estimates for error $e:=E-E$,

DESSSENRG
where $\tilde{E}$ (just) in $\mathrm{L}^{2}(\Omega)$ approximation of $E$

## Introduction：Electro－Magneto Static Maxwell Problem（special case $q=1$ ）

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## Introduction：Electro－Magneto Static Maxwell Problem（special case $q=1$ ）

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－$H \in \mathcal{H}_{\varepsilon}(\Omega)$ Dirich．$/$ Neum．－fields $\Leftrightarrow$ curl $H=0, \operatorname{div} \varepsilon H=0$ and $\tau H=0, \nu \varepsilon H=0$

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\operatorname{curl} E & =F & & \text { in } \Omega \\
& & \\
\operatorname{div} \varepsilon E & =G & & \text { in } \Omega \\
\tau E & =f & & \text { in } \Gamma_{t} \\
\nu \varepsilon E & =g & & \text { (electro-magneto static Maxwell problem) } \\
\varepsilon E & \perp \mathcal{H}_{\varepsilon}(\Omega) & &
\end{array}
$$

note：$\Gamma_{t}=\Gamma \Rightarrow$ electro static（elec．bc）；$\Gamma_{n}=\Gamma \Rightarrow$ magneto static（magn．bc）
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## Introduction: Electro-Magneto Static Maxwell Problem (special case $q=1$ )

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- $\varepsilon: \Omega \rightarrow \mathbb{R}^{3 \times 3}$ given medium property: bd., sym., unif. pos. def. matrix field
- $E$ electric or magnetic $(H)$ field
- $F, G, f, g$ given right hand side data
$■ \tau, \nu$ restr. tang. resp. norm. trace, i.e., $\tau E=n \times\left. E\right|_{\Gamma_{t}}$ resp. $\nu H=\left.n \cdot H\right|_{\Gamma_{n}}$
- $\perp$ orthogonality w.r.t. $L^{2}(\Omega)$-scalar product $\langle E, H\rangle_{\Omega}:=\int_{\Omega} E \cdot H$

```
- \(H \in \mathcal{H}_{\varepsilon}(\Omega)\) Dirich./Neum.-fields \(\Leftrightarrow\) curl \(H=0\)
\(\operatorname{curl} E=F \quad\) in \(\Omega\)
\(\operatorname{div} \varepsilon E=G \quad\) in \(\Omega\)
        \(\tau E=f \quad\) in \(\Gamma_{t} \quad\) (electro-magneto static Maxwell problem)
        \(\nu \varepsilon E=g \quad\) in \(\Gamma_{n}\)
            \(\varepsilon E \perp \mathcal{H}_{\varepsilon}(\Omega)\)
```

note: $\Gamma_{t}=\Gamma \Rightarrow$ electro static (elec. bc); $\Gamma_{n}=\Gamma \Rightarrow$ magneto static (magn. bc)

- GOAL: NON-CONFORMING estimates for error $e:=E-E$,


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$$
\begin{align*}
\operatorname{curl} E & =F & & \text { in } \Omega \\
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\tau E & =f & & \text { in } \Omega \\
& \text { in } \Gamma_{t} & & \\
\nu \varepsilon E & =g & & \text { (electro-magneto static Maxwell problem) } \\
\varepsilon E & \perp \mathcal{H}_{\varepsilon}(\Omega) & &
\end{align*}
$$

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$$

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## Introduction: Electro or Magneto Static Maxwell Problem (simplifications)

■ $\Gamma_{t}=\Gamma_{,} \Gamma_{n}=\emptyset$ or $\Gamma_{n}=\Gamma_{,} \Gamma_{t}=\emptyset$

- $\varepsilon=\mu=\mathrm{id}$
- E electric field, H magnetic field
- $D \in \mathcal{H}_{D}(\Omega)$ Dirichlet fields $\Leftrightarrow \operatorname{curl} D=0, \operatorname{div} D=0$ and $\tau D=0$

■ $N \in \mathcal{H}_{N_{1}}(\Omega)$ Neumann fields $\Leftrightarrow \operatorname{curl} N=0, \operatorname{div} N=0$ and $\nu N=0$

```
curl E=F, curl H=L in \Omega
div}E=G.\quad\operatorname{div}H=K\quad\mathrm{ in }\Omega\mathrm{ (electro or magneto
```



```
        E\perp\mp@subsup{\mathcal{H}}{D}{}(\Omega),}\quadH\perp\mp@subsup{\mathcal{H}}{N}{}(\Omega
```

- GOAL: NON-CONFORMING estimates for errors $e:=E-\tilde{E}, h:=H-\tilde{H}$, where $\tilde{E}, \tilde{H}$ (just) in $L^{2}(\Omega)$ approximations of $E, H$


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## Introduction: Electro or Magneto Static Maxwell Problem (simplifications)

■ $\Gamma_{t}=\Gamma_{,} \Gamma_{n}=\emptyset$ or $\Gamma_{n}=\Gamma_{,} \Gamma_{t}=\emptyset$

- $\varepsilon=\mu=\mathrm{id}$
- $E$ electric field, $H$ magnetic field
- $D \in \mathcal{H}_{D}(\Omega)$ Dirichlet fields $\Leftrightarrow \operatorname{curl} D=0$, $\operatorname{div} D=0$ and $\tau D=0$
-     - $N \in \mathcal{H}_{N}(\Omega)$ Neumann fields $\Leftrightarrow \operatorname{curl} N=0, \operatorname{div} N=0$ and $\nu N=0$

```
curl E=F, curl H=L in \Omega
div}E=G,\quad\operatorname{div}H=K\quad\mathrm{ in }\Omega\quad\mathrm{ (electro or magneto
```



```
        E \perp H}\mp@subsup{\mathcal{HD}}{D}{}(\Omega),\quadH\perp\mp@subsup{\mathcal{H}}{N}{}(\Omega
```

- GOAL NON-CONFORMING estimates for errors e $:=E-E, h:=H-\tilde{H}$, where $\tilde{E}, \tilde{H}$ (just) in $L^{2}(\Omega)$ approximations of $E, H$


## Introduction: Electro or Magneto Static Maxwell Problem (simplifications)

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- $D \in \mathcal{H}_{D}(\Omega)$ Dirichlet fields $\Leftrightarrow \operatorname{curl} D=0, \operatorname{div} D=0$ and $\tau D=0$
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```
curl E=F, curl H=L in \Omega
div}E=G,\quad\operatorname{div}H=K\quad\mathrm{ in }\Omega\quad\mathrm{ (electro or magneto
```



```
E \perp H}\mp@subsup{\mathcal{HD}}{D}{}(\Omega),\quadH\perp\mp@subsup{\mathcal{H}}{N}{}(\Omega
```

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## Introduction: Electro or Magneto Static Maxwell Problem (simplifications)

■ $\Gamma_{t}=\Gamma_{,} \Gamma_{n}=\emptyset$ or $\Gamma_{n}=\Gamma_{,} \Gamma_{t}=\emptyset$

- $\varepsilon=\mu=\mathrm{id}$
- $E$ electric field, $H$ magnetic field

```
| D \in H}\mp@subsup{\mathcal{L}}{D}{(\Omega)}\mathrm{ Dirichlet fields }\Leftrightarrow\operatorname{curl}D=0,\operatorname{div}D=0\mathrm{ and }\tauD=
| N N H
```

```
curl E=F, curl H=L in \Omega
```

curl E=F, curl H=L in \Omega
div}E=G,\quad\operatorname{div}H=K\quad\mathrm{ in }\Omega\quad\mathrm{ (electro or magneto

```
div}E=G,\quad\operatorname{div}H=K\quad\mathrm{ in }\Omega\quad\mathrm{ (electro or magneto
```




```
    E \perp \mathcal{H}
```

    E \perp \mathcal{H}
    - GOAL: NON-CONFORMING estimates for errors e :=E-E E, h:=H-\tilde{H},
where \tilde{E},\tilde{H}\mathrm{ (just) in L}\mp@subsup{L}{}{2}(\Omega)\mathrm{ approximations of E,H}

```

\section*{Introduction: Electro or Magneto Static Maxwell Problem (simplifications)}

■ \(\Gamma_{t}=\Gamma_{, ~} \Gamma_{n}=\emptyset\) or \(\Gamma_{n}=\Gamma_{, ~} \Gamma_{t}=\emptyset\)
- \(\varepsilon=\mu=\mathrm{id}\)
- \(E\) electric field, \(H\) magnetic field
- \(D \in \mathcal{H}_{D}(\Omega)\) Dirichlet fields \(\Leftrightarrow \operatorname{curl} D=0, \operatorname{div} D=0\) and \(\tau D=0\)
- \(N \in \mathcal{H}_{N}(\Omega)\) Neumann fields \(\Leftrightarrow\) curl \(N=0\), div \(N=0\) and \(\nu N=0\)

- GOAL: NON-CONFORMING estimates for errors \(e:=E-\tilde{E}, h:=H-\tilde{H}\), where \(\tilde{E}, \tilde{H}\) (just) in \(L^{2}(\Omega)\) approximations of \(E, H\)

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\section*{Introduction: Electro or Magneto Static Maxwell Problem (simplifications)}

■ \(\Gamma_{t}=\Gamma_{,} \Gamma_{n}=\emptyset\) or \(\Gamma_{n}=\Gamma_{,} \Gamma_{t}=\emptyset\)
- \(\varepsilon=\mu=\mathrm{id}\)
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- \(N \in \mathcal{H}_{N}(\Omega)\) Neumann fields \(\Leftrightarrow \operatorname{curl} N=0, \operatorname{div} N=0\) and \(\nu N=0\)
\[
\begin{array}{rlrlrl}
\operatorname{curl} E & =F, & & \text { curl } H & =L & \\
\operatorname{div} E & =G, & & \operatorname{div} H & =K & \\
\text { in } \Omega & & \text { (electro or magneto } \\
\tau E & =f, & \nu H & =g & & \text { in } \Gamma \\
& & & \text { static Maxwell problem) } \\
E & \perp \mathcal{H}_{D}(\Omega), & H & \perp \mathcal{H}_{N}(\Omega) & &
\end{array}
\]
- GOAL: NON-CONFORMING estimates for errors \(e:=E-\tilde{E}, h:=H-\tilde{H}\) where \(\tilde{E}, \tilde{H}\) (just) in \(L^{2}(\Omega)\) approximations of \(E, H\)

\section*{Introduction: Electro or Magneto Static Maxwell Problem (simplifications)}

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- \(\varepsilon=\mu=\mathrm{id}\)
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- \(D \in \mathcal{H}_{D}(\Omega)\) Dirichlet fields \(\Leftrightarrow \operatorname{curl} D=0, \operatorname{div} D=0\) and \(\tau D=0\)

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\[
\begin{array}{rlrlrl}
\operatorname{curl} E & =F, & & \text { curl } H & =L & \\
\operatorname{div} E & =G, & & \operatorname{div} H & =K & \\
\text { in } \Omega & & \text { (electro or magneto } \\
\tau E & =f, & \nu H & =g & & \text { in } \Gamma \\
E & & \text { static Maxwell problem) } \\
E & \mathcal{H}_{D}(\Omega), & H & \perp \mathcal{H}_{N}(\Omega) & &
\end{array}
\]
- GOAL: NON-CONFORMING estimates for errors \(e:=E-\tilde{E}, h:=H-\tilde{H}\), where \(\tilde{E}, \tilde{H}\) (just) in \(L^{2}(\Omega)\) approximations of \(E, H\)

\section*{Introduction: Dirichlet or Neumann Laplace Problems}
- \(u \in \mathrm{H}^{1}(\Omega)\) with \(\Delta u=\operatorname{div} \nabla u=G \quad\) and \(\left.\quad u\right|_{\Gamma}=v\) or \(\left.\partial_{n} u\right|_{\Gamma}=\left.n \cdot \nabla u\right|_{\Gamma}=g\)
- Set \(E:=\nabla u\) or \(H:=\nabla u\), note: \(n \times\left.\nabla u\right|_{\Gamma}=\left.\nabla_{\Gamma} u\right|_{\Gamma} \quad\) (since \(d \iota^{*}=\iota^{*} d\) )
```

curl E=0,
curl H=0 in \Omega
div}H=G in \Omega (electro or magneto
rE = \mp@subsup{\nabla}{\Gamma}{}v=:f,
static Maxwell problem)

```
- \(\Rightarrow\) NON-CONFORMING estimates for errors \(e:=E-\tilde{E}, h:=H-\tilde{H}\),
    where \(\tilde{E}, \tilde{H}\) (just) in \(L^{2}(\Omega)\) approximation of \(E=\nabla u, H=\nabla u\)
- \(\Rightarrow\) NON-CONFORMING estimates for energy norm
- note: also mixed boundary conditions possible:
    \(u \in \mathrm{H}^{1}(\Omega)\) with \(\Delta u=G \quad\) and \(\left.\quad u\right|_{\Gamma_{t}}=v\) and \(\left.\partial_{n} u\right|_{\Gamma_{n}}=\left.n \cdot \nabla u\right|_{\Gamma_{n}}=g\)
- note: also possible \(\tau E=\left.\partial_{t} u\right|_{\Gamma}=n \times\left.\nabla u\right|_{\Gamma}=f\)

\section*{Introduction: Dirichlet or Neumann Laplace Problems}
\(■ u \in \mathrm{H}^{1}(\Omega)\) with \(\Delta u=\operatorname{div} \nabla u=G \quad\) and \(\left.\quad u\right|_{\Gamma}=v\) or \(\left.\partial_{n} u\right|_{\Gamma}=\left.n \cdot \nabla u\right|_{\Gamma}=g\)
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\section*{Introduction: Dirichlet or Neumann Laplace Problems}

■ \(u \in \mathrm{H}^{1}(\Omega)\) with \(\Delta u=\operatorname{div} \nabla u=G \quad\) and \(\left.\quad u\right|_{\Gamma}=v\) or \(\left.\partial_{n} u\right|_{\Gamma}=\left.n \cdot \nabla u\right|_{\Gamma}=g\)
\(■\) Set \(E:=\nabla u\) or \(H:=\nabla u\), note: \(n \times\left.\nabla u\right|_{\Gamma}=\left.\nabla_{\Gamma} u\right|_{\Gamma} \quad\left(\right.\) since \(\left.d \iota^{*}=\iota^{*} \mathrm{~d}\right)\)
\(\left.\begin{array}{rlrlrl}\operatorname{curl} E & =0, & \operatorname{curl} H & =0 & & \text { in } \Omega \\ \operatorname{div} E & =G, & \operatorname{div} H & =G & & \\ \tau E & =\nabla_{\Gamma v}=: f, & \nu H & =g & & \text { in } \Gamma\end{array} \quad \begin{array}{l}\text { (electro or magneto } \\ E\end{array}\right)\)
- \(\Rightarrow\) NON-CONFORMING estimates for errors \(e:=E-\tilde{E}, h:=H-\tilde{H}\) where \(E, H\) (just) in \(L^{2}(\Omega)\) approximation of \(E=\nabla u, H=\nabla u\)
■ \(\Rightarrow\) NON-CONFORMING estimates for energy norm
- note: also mixed boundary conditions possible:
\(u \in H^{1}(\Omega)\) with \(\Delta u=G\) and \(\left.u\right|_{r_{t}}=v\) and \(\left.\partial_{n} u\right|_{\Gamma_{n}}=\left.n \cdot \nabla u\right|_{r_{n}}=g\)
- note: also possible \(\tau E=\left.\partial_{t} u\right|_{\Gamma}=n \times\left.\nabla u\right|_{\Gamma}=f\)

\section*{Introduction: Dirichlet or Neumann Laplace Problems}

■ \(u \in \mathrm{H}^{1}(\Omega)\) with \(\Delta u=\operatorname{div} \nabla u=G \quad\) and \(\left.\quad u\right|_{\Gamma}=v\) or \(\left.\partial_{n} u\right|_{\Gamma}=\left.n \cdot \nabla u\right|_{\Gamma}=g\)
\(■\) Set \(E:=\nabla u\) or \(H:=\nabla u\), note: \(n \times\left.\nabla u\right|_{\Gamma}=\left.\nabla_{\Gamma} u\right|_{\Gamma} \quad\left(\right.\) since \(\left.d \iota^{*}=\iota^{*} \mathrm{~d}\right)\)
\[
\begin{aligned}
\Rightarrow \quad \operatorname{curl} E & =0, & \operatorname{curl} H & =0 & & \text { in } \Omega \\
\operatorname{div} E & =G, & \operatorname{div} H & =G & & \\
\tau E & =\nabla_{\Gamma} v=: f, & \nu H & =g & & \text { in } \Gamma
\end{aligned}
\]
[ \(\quad \Rightarrow\) NON-CONFORMING estimates for errors \(e:=E-\tilde{E}, h:=H-\tilde{H}\), where \(\tilde{E}, \tilde{H}\) (just) in \(L^{2}(\Omega)\) approximation of \(E=\nabla u, H=\nabla u\)
■ \(\Rightarrow\) NON-CONFORMING estimates for energy norm
- note: also mixed boundary conditions possible:
\(u \in \mathrm{H}^{1}(\Omega)\) with \(\Delta u=G \quad\) and \(\left.\quad u\right|_{\Gamma_{t}}=v\) and \(\left.\partial_{n} u\right|_{r_{n}}=\left.n \cdot \nabla u\right|_{r_{n}}=g\)
- note: also possible \(\tau E=\left.\partial_{t} u\right|_{\Gamma}=n \times\left.\nabla u\right|_{\Gamma}=f\)

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\section*{Introduction: Dirichlet or Neumann Laplace Problems}

■ \(u \in \mathrm{H}^{1}(\Omega)\) with \(\Delta u=\operatorname{div} \nabla u=G \quad\) and \(\left.\quad u\right|_{\Gamma}=v\) or \(\left.\partial_{n} u\right|_{\Gamma}=\left.n \cdot \nabla u\right|_{\Gamma}=g\)
\(■\) Set \(E:=\nabla u\) or \(H:=\nabla u\), note: \(n \times\left.\nabla u\right|_{\Gamma}=\left.\nabla_{\Gamma} u\right|_{\Gamma} \quad\left(\right.\) since \(\left.d \iota^{*}=\iota^{*} \mathrm{~d}\right)\)
\[
\begin{array}{rlrlrl}
\Rightarrow \quad \text { curl } E & =0, & & \text { curl } H & =0 & \\
\operatorname{div} E & =G, & & \text { in } \Omega & \\
\tau E & =\nabla_{\Gamma} v=: f, & \nu H & =G & & \text { in } \Omega \\
E & & \text { (electro or magneto } \\
E & \text { proj. on } \mathcal{H}_{D}(\Omega), & H & & \text { in } \Gamma & \\
\text { static Maxwell pro }
\end{array}
\]

■ \(\Rightarrow\) NON-CONFORMING estimates for errors \(e:=E-\tilde{E}, h:=H-\tilde{H}\), where \(\tilde{E}, \tilde{H}\) (just) in \(L^{2}(\Omega)\) approximation of \(E=\nabla u, H=\nabla u\)
- \(\Rightarrow\) NON-CONFORMING estimates for energy norm
- note: also mixed boundary conditions possible:
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\section*{Introduction: Dirichlet or Neumann Laplace Problems}

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E \text { proj. on } \mathcal{H}_{D}(\Omega), & H & & \text { static Maxwell pro }
\end{array}
\]

■ \(\Rightarrow\) NON-CONFORMING estimates for errors \(e:=E-\tilde{E}, h:=H-\tilde{H}\), where \(\tilde{E}, \tilde{H}\) (just) in \(L^{2}(\Omega)\) approximation of \(E=\nabla u, H=\nabla u\)
\(■ \Rightarrow\) NON-CONFORMING estimates for energy norm
- note: also mixed boundary conditions possible:
\(u \in H^{1}(\Omega)\) with \(\Delta u=G \quad\) and \(\left.\quad u\right|_{\Gamma_{t}}=v\) and \(\left.\partial_{n} u\right|_{\Gamma_{n}}=\left.n \cdot \nabla u\right|_{\Gamma_{n}}=g\)
輏 note: also nossible \(\tau E=\left.\partial_{+} u\right|_{\Gamma}=n \times\left.\nabla u\right|_{\Gamma}=f\)
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\section*{Introduction: Dirichlet or Neumann Laplace Problems}

■ \(u \in \mathrm{H}^{1}(\Omega)\) with \(\Delta u=\operatorname{div} \nabla u=G \quad\) and \(\left.\quad u\right|_{\Gamma}=v\) or \(\left.\partial_{n} u\right|_{\Gamma}=\left.n \cdot \nabla u\right|_{\Gamma}=g\)
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\begin{aligned}
\Rightarrow \quad \operatorname{curl} E & =0, & \operatorname{curl} H & =0 & & \text { in } \Omega \\
\operatorname{div} E & =G, & \operatorname{div} H & =G & & \\
\tau E & =\nabla_{\Gamma} v=: f, & \nu H & =g & & \text { in } \Gamma
\end{aligned}
\]

■ \(\Rightarrow\) NON-CONFORMING estimates for errors \(e:=E-\tilde{E}, h:=H-\tilde{H}\), where \(\tilde{E}, \tilde{H}\) (just) in \(L^{2}(\Omega)\) approximation of \(E=\nabla u, H=\nabla u\)
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\end{aligned}
\]

■ \(\Rightarrow\) NON-CONFORMING estimates for errors \(e:=E-\tilde{E}, h:=H-\tilde{H}\), where \(\tilde{E}, \tilde{H}\) (just) in \(L^{2}(\Omega)\) approximation of \(E=\nabla u, H=\nabla u\)
\(■ \Rightarrow\) NON-CONFORMING estimates for energy norm
- note: also mixed boundary conditions possible:
\[
u \in \mathrm{H}^{1}(\Omega) \text { with } \Delta u=G \quad \text { and }\left.\quad u\right|_{\Gamma_{t}}=v \text { and }\left.\partial_{n} u\right|_{\Gamma_{n}}=\left.n \cdot \nabla u\right|_{\Gamma_{n}}=g
\]
- note: also possible \(\tau E=\left.\partial_{t} u\right|_{\Gamma}=n \times\left.\nabla u\right|_{\Gamma}=f\)

\section*{Introduction: curl curl-Problems}

回 \(\Rightarrow\) NON-CONFORMING estimates for errors \(e:=E-\tilde{E}, h:=H-\tilde{H}\), where \(\tilde{E}, \tilde{H}\) (just) in \(L^{2}(\Omega)\) approximation of \(E=\operatorname{curl} U, H=\mathrm{curl} U\)
- \(\Rightarrow\) NON-CONFORMING estimates for energy half-norm

■ note: also mixed boundary conditions possible:
\(U \in \mathrm{H}(\operatorname{curl} ; \Omega)\) with curl curl \(U=F \quad\) and \(n \times\left. U\right|_{\Gamma_{n}}=v\) or \(n \times\left.\operatorname{curl} U\right|_{r_{t}}=f\)
note: also possible \(\nu H=n \cdot\) curl \(\left.U\right|_{\Gamma_{t}}=g\)

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- \(U \in \mathrm{H}(\operatorname{curl} ; \Omega)\) with curl curl \(U=F \quad\) and \(\quad n \times\left. U\right|_{\Gamma}=v\) or \(n \times\left.\operatorname{curl} U\right|_{\Gamma}=f\)
- Set \(E:=\operatorname{curl} U\) or \(H:=\operatorname{curl} U\), note: \(\left.n \cdot \operatorname{curl} U\right|_{\Gamma}=\left.\operatorname{curl}\right|_{\Gamma} n \times\left. U\right|_{\Gamma} \quad\left(\mathrm{d} \iota^{*}=\iota^{*} \mathrm{~d}\right)\)
- \(U \in \mathrm{H}(\operatorname{curl} ; \Omega)\) with curl curl \(U=F \quad\) and \(\quad n \times\left. U\right|_{\Gamma}=v\) or \(n \times\left.\operatorname{curl} U\right|_{\Gamma}=f\)
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\(\Rightarrow\)
in \(\Omega\)
in 「
in \(\Omega \quad\) (electro or magneto
static Maxwell problem)
```

$$
E \perp \mathcal{H}_{D}(\Omega),
$$

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curl E=F.

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curl E=F.
    div}E=0,\quad\operatorname{div}H=
    div}E=0,\quad\operatorname{div}H=
        \tauE=f,}\quad\nuH=\textrm{curl}\mp@subsup{|}{\Gammav}{}=:
        \tauE=f,}\quad\nuH=\textrm{curl}\mp@subsup{|}{\Gammav}{}=:
        E \perpH}\mp@subsup{\mathcal{H}}{D}{}(\Omega),\quadH\mathrm{ proj. on }\mp@subsup{\mathcal{H}}{N}{}(\Omega
        E \perpH}\mp@subsup{\mathcal{H}}{D}{}(\Omega),\quadH\mathrm{ proj. on }\mp@subsup{\mathcal{H}}{N}{}(\Omega
curl H=F
curl H=F
    \nuH=curl |rv =: g in \Gamma
```

    \nuH=curl |rv =: g in \Gamma
    ```


\section*{Introduction: curl curl-Problems}

■ \(U \in \mathrm{H}(\) curl \(; \Omega)\) with curl curl \(U=F \quad\) and \(\quad n \times\left. U\right|_{\Gamma}=v\) or \(n \times\left.\operatorname{curl} U\right|_{\Gamma}=f\)
- Set \(E:=\operatorname{curl} U\) or \(H:=\operatorname{curl} U\), note: \(\left.n \cdot \operatorname{curl} U\right|_{\Gamma}=\left.\operatorname{curl}\right|_{\Gamma} n \times\left. U\right|_{\Gamma} \quad\left(d \iota^{*}=\iota^{*} d\right)\)
```

curl }E=F
div}E=0,\quad\operatorname{div}H=
\tauE=f,\quad \nuH= curl | }\veev=:g\quad\mathrm{ in }
E \perp H H
curl H=F
div}H=0\quad\mathrm{ in }
H proj. on }\mp@subsup{\mathcal{H}}{N}{}(\Omega

```
in \(\Omega\)
in \(\Omega \quad\) (electro or magneto
in \(\Gamma \quad\) static Maxwell problem)
- \(\Rightarrow\) NON-CONFORMING estimates for errors \(e:=E-E, h:=H-\tilde{H}\), where \(\tilde{E}, \tilde{H}\) (just) in \(L^{2}(\Omega)\) approximation of \(E=\operatorname{curl} U, H=\operatorname{curl} U\)
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\[
\begin{aligned}
& \Rightarrow \quad \text { curl } E=F, \quad \text { curl } H=F \quad \text { in } \Omega \\
& \operatorname{div} E=0, \quad \operatorname{div} H=0 \quad \text { in } \Omega \quad \text { (electro or magneto } \\
& \tau E=f, \quad \nu H=\text { curl }\left.\right|_{\Gamma} v=: g \quad \text { in } \Gamma \\
& E \perp \mathcal{H}_{D}(\Omega), \quad H \text { proj. on } \mathcal{H}_{N}(\Omega)
\end{aligned}
\]
- \(\Rightarrow\) NON-CONFORMING estimates for errors \(e:=E-\tilde{E}, h:=H-\tilde{H}\), where \(\tilde{E}, \tilde{H}\) (just) in \(L^{2}(\Omega)\) approximation of \(E=\operatorname{curl} U, H=\) curl \(U\)
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\section*{Introduction: curl curl-Problems}
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\[
\begin{aligned}
\Rightarrow \quad \operatorname{curl} E & =F, & & \text { curl } H & =F & \\
\operatorname{div} E & =0, & \operatorname{div} \Omega & =0 & & \\
\tau E & =f, & \nu H & =\left.\operatorname{curl}\right|_{\Gamma v=: g} & & \text { in } \Gamma
\end{aligned}
\]

■ \(\Rightarrow\) NON-CONFORMING estimates for errors \(e:=E-\tilde{E}, h:=H-\tilde{H}\), where \(\tilde{E}, \tilde{H}\) (just) in \(L^{2}(\Omega)\) approximation of \(E=\operatorname{curl} U, H=\operatorname{curl} U\)
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a note: also possible \(\nu H=n \cdot\) curl \(\left.U\right|_{\Gamma_{t}}=g\)
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\section*{Introduction: curl curl-Problems}

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\begin{aligned}
\Rightarrow \quad \operatorname{curl} E & =F, & & \text { curl } H & =F & \\
\operatorname{div} E & =0, & \operatorname{div} \Omega & =0 & & \\
\tau E & =f, & \nu H & =\left.\operatorname{curl}\right|_{\Gamma v=: g} & & \text { in } \Gamma
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\section*{Introduction: curl curl-Problems}

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\begin{array}{rlrlrl}
\Rightarrow \quad \operatorname{curl} E & =F, & & \text { curl } H & =F & \\
\operatorname{div} E & =0, & & \text { in } \Omega & \\
\tau E & =f, & & \operatorname{div} H & =0 & \\
E H & =\left.\operatorname{curl}\right|_{\Gamma v}=: g & & \text { in } \Gamma & & \text { (electro or magneto } \\
E & \perp \mathcal{H}_{D}(\Omega), & & H \text { proj. on } \mathcal{H}_{N}(\Omega) & &
\end{array}
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\begin{aligned}
\Rightarrow \quad \operatorname{curl} E & =F, & & \text { curl } H & =F & \\
\operatorname{div} E & =0, & & \operatorname{div} H & =0 & \\
\tau E & =f, & \nu H & =\left.\operatorname{curl}\right|_{\ulcorner v=:} g & & \text { in } \Gamma
\end{aligned}
\]

■ \(\Rightarrow\) NON-CONFORMING estimates for errors \(e:=E-\tilde{E}, h:=H-\tilde{H}\), where \(\tilde{E}, \tilde{H}\) (just) in \(\mathrm{L}^{2}(\Omega)\) approximation of \(E=\operatorname{curl} U, H=\operatorname{curl} U\)
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\section*{Simple Model Problem and Solution Theory}

introducing scalar and vector potentials \(u\) and \(U\) solving


\section*{Simple Model Problem and Solution Theory}
\[
\begin{array}{rlrlr}
\operatorname{curl} E & =F & & \text { in } \Omega & \\
\operatorname{div} E & =G & & \text { in } \Omega & \\
\tau E & =0 & & & \\
E & \perp \mathcal{H}_{D}(\Omega) & & & \\
\operatorname{curl} E_{c} & =F, & & & \\
\operatorname{div} E_{c} & =0, & \operatorname{curl} E_{d} & =0 & \\
\tau E_{c} & =0, & \operatorname{div} E_{d} & =G & \\
E_{c} & \perp \mathcal{H}_{D}(\Omega), & \tau E_{d} & =0 & \\
& & & \text { in } \Gamma & \\
E_{d} & \perp \mathcal{H}_{D}(\Omega) & &
\end{array}
\]
introducing scalar and vector potentials \(u\) and \(U\) solving
\(\Delta u=\operatorname{div} \nabla u=G \quad\) in \(\Omega\)
\(\| \perp \mathcal{H}_{D}(\Omega)\)

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\[
\begin{array}{rlrl}
\text { curl } E & =F & & \text { in } \Omega \\
\operatorname{div} E & =G & & \\
\tau E & =0 & & \text { in } \Omega \\
& & & \text { (electro static Maxwell problem) } \\
E & \perp \mathcal{H}_{D}(\Omega) & & \\
\operatorname{curl} E_{c} & =F, & \operatorname{curl} E_{d}=0 & \\
\operatorname{div} E_{c} & =0, & \operatorname{div} E_{d}=G & \\
& & \text { in } \Omega \quad \text { (2 electro static Maxwell problems) }
\end{array}
\]
introducing scalar and vector potentials \(u\) and \(U\) solving

\section*{Simple Model Problem and Solution Theory}
\[
\begin{array}{rlrlr}
\operatorname{curl} E & =F & & \text { in } \Omega & \\
\operatorname{div} E & =G & & \text { in } \Omega & \text { (electro static Maxwell problem) } \\
\tau E & =0 & & & \\
E & \perp \mathcal{H}_{D}(\Omega) & & & \\
\text { unr } E_{c} & =F, & & & \\
\operatorname{div} E_{c} & =0, & \operatorname{curl} E_{d} & =0 & \\
\tau E_{c} & =0, & \operatorname{div} \Omega & & \\
E_{c} & & \perp \mathcal{H}_{D}(\Omega), & \tau E_{d} & =0 \\
E_{d} & \perp \mathcal{H}_{D}(\Omega) & & \text { in } \Gamma &
\end{array}
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introducing scalar and vector potentials \(u\) and \(U\) solving
\[
\begin{array}{rlrl}
\Delta U=\text { curl curl } U & =F, & \Delta u=\operatorname{div} \nabla u=G & \\
\operatorname{div} U & =0 & & \text { in } \Omega \\
\tau U & =0, & & u=0 \\
& & \text { in } \Gamma \\
\tau \operatorname{curl} U & =0 & & \text { in } \Gamma \\
U & \perp \mathcal{H}_{D}(\Omega) & &
\end{array}
\]
variational formulations for \(u\) and \(U\) (right Hilbert spaces)

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\[
\begin{array}{rlrl}
\operatorname{curl} E & =F & & \text { in } \Omega \\
& & \\
\operatorname{div} E & =G & & \text { in } \Omega \\
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E & \perp \mathcal{H}_{D}(\Omega) & & \\
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\operatorname{div} E_{c} & =0, & \operatorname{curl} E_{d}=0 & \\
\tau E_{c} & =0, & \operatorname{div} E_{d}=G & \\
\text { in } \Omega \quad \text { in } \quad \text { (2 electro static Maxwell problems) } \\
& \tau E_{d}=0 & & \text { in } \Gamma
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introducing scalar and vector potentials \(u\) and \(U\) solving
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\begin{array}{rlrl}
\Delta U=\text { curl curl } U & =F, & & \Delta u=\operatorname{div} \nabla u=G \\
& & \text { in } \Omega \\
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\begin{array}{rlrlr}
\operatorname{curl} E & =F & & \text { in } \Omega & \\
\operatorname{div} E & =G & & \text { in } \Omega & \text { (electro static Maxwell problem) } \\
\tau E & =0 & & & \\
E & \perp \mathcal{H}_{D}(\Omega) & & & \\
\text { unr } E_{c} & =F, & & & \\
\operatorname{div} E_{c} & =0, & \operatorname{curl} E_{d} & =0 & \\
\tau \operatorname{div} E_{d} & =G & & \text { in } \Omega & \\
\tau E_{c} & =0, & \tau E_{d} & =0 & \\
E_{c} & \perp \mathcal{H}_{D}(\Omega), & E_{d} & \perp \mathcal{H}_{D}(\Omega) &
\end{array}
\]
introducing scalar and vector potentials \(u\) and \(U\) solving
\[
\begin{array}{rlrl}
\Delta U=\text { curl curl } U & =F, & \Delta u=\operatorname{div} \nabla u=G & \\
\operatorname{div} \Omega & =0 & & \text { in } \Omega \\
\tau U & =0, & & u=0 \\
& & \text { in } \Gamma \\
\tau \operatorname{curl} U & =0 & & \text { in } \Gamma \\
U & \perp \mathcal{H}_{D}(\Omega) & &
\end{array}
\]
variational formulations for \(u\) and \(U\) (right Hilbert spaces)
\(\Rightarrow E_{c}:=\operatorname{curl} U\) and \(E_{d}:=\nabla u\) as well as \(E:=E_{c}+E_{d}\)

\section*{Model Problem and Method for Error Estimates}
\[
\begin{aligned}
\operatorname{curl} E & =F & & \text { in } \Omega \\
\operatorname{div} E & =G & & \text { in } \Omega \\
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- method: funct. a post. error est. for linear second order elliptic problems pioneering work of Sergey Repin starting 1990's
later extended to 'all' linear and non-linear second order elliptic problems (Laplace, elastic, parabolic, hyperbolic, even order problems, ...)
- Maxwell system is first order! What to do?
- solution: Helmholtz decomposition \(\Rightarrow\) scalar and vector potentials \(\Rightarrow\) second order methods for the potentials

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\section*{Universitat}


\section*{Model Problem and Method for Error Estimates}
\[
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- Maxwell system is first order! What to do?
\(■\) solution: Helmholtz decomposition \(\Rightarrow\) scalar and vector potentials \(\Rightarrow\) second order methods for the potentials

\section*{Sobolev Spaces}
spaces:
```

    H(curl; \Omega):={E\in L' (\Omega): curl E E L'2}(\Omega)
    H(curlo;\Omega):={E\subsetH(curl;\Omega): curl E = 0}
    \circ}\overline{\circ}\textrm{H}(\textrm{curl};\Omega
    H}(\mathrm{ curl ; }\Omega):={E\inH(curl;\Omega):\tauE=0}=\mp@subsup{C}{}{\infty}(\Omega
    H}(\mp@subsup{curl}{0}{;}\Omega):=H(\operatorname{curl};\Omega)\capH(\mp@subsup{curl}{0}{\prime};\Omega
    ```
analogously:
\(H(\operatorname{div} ; \Omega), \quad H\left(\operatorname{div}_{0} ; \Omega\right), \quad H(\operatorname{div} ; \Omega), \quad H\left(\operatorname{div}_{0} ; \Omega\right)\)
and:
```

$\mathcal{H}_{D}(\Omega):=\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) \cap \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)$
$=\left\{E \subset \mathrm{~L}^{2}(\Omega) \cdot \operatorname{cutl}^{\prime} E=0\right.$, div $\left.E=0, \tau E=0\right\}$

## Sobolev Spaces

spaces:

$$
\mathrm{H}(\text { curl } ; \Omega):=\left\{E \in \mathrm{~L}^{2}(\Omega): \text { curl } E \in \mathrm{~L}^{2}(\Omega)\right\}
$$

$H\left(\right.$ curlo $\left._{0} ; \Omega\right):=\{E \in H($ curl $; \Omega): \operatorname{curl} E=0\}$
$\mathrm{H}($ curl $; \Omega):=\{E \in \mathrm{H}($ curl $; \Omega): \tau E=0\}=\mathrm{C}^{\infty}(\Omega)$

## $\mathrm{H}($ curl $0: \Omega):=\mathrm{H}($ curl $: \Omega) \cap \mathrm{H}($ curl $0: \Omega)$

analogously:
$H(\operatorname{div} ; \Omega), \quad H\left(\operatorname{div}_{0} ; \Omega\right), \quad H(\operatorname{div} ; \Omega), \quad H\left(\operatorname{div}_{0} ; \Omega\right)$
and:

$$
\begin{aligned}
\mathcal{H}_{D}(\Omega) & :=\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) \cap \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \quad \text { (finite dimension) } \\
& =\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E=0, \operatorname{div} E=0, \tau E=0\right\}
\end{aligned}
$$

## Sobolev Spaces

spaces:

$$
\mathrm{H}(\text { curl } ; \Omega):=\left\{E \in \mathrm{~L}^{2}(\Omega): \text { curl } E \in \mathrm{~L}^{2}(\Omega)\right\}
$$

$$
\mathrm{H}\left(\text { curl }_{0} ; \Omega\right):=\{E \in \mathrm{H}(\text { curl } ; \Omega): \operatorname{curl} E=0\}
$$

## $\mathrm{H}($ curl $; \Omega):=\{E \in \mathrm{H}($ curl $; \Omega): \tau E=0\}=\mathrm{C}^{\infty}(\Omega)$



## analogously

 $H(d i v ; \Omega), \quad H\left(\operatorname{div}_{0} ; \Omega\right), \quad H(\operatorname{div} ; \Omega), \quad H\left(\operatorname{div}_{0} ; \Omega\right)$and
$\mathcal{H}_{D}(\Omega):=\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) \cap \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)$

$$
=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E=0, \operatorname{div} E=0, \tau E=0\right\}
$$

## Sobolev Spaces

spaces:

$$
\begin{aligned}
\mathrm{H}(\operatorname{curl} ; \Omega) & :=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E \in \mathrm{~L}^{2}(\Omega)\right\} \\
\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) & :=\{E \in \mathrm{H}(\operatorname{curl} ; \Omega): \operatorname{curl} E=0\}
\end{aligned}
$$

$$
\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega):=\{E \in \mathrm{H}(\operatorname{curl} ; \Omega): \tau E=0\}={\bar{\circ}{ }^{\circ} \infty(\Omega)}^{\mathrm{H}(\mathrm{curl} ; \Omega)} \quad \text { (Gauß' theorem) }
$$

$$
\mathrm{H}\left(\text { curl }_{0} ; \Omega\right):=\mathrm{H}(\text { curl } ; \Omega) \cap \mathrm{H}\left(\text { curl }_{0} ; \Omega\right)
$$

## analogously

$$
\mathrm{H}(\operatorname{div} ; \Omega), \quad \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right), \quad \mathrm{H}(\operatorname{div} ; \Omega), \quad \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)
$$

and
$\mathcal{H}_{D}(\Omega):=\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) \cap \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)$

$$
=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E=0, \operatorname{div} E=0, \tau E=0\right\}
$$

## Sobolev Spaces

spaces:

$$
\begin{aligned}
& \mathrm{H}(\text { curl } ; \Omega):=\left\{E \in \mathrm{~L}^{2}(\Omega): \text { curl } E \in \mathrm{~L}^{2}(\Omega)\right\} \\
& \mathrm{H}\left(\text { curl }_{0} ; \Omega\right):=\{E \in \mathrm{H}(\text { curl } ; \Omega): \operatorname{curl} E=0\} \\
& \stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega):=\{E \in \mathrm{H}(\operatorname{curl} ; \Omega): \tau E=0\}={\bar{\circ}{ }^{\circ} \infty(\Omega)}^{\mathrm{H}(\text { curl } ; \Omega)} \quad \text { (Gauß' theorem) } \\
& \stackrel{\circ}{\mathrm{H}}\left(\text { curl }_{0} ; \Omega\right):=\stackrel{\circ}{\mathrm{H}}(\text { curl } ; \Omega) \cap \mathrm{H}\left(\text { curl }_{0} ; \Omega\right)
\end{aligned}
$$

## analogously

and
$\mathcal{H}_{D}(\Omega):=\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) \cap \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)$

$$
=\left\{E \in 1^{2}(\Omega) \cdot \operatorname{curl} E=0, \operatorname{div} E=0, \tau E=0\right\}
$$

## Sobolev Spaces

spaces:

$$
\begin{aligned}
\mathrm{H}(\operatorname{curl} ; \Omega) & :=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E \in \mathrm{~L}^{2}(\Omega)\right\} \\
\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) & :=\{E \in \mathrm{H}(\operatorname{curl} ; \Omega): \operatorname{curl} E=0\}
\end{aligned}
$$

$$
\begin{aligned}
& \stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega):=\left\{E \in \mathrm{H}\left(\operatorname{curl}^{\prime} ; \Omega\right): \tau E=0\right\}={\bar{\circ}{ }^{\infty}(\Omega)}^{\mathrm{H}\left(\text { curl }^{\infty} \Omega\right)} \quad \text { (Gauß' theorem) } \\
& \stackrel{\circ}{\mathrm{H}}\left(\operatorname{curl}_{0} ; \Omega\right):=\stackrel{\circ}{\mathrm{H}}\left(\operatorname{curl}^{\prime} ; \Omega\right) \cap \mathrm{H}\left(\text { curl }_{0} ; \Omega\right)
\end{aligned}
$$

analogously:

$$
\mathrm{H}(\operatorname{div} ; \Omega), \quad \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right), \quad \stackrel{\circ}{\mathrm{H}}(\operatorname{div} ; \Omega), \quad \stackrel{\circ}{\mathrm{H}}\left(\operatorname{div}_{0} ; \Omega\right)
$$

## and

$\mathcal{H}_{D}(\Omega):=\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) \cap \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right)$

$$
=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E=0, \operatorname{div} E=0, \tau E=0\right\}
$$

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## Sobolev Spaces

spaces:

$$
\begin{aligned}
\mathrm{H}\left(\operatorname{curl}^{\prime} \Omega\right) & :=\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E \in \mathrm{~L}^{2}(\Omega)\right\} \\
\mathrm{H}\left(\text { curl }_{0} ; \Omega\right) & :=\{E \in \mathrm{H}(\operatorname{curl} ; \Omega): \operatorname{curl} E=0\} \\
\stackrel{\circ}{\mathrm{H}}\left(\operatorname{curl}^{\prime} \Omega\right) & :=\{E \in \mathrm{H}(\operatorname{curl} ; \Omega): \tau E=0\}=\stackrel{\circ}{\circ} \infty(\Omega)_{\mathrm{H}(\text { curl } ; \Omega)}^{\circ} \quad \text { (Gauß' theorem) } \\
\mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right) & :=\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega) \cap \mathrm{H}\left(\operatorname{curl}_{0} ; \Omega\right)
\end{aligned}
$$

analogously:

$$
\mathrm{H}(\operatorname{div} ; \Omega), \quad \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right), \quad \stackrel{\circ}{\mathrm{H}}(\operatorname{div} ; \Omega), \quad \stackrel{\circ}{\mathrm{H}}\left(\operatorname{div}_{0} ; \Omega\right)
$$

and:

$$
\begin{aligned}
\mathcal{H}_{D}(\Omega) & :=\stackrel{\circ}{\mathrm{H}}\left(\operatorname{curl}_{0} ; \Omega\right) \cap \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \\
& =\left\{E \in \mathrm{~L}^{2}(\Omega): \operatorname{curl} E=0, \operatorname{div} E=0, \tau E=0\right\}
\end{aligned}
$$

## Results: Upper Bounds for Non-Conforming Approximations

## $\tilde{E} \in \mathrm{~L}^{2}(\Omega)$ approximations of $E \Rightarrow$

Theorem For all $\tilde{E} \in L^{2}(\Omega)$ and all $D \in \mathcal{H}_{D}(\Omega)$

holds. Here, $\Phi$ denotes a projection onto Dirichlet fields $\mathcal{H}_{D}(\Omega)$ and

$$
\begin{aligned}
& M_{\text {curl }}(\tilde{E} ; X):=c_{\mathrm{m}}\|F-\operatorname{curl} X\|_{L^{2}(\Omega)}+\|\tilde{E}-X\|_{L^{2}(\Omega)}, \\
& M_{\operatorname{div}}(\tilde{E} ; Y):=c_{\mathrm{p}}\|G-\operatorname{div} Y\|_{L^{2}(\Omega)}+\|\tilde{E}-Y\|_{L^{2}(\Omega)} .
\end{aligned}
$$

only natural continuity constants involved:

- $c_{p}$ Poincaré constant, i.e., $\forall u \in M^{1}(\Omega)$

$$
\|u\|_{L^{2}(\Omega)} \leq c_{\mathrm{p}}\|\nabla u\|_{\mathrm{L}^{2}(\Omega)}
$$

- $c_{\text {m }}$ Maxwell constant, i.e., $\forall E \in \mathbb{H}:=\mathrm{H}(\operatorname{curl} ; \Omega) \cap H\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{N}(\Omega) \perp$

$$
\|E\|_{L^{2}(\Omega)} \leq c_{m} \| \text { curi } E^{\prime \prime} L_{L^{2}(\Omega)}
$$

## Results: Upper Bounds for Non-Conforming Approximations

$\tilde{E} \in \mathrm{~L}^{2}(\Omega)$ approximations of $E \Rightarrow$
Theorem For all $\tilde{E} \in \mathrm{~L}^{2}(\Omega)$ and all $D \in \mathcal{H}_{D}(\Omega)$

$$
\|E-\tilde{E}-D\|_{\mathrm{L}^{2}(\Omega)}^{2} \leq \inf _{\substack{\mathcal{H}(\mathrm{curl} ; \Omega)}} M_{\text {curl }}^{2}(\tilde{E} ; X)+\inf _{Y \in \mathrm{H}(\operatorname{div} ; \Omega)} M_{\mathrm{div}}^{2}(\tilde{E} ; Y)+|\Phi(\tilde{E}-D)|^{2}
$$

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$$
\begin{aligned}
& M_{\text {curl }}(\tilde{E} ; X):=c_{\mathrm{m}}\|F-\operatorname{curl} X\|_{L^{2}(\Omega)}+\|\tilde{E}-X\|_{L^{2}(\Omega)}, \\
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$$
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\end{aligned}
$$

only natural continuity constants involved:

- $c_{\mathrm{p}}$ Poincaré constant, i.e., $\forall u \in H^{1}(\Omega)$
$\|u\|_{L^{2}(\Omega)} \leq c_{\mathrm{P}}\|\nabla u\|_{L^{2}(\Omega)}$
- $c_{\text {m }}$ Maxwell constant, i.e., $\forall E \in \mathbb{H}:=\mathrm{H}(\operatorname{curl} ; \Omega) \cap \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{N}(\Omega)$
$\|E\|_{L^{2}(\Omega)} \leq c_{\mathrm{m}} \| \operatorname{curl} E_{L^{2}(\Omega)}$


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$$
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$$

- $c_{\text {II }}$ Maxwell constant, i.e., $\forall E \in \mathbb{H}:=H(\operatorname{curl} ; \Omega) \cap H\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{N}(\Omega)$
$\|E\|_{L^{2}(\Omega)} \leq c_{\mathrm{m}}\|\operatorname{curl} E\|_{L^{2}(\Omega)}$


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$$
\|E-\tilde{E}-D\|_{\mathrm{L}^{2}(\Omega)}^{2} \leq \inf _{\substack{\dot{\mathrm{H}}(\mathrm{cur} ; \Omega)}} M_{\mathrm{curl}}^{2}(\tilde{E} ; X)+\inf _{Y \in \mathrm{H}(\text { div } ; \Omega)} M_{\mathrm{div}}^{2}(\tilde{E} ; Y)+|\Phi(\tilde{E}-D)|^{2}
$$

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\begin{aligned}
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\end{aligned}
$$

only natural continuity constants involved:

- $c_{\mathrm{p}}$ Poincaré constant, i.e., $\forall u \in \stackrel{\circ}{\mathrm{H}}^{1}(\Omega)$

$$
\|u\|_{L^{2}(\Omega)} \leq c_{\mathrm{p}}\|\nabla u\|_{\mathrm{L}^{2}(\Omega)}
$$

- $c_{\mathrm{m}}$ Maxwell constant, i.e., $\forall E \in \mathbb{H}:=\mathrm{H}(\operatorname{curl} ; \Omega) \cap \stackrel{\circ}{\mathrm{H}}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{N}(\Omega)^{\perp}$

$$
\|E\|_{L^{2}(\Omega)} \leq c_{\mathrm{m}}\|\operatorname{curl} E\|_{\mathrm{L}^{2}(\Omega)}
$$

## Proof：Tools

－Rellich＇s selection theorems，i．e．，$H^{1}(\Omega), H^{1}(\Omega) \hookrightarrow L^{2}(\Omega)$ compact
$\Rightarrow$ Poincaré estimate，i．e．

$$
\begin{array}{ll}
\forall u \in \dot{H}^{1}(\Omega) & \|u\|_{L^{2}(\Omega)} \leq c_{p}\|\nabla u\|_{L^{2}(\Omega)} \\
\forall u \in H^{1}(\Omega) \cap\{1\}^{\perp} & \|u\|_{L^{2}(\Omega)} \leq \tilde{c}_{\mathrm{p}}\|\nabla u\|_{L^{2}(\Omega)}
\end{array}
$$

－Maxwell selection theorems，i．e．，

$$
\stackrel{\circ}{\mathrm{H}}(\text { curl } ; \Omega) \cap \mathrm{H}(\text { div } ; \Omega), \mathrm{H}(\text { curl } ; \Omega) \cap \mathrm{H}(\text { div } ; \Omega) \hookrightarrow L^{2}(\Omega) \text { compact }
$$

$\Rightarrow$ Maxwell estimates，i．e．，

$$
\begin{array}{ll}
\forall E \in \tilde{\mathbb{H}}=\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega) \cap \mathrm{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{D}(\Omega)^{\perp} & \|E\|_{L^{2}(\Omega)} \leq \tilde{c}_{\mathrm{m}}\|\operatorname{curl} E\|_{\mathrm{L}^{2}(\Omega)} \\
\forall E \in \mathbb{H}=\mathrm{H}(\operatorname{curl} ; \Omega) \cap \stackrel{\circ}{\mathrm{H}}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{N}(\Omega)^{\perp} & \|E\|_{L^{2}(\Omega)} \leq c_{\mathrm{m}}\|\operatorname{curl} E\|_{L^{2}(\Omega)}
\end{array}
$$

－Maxwell selection theorems $\Rightarrow \operatorname{dim} \mathcal{H}_{D}(\Omega), \operatorname{dim} \mathcal{H}_{N}(\Omega)<\infty$（Betti numbers）
－Helmholtz decompositions（all 6 images are closed in $\mathrm{L}^{2}(\Omega)$ ）

$$
\begin{aligned}
\mathrm{L}^{2}(\Omega) & =\nabla \stackrel{\circ}{\mathrm{H}^{1}}(\Omega) \oplus \overbrace{\mathcal{H}_{D}(\Omega) \oplus \operatorname{curl} \mathrm{H}(\text { curl } ; \Omega)}^{\mathrm{H}(\text { divo } ; \Omega)}, \\
\mathrm{L}^{2}(\Omega) & =\nabla \mathrm{H}^{1}(\Omega) \oplus \underbrace{}_{=\stackrel{\mathrm{H}(\text { div } ; \Omega)}{\mathcal{H}_{N}(\Omega) \oplus \operatorname{curl} \stackrel{\circ}{\mathrm{H}}(\text { curl } ; \Omega)}},
\end{aligned}
$$

$$
\operatorname{curl} \mathrm{H}(\operatorname{curl} ; \Omega)=\operatorname{curl} \mathbb{H}
$$

$$
\operatorname{curl} \dot{\mathrm{H}}(\text { curl } ; \Omega)=\operatorname{curl} \tilde{\mathbb{H}}
$$



## Proof: Tools

- Rellich's selection theorems, i.e., $\stackrel{\mathrm{H}}{ }^{1}(\Omega), \mathrm{H}^{1}(\Omega) \hookrightarrow \mathrm{L}^{2}(\Omega)$ compact
$\Rightarrow$ Poincaré estimate, i.e.,

$$
\begin{array}{ll}
\forall u \in \stackrel{\circ}{\mathrm{H}}^{1}(\Omega) & \|u\|_{\mathrm{L}^{2}(\Omega)} \leq c_{\mathrm{p}}\|\nabla u\|_{\mathrm{L}^{2}(\Omega)} \\
\forall u \in \mathrm{H}^{1}(\Omega) \cap\{1\}^{\perp} & \|u\|_{\mathrm{L}^{2}(\Omega)} \leq \tilde{\tilde{c}_{\mathrm{p}}}\|\nabla u\|_{\mathrm{L}^{2}(\Omega)}
\end{array}
$$

- Maxwell selection theorems, i.e.
$H($ curl $; \Omega) \cap H(\operatorname{div} ; \Omega), H($ curl $; \Omega) \cap H(\operatorname{div} ; \Omega) \hookrightarrow L^{2}(\Omega)$ compact $\Rightarrow$ Maxwell estimates, i.e.,

| $\forall E \in \tilde{H}=H(\operatorname{curl} ; \Omega) \cap H\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{D}(\Omega)^{\perp}$ | $\\|E\\|_{L^{2}(\Omega)} \leq \tilde{c}_{\mathrm{m}}\\|\operatorname{curl} E\\|_{L^{2}(\Omega)}$ |
| :--- | :--- | :--- |
| $\forall E \in \mathbb{H}=H(\operatorname{curl} ; \Omega) \cap \stackrel{\circ}{H}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{N}(\Omega)^{\perp}$ | $\\|E\\|_{L^{2}(\Omega)} \leq c_{\mathrm{m}}\\|\operatorname{curl} E\\|_{L^{2}(\Omega)}$ |

- Maxwell selection theorems $\Rightarrow \operatorname{dim} \mathcal{H}_{D}(\Omega), \operatorname{dim} \mathcal{H}_{N}(\Omega)<\infty$ (Betti numbers)
- Helmholtz decompositions (all 6 images are closed in $\mathrm{L}^{2}(\Omega)$ )



## Proof: Tools

- Rellich's selection theorems, i.e., $\stackrel{H}{H}^{1}(\Omega), \mathrm{H}^{1}(\Omega) \hookrightarrow \mathrm{L}^{2}(\Omega)$ compact
$\Rightarrow$ Poincaré estimate, i.e.,

$$
\begin{array}{ll}
\forall u \in \stackrel{\circ}{\mathrm{H}}^{1}(\Omega) & \|u\|_{\mathrm{L}^{2}(\Omega)} \leq c_{\mathrm{p}}\|\nabla u\|_{\mathrm{L}^{2}(\Omega)} \\
\forall u \in \mathrm{H}^{1}(\Omega) \cap\{1\}^{\perp} & \|u\|_{\mathrm{L}^{2}(\Omega)} \leq \tilde{c_{\mathrm{p}}}\|\nabla u\|_{\mathrm{L}^{2}(\Omega)}
\end{array}
$$

■ Maxwell selection theorems, i.e.,
$\stackrel{\circ}{\mathrm{H}}($ curl $; \Omega) \cap \mathrm{H}(\operatorname{div} ; \Omega), \mathrm{H}($ curl $; \Omega) \cap \stackrel{\circ}{\mathrm{H}}(\operatorname{div} ; \Omega) \hookrightarrow \mathrm{L}^{2}(\Omega)$ compact
$\Rightarrow$ Maxwell estimates, i.e.,

$$
\begin{array}{ll}
\forall E \in \tilde{\mathbb{H}}=\stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega) \cap \dot{\mathrm{H}}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{D}(\Omega)^{\perp} & \|E\|_{L^{2}(\Omega)} \leq \tilde{c}_{\mathrm{m}}\|\operatorname{curl} E\|_{\mathrm{L}^{2}(\Omega)} \\
\forall E \in \mathbb{H}=\mathrm{H}(\operatorname{curl} ; \Omega) \cap \stackrel{\circ}{\mathrm{H}}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{N}(\Omega)^{\perp} & \|E\|_{L^{2}(\Omega)} \leq c_{\mathrm{m}}\|\operatorname{curl} E\|_{\mathrm{L}^{2}(\Omega)}
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$$

- Maxwell selection theorems $\Rightarrow \operatorname{dim} \mathcal{H}_{D}(\Omega), \operatorname{dim} \mathcal{H}_{N}(\Omega)<\infty$ (Betti numbers)
- Helmholtz decompositions (all 6 images are closed in $\mathrm{L}^{2}(\Omega)$ )



## Proof: Tools

- Rellich's selection theorems, i.e., $\stackrel{H}{H}^{1}(\Omega), \mathrm{H}^{1}(\Omega) \hookrightarrow \mathrm{L}^{2}(\Omega)$ compact
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\forall E \in \mathbb{H}=\mathrm{H}(\operatorname{curl} ; \Omega) \cap \stackrel{\circ}{\mathrm{H}}\left(\operatorname{div}_{0} ; \Omega\right) \cap \mathcal{H}_{N}(\Omega)^{\perp} & \|E\|_{\mathrm{L}^{2}(\Omega)} \leq c_{\mathrm{m}}\|\operatorname{curl} E\|_{\mathrm{L}^{2}(\Omega)}
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- Maxwell selection theorems, i.e.,

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$$

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$$

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- Helmholtz decompositions (all 6 images are closed in $\mathrm{L}^{2}(\Omega)$ )

$$
\begin{aligned}
\mathrm{L}^{2}(\Omega)=\nabla \stackrel{\circ}{\mathrm{H}}^{1}(\Omega) \oplus \overbrace{\mathcal{H}_{D}(\Omega) \oplus \operatorname{curl} \mathrm{H}(\text { curl } ; \Omega)}^{=\mathrm{H}(\text { divo } ; \Omega)}, & \text { curl } \mathrm{H}(\text { curl } ; \Omega)=\operatorname{curl} \mathbb{H} \\
\mathrm{L}^{2}(\Omega)=\nabla \mathrm{H}^{1}(\Omega) \oplus \underbrace{\mathcal{H}_{N}(\Omega) \oplus \operatorname{curl} \stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega)}_{=\stackrel{\circ}{\mathrm{H}}\left(\text { div }^{2} ; \Omega\right)}, & \text { curl } \stackrel{\circ}{\mathrm{H}}(\text { curl } ; \Omega)=\operatorname{curl} \tilde{\mathbb{H}}
\end{aligned}
$$

## Proof

$$
e=E-\tilde{E} \in \mathrm{~L}^{2}(\Omega)
$$

- Helmholtz decomposition of error $\Rightarrow$
- $e_{\nabla}=\nabla u$ with scalar potential $u \in \mathrm{H}^{1}(\Omega)$
- $e_{\text {curl }}=\operatorname{curl} U$ with vector potential $U \in \mathbb{H}$
$=\|e\|_{L^{2}(\Omega)}^{2}=\left\|e_{\nabla}\right\|_{L^{2}(\Omega)}^{2}+\left\|e_{\mathcal{H}}\right\|_{L^{2}(\Omega)}^{2}+\left\|e_{\text {curr }}\right\|_{L^{2}(\Omega)}^{2}$
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## Proof

$e=E-\tilde{E} \in \mathrm{~L}^{2}(\Omega)$
■ Helmholtz decomposition of error $\Rightarrow$

$$
e=e_{\nabla}+e_{\mathcal{H}}+e_{\text {curl }} \in \nabla \dot{H}^{1}(\Omega) \oplus \mathcal{H}_{D}(\Omega) \oplus \text { curl } \mathbb{H}
$$

- $e_{\nabla}=\nabla u$ with scalar potential $u \in \mathrm{H}^{1}(\Omega)$
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- $e_{\nabla}=\nabla u$ with scalar potential $u \in \stackrel{\circ}{H}^{1}(\Omega)$
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- $\|e\|_{\mathrm{L}^{2}(\Omega)}^{2}=\left\|e_{\nabla}\right\|_{\mathrm{L}^{2}(\Omega)}^{2}+\left\|e_{\mathcal{H}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2}+\left\|e_{\mathrm{cur}}\right\|_{\mathrm{L}^{2}(\Omega)}^{2}$


## Proof

- $e_{\nabla}=\nabla u, u \in{\stackrel{\circ}{H^{1}}(\Omega): \quad \forall \varphi \in \dot{H}^{1}(\Omega) \quad \forall \gamma \in H(d i v ; \Omega)}^{(\Omega)}$

$$
\begin{aligned}
&\left\langle e_{\nabla}, \nabla \varphi\right\rangle_{L^{2}(\Omega)}=\langle e, \nabla \varphi\rangle_{L^{2}(\Omega)}=\langle E, \nabla \varphi\rangle_{L^{2}(\Omega)}-\langle\tilde{E}, \nabla \varphi\rangle_{L^{2}(\Omega)} \\
&=\langle\operatorname{div} Y-G, \varphi\rangle_{L^{2}(\Omega)}+\langle Y-\tilde{E}, \nabla \varphi\rangle_{L^{2}(\Omega)} \\
& \leq\|\operatorname{div} Y-G\|_{L^{2}(\Omega)} \underbrace{\|\varphi\|_{L^{2}(\Omega)}}+\|Y-\tilde{E}\|_{L^{2}(\Omega)}\|\nabla \varphi\|_{L^{2}(\Omega)}
\end{aligned}
$$

$$
\leq c_{p}\|\nabla \varphi\|_{L^{2}(\Omega)}
$$

$$
\varphi:=u \quad \Rightarrow \quad\left\|e_{\nabla}\right\|_{L^{2}(\Omega)} \leq c_{p}\|\operatorname{div} Y-G\|_{L^{2}(\Omega)}+\|Y-\tilde{E}\|_{L^{2}(\Omega)}
$$

$$
e_{\text {curl }}=\operatorname{curl} U, U \in \mathbb{H}:
$$

## Proof

- $e_{\nabla}=\nabla u, u \in \dot{H}^{1}(\Omega): \quad \forall \varphi \in \dot{H}^{1}(\Omega) \quad \forall Y \in H(\operatorname{div} ; \Omega)$

$$
\left\langle e_{\nabla}, \nabla \varphi\right\rangle_{L^{2}(\Omega)}=\langle e, \nabla \varphi\rangle_{L^{2}(\Omega)}=\langle E, \nabla \varphi\rangle_{L^{2}(\Omega)}-\langle\tilde{E}, \nabla \varphi\rangle_{L^{2}(\Omega)}
$$

$$
=\langle\operatorname{div} Y-G, \varphi\rangle_{L^{2}(\Omega)}+\langle Y-\tilde{E}, \nabla \varphi\rangle_{L^{2}(\Omega)}
$$

$$
\leq\|\operatorname{div} Y-G\|_{L^{2}(\Omega)} \underbrace{\|\varphi\|_{L^{2}}}_{\leq c_{\mathrm{P}}\left\|\nabla \varphi_{L^{2}}\right\|_{L^{2}(\Omega)}}+\|Y-\tilde{E}\|_{L^{2}(\Omega)}\|\nabla \varphi\|_{L^{2}(\Omega)}
$$

- $e_{\text {curl }}=\operatorname{curl} U, U \in \mathbb{H}:$


## Proof

- $e_{\nabla}=\nabla u, u \in \dot{\mathrm{H}}^{1}(\Omega): \quad \forall \varphi \in \dot{\mathrm{H}}^{1}(\Omega) \quad \forall Y \in \mathrm{H}(\operatorname{div} ; \Omega)$

$$
\begin{aligned}
&\left\langle e_{\nabla}, \nabla \varphi\right\rangle_{L^{2}(\Omega)}=\langle e, \nabla \varphi\rangle_{L^{2}(\Omega)}=\langle E, \nabla \varphi\rangle_{L^{2}(\Omega)}-\langle\tilde{E}, \nabla \varphi\rangle_{L^{2}(\Omega)} \\
&=\langle\operatorname{div} Y-G, \varphi\rangle_{L^{2}(\Omega)}+\langle Y-\tilde{E}, \nabla \varphi\rangle_{L^{2}(\Omega)} \\
& \leq\|\operatorname{div} Y-G\|_{L^{2}(\Omega)} \underbrace{\| \|_{L^{2}(\Omega)}}_{\leq c_{\mathrm{p}}\|\nabla\|_{L^{2}(\Omega)}}+\|Y-\tilde{E}\|_{L^{2}(\Omega)}\|\nabla \varphi\|_{L^{2}(\Omega)} \\
& \varphi:=u \Rightarrow\left\|e_{\nabla}\right\|_{L^{2}(\Omega)} \leq c_{\mathrm{p}}\|\operatorname{div} Y-G\|_{L^{2}(\Omega)}+\|Y-\tilde{E}\|_{L^{2}(\Omega)}
\end{aligned}
$$

## Proof

- $e_{\nabla}=\nabla u, u \in \dot{\mathrm{H}}^{1}(\Omega): \quad \forall \varphi \in \dot{\mathrm{H}}^{1}(\Omega) \quad \forall Y \in \mathrm{H}(\operatorname{div} ; \Omega)$

$$
\left\langle e_{\nabla}, \nabla \varphi\right\rangle_{L^{2}(\Omega)}=\langle e, \nabla \varphi\rangle_{L^{2}(\Omega)}=\langle E, \nabla \varphi\rangle_{L^{2}(\Omega)}-\langle\tilde{E}, \nabla \varphi\rangle_{L^{2}(\Omega)}
$$

$$
=\langle\operatorname{div} Y-G, \varphi\rangle_{L^{2}(\Omega)}+\langle Y-\tilde{E}, \nabla \varphi\rangle_{L^{2}(\Omega)}
$$

$$
\leq\|\operatorname{div} Y-G\|_{L^{2}(\Omega)} \underbrace{\|\varphi\|_{L^{2}(\Omega)}}_{\leq c_{\mathrm{C}}\|\nabla\|_{L_{L^{2}}(\Omega)}}+\|Y-\tilde{E}\|_{L^{2}(\Omega)}\|\nabla \varphi\|_{L^{2}(\Omega)}
$$

$$
\varphi:=u \quad \Rightarrow \quad\left\|e_{\nabla}\right\|_{L^{2}(\Omega)} \leq c_{\mathrm{p}}\|\operatorname{div} Y-G\|_{L^{2}(\Omega)}+\|Y-\tilde{E}\|_{L^{2}(\Omega)}
$$

- $e_{\text {curl }}=\operatorname{curl} U, U \in \mathbb{H}:$
$\left\langle e_{\text {curl }}, \text { curl } \mid \Phi\right\rangle_{L^{2}(\Omega)}=\langle e, \operatorname{curl} \mid \Phi\rangle_{L^{2}(\Omega)}=\langle E, \text { curl } \Phi\rangle_{L^{2}(\Omega)}-\langle\tilde{E}, \text { curl } \Phi\rangle_{L^{2}(\Omega)}$
$=\langle F-\operatorname{curl} X, \Phi\rangle_{L^{2}(\Omega)}+\langle X-\tilde{E}, \operatorname{curl} \Phi\rangle_{L^{2}(\Omega)}$

$\Phi:=U \quad \Rightarrow$
$\left\|e_{\text {curl }}\right\|_{\mathrm{L}^{2}(\Omega)} \leq c_{\mathrm{m}}\|F-\operatorname{curl} X\|_{\mathrm{L}^{2}(\Omega)}+\|X-E\|_{\mathrm{L}^{2}(\Omega)}$
- $e_{\mathcal{H}}$ : simple algebraic manipulation


## Proof

- $e_{\nabla}=\nabla u, u \in \dot{H}^{1}(\Omega): \quad \forall \varphi \in \dot{H}^{1}(\Omega) \quad \forall Y \in H(\operatorname{div} ; \Omega)$

$$
\left\langle e_{\nabla}, \nabla \varphi\right\rangle_{\mathrm{L}^{2}(\Omega)}=\langle e, \nabla \varphi\rangle_{\mathrm{L}^{2}(\Omega)}=\langle E, \nabla \varphi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\tilde{E}, \nabla \varphi\rangle_{\mathrm{L}^{2}(\Omega)}
$$

$$
=\langle\operatorname{div} Y-G, \varphi\rangle_{L^{2}(\Omega)}+\langle Y-\tilde{E}, \nabla \varphi\rangle_{L^{2}(\Omega)}
$$

$$
\leq\|\operatorname{div} Y-G\|_{L^{2}(\Omega)} \underbrace{\|\varphi\|_{L^{2}(\Omega)}}_{\leq c_{\mathrm{p}}\|\nabla\|_{L_{L^{2}}(\Omega)}}+\|Y-\tilde{E}\|_{L^{2}(\Omega)}\|\nabla \varphi\|_{L^{2}(\Omega)}
$$

$$
\varphi:=u \quad \Rightarrow \quad\left\|e_{\nabla}\right\|_{L^{2}(\Omega)} \leq c_{p}\|\operatorname{div} Y-G\|_{L^{2}(\Omega)}+\|Y-\tilde{E}\|_{L^{2}(\Omega)}
$$

■ $e_{\text {curl }}=\operatorname{curl} U, U \in \mathbb{H}: \quad \forall \Phi \in \mathbb{H} \quad \forall X \in \stackrel{\circ}{\mathrm{H}}($ curl $; \Omega)$

$$
\begin{aligned}
&\left\langle e_{\text {curl }}, \operatorname{curl} \mid \Phi\right\rangle_{L^{2}(\Omega)}=\langle e, \operatorname{curl} \mid \Phi\rangle_{L^{2}(\Omega)}=\langle E, \operatorname{curl} \Phi\rangle_{L^{2}(\Omega)}-\langle\tilde{E}, \operatorname{curl} \mid \Phi\rangle_{L^{2}(\Omega)} \\
&=\langle F-\operatorname{curl} X, \Phi\rangle_{L^{2}(\Omega)}+\langle X-\tilde{E}, \operatorname{curl} \Phi\rangle_{L^{2}(\Omega)} \\
& \leq\|F-\operatorname{curl} X\|_{L^{2}(\Omega)} \underbrace{\left\|\operatorname{curl}^{2} \mid\right\|_{L^{2}(\Omega)}}_{\leq c_{i n}\| \|_{L^{2}}(\Omega)}+\|X-\tilde{E}\|_{L^{2}(\Omega)}\|\operatorname{curl} \mid \Phi\|_{L^{2}(\Omega)}
\end{aligned}
$$

$\Phi:=U \quad \Rightarrow \quad\left\|e_{\text {curr }}\right\|_{L^{2}(\Omega)} \leq c_{\|}\|F-\operatorname{curl} X\|_{L^{2}(\Omega)}+\|X-\tilde{E}\|_{L^{2}}(\Omega)$

- $e_{\mathcal{H}}$ : simple algebraic manipulation
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$\Rightarrow\left\|e_{\mathcal{H}}\right\|_{L^{2}(\Omega)} \leq|\Phi(\tilde{E}-D)|$
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## Proof

- $e_{\nabla}=\nabla u, u \in \dot{H}^{1}(\Omega): \quad \forall \varphi \in \dot{H}^{1}(\Omega) \quad \forall Y \in H(\operatorname{div} ; \Omega)$

$$
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&\left\langle e_{\nabla}, \nabla \varphi\right\rangle_{L^{2}(\Omega)}=\langle e, \nabla \varphi\rangle_{L^{2}(\Omega)}=\langle E, \nabla \varphi\rangle_{L^{2}(\Omega)}-\langle\tilde{E}, \nabla \varphi\rangle_{L^{2}(\Omega)} \\
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$$

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$$
\left\langle e_{\text {curl }}, \operatorname{curl} \Phi\right\rangle_{L^{2}(\Omega)}=\langle e, \operatorname{curl} \Phi\rangle_{L^{2}(\Omega)}=\langle E, \operatorname{curl} \mid \Phi\rangle_{L^{2}(\Omega)}-\langle\tilde{E}, \operatorname{curl} \mid \Phi\rangle_{L^{2}(\Omega)}
$$

$$
=\langle F-\operatorname{curl} X, \Phi\rangle_{L^{2}(\Omega)}+\langle X-\tilde{E}, \operatorname{curl} \Phi\rangle_{L^{2}(\Omega)}
$$

$$
\leq\|F-\operatorname{curl} X\|_{L^{2}(\Omega)} \underbrace{\|\Phi\|_{L^{2}(\Omega)}}_{\leq c_{\mathrm{m}}\|\operatorname{cur} \mid \Phi\|_{L^{2}(\Omega)}}+\|X-\tilde{E}\|_{L^{2}(\Omega)}\|\operatorname{curl} \Phi\|_{L^{2}(\Omega)}
$$

$$
\Phi:=U \quad \Rightarrow \quad\left\|e_{\text {curl }}\right\|_{L^{2}(\Omega)} \leq c_{m}\|F-\operatorname{curl} X\|_{L^{2}(\Omega)}+\|X-\tilde{E}\|_{L^{2}(\Omega)}
$$

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## - $e_{\mathcal{H}}$ : simple algebraic manipulation

## Proof



$$
\begin{aligned}
&\left\langle e_{\nabla}, \nabla \varphi\right\rangle_{\mathrm{L}^{2}(\Omega)}=\langle e, \nabla \varphi\rangle_{\mathrm{L}^{2}(\Omega)}=\langle E, \nabla \varphi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\tilde{E}, \nabla \varphi\rangle_{\mathrm{L}^{2}(\Omega)} \\
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\end{aligned}
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$$
\left\langle e_{\mathrm{curl}}, \operatorname{curl} \Phi\right\rangle_{\mathrm{L}^{2}(\Omega)}=\langle e, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)}=\langle E, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\langle\tilde{E}, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)}
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$$

$$
\leq\|F-\operatorname{curl} X\|_{L^{2}(\Omega)} \underbrace{\|\Phi\|_{L^{2}(\Omega)}}_{\leq c_{\pi}\|\operatorname{curl} \mid\|_{L^{2}(\Omega)}}+\|X-\tilde{E}\|_{L^{2}(\Omega)}\|\operatorname{curl} \Phi\|_{L^{2}(\Omega)}
$$

$$
\Phi:=U \quad \Rightarrow \quad\left\|e_{\text {curl }}\right\|_{L^{2}(\Omega)} \leq c_{m}\|F-\operatorname{curl} X\|_{L^{2}(\Omega)}+\|X-\tilde{E}\|_{L^{2}(\Omega)}
$$

- $e_{\mathcal{H}}$ : simple algebraic manipulation

$$
\Rightarrow\left\|e_{\mathcal{H}}\right\|_{L^{2}(\Omega)} \leq|\Phi(\tilde{E}-D)|
$$

## Eddy Current: Problem Formulation and Solution

much simpler problem! no projection on solenoidal or Dirichlet fields, coercivity immediately implied by equation


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$$
\begin{array}{rlrl}
\partial_{t} B+\operatorname{curl} E & =J, & \operatorname{div} B & =\kappa, \\
& & B=\mu H \\
-\partial_{t} D+\operatorname{curl} H & =\sigma E+j, & \operatorname{div} D=\rho, & \\
=\varepsilon E
\end{array}
$$

assume $\sigma, \mu$ to be time independent
time-harmonic ansatz for $E, F$, i.e., $U(t, x)=\exp (i n \omega t) U_{n}(x)$ with frequency $\omega>0$ curl curl $E_{n}+i n \omega E_{n}=F_{n} \quad(\sigma=\mu=$ id for simplicity $)$

Hilbert space for Lax-Milgram: $\mathrm{H}($ curl $; \Omega)$; denote $E:=E_{n}$ variational formulation: find $E \in \mathrm{H}($ curl $; \Omega)$ s.t. $\forall \Phi \in \mathrm{H}($ curl $; \Omega)$ $b(E, \phi):=\langle\operatorname{curl} E, \operatorname{curl} \phi\rangle_{L^{2}(\Omega)}+i n \omega\langle E, \Phi\rangle_{L^{2}(\Omega)}=f(\Phi):=\langle F, \Phi\rangle_{L^{2}(\Omega)}$
$c_{b}$ depending on $n$ and $\omega$

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Hilbert space for Lax-Milgram: $\mathrm{H}($ curl $; \Omega)$; denote $E:=E_{n}$ variational formulation: find $E \in \mathrm{H}($ curl $; \Omega)$ s.t. $\forall \Phi \in \mathrm{H}($ curl $; \Omega)$ $b(E, \phi):=\langle\operatorname{curl} E, \operatorname{curl} \phi\rangle_{L^{2}(\Omega)}+i n \omega\langle E, \Phi\rangle_{L^{2}(\Omega)}=f(\Phi):=\langle F, \Phi\rangle_{L^{2}(\Omega)}$

## Eddy Current: Problem Formulation and Solution

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$$
\begin{aligned}
\partial_{t} B+\operatorname{curl} E & =J, & \operatorname{div} B & =\kappa,
\end{aligned}
$$

assume $\sigma, \mu$ to be time independent

$$
\Rightarrow \quad \sigma \partial_{t} E=\operatorname{curl} \partial_{t} H-\partial_{t} j=-\operatorname{curl} \mu^{-1} \operatorname{curl} E+\operatorname{curl} \mu^{-1} J-\partial_{t} j
$$

time-harmonic ansatz for $E, F$, i.e., $U(t, x)=\exp (i n \omega t) U_{n}(x)$ with frequency $\omega>0$ $\Rightarrow \quad$ curl curl $E_{n}+$ inw $E_{n}=F_{n} \quad(\sigma=\mu=i d$ for simplicity $)$

Hilbert space for Lax-Milgram: $\mathrm{H}($ curl ; $\Omega)$; denote $E:=E_{n}$ variational formulation: find $E \in \mathrm{H}($ curl $; \Omega)$ s.t. $\forall \Phi \in \mathrm{H}($ curl $; \Omega)$ $b(E, \phi):=\langle\text { curl } E, \operatorname{curl} \phi\rangle_{L^{2}(\Omega)}+i n \omega\langle E, \phi\rangle_{L^{2}(\Omega)}=f(\phi):=\langle F, \Phi\rangle_{L^{2}(\Omega)}$

## Eddy Current: Problem Formulation and Solution

much simpler problem! no projection on solenoidal or Dirichlet fields, coercivity immediately implied by equation

$$
\begin{aligned}
\partial_{t} B+\operatorname{curl} E & =J, & & \operatorname{div} B
\end{aligned}=\kappa, \quad B=\mu H
$$

assume $\sigma, \mu$ to be time independent

$$
\begin{gathered}
\Rightarrow \quad \sigma \partial_{t} E=\operatorname{curl} \partial_{t} H-\partial_{t} j=-\operatorname{curl} \mu^{-1} \operatorname{curl} E+\operatorname{curl} \mu^{-1} J-\partial_{t} j \\
\Rightarrow \quad \operatorname{curl} \mu^{-1} \operatorname{curl} E+\sigma \partial_{t} E=F, \quad F:=\operatorname{curl} \mu^{-1} J-\partial_{t} j
\end{gathered}
$$

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much simpler problem! no projection on solenoidal or Dirichlet fields, coercivity immediately implied by equation

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\begin{array}{rlrl}
\partial_{t} B+\operatorname{curl} E & =J, & \operatorname{div} B & =\kappa, \\
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assume $\sigma, \mu$ to be time independent

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$$

$b$ coercive since $|b(\Phi, \Phi)|^{2}=\|\operatorname{curl} \Phi\|_{\mathrm{L}^{2}(\Omega)}^{2}+n^{2} \omega^{2}\|\Phi\|_{\mathrm{L}^{2}(\Omega)}^{2} \geq c_{\mathrm{b}}\|\Phi\|_{\mathrm{H}(\text { curl } ; \Omega)}^{2}$
$c_{\mathrm{b}}$ depending on $n$ and $\omega$

## Eddy Current: Upper and Lower Bounds

## $\tilde{E} \in \stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega)$ approximation of $E \in \stackrel{\circ}{\mathrm{H}}(\operatorname{curl} ; \Omega) \Rightarrow e:=E-\tilde{E} \in \stackrel{\circ}{\mathrm{H}}(\mathrm{curl} ; \Omega)$

pick $\Phi \in H(c u r l ; \Omega)$ and $X \in H(c u r l ; \Omega)$

$$
\begin{aligned}
& b(e, \Phi)=f(\Phi)-b(\tilde{E}, \Phi) \\
= & \langle F, \Phi\rangle_{L^{2}(\Omega)}-\langle\operatorname{curl} \tilde{E}, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)}-\operatorname{in} \omega\langle\tilde{E}, \Phi\rangle_{\mathrm{L}^{2}(\Omega)} \\
= & \langle F-\operatorname{curl} X-\operatorname{in} \omega \tilde{E}, \Phi\rangle_{\mathrm{L}^{2}(\Omega)}+\langle X-\operatorname{curl} \tilde{E}, \operatorname{curl} \Phi\rangle_{\mathrm{L}^{2}(\Omega)} \\
\leq & (\underbrace{\|F-\operatorname{curl} X-\operatorname{in\omega } \tilde{E}\|_{\mathrm{L}^{2}(\Omega)}+\|X-\operatorname{curl} \tilde{E}\|_{\mathrm{L}^{2}(\Omega)}})\|\Phi\|_{\mathrm{H}(\operatorname{curl} ; \Omega)}
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$: M_{+}(E, X)$
$\phi:=e \Rightarrow$


Remark:

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## more results:

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- differential forms, $\Omega \subset \mathbb{R}^{N}, \Omega$ Riemannian manifold
- hyperbolic problems full time-dependent Maxwell system, eddy current,
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- mixed boundary conditions
- first papers published with Sergey Repin, soon more...
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## Last Slide!

more results:

- lower bounds
- usual features of Sergey's estimates: sharpness, only natural constants, simple implementation, ...
$■ \Omega$ exterior domain, polynomially weighted estimates
- differential forms, $\Omega \subset \mathbb{R}^{N}, \Omega$ Riemannian manifold

■ hyperbolic problems, full time-dependent Maxwell system, eddy current, ...
■ include $\varepsilon, \mu$

- diffusion problem, elasticity, ...
- mixed boundary conditions
- first papers published with Sergey Repin, soon more...


# Thank You! 

