Functional A Posteriori Error Estimates for Static and Eddy Current Maxwell Type Problems

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joint work with

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Introduction: General Electro-Magneto Static Maxwell Type Problem

- Ω smooth N-dim. Riemann. manifold. with comp. cl. and Lip. bound. $\Gamma = \partial\Omega$
- Γ is decomposed in tangential and normal parts divided by a Lipschitz interface, i.e., $\Gamma_t \subset \Gamma$, $\Gamma_n := \Gamma \setminus \overline{\Gamma_t}$, where $\gamma := \overline{\Gamma_t} \cap \overline{\Gamma_n}$ Lipschitz
- ε given medium property: bd., sym., unif. pos. def., lin. mapping on q-forms
- E 'electric field': differential form (q-form) on Ω
- F, G, f, g given right hand side data: diff. forms on Ω resp. Γ, $\Gamma_{t/n}$
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- d, $\delta = \mathrm{d}' = \pm * \mathrm{d} *$ exterior derivative and co-derivative
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- note: $\Gamma_t = \Gamma \Rightarrow$ electro static (elec. bc); $\Gamma_n = \Gamma \Rightarrow$ magneto static (magn. bc)
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Introduction: Dirichlet or Neumann Laplace Problems

- $u \in \mathrm{H}^1(\Omega)$ with $\Delta u = \operatorname{div} \nabla u = G$ and $u|_\Gamma = v$ or $\partial_n u|_\Gamma = n \cdot \nabla u|_\Gamma = g$
- Set $E := \nabla u$ or $H := \nabla u$, note: $n \times \nabla u|_\Gamma = \nabla_\Gamma u|_\Gamma$ (since $\mathrm{d}\, \iota^* = \iota^* \mathrm{d}$)

$$\Rightarrow \quad \begin{array}{lll} \operatorname{curl} E = 0, & \operatorname{curl} H = 0 & \text{in } \Omega \\ \operatorname{div} E = G, & \operatorname{div} H = G & \text{in } \Omega \\ \tau E = \nabla_\Gamma v =: f, & \nu H = g & \text{in } \Gamma \\ E \text{ proj. on } \mathcal{H}_D(\Omega), & H \perp \mathcal{H}_N(\Omega) & \end{array}$$

(electro or magneto static Maxwell problem)

- $\Rightarrow$ NON-CONFORMING estimates for errors $e := E - \tilde{E}$, $h := H - \tilde{H}$, where $\tilde{E}$, $\tilde{H}$ (just) in $\mathrm{L}^2(\Omega)$ approximation of $E = \nabla u$, $H = \nabla u$
- $\Rightarrow$ NON-CONFORMING estimates for energy norm
- note: also mixed boundary conditions possible:
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- note: also possible $\tau E = \partial_t u|_\Gamma = n \times \nabla u|_\Gamma = f$

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Simple Model Problem and Solution Theory

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$\Rightarrow E_c := \operatorname{curl} U$ and $E_d := \nabla u$ as well as $E := E_c + E_d$

Simple Model Problem and Solution Theory

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Sobolev Spaces

spaces:

$$\mathrm{H}(\mathrm{curl};\Omega) := \{E \in \mathrm{L}^2(\Omega) \,:\, \mathrm{curl}\, E \in \mathrm{L}^2(\Omega)\}$$

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Results: Upper Bounds for Non-Conforming Approximations

$\tilde{E} \in \mathrm{L}^2(\Omega)$ approximations of $E \Rightarrow$

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holds. Here, Φ denotes a projection onto Dirichlet fields $\mathcal{H}_D(\Omega)$ and

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only natural continuity constants involved:

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Proof: Tools

- Rellich's selection theorems, i.e., $\mathring{\mathrm{H}}^1(\Omega), \mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$ compact
  $\Rightarrow$ Poincaré estimate, i.e.,

$$\forall\, u \in \mathring{\mathrm{H}}^1(\Omega) \qquad |u|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{p}} |\nabla u|_{\mathrm{L}^2(\Omega)}$$

$$\forall\, u \in \mathrm{H}^1(\Omega) \cap \{1\}^{\perp} \qquad |u|_{\mathrm{L}^2(\Omega)} \leq \tilde{c}_{\mathrm{p}} |\nabla u|_{\mathrm{L}^2(\Omega)}$$

- Maxwell selection theorems, i.e.,

$$\mathring{\mathrm{H}}(\mathrm{curl};\Omega) \cap \mathrm{H}(\mathrm{div};\Omega), \mathrm{H}(\mathrm{curl};\Omega) \cap \mathring{\mathrm{H}}(\mathrm{div};\Omega) \hookrightarrow \mathrm{L}^2(\Omega) \text{ compact}$$

  $\Rightarrow$ Maxwell estimates, i.e.,

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- Maxwell selection theorems $\Rightarrow \dim \mathcal{H}_D(\Omega), \dim \mathcal{H}_N(\Omega) < \infty$ (Betti numbers)
- Helmholtz decompositions (all 6 images are closed in $\mathrm{L}^2(\Omega)$)

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- Rellich's selection theorems, i.e., $\overset{\circ}{\mathrm{H}}{}^1(\Omega), \mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$ compact
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Proof

$e = E - \tilde{E} \in \mathrm{L}^2(\Omega)$

- Helmholtz decomposition of error $\Rightarrow$

$$e = e_{\nabla} + e_{\mathcal{H}} + e_{\mathrm{curl}} \in \nabla \mathring{\mathrm{H}}^1(\Omega) \oplus \mathcal{H}_D(\Omega) \oplus \operatorname{curl} \mathbb{H}$$

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- $e_{\mathcal{H}}$: simple algebraic manipulation
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$$\leq \|F - \operatorname{curl} X\|_{\mathrm{L}^2(\Omega)} \underbrace{\|\Phi\|_{\mathrm{L}^2(\Omega)}}_{\leq c_{\mathrm{m}}\|\operatorname{curl}\Phi\|_{\mathrm{L}^2(\Omega)}} + \|X - \tilde{E}\|_{\mathrm{L}^2(\Omega)}\|\operatorname{curl}\Phi\|_{\mathrm{L}^2(\Omega)}$$

$$\Phi := U \quad \Rightarrow \quad \|e_{\mathrm{curl}}\|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{m}}\|F - \operatorname{curl} X\|_{\mathrm{L}^2(\Omega)} + \|X - \tilde{E}\|_{\mathrm{L}^2(\Omega)}$$

- $e_{\mathcal{H}}$: simple algebraic manipulation
$\Rightarrow \|e_{\mathcal{H}}\|_{\mathrm{L}^2(\Omega)} \leq |\Phi(\tilde{E} - D)|$

Proof

- $e_\nabla = \nabla u,\ u \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)$: $\quad \forall \varphi \in \overset{\circ}{\mathsf{H}}{}^1(\Omega) \quad \forall Y \in \mathsf{H}(\operatorname{div};\Omega)$

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$$= \langle \operatorname{div} Y - G, \varphi\rangle_{\mathsf{L}^2(\Omega)} + \langle Y - \tilde{E}, \nabla\varphi\rangle_{\mathsf{L}^2(\Omega)}$$

$$\leq \|\operatorname{div} Y - G\|_{\mathsf{L}^2(\Omega)} \underbrace{\|\varphi\|_{\mathsf{L}^2(\Omega)}}_{\leq c_{\mathrm{p}}\|\nabla\varphi\|_{\mathsf{L}^2(\Omega)}} + \|Y - \tilde{E}\|_{\mathsf{L}^2(\Omega)} \|\nabla\varphi\|_{\mathsf{L}^2(\Omega)}$$

$$\varphi := u \quad \Rightarrow \quad \|e_\nabla\|_{\mathsf{L}^2(\Omega)} \leq c_{\mathrm{p}} \|\operatorname{div} Y - G\|_{\mathsf{L}^2(\Omega)} + \|Y - \tilde{E}\|_{\mathsf{L}^2(\Omega)}$$

- $e_{\operatorname{curl}} = \operatorname{curl} U,\ U \in \mathbb{H}$: $\quad \forall \Phi \in \mathbb{H} \quad \forall X \in \overset{\circ}{\mathsf{H}}(\operatorname{curl};\Omega)$

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  $\Rightarrow \|e_{\mathcal{H}}\|_{\mathsf{L}^2(\Omega)} \leq |\Phi(\tilde{E} - D)|$ □

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- $e_{\mathcal{H}}$: simple algebraic manipulation $\square$
  $\Rightarrow \|e_{\mathcal{H}}\|_{\mathrm{L}^2(\Omega)} \leq \left|\Phi(\tilde{E} - D)\right|$

Eddy Current: Problem Formulation and Solution

much simpler problem! no projection on solenoidal or Dirichlet fields, coercivity immediately implied by equation

$$\partial_t B + \operatorname{curl} E = J, \qquad \operatorname{div} B = \kappa, \qquad B = \mu H$$
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assume σ, μ to be time independent

$$\Rightarrow \qquad \sigma\,\partial_t E = \operatorname{curl} \partial_t H - \partial_t j = -\operatorname{curl} \mu^{-1} \operatorname{curl} E + \operatorname{curl} \mu^{-1} J - \partial_t j$$

$$\Rightarrow \qquad \operatorname{curl} \mu^{-1} \operatorname{curl} E + \sigma\,\partial_t E = F, \qquad F := \operatorname{curl} \mu^{-1} J - \partial_t j$$

time-harmonic ansatz for E, F, i.e., $U(t,x) = \exp(in\omega t) U_n(x)$ with frequency $\omega > 0$

$$\Rightarrow \qquad \operatorname{curl} \operatorname{curl} E_n + in\omega E_n = F_n \qquad (\sigma = \mu = \mathrm{id} \text{ for simplicity})$$

Hilbert space for Lax-Milgram: $\overset{\circ}{\mathrm{H}}(\operatorname{curl};\Omega)$; denote $E := E_n$

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Eddy Current: Upper and Lower Bounds

$\tilde{E} \in \overset{\circ}{\mathrm{H}}(\mathrm{curl};\Omega)$ approximation of $E \in \overset{\circ}{\mathrm{H}}(\mathrm{curl};\Omega) \Rightarrow e := E - \tilde{E} \in \overset{\circ}{\mathrm{H}}(\mathrm{curl};\Omega)$

pick $\Phi \in \overset{\circ}{\mathrm{H}}(\mathrm{curl};\Omega)$ and $X \in \mathrm{H}(\mathrm{curl};\Omega)$

$$\Rightarrow \qquad b(e,\Phi) = f(\Phi) - b(\tilde{E},\Phi)$$
$$= \langle F,\Phi\rangle_{\mathrm{L}^2(\Omega)} - \langle \mathrm{curl}\,\tilde{E}, \mathrm{curl}\,\Phi\rangle_{\mathrm{L}^2(\Omega)} - in\omega\langle \tilde{E},\Phi\rangle_{\mathrm{L}^2(\Omega)}$$
$$= \langle F - \mathrm{curl}\,X - in\omega\tilde{E},\Phi\rangle_{\mathrm{L}^2(\Omega)} + \langle X - \mathrm{curl}\,\tilde{E}, \mathrm{curl}\,\Phi\rangle_{\mathrm{L}^2(\Omega)}$$
$$\leq \underbrace{\left(\|F - \mathrm{curl}\,X - in\omega\tilde{E}\|_{\mathrm{L}^2(\Omega)} + \|X - \mathrm{curl}\,\tilde{E}\|_{\mathrm{L}^2(\Omega)}\right)}_{=:M_+(\tilde{E},X)}\|\Phi\|_{\mathrm{H}(\mathrm{curl};\Omega)}$$

$\Phi := e \Rightarrow$

$$\|e\|_{\mathrm{H}(\mathrm{curl};\Omega)} \leq c_b^{-1} \inf_{X\in\mathrm{H}(\mathrm{curl};\Omega)} M_+(\tilde{E},X)$$

Remark:

- also lower bounds
- also mixed boundary conditions possible
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Eddy Current: Upper and Lower Bounds

$\tilde{E} \in \overset{\circ}{\mathrm{H}}(\mathrm{curl};\Omega)$ approximation of $E \in \overset{\circ}{\mathrm{H}}(\mathrm{curl};\Omega) \Rightarrow e := E - \tilde{E} \in \overset{\circ}{\mathrm{H}}(\mathrm{curl};\Omega)$

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$$\begin{aligned}\Rightarrow \qquad b(e,\Phi) &= f(\Phi) - b(\tilde{E},\Phi)\\ &= \langle F,\Phi\rangle_{\mathrm{L}^2(\Omega)} - \langle \operatorname{curl}\tilde{E}, \operatorname{curl}\Phi\rangle_{\mathrm{L}^2(\Omega)} - in\omega\langle \tilde{E},\Phi\rangle_{\mathrm{L}^2(\Omega)}\\ &= \langle F - \operatorname{curl} X - in\omega\tilde{E},\Phi\rangle_{\mathrm{L}^2(\Omega)} + \langle X - \operatorname{curl}\tilde{E}, \operatorname{curl}\Phi\rangle_{\mathrm{L}^2(\Omega)}\\ &\le \underbrace{\Big(\|F - \operatorname{curl} X - in\omega\tilde{E}\|_{\mathrm{L}^2(\Omega)} + \|X - \operatorname{curl}\tilde{E}\|_{\mathrm{L}^2(\Omega)}\Big)}_{=:M_+(\tilde{E},X)}\|\Phi\|_{\mathrm{H}(\mathrm{curl};\Omega)}\end{aligned}$$

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$$
\begin{aligned}
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