Old and New Results for Hilbert Complexes and (Linear) First Order Systems

Dirk Pauly

Fakultät für Mathematik

UNIVERSITÄT DUISBURG ESSEN

Open-Minded ;-)

CMAM-8 Computational Methods in Applied Mathematics
July 2-6, 2018

Institute of Mathematics, National Academy of Sciences

Minsk, Belarus

general observations

$$\mathrm{A}\,x = f$$

general observations

$$Ax = f$$

general theory

- solution theory
- Friedrichs/Poincaré estimates and constants
- Helmholtz/Hodge/Weyl decompositions
- compact embeddings
- continuous and compact inverse operators
- closed ranges
- variational formulations
- functional a posteriori error estimates
- generalized div-curl-lemma
- ...

idea: solve problem with general and simple linear functional analysis

$\Rightarrow$ functional analysis toolbox (fa-toolbox) ...

general observations

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ linear, H_0, H_1 Hilbert spaces (for simplicity)

solve and provide tools

$$\mathrm{A}x = f$$

examples

- $\mathrm{A} \in \mathbb{R}^{m \times n}$ (matrix eq)
- $\mathrm{A} = \partial_t - \mathring{\Delta}$ (so heat/diffusion eq)
- $\mathrm{A} = \partial_t^2 - \mathring{\Delta}$ (so wave eq)
- $\mathrm{A} = -\mathring{\Delta} - \omega^2$ (so red wave/Helmholtz eq)
- $\mathrm{A} = -\mathring{\Delta}$ (so Laplace eq)

right hand sides and solutions (typically)

- $f \in \mathbb{R}^m$ $x \in \mathbb{R}^n$
- $f \in \mathrm{L}^2(I \times \Omega)$ $x \in \mathrm{L}^2(I) \times \mathring{\mathrm{H}}^1(\Omega)$
- $f \in \mathrm{L}^2(\Omega)$ $x \in \mathring{\mathrm{H}}^1(\Omega)$

or in (closed) subspaces $R(\mathrm{A})$ $\big(\text{here } \mathring{\mathrm{H}}^1(\Omega) = \mathrm{H}^1_0(\Omega)\big)$

general observations

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ linear, $\quad \mathrm{H}_0$, H_1 Hilbert spaces (for simplicity)

solve and provide tools

$$\mathrm{A}x = f$$

examples

- $\mathrm{A} = \partial_t^2 - \mathring{\Delta} = \partial_t^2 - \operatorname{div} \mathring{\nabla}$ (so wave eq)
- $\mathrm{A} = \partial_t - \begin{bmatrix} 0 & \operatorname{div} \\ \mathring{\nabla} & 0 \end{bmatrix}$ (fo wave eq, pref form, acoustics)
- $\mathrm{A} = \begin{bmatrix} 0 & \operatorname{div} \\ \mathring{\nabla} & 0 \end{bmatrix} - \omega$ (fo red wave/Helmholtz eq, time-harm acoustics)

right hand sides and solutions (typically)

- $f \in \mathrm{L}^2(I \times \Omega)$ $\qquad x \in \mathrm{L}^2(I) \times \mathring{\mathrm{H}}^1(\Omega)$
- $f \in \mathrm{L}^2(I \times \Omega) \times \mathrm{L}^2(I \times \Omega)$ $\qquad x \in \big(\mathrm{L}^2(I) \times \mathring{\mathrm{H}}^1(\Omega)\big) \times \big(\mathrm{L}^2(I) \times \mathrm{D}(\Omega)\big)$
- $f \in \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega)$ $\qquad x \in \mathring{\mathrm{H}}^1(\Omega) \times \mathrm{D}(\Omega)$

or in (closed) subspaces $R(\mathrm{A})$ $\qquad \big(\text{here } \mathrm{D}(\Omega) = \mathrm{H}(\operatorname{div}, \Omega)\big)$

general observations

$A : D(A) \subset H_0 \to H_1$ linear, H_0, H_1 Hilbert spaces (for simplicity)

solve and provide tools

$$Ax = f$$

examples

- $A = \partial_t^2 + \operatorname{rot} \mathring{\operatorname{rot}}$ (so Maxwell/wave eq)
- $A = \partial_t - \begin{bmatrix} 0 & -\operatorname{rot} \\ \mathring{\operatorname{rot}} & 0 \end{bmatrix}$ (fo Maxwell/wave eq, pref form)
- $A = \begin{bmatrix} 0 & -\operatorname{rot} \\ \mathring{\operatorname{rot}} & 0 \end{bmatrix} - \omega$ (fo time-harm Maxwell eq)
- $A = \begin{bmatrix} 0 & -\operatorname{rot} \\ \mathring{\operatorname{rot}} & 0 \end{bmatrix}$ (fo stat Maxwell eq) $\Rightarrow$ $\boxed{\mathring{\operatorname{rot}} - \operatorname{div} / \operatorname{rot} - \mathring{\operatorname{div}} \text{ sys}}$
- $A = \partial_t + \operatorname{rot} \mathring{\operatorname{rot}}$ (so eddy current Maxwell eq)
- $A = \operatorname{rot} \mathring{\operatorname{rot}} - \omega$ (so time-harm eddy current Maxwell eq)

right hand sides and solutions (typically)

- $f \in L^2(I \times \Omega)$ $\quad x \in L^2(I) \times \mathring{R}(\Omega)$
- $f \in L^2(I \times \Omega) \times L^2(I \times \Omega)$ $\quad x \in \big(L^2(I) \times \mathring{R}(\Omega)\big) \times \big(L^2(I) \times R(\Omega)\big)$
- $f \in L^2(\Omega) \times L^2(\Omega)$ $\quad x \in \mathring{R}(\Omega) \times R(\Omega)$

or in (closed) subspaces $R(A)$ $\quad \big(\text{here } R(\Omega) = H(\operatorname{rot}, \Omega) \quad \mathring{R}(\Omega) = H_0(\operatorname{rot}, \Omega)\big)$

general observations

so far all equations form the classical de Rham complex in 3D (∇-rot-div-complex)

($\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain)

electro-magneto dynamics/time-harm/statics, Maxwell's equations, acoustics

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{-\operatorname{div}}{\overset{\mathring{\nabla}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{\operatorname{rot}}{\overset{\mathring{\operatorname{rot}}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{-\nabla}{\overset{\mathring{\operatorname{div}}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}} \mathbb{R}$$

complex: $\operatorname{rot} \nabla = 0$ $\qquad \operatorname{div} \operatorname{rot} = 0$

general observations

other possible complexes:

elasticity complex in 3D ($\operatorname{sym}\nabla$-$\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}$-$\operatorname{Div}_{\mathbb{S}}$-complex)

($\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain)

elasticity, $\operatorname{Rot}\operatorname{Rot}^{\perp}$ $\operatorname{Rot}\operatorname{Rot}^{\perp}$ eq

$$\{0\} \;\underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}}\; \mathsf{L}^2(\Omega) \;\underset{-\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\operatorname{sym}}\nabla}{\rightleftarrows}}\; \mathsf{L}^2_{\mathbb{S}}(\Omega) \;\underset{\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}}{\overset{\mathring{\operatorname{Rot}}\operatorname{Rot}_{\mathbb{S}}^{\top}}{\rightleftarrows}}\; \mathsf{L}^2_{\mathbb{S}}(\Omega) \;\underset{-\operatorname{sym}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{S}}}{\rightleftarrows}}\; \mathsf{L}^2(\Omega) \;\underset{\iota_{\mathsf{RM}}}{\overset{\pi_{\mathsf{RM}}}{\rightleftarrows}}\; \mathsf{RM}$$

complex: $\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top}\operatorname{sym}\nabla = 0$ $\quad$ $\operatorname{Div}_{\mathbb{S}}\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top} = 0$

general observations

other possible complexes:

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\mathrm{Rot}_{\mathbb{S}}$-$\mathrm{Div}_{\mathbb{T}}$-complex)

($\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain)

biharmonic / general relativity

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\nabla\nabla}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}}(\Omega) \underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}}(\Omega) \underset{-\operatorname{dev}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}} \mathsf{L}^2(\Omega) \underset{\iota_{\mathsf{RT}}}{\overset{\pi_{\mathsf{RT}}}{\rightleftarrows}} \mathsf{RT}$$

complex: $\operatorname{Rot}_{\mathbb{S}} \nabla\nabla = 0$ $\quad$ $\operatorname{Div}_{\mathbb{T}} \operatorname{Rot}_{\mathbb{S}} = 0$

general observations

H_0, H_1, H_2 Hilbert spaces (for simplicity)

$A_0 : D(A_0) \subset H_0 \to H_1$ lddc (lin, den def, cl)

$A_1 : D(A_1) \subset H_1 \to H_2$ lddc

$A_0^* : D(A_0^*) \subset H_1 \to H_0$ lddc (Hilbert space adjoint)

$A_1^* : D(A_1^*) \subset H_2 \to H_1$ lddc (Hilbert space adjoint)

general complex

$$\cdots \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad H_0 \quad \underset{A_0^*}{\overset{A_0}{\rightleftarrows}} \quad H_1 \quad \underset{A_1^*}{\overset{A_1}{\rightleftarrows}} \quad H_2 \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad \cdots$$

complex: $A_1 A_0 = 0 \qquad \big(\Leftrightarrow \quad A_0^* A_1^* = 0 \big)$

more precisely: $R(A_0) \subset N(A_1) \quad \big(\Leftrightarrow \quad R(A_1^*) \subset N(A_0^*) \big)$

general observations

$A_0 : D(A_0) \subset H_0 \to H_1,\ \ A_1 : D(A_1) \subset H_1 \to H_2$ lddc

$A_0^* : D(A_0^*) \subset H_1 \to H_0,\ A_1^* : D(A_1^*) \subset H_2 \to H_1$ lddc (Hilbert space adjoints)

general complex ($A_1 A_0 = 0$)

$$\cdots \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad H_0 \quad \underset{A_0^*}{\overset{A_0}{\rightleftarrows}} \quad H_1 \quad \underset{A_1^*}{\overset{A_1}{\rightleftarrows}} \quad H_2 \quad \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \quad \cdots$$

$$Ax = f$$

typical equations/systems

(stat fos)

$$A = A_1$$

(stat sos (sa))

$$A = A_1^* A_1$$

(stat fos (sa/ssa))

$$A = \begin{bmatrix} 0 & \pm A_1^* \\ A_1 & 0 \end{bmatrix}$$

(time-harm fos (sa/ssa))

$$A = \omega - \begin{bmatrix} 0 & \pm A_1^* \\ A_1 & 0 \end{bmatrix}$$

(diff sos)

$$A = \partial_t + A_1^* A_1$$

(wave sos)

$$A = \partial_t^2 + A_1^* A_1$$

(wave fos)

$$A = \partial_t - \begin{bmatrix} 0 & -A_1^* \\ A_1 & 0 \end{bmatrix}$$

general observations

$$\mathrm{A}x = f$$

let's say $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ linear and H_0, H_1 Hilbert spaces

question: How to solve?

general observations

$$\mathrm{A}x = f$$

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ linear

solution theory in the sense of Hadamard

- existence $\Leftrightarrow$ $f \in R(\mathrm{A})$
- uniqueness $\Leftrightarrow$ A inj $\Leftrightarrow$ $N(\mathrm{A}) = \{0\}$ $\Leftrightarrow$ A^{-1} exists
- cont dep on f $\Leftrightarrow$ A^{-1} cont

$\Rightarrow$ $x = \mathrm{A}^{-1} f \in D(\mathrm{A})$ and cont estimate (Friedrichs/Poincaré type estimate)

$$|x|_{\mathrm{H}_0} = |\mathrm{A}^{-1} f|_{\mathrm{H}_0} \leq c_\mathrm{A} |f|_{\mathrm{H}_1} = c_\mathrm{A} |\mathrm{A}x|_{\mathrm{H}_1}$$

$\Rightarrow$ best constant $c_\mathrm{A} = |\mathrm{A}^{-1}|_{R(\mathrm{A}),\mathrm{H}_0}$ $\quad |\mathrm{A}^{-1}|_{R(\mathrm{A}),D(\mathrm{A})} = (c_\mathrm{A}^2 + 1)^{1/2}$

general observations

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$

$\mathrm{A}^* : D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0$ Hilbert space adjoint

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$\mathrm{H}_1 = \overline{R(\mathrm{A})} \oplus N(\mathrm{A}^*) \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

$$\mathrm{A}x = f$$

solution theory in the sense of Hadamard

- existence $\Leftrightarrow$ $f \in R(\mathrm{A}) = N(\mathrm{A}^*)^\perp$
- uniqueness $\Leftrightarrow$ A inj $\Leftrightarrow$ $N(\mathrm{A}) = \{0\}$ $\Leftrightarrow$ A^{-1} exists
- cont dep on f $\Leftrightarrow$ A^{-1} cont $\Leftrightarrow$ $R(\mathrm{A})$ cl (cl range theo)

fund range cond: $\boxed{R(\mathrm{A}) = \overline{R(\mathrm{A})} \text{ closed}}$ (must hold $\rightsquigarrow$ right setting!)

kernel cond: $\boxed{N(\mathrm{A}) = \{0\}}$ (fails in gen $\rightsquigarrow$ proj onto $N(\mathrm{A})^\perp = \overline{R(\mathrm{A}^*)}$)

general observations

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$\mathrm{H}_1 = \overline{R(\mathrm{A})} \oplus N(\mathrm{A}^*) \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

remarkable observations

- time-dependent problems are simple
  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H} \to \mathrm{H}$, $\mathrm{A} = \partial_t + \mathrm{T}$ (gen T skew-sa, or alt lsast Re $\mathrm{T} \geq 0$)

  $$N(\mathrm{A}) = \{0\} \qquad N(\mathrm{A}^*) = \{0\} \qquad R(\mathrm{A}) \text{ (cl) } = N(\mathrm{A}^*)^\perp = \mathrm{H}$$

- time-harmonic problems are more complicated
  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H} \to \mathrm{H}$, $\mathrm{A} = -\omega + \mathrm{T}$

  $$N(\mathrm{A}),\ N(\mathrm{A}^*) \text{ (fin dim)} \qquad R(\mathrm{A}) \text{ (cl, fin co-dim) } = N(\mathrm{A}^*)^\perp$$

  (Fredholm alternative)

- stat problems are most complicated
  in gen $\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$

  $$\dim N(\mathrm{A}) = \dim N(\mathrm{A}^*) = \infty \text{ (possibly)} \qquad R(\mathrm{A}) \text{ (cl, infin co-dim) } = N(\mathrm{A}^*)^\perp$$

fa-toolbox for linear (first order) problems/systems

$$Ax = f$$

general theory

- solution theory
- Friedrichs/Poincaré estimates and constants
- Helmholtz/Hodge/Weyl decompositions
- compact embeddings
- continuous and compact inverse operators
- closed ranges
- variational formulations
- functional a posteriori error estimates
- generalized div-curl-lemma
- ...

idea: solve problem with general and simple linear functional analysis
($\Rightarrow$ fa-toolbox) ...

literature: probably very well known for ages, but hard to find ...

Friedrichs, Weyl, Hörmander, Fredholm, von Neumann, Riesz, Banach, ... ?

Why not rediscover and extend/modify for our purposes?

1st fundamental observations

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ lddc, $\mathrm{A}^* : D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0$ Hilbert space adjoint

$(\mathrm{A}, \mathrm{A}^*)$ dual pair as $(\mathrm{A}^*)^* = \overline{\mathrm{A}} = \mathrm{A}$

A, A^* may not be inj

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$\mathrm{H}_1 = N(\mathrm{A}^*) \oplus \overline{R(\mathrm{A})} \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

reduced operators restr to $N(\mathrm{A})^\perp$ and $N(\mathrm{A}^*)^\perp$

$$\mathcal{A} := \mathrm{A}|_{N(\mathrm{A})^\perp} = \mathrm{A}|_{\overline{R(\mathrm{A}^*)}} \qquad \mathcal{A}^* := \mathrm{A}^*|_{N(\mathrm{A}^*)^\perp} = \mathrm{A}^*|_{\overline{R(\mathrm{A})}}$$

$\mathcal{A}$, $\mathcal{A}^*$ inj $\quad \Rightarrow \quad \mathcal{A}^{-1}$, $(\mathcal{A}^*)^{-1}$ ex

1st fundamental observations

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1, \quad \mathrm{A}^* : D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0$ lddc $\qquad (\mathrm{A}, \mathrm{A}^*)$ dual pair

$$\mathrm{H}_1 = N(\mathrm{A}^*) \oplus \overline{R(\mathrm{A})} \qquad \mathrm{H}_0 = N(\mathrm{A}) \oplus \overline{R(\mathrm{A}^*)}$$

more precisely

$$\mathcal{A} := \mathrm{A}|_{\overline{R(\mathrm{A}^*)}} : D(\mathcal{A}) \subset \overline{R(\mathrm{A}^*)} \to \overline{R(\mathrm{A})}, \qquad D(\mathcal{A}) := D(\mathrm{A}) \cap N(\mathrm{A})^\perp = D(\mathrm{A}) \cap \overline{R(\mathrm{A}^*)}$$
$$\mathcal{A}^* := \mathrm{A}^*|_{\overline{R(\mathrm{A})}} : D(\mathcal{A}^*) \subset \overline{R(\mathrm{A})} \to \overline{R(\mathrm{A}^*)}, \quad D(\mathcal{A}^*) := D(\mathrm{A}^*) \cap N(\mathrm{A}^*)^\perp = D(\mathrm{A}^*) \cap \overline{R(\mathrm{A})}$$

$(\mathcal{A}, \mathcal{A}^*)$ dual pair and $\mathcal{A}$, $\mathcal{A}^*$ inj $\Rightarrow$

inverse ops exist (and bij)

$$\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A}) \qquad (\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$$

refined decompositions

$$D(\mathrm{A}) = N(\mathrm{A}) \oplus D(\mathcal{A}) \qquad D(\mathrm{A}^*) = N(\mathrm{A}^*) \oplus D(\mathcal{A}^*)$$

$\Rightarrow$

$$R(\mathrm{A}) = R(\mathcal{A}) \qquad R(\mathrm{A}^*) = R(\mathcal{A}^*)$$

1st fundamental observations

closed range theorem & closed graph theorem $\Rightarrow$

Lemma (Friedrichs-Poincaré type est/cl range/cont inv)

The following assertions are equivalent:

- **(i)** $\exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathrm{H}_0} \leq c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$
- **(i*)** $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$
- **(ii)** $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_1.
- **(ii*)** $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_0.
- **(iii)** $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is continuous and bijective.*
- **(iii*)** $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is continuous and bijective.*

In case that one of the latter assertions is true, e.g., (ii), $R(\mathrm{A})$ *is closed, we have*

$$\mathrm{H}_0 = N(\mathrm{A}) \oplus R(\mathrm{A}^*) \qquad \mathrm{H}_1 = N(\mathrm{A}^*) \oplus R(\mathrm{A})$$
$$D(\mathrm{A}) = N(\mathrm{A}) \oplus D(\mathcal{A}) \qquad D(\mathrm{A}^*) = N(\mathrm{A}^*) \oplus D(\mathcal{A}^*)$$
$$D(\mathcal{A}) = D(\mathrm{A}) \cap R(\mathrm{A}^*) \qquad D(\mathcal{A}^*) = D(\mathrm{A}^*) \cap R(\mathrm{A})$$

and $\quad \mathcal{A} : D(\mathcal{A}) \subset R(\mathrm{A}^*) \to R(\mathrm{A}), \quad \mathcal{A}^* : D(\mathcal{A}^*) \subset R(\mathrm{A}) \to R(\mathrm{A}^*)$.

1st fundamental observations

recall

(i) $\exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \quad |x|_{\mathrm{H}_0} \leq c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$

(i*) $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \quad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$

'best' consts in **(i)** and **(i*)** equal norms of the inv ops and Rayleigh quotients

$$c_{\mathrm{A}} = |\mathcal{A}^{-1}|_{R(\mathrm{A}),R(\mathrm{A}^*)} \qquad c_{\mathrm{A}^*} = |(\mathcal{A}^*)^{-1}|_{R(\mathrm{A}^*),R(\mathrm{A})}$$

$$\frac{1}{c_{\mathrm{A}}} = \inf_{0\neq x\in D(\mathcal{A})} \frac{|\mathrm{A}x|_{\mathrm{H}_1}}{|x|_{\mathrm{H}_0}} \qquad \frac{1}{c_{\mathrm{A}^*}} = \inf_{0\neq y\in D(\mathcal{A}^*)} \frac{|\mathrm{A}^*y|_{\mathrm{H}_0}}{|y|_{\mathrm{H}_1}}$$

Lemma (Friedrichs-Poincaré type const)

$c_{\mathrm{A}} = c_{\mathrm{A}^*}$

1st fundamental observations

Lemma (cpt emb/cpt inv)

The following assertions are equivalent:

(i) $D(\mathcal{A}) \hookrightarrow\!\!\!\to \mathrm{H}_0$ *is compact.*

(i*) $D(\mathcal{A}^*) \hookrightarrow\!\!\!\to \mathrm{H}_1$ *is compact.*

(ii) $\mathcal{A}^{-1} : R(\mathrm{A}) \to R(\mathrm{A}^*)$ *is compact.*

(ii*) $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to R(\mathrm{A})$ *is compact.*

Lemma (Friedrichs-Poincaré type est/cl range/cont inv)

$\Downarrow$ $\boxed{D(\mathcal{A}) \hookrightarrow\!\!\!\to \mathrm{H}_0 \text{ compact}}$

(i) $\exists\, c_{\mathrm{A}} \in (0,\infty) \quad \forall\, x \in D(\mathcal{A}) \quad |x|_{\mathrm{H}_0} \leq c_{\mathrm{A}} |\mathrm{A}x|_{\mathrm{H}_1}$

(i*) $\exists\, c_{\mathrm{A}^*} \in (0,\infty) \quad \forall\, y \in D(\mathcal{A}^*) \quad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathrm{H}_0}$

(ii) $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_1.

(ii*) $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_0.

(iii) $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is continuous and bijective.*

(iii*) $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is continuous and bijective.*

(i)-(iii*) *equi* & *the resp Helm deco hold* & $|\mathcal{A}^{-1}| = c_{\mathrm{A}} = c_{\mathrm{A}^*} = |(\mathcal{A}^*)^{-1}|$

2nd fundamental observations

So far no complex...

$$A_0 : D(A_0) \subset H_0 \to H_1, \quad A_1 : D(A_1) \subset H_1 \to H_2 \text{ (lddc)}$$
$$A_0^* : D(A_0^*) \subset H_1 \to H_0, \quad A_1^* : D(A_1^*) \subset H_2 \to H_1 \text{ (lddc)}$$

general complex ($A_1 A_0 = 0$, i.e., $R(A_0) \subset N(A_1)$ and $R(A_1^*) \subset N(A_0^*)$)

$$\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} H_0 \underset{A_0^*}{\overset{A_0}{\rightleftarrows}} H_1 \underset{A_1^*}{\overset{A_1}{\rightleftarrows}} H_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$$

recall Helmholtz deco

$$\begin{aligned} H_1 &= \overline{R(A_0)} \oplus N(A_0^*) \\ &\quad\;\; \bigcap \qquad\quad \bigcup \\ &= N(A_1) \oplus \overline{R(A_1^*)} \end{aligned} \qquad \Rightarrow \text{ (e.g.) } N(A_1) = \overline{R(A_0)} \oplus \big(\underbrace{N(A_1) \cap N(A_0^*)}_{=:K_1}\big)$$

$\Rightarrow$ refined Helmholtz deco

$$H_1 = \overline{R(A_0)} \oplus K_1 \oplus \overline{R(A_1^*)}$$

2nd fundamental observations

recall

$$D(\mathrm{A}_1) = D(\mathcal{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \qquad R(\mathrm{A}_1) = R(\mathcal{A}_1) \qquad R(\mathrm{A}_1^*) = R(\mathcal{A}_1^*)$$

$$D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \cap \overline{R(\mathrm{A}_0)} \qquad R(\mathrm{A}_0^*) = R(\mathcal{A}_0^*) \qquad R(\mathrm{A}_0) = R(\mathcal{A}_0)$$

cohomology group $K_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)$

Lemma (Helmholtz deco I)

$$\begin{aligned} \mathrm{H}_1 &= \overline{R(\mathrm{A}_0)} \oplus N(\mathrm{A}_0^*) & \mathrm{H}_1 &= \overline{R(\mathrm{A}_1^*)} \oplus N(\mathrm{A}_1) \\ D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus N(\mathrm{A}_0^*) & D(\mathrm{A}_1) &= D(\mathcal{A}_1) \oplus N(\mathrm{A}_1) \\ N(\mathrm{A}_1) &= D(\mathcal{A}_0^*) \oplus K_1 & N(\mathrm{A}_0^*) &= D(\mathcal{A}_1) \oplus K_1 \\ D(\mathrm{A}_1) &= \overline{R(\mathrm{A}_0)} \oplus \big(D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)\big) & D(\mathrm{A}_0^*) &= \overline{R(\mathrm{A}_1^*)} \oplus \big(D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1)\big) \end{aligned}$$

Lemma (Helmholtz deco II)

$$\begin{aligned} \mathrm{H}_1 &= \overline{R(\mathrm{A}_0)} \oplus K_1 \oplus \overline{R(\mathrm{A}_1^*)} \\ D(\mathrm{A}_1) &= \overline{R(\mathrm{A}_0)} \oplus K_1 \oplus D(\mathcal{A}_1) \\ D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus K_1 \oplus \overline{R(\mathrm{A}_1^*)} \\ D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus K_1 \oplus D(\mathcal{A}_1) \end{aligned}$$

2nd fundamental observations

$$K_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \qquad D(\mathrm{A}_1) = D(\mathcal{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \qquad D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \cap \overline{R(\mathrm{A}_0)}$$

Lemma (cpt emb II)

The following assertions are equivalent:

(i) $D(\mathcal{A}_0) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_0$, $D(\mathcal{A}_1) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$, *and* $K_1 \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$ *are compact.*

(ii) $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$ *is compact.*

In this case $K_1 < \infty$.

Theorem (fa-toolbox I)

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1 \text{ compact}}$

(i) *all emb cpt, i.e.,* $D(\mathcal{A}_0) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_0$, $D(\mathcal{A}_1) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$, $D(\mathcal{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$, $D(\mathcal{A}_1^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_2$ *cpt*

(ii) *cohomology group* K_1 *finite dim*

(iii) *all ranges closed, i.e.,* $R(\mathrm{A}_0)$, $R(\mathrm{A}_0^*)$, $R(\mathrm{A}_1)$, $R(\mathrm{A}_1^*)$ *cl*

(iv) *all Friedrichs-Poincaré type est hold*

(v) *all Hodge-Helmholtz-Weyl type deco I & II hold with closed ranges*

2nd fundamental observations

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

Theorem (fa-toolbox I (Friedrichs-Poincaré type est))

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1 \ \textit{compact}}$ $\Rightarrow$ $\exists$ $|\mathcal{A}_i^{-1}| = c_{\mathrm{A}_i} = c_{\mathrm{A}_i^*} = |(\mathcal{A}_i^*)^{-1}| \in (0, \infty)$

- **(i)** $\forall\, x \in D(\mathcal{A}_0)$ $\quad |x|_{\mathrm{H}_0} \leq c_{\mathrm{A}_0} |\mathrm{A}_0 x|_{\mathrm{H}_1}$
- **(i*)** $\forall\, y \in D(\mathcal{A}_0^*)$ $\quad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}_0} |\mathrm{A}_0^* y|_{\mathrm{H}_0}$
- **(ii)** $\forall\, y \in D(\mathcal{A}_1)$ $\quad |y|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |\mathrm{A}_1 y|_{\mathrm{H}_2}$
- **(ii*)** $\forall\, z \in D(\mathcal{A}_1^*)$ $\quad |z|_{\mathrm{H}_2} \leq c_{\mathrm{A}_1} |\mathrm{A}_1^* z|_{\mathrm{H}_1}$
- **(iii)** $\forall\, y \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ $\quad |(1 - \pi_{K_1}) y|_{\mathrm{H}_1} \leq c_{\mathrm{A}_1} |\mathrm{A}_1 y|_{\mathrm{H}_2} + c_{\mathrm{A}_0} |\mathrm{A}_0^* y|_{\mathrm{H}_0}$

note $\pi_{K_1} y \in K_1$ and $(1 - \pi_{K_1}) y \in K_1^{\perp}$

Remark

enough $R(\mathrm{A}_0)$ *and* $R(\mathrm{A}_1)$ *cl*

2nd fundamental observations

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

Theorem (fa-toolbox I (Helmholtz deco))

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\rightarrow \mathrm{H}_1 \text{ compact}}$

$$\mathrm{H}_1 = R(\mathrm{A}_0) \oplus N(\mathrm{A}_0^*) \qquad \mathrm{H}_1 = R(\mathrm{A}_1^*) \oplus N(\mathrm{A}_1)$$

$$D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus N(\mathrm{A}_0^*) \qquad D(\mathrm{A}_1) = D(\mathcal{A}_1) \oplus N(\mathrm{A}_1)$$

$$N(\mathrm{A}_1) = D(\mathcal{A}_0^*) \oplus K_1 \qquad N(\mathrm{A}_0^*) = D(\mathcal{A}_1) \oplus K_1$$

$$D(\mathrm{A}_1) = R(\mathrm{A}_0) \oplus \big(D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)\big) \qquad D(\mathrm{A}_0^*) = R(\mathrm{A}_1^*) \oplus \big(D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1)\big)$$

$$\mathrm{H}_1 = R(\mathrm{A}_0) \oplus K_1 \oplus R(\mathrm{A}_1^*)$$

$$D(\mathrm{A}_1) = R(\mathrm{A}_0) \oplus K_1 \oplus D(\mathcal{A}_1)$$

$$D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus K_1 \oplus R(\mathrm{A}_1^*)$$

$$D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) = D(\mathcal{A}_0^*) \oplus K_1 \oplus D(\mathcal{A}_1)$$

Remark

enough $R(\mathrm{A}_0)$ *and* $R(\mathrm{A}_1)$ *cl*

(stat) first order system - solution theory

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

$\boxed{\mathrm{A}_1 x = f}$ $\dim N(\mathrm{A}_1) = \infty$

find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ such that the fos

$$\begin{aligned} \mathrm{A}_1 x &= f \\ \mathrm{A}_0^* x &= g \\ \pi_{K_1} x &= k \end{aligned} \qquad \text{think of} \qquad \begin{aligned} (\mathring{\mathrm{rot}}\, E &= F) \\ (-\operatorname{div} E &= g) \\ (\pi_{\mathrm{D}} E &= K) \end{aligned}$$

kernel = cohomology group = $K_1 = N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*)$

trivially necessary $f \in R(\mathrm{A}_1)$ $g \in R(\mathrm{A}_0^*)$ $k \in K_1$

apply fa-toolbox

(stat) first order system - solution theory

complex $\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$

find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ st fos $\quad \mathrm{A}_1 x = f \quad \mathrm{A}_0^* x = g \quad \pi_{K_1} x = k$

Theorem (fa-toolbox II (solution theory))

$\Downarrow$ $\boxed{D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\to \mathrm{H}_1 \text{ compact}}$

fos is uniq sol $\Leftrightarrow$ $f \in R(\mathrm{A}_1) \quad g \in R(\mathrm{A}_0^*) \quad k \in K_1$

$$x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus K_1 = D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$$

$$\boxed{x_f := \mathcal{A}_1^{-1} f} \in D(\mathcal{A}_1)$$

$$\boxed{x_g := (\mathcal{A}_0^*)^{-1} g} \in D(\mathcal{A}_0^*)$$

dep cont on data $|x|_{\mathrm{H}_1} \le |x_f|_{\mathrm{H}_1} + |x_g|_{\mathrm{H}_1} + |k|_{\mathrm{H}_1} \le c_{\mathrm{A}_1} |f|_{\mathrm{H}_2} + c_{\mathrm{A}_0} |g|_{\mathrm{H}_0} + |k|_{\mathrm{H}_1}$

moreover

$$\pi_{R(\mathrm{A}_1^*)} x = x_f \qquad \pi_{R(\mathrm{A}_0)} x = x_g \qquad \pi_{K_1} x = k \qquad |x|_{\mathrm{H}_1}^2 = |x_f|_{\mathrm{H}_1}^2 + |x_g|_{\mathrm{H}_1}^2 + |k|_{\mathrm{H}_1}^2$$

Remark

enough $R(\mathrm{A}_0)$ *and* $R(\mathrm{A}_1)$ *cl*

(stat) first order system - variational formulations

$$
\begin{aligned}
x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus K_1 &= D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \\
x_f := \mathcal{A}_1^{-1} f \in D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap R(\mathrm{A}_1^*) &= D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \cap K_1^{\perp} \\
x_g := (\mathcal{A}_0^*)^{-1} g \in D(\mathcal{A}_0^*) = D(\mathrm{A}_0^*) \cap R(\mathrm{A}_0) &= D(\mathrm{A}_0^*) \cap N(\mathrm{A}_1) \cap K_1^{\perp}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{A}_1 x &= f & \mathrm{A}_1 x_f &= f & \mathrm{A}_1 x_g &= 0 & \mathrm{A}_1 k &= 0 \\
\mathrm{A}_0^* x &= g & \mathrm{A}_0^* x_f &= 0 & \mathrm{A}_0^* x_g &= g & \mathrm{A}_0^* k &= 0 \\
\pi_{K_1} x &= k & \pi_{K_1} x_f &= 0 & \pi_{K_1} x_g &= 0 & \pi_{K_1} k &= k
\end{aligned}
$$

- option I: find x_f and x_g separately $\Rightarrow$ $x = x_f + x_g + k$
- option II: find x directly

(stat) first order system - variational formulations I

finding

$$x_f := \mathcal{A}_1^{-1} f \in D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap \underbrace{R(\mathrm{A}_1^*)}_{=R(\mathcal{A}_1^*)} = D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \cap K_1^{\perp}$$

$$\mathrm{A}_1 x_f = f$$
$$\mathrm{A}_0^* x_f = 0$$
$$\pi_{K_1} x_f = 0$$

at least two options

- option Ia: multiply $\mathrm{A}_1 x_f = f$ by $\mathrm{A}_1 \xi$ $\Rightarrow$

$$\forall\, \xi \in D(\mathcal{A}_1) \qquad \langle \mathrm{A}_1 x_f, \mathrm{A}_1 \xi \rangle_{\mathrm{H}_2} = \langle f, \mathrm{A}_1 \xi \rangle_{\mathrm{H}_2}$$

  weak form of $\boxed{\mathrm{A}_1^* \mathrm{A}_1 x_f = \mathrm{A}_1^* f}$

- option Ib: repr $x_f = \mathrm{A}_1^* y_f$ with potential $y_f = (\mathcal{A}_1^*)^{-1} x_f \in D(\mathcal{A}_1^*)$ and mult by x_f by $\mathrm{A}_1^* \phi$ $\Rightarrow$

$$\forall\, \phi \in D(\mathcal{A}_1^*) \qquad \langle \mathrm{A}_1^* y_f, \mathrm{A}_1^* \phi \rangle_{\mathrm{H}_1} = \langle x_f, \mathrm{A}_1^* \phi \rangle_{\mathrm{H}_1} = \langle \mathrm{A}_1 x_f, \phi \rangle_{\mathrm{H}_2} = \langle f, \phi \rangle_{\mathrm{H}_2}$$

  weak form of $\boxed{\mathrm{A}_1 x_f = f}$ and $\boxed{\mathrm{A}_1 \mathrm{A}_1^* y_f = f}$

analogously for x_g

(stat) first order system - variational formulations I

Theorem

Let $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathrm{H}_1$ *be compact and let* $f \in R(\mathrm{A}_1)$ *and* $g \in R(\mathrm{A}_0^*)$*. The part sol* x_f *and* x_g *can be found by the following 4 var form:*

(i) $\exists^1\, \tilde{x}_f \in D(\mathcal{A}_1)$ *st* $\forall\, \xi \in D(\mathcal{A}_1)$ $\quad \langle \mathrm{A}_1 \tilde{x}_f, \mathrm{A}_1 \xi \rangle_{\mathrm{H}_2} = \langle f, \mathrm{A}_1 \xi \rangle_{\mathrm{H}_2}$

which even holds for all $\xi \in D(\mathrm{A}_1)$. $\Rightarrow$ $\boxed{\tilde{x}_f = x_f}$

(i') $\exists^1\, y_f \in D(\mathcal{A}_1^*)$ *st* $\forall\, \phi \in D(\mathcal{A}_1^*)$ $\quad \langle \mathrm{A}_1^* y_f, \mathrm{A}_1^* \phi \rangle_{\mathrm{H}_1} = \langle f, \phi \rangle_{\mathrm{H}_2}$

which even holds for all $\phi \in D(\mathrm{A}_1^*)$. $\Rightarrow$ $\boxed{\mathrm{A}_1^* y_f = x_f}$

(ii) $\exists^1\, \tilde{x}_g \in D(\mathcal{A}_0^*)$ *st* $\forall\, \zeta \in D(\mathcal{A}_0^*)$ $\quad \langle \mathrm{A}_0^* \tilde{x}_g, \mathrm{A}_0^* \zeta \rangle_{\mathrm{H}_0} = \langle g, \mathrm{A}_0^* \zeta \rangle_{\mathrm{H}_0}$

which even holds for all $\zeta \in D(\mathrm{A}_0^*)$. $\Rightarrow$ $\boxed{\tilde{x}_g = x_g}$

(ii') $\exists^1\, z_g \in D(\mathcal{A}_0)$ *st* $\forall\, \varphi \in D(\mathcal{A}_0)$ $\quad \langle \mathrm{A}_0 z_g, \mathrm{A}_0 \varphi \rangle_{\mathrm{H}_1} = \langle g, \varphi \rangle_{\mathrm{H}_0}$

which even holds for all $\varphi \in D(\mathrm{A}_0)$. $\Rightarrow$ $\boxed{\mathrm{A}_0 z_g = x_g}$

(stat) first order system - variational formulations I

e.g. $\exists^1\, \tilde{x}_f \in D(\mathcal{A}_1)$ st $\forall\, \xi \in D(\mathrm{A}_1)$ $\langle \mathrm{A}_1 \tilde{x}_f, \mathrm{A}_1 \xi \rangle_{\mathrm{H}_2} = \langle f, \mathrm{A}_1 \xi \rangle_{\mathrm{H}_2}$ $\Rightarrow$ $\boxed{\tilde{x}_f = x_f}$

Helmholtz deco $\Rightarrow$

$$\tilde{x}_f \in D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap R(\mathrm{A}_1^*) = D(\mathrm{A}_1) \cap N(\mathrm{A}_1)^\perp = D(\mathrm{A}_1) \cap \big(R(\mathcal{A}_0) \oplus K_1\big)^\perp$$
$$= D(\mathrm{A}_1) \cap R(\mathcal{A}_0)^\perp \cap K_1^\perp$$

$\Rightarrow$ saddle point formulations/double (multiple) saddle point formulations

Theorem

Let $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$ be compact and let $f \in R(\mathrm{A}_1)$ and $g \in R(\mathrm{A}_0^*)$. The part sol x_f and x_g can be found by the following 4 var form:

(i) $\exists^1\, (\tilde{x}_f, u, h) \in D(\mathrm{A}_1) \times D(\mathcal{A}_0) \times K_1$ *st* $\forall\, (\xi, \varphi, \kappa) \in D(\mathrm{A}_1) \times D(\mathrm{A}_0) \times K_1$

$$\langle \mathrm{A}_1 \tilde{x}_f, \mathrm{A}_1 \xi \rangle_{\mathrm{H}_2} + \langle \mathrm{A}_0 u, \xi \rangle_{\mathrm{H}_1} + \langle h, \xi \rangle_{\mathrm{H}_1} = \langle f, \mathrm{A}_1 \xi \rangle_{\mathrm{H}_2}$$
$$\langle \tilde{x}_f, \mathrm{A}_0 \varphi \rangle_{\mathrm{H}_1} = 0$$
$$\langle \tilde{x}_f, \kappa \rangle_{\mathrm{H}_1} = 0$$

$\Rightarrow$ $u = 0$ $h = 0$ $\boxed{\tilde{x}_f = x_f}$

(i') *analogously for* y_f

(ii) *analogously for* $\tilde{x}_g$

(ii') *analogously for* z_g

(stat) first order system - variational formulations I

latter tripple saddle point formulation

$$\exists^1\,(\tilde{x}_f,u,h)\in D(\mathrm{A}_1)\times D(\mathcal{A}_0)\times K_1\quad\text{st}\quad\forall\,(\xi,\varphi,\kappa)\in D(\mathrm{A}_1)\times D(\mathrm{A}_0)\times K_1$$

$$\begin{aligned}\langle \mathrm{A}_1\tilde{x}_f,\mathrm{A}_1\xi\rangle_{\mathrm{H}_2}+\langle \mathrm{A}_0 u,\xi\rangle_{\mathrm{H}_1}+\langle h,\xi\rangle_{\mathrm{H}_1}&=\langle f,\mathrm{A}_1\xi\rangle_{\mathrm{H}_2}\\ \langle\tilde{x}_f,\mathrm{A}_0\varphi\rangle_{\mathrm{H}_1}&=0\\ \langle\tilde{x}_f,\kappa\rangle_{\mathrm{H}_1}&=0\end{aligned}$$

is weak formulation of

$$\mathrm{A}_1^*\mathrm{A}_1\tilde{x}_f+\mathrm{A}_0 u+h=\mathrm{A}_1^* f\qquad \mathrm{A}_0^*\tilde{x}_f=0\qquad \pi_{K_1}\tilde{x}_f=0$$

i.e., in formal matrix notation

$$\begin{bmatrix}\mathrm{A}_1^*\mathrm{A}_1 & \mathrm{A}_0 & \iota_{K_1}\\ \mathrm{A}_0^* & 0 & 0\\ \pi_{K_1}=\iota_{K_1}^* & 0 & 0\end{bmatrix}\begin{bmatrix}\tilde{x}_f\\ u\\ h\end{bmatrix}=\begin{bmatrix}\mathrm{A}_1^* f\\ 0\\ 0\end{bmatrix}$$

Note $\quad u=0,\quad h=0,\quad \boxed{\tilde{x}_f=x_f}$

potential y_f

$$\begin{bmatrix}\mathrm{A}_1\mathrm{A}_1^* & \mathrm{A}_2^* & \iota_{K_2}\\ \mathrm{A}_2 & 0 & 0\\ \pi_{K_2}=\iota_{K_2}^* & 0 & 0\end{bmatrix}\begin{bmatrix}y_f\\ v\\ h_2\end{bmatrix}=\begin{bmatrix}f\\ 0\\ 0\end{bmatrix}$$

Note $\quad v=0,\quad h_2=0,\quad \boxed{\mathrm{A}_1^* y_f=x_f}$

(stat) first order system - variational formulations II

$$\begin{bmatrix} A_1^* A_1 & A_0 & \iota_{K_1} \\ A_0^* & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_f \\ u \\ h \end{bmatrix} = \begin{bmatrix} A_1^* f \\ 0 \\ 0 \end{bmatrix}$$

Note $u = 0, \quad h = 0, \quad \boxed{\tilde{x}_f = x_f}$

SAME formulation can be used to compute $x = x_f + x_g + k$ directly!

$$\begin{bmatrix} A_1^* A_1 & A_0 & \iota_{K_1} \\ A_0^* & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ u \\ h \end{bmatrix} = \begin{bmatrix} A_1^* f \\ g \\ k \end{bmatrix}$$

$$\Rightarrow \quad u = 0, \quad h = 0, \quad \boxed{\tilde{x} = x}$$

Remark

special case $A_0 = \mathring{\nabla}$ $\quad A_0^* = -\operatorname{div}$ $\quad$ *or* $\quad A_0 = \nabla$ $\quad A_0^* = -\mathring{\operatorname{div}}$

$A_1 = \mathring{\operatorname{rot}}$ $\quad A_1^* = \operatorname{rot}$ $\quad\quad A_0 = \operatorname{rot}$ $\quad A_0^* = \mathring{\operatorname{rot}}$

var form recently proposed by
Alonso Rodriguez, A., Bertolazzi E., and Valli A.: The curl-div system: theory and finite element approximation, talk/preprint, 2018

(stat) first order system - variational formulations II

Theorem

Let $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$ *be compact and let* $f \in R(\mathrm{A}_1)$ *and* $g \in R(\mathrm{A}_0^*)$*.*
x *can be found by the following 2 double saddle point var form:*

(i) $\exists^1\, (\tilde{x}, u, h_1) \in D(\mathrm{A}_1) \times D(\mathcal{A}_0) \times K_1 \quad st \quad \forall\, (\xi, \varphi, \kappa) \in D(\mathrm{A}_1) \times D(\mathrm{A}_0) \times K_1$

$$\begin{aligned} \langle \mathrm{A}_1\tilde{x}, \mathrm{A}_1\xi\rangle_{\mathrm{H}_2} + \langle \mathrm{A}_0 u, \xi\rangle_{\mathrm{H}_1} + \langle h_1, \xi\rangle_{\mathrm{H}_1} &= \langle f, \mathrm{A}_1\xi\rangle_{\mathrm{H}_2} \\ \langle \tilde{x}, \mathrm{A}_0\varphi\rangle_{\mathrm{H}_1} &= \langle g, \varphi\rangle_{\mathrm{H}_0} \\ \langle \tilde{x}, \kappa\rangle_{\mathrm{H}_1} &= \langle k, \kappa\rangle_{\mathrm{H}_1} \end{aligned}$$

$$\Rightarrow \quad u = 0, \quad h_1 = 0, \quad \boxed{\tilde{x} = x}$$

(ii) $\exists^1\, (\hat{x}, v, h_2) \in D(\mathrm{A}_0^*) \times D(\mathcal{A}_1^*) \times K_1 \quad st \quad \forall\, (\zeta, \phi, \kappa) \in D(\mathrm{A}_0^*) \times D(\mathrm{A}_1^*) \times K_1$

$$\begin{aligned} \langle \mathrm{A}_0^*\hat{x}, \mathrm{A}_0^*\zeta\rangle_{\mathrm{H}_0} + \langle \mathrm{A}_1^* v, \zeta\rangle_{\mathrm{H}_1} + \langle h_2, \zeta\rangle_{\mathrm{H}_1} &= \langle g, \mathrm{A}_0^*\zeta\rangle_{\mathrm{H}_0} \\ \langle \hat{x}, \mathrm{A}_1^*\phi\rangle_{\mathrm{H}_1} &= \langle f, \phi\rangle_{\mathrm{H}_2} \\ \langle \hat{x}, \kappa\rangle_{\mathrm{H}_1} &= \langle k, \kappa\rangle_{\mathrm{H}_1} \end{aligned}$$

$$\Rightarrow \quad v = 0, \quad h_2 = 0, \quad \boxed{\hat{x} = x}$$

(stat) first order system - variational formulations II

form matrix not

$$\begin{bmatrix} \mathrm{A}_1^*\mathrm{A}_1 & \mathrm{A}_0 & \iota_{K_1} \\ \mathrm{A}_0^* & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ u \\ h_1 \end{bmatrix} = \begin{bmatrix} \mathrm{A}_1^* f \\ g \\ k \end{bmatrix}$$

$$\Rightarrow \quad u = 0, \quad h_1 = 0, \quad \boxed{\tilde{x} = x}$$

$$\begin{bmatrix} \mathrm{A}_0\mathrm{A}_0^* & \mathrm{A}_1^* & \iota_{K_1} \\ \mathrm{A}_1 & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ v \\ h_2 \end{bmatrix} = \begin{bmatrix} \mathrm{A}_0 g \\ f \\ k \end{bmatrix}$$

$$\Rightarrow \quad v = 0, \quad h_2 = 0, \quad \boxed{\hat{x} = x}$$

(stat) first order system - variational formulations II

$$\begin{aligned} D(\mathcal{A}_0) &= D(\mathrm{A}_0) \cap R(\mathrm{A}_0^*) = D(\mathrm{A}_0) \cap N(\mathrm{A}_0)^\perp = D(\mathrm{A}_0) \cap \big(R(\mathcal{A}_{-1}) \oplus K_0\big)^\perp \\ &= D(\mathrm{A}_0) \cap R(\mathcal{A}_{-1})^\perp \cap K_0^\perp \end{aligned}$$

$$\begin{aligned} D(\mathcal{A}_1^*) &= D(\mathrm{A}_1^*) \cap R(\mathrm{A}_1) = D(\mathrm{A}_1^*) \cap N(\mathrm{A}_1^*)^\perp = D(\mathrm{A}_1^*) \cap \big(R(\mathcal{A}_2^*) \oplus K_2\big)^\perp \\ &= D(\mathrm{A}_1^*) \cap R(\mathcal{A}_2^*)^\perp \cap K_2^\perp \end{aligned}$$

(stat) first order system - variational formulations II

$$D(\mathcal{A}_0) = D(\mathrm{A}_0) \cap R(\mathcal{A}_{-1})^\perp \cap K_0^\perp \qquad D(\mathcal{A}_1^*) = D(\mathrm{A}_1^*) \cap R(\mathcal{A}_2^*)^\perp \cap K_2^\perp$$

Theorem

Let $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$ *be compact and let* $f \in R(\mathrm{A}_1)$ *and* $g \in R(\mathrm{A}_0^*)$*.*
x *can be found by the following quadruple saddle point var form:*

$\exists^1\, (\tilde{x}, u, y, h_1, h_0) \in D(\mathrm{A}_1) \times D(\mathrm{A}_0) \times D(\mathcal{A}_{-1}) \times K_1 \times K_0$ *st*
$\forall\, (\xi, \varphi, \vartheta, \kappa, \lambda) \in D(\mathrm{A}_1) \times D(\mathrm{A}_0) \times D(\mathrm{A}_{-1}) \times K_1 \times K_0$

$$\begin{aligned}
\langle \mathrm{A}_1\tilde{x}, \mathrm{A}_1\xi\rangle_{\mathrm{H}_2} + \langle \mathrm{A}_0 u, \xi\rangle_{\mathrm{H}_1} + \langle h_1, \xi\rangle_{\mathrm{H}_1} &= \langle f, \mathrm{A}_1\xi\rangle_{\mathrm{H}_2} \\
\langle \tilde{x}, \mathrm{A}_0\varphi\rangle_{\mathrm{H}_1} + \langle \mathrm{A}_{-1} y, \varphi\rangle_{\mathrm{H}_0} + \langle h_0, \varphi\rangle_{\mathrm{H}_0} &= \langle g, \varphi\rangle_{\mathrm{H}_0} \\
\langle u, \mathrm{A}_{-1}\vartheta\rangle_{\mathrm{H}_0} &= 0 \\
\langle \tilde{x}, \kappa\rangle_{\mathrm{H}_1} &= \langle k, \kappa\rangle_{\mathrm{H}_1} \\
\langle u, \lambda\rangle_{\mathrm{H}_0} &= 0
\end{aligned}$$

$$\Rightarrow \quad u = 0, \quad y = 0, \quad h_1 = 0, \quad h_0 = 0, \quad \boxed{\tilde{x} = x}$$

(stat) first order system - variational formulations II

$$D(\mathcal{A}_0) = D(\mathrm{A}_0) \cap R(\mathcal{A}_{-1})^\perp \cap K_0^\perp \qquad D(\mathcal{A}_1^*) = D(\mathrm{A}_1^*) \cap R(\mathcal{A}_2^*)^\perp \cap K_2^\perp$$

Theorem

Let $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow\!\!\!\!\rightarrow \mathrm{H}_1$ be compact and let $f \in R(\mathrm{A}_1)$ and $g \in R(\mathrm{A}_0^*)$. x can be found by the following quadruple saddle point var form:

$\exists^1\,(\hat{x}, v, z, h_1, h_2) \in D(\mathrm{A}_0^*) \times D(\mathrm{A}_1^*) \times D(\mathcal{A}_2^*) \times K_1 \times K_2$ *st*

$\forall\,(\zeta, \phi, \theta, \kappa, \lambda) \in D(\mathrm{A}_0^*) \times D(\mathrm{A}_1^*) \times D(\mathrm{A}_2^*) \times K_1 \times K_2$

$$\begin{aligned}
\langle \mathrm{A}_0^*\hat{x}, \mathrm{A}_0^*\zeta\rangle_{\mathrm{H}_0} + \langle \mathrm{A}_1^* v, \zeta\rangle_{\mathrm{H}_1} + \langle h_1, \zeta\rangle_{\mathrm{H}_1} &= \langle g, \mathrm{A}_0^*\zeta\rangle_{\mathrm{H}_0}\\
\langle \hat{x}, \mathrm{A}_1^*\phi\rangle_{\mathrm{H}_1} + \langle \mathrm{A}_2^* z, \phi\rangle_{\mathrm{H}_2} + \langle h_2, \phi\rangle_{\mathrm{H}_2} &= \langle f, \phi\rangle_{\mathrm{H}_2}\\
\langle v, \mathrm{A}_2^*\theta\rangle_{\mathrm{H}_2} &= 0\\
\langle \hat{x}, \kappa\rangle_{\mathrm{H}_1} &= \langle k, \kappa\rangle_{\mathrm{H}_1}\\
\langle v, \lambda\rangle_{\mathrm{H}_2} &= 0
\end{aligned}$$

$$\Rightarrow \quad v = 0, \quad z = 0, \quad h_1 = 0, \quad h_2 = 0, \quad \boxed{\hat{x} = x}$$

(stat) first order system - variational formulations II

form matrix not

$$\begin{bmatrix} \mathrm{A}_1^*\mathrm{A}_1 & \mathrm{A}_0 & 0 & \iota_{K_1} & 0 \\ \mathrm{A}_0^* & 0 & \mathrm{A}_{-1} & 0 & \iota_{K_0} \\ 0 & \mathrm{A}_{-1}^* & 0 & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 & 0 & 0 \\ 0 & \pi_{K_0} = \iota_{K_0}^* & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ u \\ y \\ h_1 \\ h_0 \end{bmatrix} = \begin{bmatrix} \mathrm{A}_1^* f \\ g \\ 0 \\ k \\ 0 \end{bmatrix}$$

note $\quad u = 0, \quad y = 0, \quad h_1 = 0, \quad h_0 = 0, \quad \boxed{\tilde{x} = x}$

$$\begin{bmatrix} \mathrm{A}_0\mathrm{A}_0^* & \mathrm{A}_1^* & 0 & \iota_{K_1} & 0 \\ \mathrm{A}_1 & 0 & \mathrm{A}_2^* & 0 & \iota_{K_2} \\ 0 & \mathrm{A}_2 & 0 & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 & 0 & 0 \\ 0 & \pi_{K_2} = \iota_{K_2}^* & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ v \\ z \\ h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} \mathrm{A}_0 g \\ f \\ 0 \\ k \\ 0 \end{bmatrix}$$

note $\quad v = 0, \quad z = 0, \quad h_1 = 0, \quad h_2 = 0, \quad \boxed{\hat{x} = x}$

typical situation in 3D:

- K_{-1}, K_0, K_3, K_4 trivial only K_1, K_2 non-trivial (Dirichlet/Neumann fields)
- $\mathrm{A}_{-2}^* = 0 \quad \Rightarrow \quad N(\mathrm{A}_{-2}^*) = \mathrm{H}_{-1}$
- $\mathrm{A}_4 = 0 \quad \Rightarrow \quad N(\mathrm{A}_4) = \mathrm{H}_4$
- $N(\mathrm{A}_3)$, $N(\mathrm{A}_{-1}^*)$ finite co-dim

(stat) first order system - variational formulations II

typical situation in 3D:

- K_{-1}, K_0, K_3, K_4 trivial only K_1, K_2 non-trivial (Dirichlet/Neumann fields)
- $\mathrm{A}_{-2}^* = 0 \quad \Rightarrow \quad N(\mathrm{A}_{-2}^*) = \mathrm{H}_{-1}$
- $\mathrm{A}_4 = 0 \quad \Rightarrow \quad N(\mathrm{A}_4) = \mathrm{H}_4$
- $N(\mathrm{A}_3)$, $N(\mathrm{A}_{-1}^*)$ finite co-dim

recall

$$D(\mathcal{A}_i) = D(\mathrm{A}_i) \cap R(\mathrm{A}_{i-1})^\perp \cap K_i^\perp = D(\mathrm{A}_i) \cap N(\mathrm{A}_{i-1}^*) \cap K_i^\perp$$

$$D(\mathcal{A}_i^*) = D(\mathrm{A}_i^*) \cap R(\mathrm{A}_{i+1}^*)^\perp \cap K_{i+1}^\perp = D(\mathrm{A}_i^*) \cap N(\mathrm{A}_{i+1}) \cap K_{i+1}^\perp$$

always in 3D

$$D(\mathcal{A}_{-1}) = D(\mathrm{A}_{-1}) \qquad D(\mathcal{A}_3^*) = D(\mathrm{A}_3^*)$$

$$D(\mathcal{A}_0) = D(\mathrm{A}_0) \cap N(\mathrm{A}_{-1}^*) \qquad D(\mathcal{A}_2^*) = D(\mathrm{A}_2^*) \cap N(\mathrm{A}_3)$$

often in 3D

$$D(\mathcal{A}_0) = D(\mathrm{A}_0) \qquad D(\mathcal{A}_2^*) = D(\mathrm{A}_2^*)$$

(stat) first order system - variational formulations II

always in 3D

$$D(\mathcal{A}_{-1}) = D(\mathrm{A}_{-1}) \qquad D(\mathcal{A}_3^*) = D(\mathrm{A}_3^*)$$

$$D(\mathcal{A}_0) = D(\mathrm{A}_0) \cap N(\mathrm{A}_{-1}^*) \qquad D(\mathcal{A}_2^*) = D(\mathrm{A}_2^*) \cap N(\mathrm{A}_3)$$

often in 3D

$$D(\mathcal{A}_0) = D(\mathrm{A}_0) \qquad D(\mathcal{A}_2^*) = D(\mathrm{A}_2^*)$$

always in 3D: test spaces $(K_0$ trivial)

$D(\mathrm{A}_1) \times D(\mathrm{A}_0) \times D(\mathcal{A}_{-1}) \times K_1 = D(\mathrm{A}_1) \times D(\mathrm{A}_0) \times D(\mathrm{A}_{-1}) \times K_1$ OK

$$\begin{bmatrix} \mathrm{A}_1^*\mathrm{A}_1 & \mathrm{A}_0 & 0 & \iota_{K_1} \\ \mathrm{A}_0^* & 0 & \mathrm{A}_{-1} & 0 \\ 0 & \mathrm{A}_{-1}^* & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ u \\ y \\ h_1 \end{bmatrix} = \begin{bmatrix} \mathrm{A}_1^* f \\ g \\ 0 \\ k \end{bmatrix}$$

often in 3D: test spaces

$D(\mathrm{A}_0^*) \times D(\mathrm{A}_1^*) \times D(\mathcal{A}_2^*) \times K_1 \times K_2 = D(\mathrm{A}_0^*) \times D(\mathrm{A}_1^*) \times D(\mathrm{A}_2^*) \times K_1 \times K_2$ OK

$$\begin{bmatrix} \mathrm{A}_0\mathrm{A}_0^* & \mathrm{A}_1^* & 0 & \iota_{K_1} & 0 \\ \mathrm{A}_1 & 0 & \mathrm{A}_2^* & 0 & \iota_{K_2} \\ 0 & \mathrm{A}_2 & 0 & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 & 0 & 0 \\ 0 & \pi_{K_2} = \iota_{K_2}^* & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ v \\ z \\ h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} \mathrm{A}_0 g \\ f \\ 0 \\ k \\ 0 \end{bmatrix}$$

(stat) first order system - variational formulations II

always in 3D

$$D(\mathcal{A}_{-1}) = D(\mathrm{A}_{-1}) \qquad\qquad D(\mathcal{A}_3^*) = D(\mathrm{A}_3^*)$$

always in 3D: test spaces (K_0 trivial)

$D(\mathrm{A}_1) \times D(\mathrm{A}_0) \times D(\mathcal{A}_{-1}) \times K_1 = D(\mathrm{A}_1) \times D(\mathrm{A}_0) \times D(\mathrm{A}_{-1}) \times K_1$ OK

$$\begin{bmatrix} \mathrm{A}_1^*\mathrm{A}_1 & \mathrm{A}_0 & 0 & \iota_{K_1} \\ \mathrm{A}_0^* & 0 & \mathrm{A}_{-1} & 0 \\ 0 & \mathrm{A}_{-1}^* & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ u \\ y \\ h_1 \end{bmatrix} = \begin{bmatrix} \mathrm{A}_1^* f \\ g \\ 0 \\ k \end{bmatrix}$$

always in 3D: test spaces (K_3 trivial)

$D(\mathrm{A}_0^*) \times D(\mathrm{A}_1^*) \times D(\mathrm{A}_2^*) \times D(\mathcal{A}_3^*) \times K_1 \times K_2 = D(\mathrm{A}_0^*) \times D(\mathrm{A}_1^*) \times D(\mathrm{A}_2^*) \times D(\mathrm{A}_3^*) \times K_1 \times K_2$

$$\begin{bmatrix} \mathrm{A}_0\mathrm{A}_0^* & \mathrm{A}_1^* & 0 & 0 & \iota_{K_1} & 0 \\ \mathrm{A}_1 & 0 & \mathrm{A}_2^* & 0 & 0 & \iota_{K_2} \\ 0 & \mathrm{A}_2 & 0 & \mathrm{A}_3^* & 0 & 0 \\ 0 & 0 & \mathrm{A}_3 & 0 & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi_{K_2} = \iota_{K_2}^* & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ v \\ z \\ w \\ h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} \mathrm{A}_0 g \\ f \\ 0 \\ 0 \\ k \\ 0 \end{bmatrix}$$

(stat) first order system - a posteriori error estimates

problem: find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ st $\mathrm{A}_1 x = f$ $\mathrm{A}_0^* x = g$ $\pi_{K_1} x = k$

'very' non-conforming 'approximation' of x: $\tilde{x} \in \mathrm{H}_1$

def., dcmp. err. $e = x - \tilde{x} = \pi_{R(\mathrm{A}_0)} e + \pi_{K_1} e + \pi_{R(\mathrm{A}_1^*)} e \in \mathrm{H}_1 = R(\mathrm{A}_0) \oplus K_1 \oplus R(\mathrm{A}_1^*)$

Theorem (sharp upper bounds)

Let $\tilde{x} \in \mathrm{H}_1$ *and* $e = x - \tilde{x}$. *Then*

$$|e|_{\mathrm{H}_1}^2 = |\pi_{R(\mathrm{A}_0)} e|_{\mathrm{H}_1}^2 + |\pi_{K_1} e|_{\mathrm{H}_1}^2 + |\pi_{R(\mathrm{A}_1^*)} e|_{\mathrm{H}_1}^2$$

$$|\pi_{R(\mathrm{A}_0)} e|_{\mathrm{H}_1} = \min_{\phi \in D(\mathrm{A}_0^*)} \big(c_{\mathrm{A}_0} |\mathrm{A}_0^* \phi - g|_{\mathrm{H}_0} + |\phi - \tilde{x}|_{\mathrm{H}_1} \big)$$

reg $(\mathrm{A}_0 \mathrm{A}_0^* + 1)$*-prbl in* $D(\mathrm{A}_0^*)$

$$|\pi_{R(\mathrm{A}_1^*)} e|_{\mathrm{H}_1} = \min_{\varphi \in D(\mathrm{A}_1)} \big(c_{\mathrm{A}_1} |\mathrm{A}_1 \varphi - f|_{\mathrm{H}_2} + |\varphi - \tilde{x}|_{\mathrm{H}_1} \big)$$

reg $(\mathrm{A}_1^* \mathrm{A}_1 + 1)$*-prbl in* $D(\mathrm{A}_1)$

$$|\pi_{K_1} e|_{\mathrm{H}_1} = |\pi_{K_1} \tilde{x} - k|_{\mathrm{H}_1} = \min_{\substack{\xi \in D(\mathrm{A}_0) \\ \zeta \in D(\mathrm{A}_1^*)}} |\mathrm{A}_0 \xi + \mathrm{A}_1^* \zeta + \tilde{x} - k|_{\mathrm{H}_1}$$

cpld $(\mathrm{A}_0^* \mathrm{A}_0)$-$(\mathrm{A}_1 \mathrm{A}_1^*)$*-sys in* $D(\mathcal{A}_0)$-$D(\mathcal{A}_1^*)$

Remark

Even $\pi_{K_1} e = k - \pi_{K_1} \tilde{x}$ *and the minima are attained at*

$$\hat{\phi} = \pi_{R(\mathrm{A}_0)} e + \tilde{x}, \qquad \hat{\varphi} = \pi_{R(\mathrm{A}_1^*)} e + \tilde{x}, \qquad \mathrm{A}_0 \hat{\xi} + \mathrm{A}_1^* \hat{\zeta} = (\pi_{K_1} - 1)\tilde{x}.$$

(stat) first order system - a posteriori error estimates

problem: find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$ st $\mathrm{A}_1 x = f$ $\mathrm{A}_0^* x = g$ $\pi_{K_1} x = k$

'very' non-conforming 'approximation' of x: $\tilde{x} \in \mathrm{H}_1$

def., dcmp. err. $e = x - \tilde{x} = \pi_{R(\mathrm{A}_0)} e + \pi_{K_1} e + \pi_{R(\mathrm{A}_1^*)} e \in \mathrm{H}_1 = R(\mathrm{A}_0) \oplus K_1 \oplus R(\mathrm{A}_1^*)$

Theorem (sharp lower bounds)

Let $\tilde{x} \in \mathrm{H}_1$ *and* $e = x - \tilde{x}$. *Then*

$$|e|_{\mathrm{H}_1}^2 = |\pi_{R(\mathrm{A}_0)} e|_{\mathrm{H}_1}^2 + |\pi_{K_1} e|_{\mathrm{H}_1}^2 + |\pi_{R(\mathrm{A}_1^*)} e|_{\mathrm{H}_1}^2$$

$$|\pi_{R(\mathrm{A}_0)} e|_{\mathrm{H}_1}^2 = \max_{\phi \in D(\mathrm{A}_0)} \big(2\langle g, \phi\rangle_{\mathrm{H}_0} - \langle 2\tilde{x} + \mathrm{A}_0\phi, \mathrm{A}_0\phi\rangle_{\mathrm{H}_1}\big)$$

$(\mathrm{A}_0^*\mathrm{A}_0)$-*prbl in* $D(\mathcal{A}_0)$

$$|\pi_{R(\mathrm{A}_1^*)} e|_{\mathrm{H}_1}^2 = \max_{\varphi \in D(\mathrm{A}_1^*)} \big(2\langle f, \varphi\rangle_{\mathrm{H}_2} - \langle 2\tilde{x} + \mathrm{A}_1^*\varphi, \mathrm{A}_1^*\varphi\rangle_{\mathrm{H}_1}\big)$$

$(\mathrm{A}_1\mathrm{A}_0^*)$-*prbl in* $D(\mathcal{A}_1^*)$

$$|\pi_{K_1} e|_{\mathrm{H}_1}^2 = \max_{\psi \in K_1} \langle 2(k - \tilde{x}) - \psi, \psi\rangle_{\mathrm{H}_1}$$

$$\pi_{K_1} e = k - \pi_{K_1}\tilde{x}$$

Remark

The maxima are attained at $\phi \in D(\mathrm{A}_0)$ *with* $\mathrm{A}_0\phi = \pi_{\mathrm{A}_0} e$ *and* $\varphi \in D(\mathrm{A}_1^*)$ *with* $\mathrm{A}_1^*\varphi = \pi_{R(\mathrm{A}_1^*)} e$ *and* $\psi = \pi_{K_1} e$

A_0^*-A_1-lemma (generalized global div-curl-lemma)

Lemma (A_0^*-A_1-lemma)

Let $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathrm{H}_1$ *be compact, and*

(i) (x_n) *bounded in* $D(\mathrm{A}_1)$,

(ii) (y_n) *bounded in* $D(\mathrm{A}_0^*)$.

$\Rightarrow \exists\ x \in D(\mathrm{A}_1)$, $y \in D(\mathrm{A}_0^*)$ *and subsequences st*

$x_n \rightharpoonup x$ *in* $D(\mathrm{A}_1)$ *and* $y_n \rightharpoonup y$ *in* $D(\mathrm{A}_0^*)$ *as well as*

$$\langle x_n, y_n \rangle_{\mathrm{H}_1} \to \langle x, y \rangle_{\mathrm{H}_1}.$$

classical de Rham complex in 3D (∇-rot-div-complex)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot\cup \Gamma_n}$

(electro-magneto dynamics, Maxwell's equations)

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{div}}{\overset{\mathring{\nabla}}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{rot}}{\overset{\mathring{\operatorname{rot}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\nabla}{\overset{\mathring{\operatorname{div}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}} \mathbb{R}$$

mixed boundary conditions and inhomogeneous and anisotropic media

$$\{0\} \text{ or } \mathbb{R} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{div}_{\Gamma_n} \varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_\varepsilon \underset{\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathbb{R} \text{ or } \{0\}$$

classical de Rham complex in 3D (∇-rot-div-complex)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot\cup \Gamma_n}$

(electro-magneto dynamics, Maxwell's equations with mixed boundary conditions)

$$\{0\} \text{ or } \mathbb{R} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{div}_{\Gamma_n}\varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_\varepsilon \underset{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathbb{R} \text{ or } \{0\}$$

related fos

$\nabla_{\Gamma_t} u = A$ in Ω	$\operatorname{rot}_{\Gamma_t} E = J$ in Ω	$\operatorname{div}_{\Gamma_t} H = k$ in Ω	$\pi v = b$ in Ω
$\pi u = a$ in Ω	$-\operatorname{div}_{\Gamma_n} \varepsilon E = j$ in Ω	$\varepsilon^{-1}\operatorname{rot}_{\Gamma_n} H = K$ in Ω	$-\nabla_{\Gamma_n} v = B$ in Ω

related sos

$-\operatorname{div}_{\Gamma_n} \varepsilon \nabla_{\Gamma_t} u = j$ in Ω	$\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}\operatorname{rot}_{\Gamma_t} E = K$ in Ω	$-\nabla_{\Gamma_n}\operatorname{div}_{\Gamma_t} H = B$ in Ω
$\pi u = a$ in Ω	$-\operatorname{div}_{\Gamma_n} \varepsilon E = j$ in Ω	$\varepsilon^{-1}\operatorname{rot}_{\Gamma_n} H = K$ in Ω

corresponding compact embeddings:

$$D(\nabla_{\Gamma_t}) \cap D(\pi) = D(\nabla_{\Gamma_t}) = \mathsf{H}^1_{\Gamma_t} \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

$$D(\operatorname{rot}_{\Gamma_t}) \cap D(-\operatorname{div}_{\Gamma_n}\varepsilon) = \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n} \hookrightarrow \mathsf{L}^2_\varepsilon \qquad \text{(Weck's selection theorem, '74)}$$

$$D(\operatorname{div}_{\Gamma_t}) \cap D(\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}) = \mathsf{D}_{\Gamma_t} \cap \mathsf{R}_{\Gamma_n} \hookrightarrow \mathsf{L}^2 \qquad \text{(Weck's selection theorem, '74)}$$

$$D(\nabla_{\Gamma_n}) \cap D(\pi) = D(\nabla_{\Gamma_n}) = \mathsf{H}^1_{\Gamma_n} \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

Weck's selection theorem for weak Lip. dom. and mixed bc: Bauer/Py/Schomburg ('16)

Weck's selection theorem (Weck '74, (Habil. '72) stimulated by Rolf Leis)
(Weber '80, Picard '84, Costabel '90, Witsch '93, Jochmann '97, Kuhn '99, Picard/Weck/Witsch '01, Py '96, '03, '06, '07, '08)

classical de Rham complex in 3D (∇-rot-div-complex)

$$
\begin{aligned}
\operatorname{rot} E &= F && \text{in } \Omega \\
-\operatorname{div} \varepsilon E &= g && \text{in } \Omega \\
\nu \times E &= 0 && \text{at } \Gamma_t \\
\nu \cdot \varepsilon E &= 0 && \text{at } \Gamma_n
\end{aligned}
$$

non-trivial kernel $\mathcal{H}_{\mathrm{D},\varepsilon} = \{H \in \mathrm{L}^2 : \operatorname{rot} H = 0,\ \operatorname{div} \varepsilon H = 0,\ \nu \times H|_{\Gamma_t} = 0,\ \nu \cdot \varepsilon H|_{\Gamma_n} = 0\}$
additional condition on Dirichlet/Neumann fields for uniqueness

$$\pi_{\mathrm{D}} E = K \in \mathcal{H}_{\mathrm{D},\varepsilon}$$

$$\{0\} \text{ or } \mathbb{R} \ \underset{\pi}{\overset{\iota}{\rightleftarrows}} \ \mathrm{L}^2 \ \underset{-\operatorname{div}_{\Gamma_n} \varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}} \ \mathrm{L}^2_\varepsilon \ \underset{\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}} \ \mathrm{L}^2 \ \underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}} \ \mathrm{L}^2 \ \underset{\iota}{\overset{\pi}{\rightleftarrows}} \ \mathbb{R} \text{ or } \{0\}$$

$$\cdots \ \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \ \mathrm{H}_{-1} \ \underset{\mathrm{A}^*_{-1}}{\overset{\mathrm{A}_{-1}}{\rightleftarrows}} \ \mathrm{H}_0 \ \underset{\mathrm{A}^*_0}{\overset{\mathrm{A}_0}{\rightleftarrows}} \ \mathrm{H}_1 \ \underset{\mathrm{A}^*_1}{\overset{\mathrm{A}_1}{\rightleftarrows}} \ \mathrm{H}_2 \ \underset{\mathrm{A}^*_2}{\overset{\mathrm{A}_2}{\rightleftarrows}} \ \mathrm{H}_3 \ \underset{\mathrm{A}^*_3}{\overset{\mathrm{A}_3}{\rightleftarrows}} \ \mathrm{H}_4 \ \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \ \cdots$$

find $E \in \mathrm{R}_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}\mathrm{D}_{\Gamma_n}(\Omega)$ st (fos) find $x \in D(\mathrm{A}_1) \cap D(\mathrm{A}^*_0)$ st

$$
\begin{aligned}
\operatorname{rot}_{\Gamma_t} E &= F & \qquad & & \mathrm{A}_1 x &= f \\
-\operatorname{div}_{\Gamma_n} \varepsilon E &= g & \qquad & \text{translation} & \mathrm{A}^*_0 x &= g \\
\pi_{\mathrm{D/N}} E &= K & \qquad & & \pi_{K_1} x &= k
\end{aligned}
$$

classical de Rham complex in 3D (∇-rot-div-complex)

$c_{A_0} = c_{fp}$ (Friedrichs/Poincaré constant) and $c_{A_1} = c_m$ (Maxwell constant)

Lemma/Theorem $\Downarrow$ $\boxed{D(A_1) \cap D(A_0^*) \hookrightarrow\!\!\!\rightarrow L^2_\varepsilon(\Omega) \text{ compact}}$

(i) all Friedrichs-Poincaré type est hold

$$\forall\, \varphi \in D(\mathcal{A}_0) \quad |\varphi|_{H_0} \le c_{A_0} |A_0 \varphi|_{H_1} \quad \Leftrightarrow \quad \forall\, \varphi \in H^1_{\Gamma_t} \quad |\varphi|_{L^2} \le c_{fp} |\nabla \varphi|_{L^2_\varepsilon}$$

$$\forall\, \phi \in D(\mathcal{A}_0^*) \quad |\phi|_{H_1} \le c_{A_0} |A_0^* \phi|_{H_0} \quad \Leftrightarrow \quad \forall\, \Phi \in \varepsilon^{-1} D_{\Gamma_n} \cap \nabla H^1_{\Gamma_t} \quad |\Phi|_{L^2_\varepsilon} \le c_{fp} |\operatorname{div} \varepsilon \Phi|_{L^2}$$

$$\forall\, \varphi \in D(\mathcal{A}_1) \quad |\varphi|_{H_1} \le c_{A_1} |A_1 \varphi|_{H_2} \quad \Leftrightarrow \quad \forall\, \Phi \in R_{\Gamma_t} \cap \varepsilon^{-1} \operatorname{rot} R_{\Gamma_n} \quad |\Phi|_{L^2_\varepsilon} \le c_m |\operatorname{rot} \Phi|_{L^2}$$

$$\forall\, \psi \in D(\mathcal{A}_1^*) \quad |\psi|_{H_2} \le c_{A_1} |A_1^* \psi|_{H_1} \quad \Leftrightarrow \quad \forall\, \Psi \in R_{\Gamma_n} \cap \operatorname{rot} R_{\Gamma_t} \quad |\Psi|_{L^2} \le c_m |\operatorname{rot} \Psi|_{L^2_\varepsilon}$$

(ii) all ranges $R(A_0) = \nabla H^1_{\Gamma_t}$, $R(A_1) = \operatorname{rot} R_{\Gamma_t}$, $R(A_0^*) = \operatorname{div} D_{\Gamma_n}$ are cl in L^2

(iii) the inverse ops $(\widetilde{\nabla}_{\Gamma_t})^{-1}$, $(\widetilde{\operatorname{div}_{\Gamma_n} \varepsilon})^{-1}$, $(\widetilde{\operatorname{rot}}_{\Gamma_t})^{-1}$, $(\widetilde{\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}})^{-1}$ are cont, even cpt

(iv) all Helmholtz decomposition hold, e.g.,

$$H_1 = R(A_0) \oplus K_1 \oplus R(A_1^*) \quad \Leftrightarrow \quad L^2_\varepsilon = \nabla H^1_{\Gamma_t} \oplus_{L^2_\varepsilon} \mathcal{H}_{D,\varepsilon} \oplus_{L^2_\varepsilon} \varepsilon^{-1} \operatorname{rot} R_{\Gamma_n}$$

(v) solution theory

(vi) variational formulations

(vii) functional a posteriori error estimates

(viii) div-curl-lemma

(ix) ...

classical de Rham complex in 3D (∇-rot-div-complex)

$$\{0\} \text{ or } \mathbb{R} \;\underset{\pi}{\overset{\iota}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\operatorname{div}_{\Gamma_n}\varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2_\varepsilon \;\underset{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota}{\overset{\pi}{\rightleftarrows}}\; \mathbb{R} \text{ or } \{0\}$$

variational formulations

var space $\quad (\tilde{E}, u, r, H) \in \boxed{\mathsf{R}_{\Gamma_t} \times \mathsf{H}^1_{\Gamma_t} \times \{0\}/\mathbb{R} \times \mathcal{H}_{\mathtt{D},\varepsilon}}$

$$\begin{bmatrix} \mu\operatorname{rot}_{\Gamma_n}\operatorname{rot}_{\Gamma_t} & \operatorname{grad}_{\Gamma_t} & 0 & \iota_{\mathcal{H}_{\mathtt{D},\varepsilon}} \\ -\operatorname{div}_{\Gamma_n}\varepsilon & 0 & \iota_{\{0\}/\mathbb{R}} & 0 \\ 0 & \pi_{\{0\}/\mathbb{R}} & 0 & 0 \\ \pi_{\mathcal{H}_{\mathtt{D},\varepsilon}} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{E} \\ u \\ r \\ H \end{bmatrix} = \begin{bmatrix} \mu\operatorname{rot}_{\Gamma_n} F \\ g \\ 0 \\ K \end{bmatrix}$$

var space $\quad (\hat{E}, U, v, r, H, \tilde{H}) \in \boxed{\varepsilon^{-1}\mathsf{D}_{\Gamma_n} \times \mathsf{R}_{\Gamma_n} \times \mathsf{H}^1_{\Gamma_n} \times \{0\}/\mathbb{R} \times \mathcal{H}_{\mathtt{D},\varepsilon} \times \mathcal{H}_{\mathtt{N}}}$

$$\begin{bmatrix} -\operatorname{grad}_{\Gamma_t}\operatorname{div}_{\Gamma_n}\varepsilon & \mu\operatorname{rot}_{\Gamma_n} & 0 & 0 & \iota_{\mathcal{H}_{\mathtt{D},\varepsilon}} & 0 \\ \operatorname{rot}_{\Gamma_t} & 0 & -\operatorname{grad}_{\Gamma_n} & 0 & 0 & \iota_{\mathcal{H}_{\mathtt{N}}} \\ 0 & \operatorname{div}_{\Gamma_t} & 0 & \iota_{\{0\}/\mathbb{R}} & 0 & 0 \\ 0 & 0 & \pi_{\{0\}/\mathbb{R}} & 0 & 0 & 0 \\ \pi_{\mathcal{H}_{\mathtt{D},\varepsilon}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi_{\mathcal{H}_{\mathtt{N}}} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{E} \\ U \\ v \\ r \\ H \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} \operatorname{grad}_{\Gamma_t} g \\ F \\ 0 \\ 0 \\ K \\ 0 \end{bmatrix}$$

note $\quad u = v = 0, \quad r = 0, \quad U = H = \tilde{H} = 0, \quad \boxed{\tilde{E} = \hat{E} = E}$

classical de Rham complex in 3D (∇-rot-div-complex)

Theorem (sharp upper bounds)

Let $\boxed{\tilde{E} \in \mathsf{L}^2_\varepsilon}$ *(very non-conforming approximation of* E*!) and* $\boxed{e := E - \tilde{E}}$. *Then*

$$|e|^2_{\mathsf{L}^2_\varepsilon} = |\pi_{R(\nabla_{\Gamma_t})} e|^2_{\mathsf{L}^2_\varepsilon} + |\pi_{R(\varepsilon^{-1}\operatorname{rot}_{\Gamma_n})} e|^2_{\mathsf{L}^2_\varepsilon} + |\pi_{\mathcal{H}_{\mathsf{D},\varepsilon}} e|^2_{\mathsf{L}^2_\varepsilon}$$

$$= \min_{\Phi \in \varepsilon^{-1}\mathsf{D}_{\Gamma_n}} \big(c_{\mathrm{fp}} |\operatorname{div} \varepsilon\Phi + g|_{\mathsf{L}^2} + |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon} \big)^2$$

$\boxed{\textit{reg } (-\nabla_{\Gamma_t} \operatorname{div}_{\Gamma_n} + 1)\textit{-prbl in } \mathsf{D}_{\Gamma_n}}$

$$+ \min_{\Phi \in \mathsf{R}_{\Gamma_t}} \big(c_{\mathrm{m}} |\operatorname{rot} \Phi - F|_{\mathsf{L}^2} + |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon} \big)^2$$

$\boxed{\textit{reg } (\operatorname{rot}_{\Gamma_n} \operatorname{rot}_{\Gamma_t} + 1)\textit{-prbl in } \mathsf{R}_{\Gamma_t}}$

$$+ \min_{\phi \in \mathsf{H}^1_{\Gamma_t}, \Psi \in \mathsf{R}_{\Gamma_n}} |\nabla\phi + \varepsilon^{-1} \operatorname{rot} \Psi + \tilde{E} - K|^2_{\mathsf{L}^2_\varepsilon}$$

$\boxed{\textit{cpld } (-\operatorname{div}_{\Gamma_n} \nabla_{\Gamma_t})\text{-}(\operatorname{rot}_{\Gamma_t} \operatorname{rot}_{\Gamma_n})\textit{-sys in } \mathsf{H}^1_{\Gamma_t}\text{-}\mathsf{R}_{\Gamma_n}}$

Remark

- $(\operatorname{rot}_{\Gamma_t} \operatorname{rot}_{\Gamma_n})$*-prbl needs saddle point formulation*
- Ω *top trv* $\Rightarrow \pi_{\mathsf{D}} = 0$ *and* $\mathsf{R}_{\Gamma_t,0} = \nabla \mathsf{H}^1_{\Gamma_t}$ *and* $\mathsf{D}_{\Gamma_n,0} = \operatorname{rot} \mathsf{R}_{\Gamma_n}$
- $\boxed{\Omega \textit{ convex and } \varepsilon = \mu = 1 \textit{ and } \Gamma_t = \Gamma \textit{ or } \Gamma_n = \Gamma \Rightarrow c_{\mathrm{f}} \le c_{\mathrm{m}} \le c_{\mathrm{p}} \le \frac{\operatorname{diam}_\Omega}{\pi}}$

classical de Rham complex in 3D (∇-rot-div-complex)

Theorem (sharp lower bounds)

Let $\boxed{\tilde{E} \in \mathsf{L}^2_\varepsilon}$ *(very non-conforming approximation of E!) and* $\boxed{e := E - \tilde{E}}$. *Then*

$$|e|^2_{\mathsf{L}^2_\varepsilon} = |\pi_{R(\nabla_{\Gamma_t})} e|^2_{\mathsf{L}^2_\varepsilon} + |\pi_{R(\varepsilon^{-1}\operatorname{rot}_{\Gamma_n})} e|^2_{\mathsf{L}^2_\varepsilon} + |\pi_{\mathcal{H}_{\mathrm{D},\varepsilon}} e|^2_{\mathsf{L}^2_\varepsilon}$$

$$= \max_{\varphi \in \mathsf{H}^1_{\Gamma_t}} \big(2\langle g, \varphi\rangle_{\mathsf{L}^2} - \langle 2\tilde{E} + \operatorname{grad}\varphi, \varepsilon \operatorname{grad}\varphi\rangle_{\mathsf{L}^2}\big)$$

reg $(-\nabla_{\Gamma_t} \operatorname{div}_{\Gamma_n} + 1)$*-prbl in* D_{Γ_n}

$$+ \max_{\Psi \in \mathsf{R}_{\Gamma_n}} \big(2\langle F, \Psi\rangle_{\mathsf{L}^2} - \langle 2\tilde{E} + \mu \operatorname{rot}\Psi, \operatorname{rot}\Psi\rangle_{\mathsf{L}^2}\big)$$

reg $(\operatorname{rot}_{\Gamma_n} \operatorname{rot}_{\Gamma_t} + 1)$*-prbl in* R_{Γ_t}

$$+ \max_{\Psi \in \mathcal{H}_{\mathrm{D},\varepsilon}} \langle 2(K - \tilde{E}) - \Psi, \Psi\rangle_{\mathsf{L}^2_\varepsilon}$$

div-curl-lemma

Lemma (div-curl-lemma (global version))

Assumptions:

(i) (E_n) *bounded in* $\mathsf{L}^2(\Omega)$

(i') (H_n) *bounded in* $\mathsf{L}^2(\Omega)$

(ii) $(\operatorname{rot} E_n)$ *bounded in* $\mathsf{L}^2(\Omega)$

(ii') $(\operatorname{div} \varepsilon H_n)$ *bounded in* $\mathsf{L}^2(\Omega)$

(iii) $\nu \times E_n = 0$ *on* Γ_t, *i.e.,* $E_n \in \mathsf{R}_{\Gamma_t}(\Omega)$

(iii') $\nu \cdot \varepsilon H_n = 0$ *on* Γ_n, *i.e.,* $H_n \in \varepsilon^{-1}\mathsf{D}_{\Gamma_n}(\Omega)$

$\Rightarrow \exists\ E, H$ *and subsequences st*

$E_n \rightharpoonup E$, $\operatorname{rot} E_n \rightharpoonup \operatorname{rot} E$ *and* $H_n \rightharpoonup H$, $\operatorname{div} H_n \rightharpoonup \operatorname{div} H$ *in* $\mathsf{L}^2(\Omega)$ *and*

$$\langle E_n, H_n \rangle_{\mathsf{L}^2_\varepsilon(\Omega)} \to \langle E, H \rangle_{\mathsf{L}^2_\varepsilon(\Omega)}$$

de Rham complex in ND or on Riemannian manifolds (d-complex)

$\Omega \subset \mathbb{R}^N$ bd w. Lip. dom. or Ω Riemannian manifold with cpt cl. and Lip. boundary Γ

(generalized Maxwell equations)

$$\{0\} \overset{\iota_{\{0\}}}{\underset{\pi_{\{0\}}}{\rightleftarrows}} \mathsf{L}^{2,0} \overset{\mathring{\mathrm{d}}}{\underset{-\delta}{\rightleftarrows}} \mathsf{L}^{2,1} \overset{\mathring{\mathrm{d}}}{\underset{-\delta}{\rightleftarrows}} \cdots \mathsf{L}^{2,q} \overset{\mathring{\mathrm{d}}}{\underset{-\delta}{\rightleftarrows}} \mathsf{L}^{2,q+1} \ldots \mathsf{L}^{2,N-1} \overset{\mathring{\mathrm{d}}}{\underset{-\delta}{\rightleftarrows}} \mathsf{L}^{2,N} \overset{\pi_{\mathbb{R}}}{\underset{\iota_{\mathbb{R}}}{\rightleftarrows}} \mathbb{R}$$

de Rham complex in ND or on Riemannian manifolds (d-complex)

$\Omega \subset \mathbb{R}^N$ bd w. Lip. dom. or Ω Riemannian manifold with cpt cl. and Lip. boundary Γ

(generalized Maxwell equations)

$$\{0\} \text{ or } \mathbb{R} \overset{\iota}{\underset{\pi}{\rightleftarrows}} \mathsf{L}^{2,0} \overset{\mathrm{d}^0_{\Gamma_t}}{\underset{-\delta^1_{\Gamma_n}}{\rightleftarrows}} \mathsf{L}^{2,1} \overset{\mathrm{d}^1_{\Gamma_t}}{\underset{-\delta^2_{\Gamma_n}}{\rightleftarrows}} \cdots \mathsf{L}^{2,q} \overset{\mathrm{d}^q_{\Gamma_t}}{\underset{-\delta^{q+1}_{\Gamma_n}}{\rightleftarrows}} \mathsf{L}^{2,q+1} \cdots \mathsf{L}^{2,N-1} \overset{\mathrm{d}^{N-1}_{\Gamma_t}}{\underset{-\delta^N_{\Gamma_n}}{\rightleftarrows}} \mathsf{L}^{2,N} \overset{\pi}{\underset{\iota}{\rightleftarrows}} \mathbb{R} \text{ or } \{0\}$$

related fos

$$\mathrm{d}^q_{\Gamma_t} E = F \qquad \text{in } \Omega$$

$$-\delta^q_{\Gamma_n} E = G \qquad \text{in } \Omega$$

related sos

$$-\delta^{q+1}_{\Gamma_n} \mathrm{d}^q_{\Gamma_t} E = F \qquad \text{in } \Omega$$

$$-\delta^q_{\Gamma_n} E = G \qquad \text{in } \Omega$$

includes: EMS rot / div, Laplacian, rot rot, and more...
corresponding compact embeddings:

$$D(\mathrm{d}^q_{\Gamma_t}) \cap D(\delta^q_{\Gamma_n}) \hookrightarrow \mathsf{L}^{2,q} \qquad \text{(Weck's selection theorems, '74)}$$

Weck's selection theorem for Lip. manifolds and mixed bc: Bauer/Py/Schomburg ('17)

elasticity complex in 3D (sym ∇-Rot $\text{Rot}_{\mathbb{S}}^{\top}$-$\text{Div}_{\mathbb{S}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \quad \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{-\,\text{Div}_{\mathbb{S}}}{\overset{\mathring{\text{sym}}\,\nabla}{\rightleftarrows}} \quad \mathsf{L}^2_{\mathbb{S}} \quad \underset{\text{Rot}\,\text{Rot}_{\mathbb{S}}^{\top}}{\overset{\text{Rot}\,\mathring{\text{Rot}}_{\mathbb{S}}^{\top}}{\rightleftarrows}} \quad \mathsf{L}^2_{\mathbb{S}} \quad \underset{-\,\text{sym}\,\nabla}{\overset{\mathring{\text{Div}}_{\mathbb{S}}}{\rightleftarrows}} \quad \mathsf{L}^2 \quad \underset{\iota_{\text{RM}}}{\overset{\pi_{\text{RM}}}{\rightleftarrows}} \quad \text{RM}$$

elasticity complex in 3D (sym ∇-Rot $\text{Rot}_{\mathbb{S}}^{\top}$-$\text{Div}_{\mathbb{S}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\text{Div}_{\mathbb{S}}}{\overset{\mathring{\text{sym}}\,\nabla}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\text{Rot}\,\text{Rot}_{\mathbb{S}}^{\top}}{\overset{\text{Rot}\,\mathring{\text{Rot}}_{\mathbb{S}}^{\top}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{-\text{sym}\,\nabla}{\overset{\mathring{\text{Div}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathsf{RM}}}{\overset{\pi_{\mathsf{RM}}}{\rightleftarrows}} \mathsf{RM}$$

related fos ($\text{Rot}\,\text{Rot}_{\mathbb{S},\Gamma}^{\top}$, $\text{Rot}\,\text{Rot}_{\mathbb{S}}^{\top}$ first order operators!)

$\text{sym}\,\nabla_{\Gamma} v = M$ in Ω	$\text{Rot}\,\text{Rot}_{\mathbb{S},\Gamma}^{\top} M = F$ in Ω	$\text{Div}_{\mathbb{S},\Gamma} N = g$ in Ω	$\pi v = r$ in Ω
$\pi v = 0$ in Ω	$-\text{Div}_{\mathbb{S}} M = f$ in Ω	$\text{Rot}\,\text{Rot}_{\mathbb{S}}^{\top} N = G$ in Ω	$-\text{sym}\,\nabla v = M$ in Ω

related sos ($\text{Rot}\,\text{Rot}_{\mathbb{S}}^{\top}\,\text{Rot}\,\text{Rot}_{\mathbb{S},\Gamma}^{\top}$ second order operator!)

$-\text{Div}_{\mathbb{S}}\,\text{sym}\,\nabla_{\Gamma} v = f$ in Ω	$\text{Rot}\,\text{Rot}_{\mathbb{S}}^{\top}\,\text{Rot}\,\text{Rot}_{\mathbb{S},\Gamma}^{\top} M = G$ in Ω	$-\text{sym}\,\nabla\,\text{Div}_{\mathbb{S},\Gamma} N = M$ in Ω
$\pi v = 0$ in Ω	$-\text{Div}_{\mathbb{S}} M = f$ in Ω	$\text{Rot}\,\text{Rot}_{\mathbb{S}}^{\top} N = G$ in Ω

corresponding compact embeddings:

$$D(\text{sym}\,\nabla_{\Gamma}) \cap D(\pi) = D(\nabla_{\Gamma}) = \mathsf{H}^1_{\Gamma} \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem and Korn ineq.)}$$

$$D(\text{Rot}\,\text{Rot}_{\mathbb{S},\Gamma}^{\top}) \cap D(\text{Div}_{\mathbb{S}}) \hookrightarrow \mathsf{L}^2_{\mathbb{S}} \qquad \text{(new selection theorem)}$$

$$D(\text{Div}_{\mathbb{S},\Gamma}) \cap D(\text{Rot}\,\text{Rot}_{\mathbb{S}}^{\top}) \hookrightarrow \mathsf{L}^2_{\mathbb{S}} \qquad \text{(new selection theorem)}$$

$$D(\pi) \cap D(\text{sym}\,\nabla) = D(\nabla) = \mathsf{H}^1 \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem and Korn ineq.)}$$

two new selection theorems for strong Lip. dom.: Py/Schomburg/Zulehner ('18)

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\mathrm{Rot}_{\mathbb{S}}$-$\mathrm{Div}_{\mathbb{T}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \; \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \; \mathsf{L}^2 \; \underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\nabla\nabla}}{\rightleftarrows}} \; \mathsf{L}^2_{\mathbb{S}} \; \underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \; \mathsf{L}^2_{\mathbb{T}} \; \underset{-\operatorname{dev}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}} \; \mathsf{L}^2 \; \underset{\iota_{\mathsf{RT}}}{\overset{\pi_{\mathsf{RT}}}{\rightleftarrows}} \; \mathsf{RT}$$

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\mathrm{Rot}_{\mathbb{S}}$-$\mathrm{Div}_{\mathbb{T}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\nabla}\nabla}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}} \underset{-\operatorname{dev}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathsf{RT}}}{\overset{\pi_{\mathsf{RT}}}{\rightleftarrows}} \mathsf{RT}$$

related fos ($\nabla\nabla_\Gamma$, $\operatorname{div}\operatorname{Div}_{\mathbb{S}}$ first order operators!)

$$\begin{aligned} \nabla\nabla_\Gamma u &= M \quad \text{in } \Omega \quad | \quad & \operatorname{Rot}_{\mathbb{S},\Gamma} M &= F \quad \text{in } \Omega \quad | \quad & \operatorname{Div}_{\mathbb{T},\Gamma} N &= g \quad \text{in } \Omega \quad | \quad & \pi v &= r \quad \text{in } \Omega \\ \pi u &= 0 \quad \text{in } \Omega \quad | \quad & \operatorname{div}\operatorname{Div}_{\mathbb{S}} M &= f \quad \text{in } \Omega \quad | \quad & \operatorname{sym}\operatorname{Rot}_{\mathbb{T}} N &= G \quad \text{in } \Omega \quad | \quad & -\operatorname{dev}\nabla v &= T \quad \text{in } \Omega \end{aligned}$$

related sos ($\operatorname{div}\operatorname{Div}_{\mathbb{S}}\nabla\nabla_\Gamma = \Delta^2_\Gamma$ second order operator!)

$$\begin{aligned} \operatorname{div}\operatorname{Div}_{\mathbb{S}}\nabla\nabla_\Gamma u = \Delta^2_\Gamma u &= f \quad \text{in } \Omega \quad | \quad & \operatorname{sym}\operatorname{Rot}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma} M &= G \quad \text{in } \Omega \quad | \quad & -\operatorname{dev}\nabla\operatorname{Div}_{\mathbb{T},\Gamma} N &= T \quad \text{in } \Omega \\ \pi u &= 0 \quad \text{in } \Omega \quad | \quad & \operatorname{div}\operatorname{Div}_{\mathbb{S}} M &= f \quad \text{in } \Omega \quad | \quad & \operatorname{sym}\operatorname{Rot}_{\mathbb{T}} N &= G \quad \text{in } \Omega \end{aligned}$$

corresponding compact embeddings:

$$D(\nabla\nabla_\Gamma) \cap D(\pi) = D(\nabla\nabla_\Gamma) = \mathsf{H}^2_\Gamma \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

$$D(\operatorname{Rot}_{\mathbb{S},\Gamma}) \cap D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) \hookrightarrow \mathsf{L}^2_{\mathbb{S}} \qquad \text{(new selection theorem)}$$

$$D(\operatorname{Div}_{\mathbb{T},\Gamma}) \cap D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) \hookrightarrow \mathsf{L}^2_{\mathbb{T}} \qquad \text{(new selection theorem)}$$

$$D(\pi) \cap D(\operatorname{dev}\nabla) = D(\operatorname{dev}\nabla) = D(\nabla) = \mathsf{H}^1 \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem and Korn type ineq.)}$$

two new selection theorems for strong Lip. dom. and Korn Type ineq.: Py/Zulehner ('16)

literature (fa-toolbox, complexes, a posteriori error estimates, ...)

results of this talk:

- Py: *Solution Theory and Functional A Posteriori Error Estimates for General First Order Systems with Applications to Electro-Magneto-Statics*, (NFAO) Numerical Functional Analysis and Optimization, 2018

(paper contains main results of this talk)

literature (a posteriori error estimates of functional type)

- Repin, S.: *A posteriori error estimates for variational problems with uniformly convex functionals*,
  (MC) Mathematics of Computation, 2000
- Neittaanmäki, P., Repin, S.: *Reliable methods for computer simulation, error control and a posteriori estimates*,
  Elsevier, 2004
- Repin, S.: *A posteriori estimates for partial differential equations*,
  Radon Series on Applied Mathematics, De Gruyter, 2008
- Py, Repin, S.: *Functional A Posteriori Error Estimates for Elliptic Problems in Exterior Domains*,
  (PMA) Problemy Matematicheskogo Analiza/ (JMS)Journal of Mathematical Sciences (Springer New York), 2009
- Py, Repin, S.: *Two-sided a posteriori error bounds for electro-magneto static problems*,
  Zapiski POMI/ (JMS)Journal of Mathematical Sciences (Springer New York), 2009
- Mali, O., Neittaanmäki, P., Repin, S.: *Accuracy verification methods, theory and algorithms*,
  Springer, 2014

literature (a posteriori error equations)

- Anjam, I., Py: *Functional a posteriori error control for conforming mixed approximations of coercive problems with lower order terms*, (CMAM) Computational Methods in Applied Mathematics, 2016
- Anjam, I., Py: *An Elementary Method of Deriving A Posteriori Error Equalities and Estimates for Linear Partial Differential Equations*, (CMAM) Computational Methods in Applied Mathematics, 2018

literature (complexes, Friedrichs type constants, Maxwell constants)

results of this talk:

- Py: *On Constants in Maxwell Inequalities for Bounded and Convex Domains*, Zapiski POMI/ (JMS)Journal of Mathematical Sciences (Springer New York), 2015
- Py: *On Maxwell's and Poincare's Constants*, (DCDS) Discrete and Continuous Dynamical Systems - Series S, 2015
- Py: *On the Maxwell Constants in 3D*, (M2AS) Mathematical Methods in the Applied Sciences, 2017
- Py: *On the Maxwell and Friedrichs/Poincare Constants in ND*, submitted, 2017

literature (complexes, Friedrichs type constants, compact embeddings)

- Weck, N.: *Maxwell's boundary value problems on Riemannian manifolds with nonsmooth boundaries*,
  (JMA2) Journal of Mathematical Analysis and Applications, 1974 (1972)
- Picard, R.: *An elementary proof for a compact imbedding result in generalized electromagnetic theory*,
  (MZ) Mathematische Zeitschrift, 1984
- Witsch, K.-J.: *A remark on a compactness result in electromagnetic theory*,
  (M2AS) Mathematical Methods in the Applied Sciences, 1993

results of this talk:

- Bauer, S., Py, Schomburg, M.: *The Maxwell Compactness Property in Bounded Weak Lipschitz Domains with Mixed Boundary Conditions*,
  (SIMA) SIAM Journal on Mathematical Analysis, 2016
- Zulehner, W., Py: *On Closed and Exact Grad grad- and div Div-Complexes, Corresponding Compact Embeddings for Tensor Rotations, and a Related Decomposition Result for Biharmonic Problems in 3D*,
  submitted, 2016
- Py, Schomburg, M., Zulehner, W.: *Compact Embeddings, Friedrichs/Poincaré Type Estimates, Helmholtz Type Decompositions, and a General Toolbox for the Elasticity Complex in 3D*,
  in preparation, 2018

literature (div-curl-lemma)

original papers (local div-curl-lemma):

- Murat, F.: *Compacité par compensation*,
  Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 1978
- Tartar, L.: *Compensated compactness and applications to partial differential equations*,
  Nonlinear analysis and mechanics, Heriot-Watt symposium, 1979

literature (div-curl-lemma)

recent papers (global div-curl-lemma, H^1-detour):

- Gloria, A., Neukamm, S., Otto, F.: *Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics*, (IM) Invent. Math., 2015
- Kozono, H., Yanagisawa, T.: *Global compensated compactness theorem for general differential operators of first order*, (ARMA) Arch. Ration. Mech. Anal., 2013
- Schweizer, B.: *On Friedrichs inequality, Helmholtz decomposition, vector potentials, and the div-curl lemma*, accepted preprint, 2018

recent papers (global div-curl-lemma, general results/this talk):

- Waurick, M.: *A Functional Analytic Perspective to the div-curl Lemma*, (JOP) J. Operator Theory, 2018
- Py: *A Global div-curl-Lemma for Mixed Boundary Conditions in Weak Lipschitz Domains and a Corresponding Generalized* A_0^*-A_1*-Lemma in Hilbert Spaces*, (ANA) Analysis (Munich), 2018

literature (complexes, optimization, and a posteriori error estimates for degenerated magneto statics)

- Py, Yousept, I.: *A Posteriori Error Analysis for the Optimal Control of Magneto-Static Fields*,
  (M2NA) ESAIM: Mathematical Modelling and Numerical Analysis, 2017

literature (full time-dependent Maxwell equations)

- Py, Picard, R.: *A Note on the Justification of the Eddy Current Model in Electrodynamics*,
  (M2AS) Mathematical Methods in the Applied Sciences, 2017

literature (Maxwell's equations and more...)

upcoming books:

- Langer, U., Py, Repin, S. (Eds): *Maxwell's equations. Analysis and numerics*, Radon Series on Applied Mathematics, De Gruyter, 2018
- Py: *Maxwell's Equations: Hilbert Space Methods for the Theory of Electromagnetism*, Radon Series on Applied Mathematics, De Gruyter, 2020

(last book: contains all results of this talk and more...)

. . . the world is full of complexes . . . ;)

⇒ relaxing at

AANMPDE 11

11th Workshop on Analysis and Advanced Numerical Methods for Partial Differential Equations (not only) for Junior Scientists

`http://www.mit.jyu.fi/scoma/AANMPDE11`

August 6–10 2018, Särkisaari, Finland

organizers: Ulrich Langer, Dirk Pauly, Sergey Repin