

Old and New Results for Hilbert Complexes and (Linear) First Order Systems

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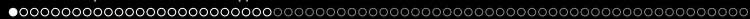
Open-Minded :-)

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linear equation

general observations

$$A x = f$$



general observations

$$Ax = f$$

general theory

- solution theory
- Friedrichs/Poincaré estimates and constants
- Helmholtz/Hodge/Weyl decompositions
- compact embeddings
- continuous and compact inverse operators
- closed ranges
- variational formulations
- functional a posteriori error estimates
- generalized div-curl-lemma
- ...

idea: solve problem with general and simple linear functional analysis

⇒ functional analysis toolbox (fa-toolbox) ...



general observations

$A : D(A) \subset H_0 \rightarrow H_1$ linear, H_0, H_1 Hilbert spaces (for simplicity)

solve and provide tools

$$Ax = f$$

examples

- $A \in \mathbb{R}^{m \times n}$ (matrix eq)
- $A = \partial_t - \Delta$ (so heat/diffusion eq)
- $A = \partial_t^2 - \Delta$ (so wave eq)
- $A = -\Delta - \omega^2$ (so red wave/Helmholtz eq)
- $A = -\Delta$ (so Laplace eq)

right hand sides and solutions (typically)

- $f \in \mathbb{R}^m$ $x \in \mathbb{R}^n$
- $f \in L^2(I \times \Omega)$ $x \in L^2(I) \times \dot{H}^1(\Omega)$
- $f \in L^2(\Omega)$ $x \in \dot{H}^1(\Omega)$

or in (closed) subspaces $R(A)$

(here $\dot{H}^1(\Omega) = H_0^1(\Omega)$)



general observations

$A : D(A) \subset H_0 \rightarrow H_1$ linear, H_0, H_1 Hilbert spaces (for simplicity)

solve and provide tools

$$Ax = f$$

examples

- $A = \partial_t^2 - \overset{\circ}{\Delta} = \partial_t^2 - \text{div } \overset{\circ}{\nabla}$ (so wave eq)
- $A = \partial_t - \begin{bmatrix} 0 & \text{div} \\ \overset{\circ}{\nabla} & 0 \end{bmatrix}$ (fo wave eq, pref form, acoustics)
- $A = \begin{bmatrix} 0 & \text{div} \\ \overset{\circ}{\nabla} & 0 \end{bmatrix} - \omega$ (fo red wave/Helmholtz eq, time-harm acoustics)

right hand sides and solutions (typically)

- $f \in L^2(I \times \Omega)$ $x \in L^2(I) \times \overset{\circ}{H}^1(\Omega)$
- $f \in L^2(I \times \Omega) \times L^2(I \times \Omega)$ $x \in (L^2(I) \times \overset{\circ}{H}^1(\Omega)) \times (L^2(I) \times D(\Omega))$
- $f \in L^2(\Omega) \times L^2(\Omega)$ $x \in \overset{\circ}{H}^1(\Omega) \times D(\Omega)$

or in (closed) subspaces $R(A)$

(here $D(\Omega) = H(\text{div}, \Omega)$)



general observations

$A : D(A) \subset H_0 \rightarrow H_1$ linear, H_0, H_1 Hilbert spaces (for simplicity)

solve and provide tools

$$Ax = f$$

examples

- $A = \partial_t^2 + \text{rot } \dot{\text{rot}}$ (so Maxwell/wave eq)
- $A = \partial_t - \begin{bmatrix} 0 & -\text{rot} \\ \dot{\text{rot}} & 0 \end{bmatrix}$ (fo Maxwell/wave eq, pref form)
- $A = \begin{bmatrix} 0 & -\text{rot} \\ \dot{\text{rot}} & 0 \end{bmatrix} - \omega$ (fo time-harm Maxwell eq)
- $A = \begin{bmatrix} 0 & -\text{rot} \\ \dot{\text{rot}} & 0 \end{bmatrix}$ (fo stat Maxwell eq) \Rightarrow $\dot{\text{rot}} - \text{div} / \text{rot} - \text{div}$ sys
- $A = \partial_t + \text{rot } \dot{\text{rot}}$ (so eddy current Maxwell eq)
- $A = \text{rot } \dot{\text{rot}} - \omega$ (so time-harm eddy current Maxwell eq)

right hand sides and solutions (typically)

- $f \in L^2(I \times \Omega)$ $x \in L^2(I) \times \dot{R}(\Omega)$
- $f \in L^2(I \times \Omega) \times L^2(I \times \Omega)$ $x \in (L^2(I) \times \dot{R}(\Omega)) \times (L^2(I) \times R(\Omega))$
- $f \in L^2(\Omega) \times L^2(\Omega)$ $x \in \dot{R}(\Omega) \times R(\Omega)$

or in (closed) subspaces $R(A)$ (here $R(\Omega) = H(\text{rot}, \Omega)$ $\dot{R}(\Omega) = H_0(\text{rot}, \Omega)$)



general observations

so far all equations form the classical de Rham complex in 3D (∇ -rot-div-complex)

($\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain)

electro-magneto dynamics/time-harm/statics, Maxwell's equations, acoustics

$$\{0\} \begin{array}{c} \xleftarrow{\iota_{\{0\}}} \\ \xrightarrow{\pi_{\{0\}}} \end{array} L^2(\Omega) \begin{array}{c} \xleftarrow{\dot{\nabla}} \\ \xrightarrow{-\operatorname{div}} \end{array} L^2(\Omega) \begin{array}{c} \xleftarrow{\operatorname{rot}} \\ \xrightarrow{\operatorname{rot}} \end{array} L^2(\Omega) \begin{array}{c} \xleftarrow{\operatorname{div}} \\ \xrightarrow{-\nabla} \end{array} L^2(\Omega) \begin{array}{c} \xleftarrow{\pi_{\mathbb{R}}} \\ \xrightarrow{\iota_{\mathbb{R}}} \end{array} \mathbb{R}$$

complex: $\operatorname{rot} \nabla = 0$ $\operatorname{div} \operatorname{rot} = 0$



general observations

other possible complexes:

elasticity complex in 3D (sym ∇ -Rot Rot $_{\mathbb{S}}^T$ -Div $_{\mathbb{S}}$ -complex)

($\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain)

elasticity, Rot Rot $^{\perp}$ Rot Rot $^{\perp}$ eq

$$\{0\} \begin{array}{c} \overset{\mathcal{L}\{0\}}{\rightleftarrows} \\ \underset{\pi\{0\}}{\rightleftarrows} \end{array} L^2(\Omega) \begin{array}{c} \overset{\text{sym } \nabla}{\rightleftarrows} \\ \underset{-\text{Div}_{\mathbb{S}}}{\rightleftarrows} \end{array} L^2_{\mathbb{S}}(\Omega) \begin{array}{c} \overset{\text{Rot Rot}_{\mathbb{S}}^T}{\rightleftarrows} \\ \underset{\text{Rot Rot}_{\mathbb{S}}^T}{\rightleftarrows} \end{array} L^2_{\mathbb{S}}(\Omega) \begin{array}{c} \overset{\text{Div}_{\mathbb{S}}}{\rightleftarrows} \\ \underset{-\text{sym } \nabla}{\rightleftarrows} \end{array} L^2(\Omega) \begin{array}{c} \overset{\pi_{\text{RM}}}{\rightleftarrows} \\ \underset{\mathcal{L}_{\text{RM}}}{\rightleftarrows} \end{array} \text{RM}$$

complex: Rot Rot $_{\mathbb{S}}^T$ sym $\nabla = 0$ Div $_{\mathbb{S}}$ Rot Rot $_{\mathbb{S}}^T = 0$



general observations

other possible complexes:

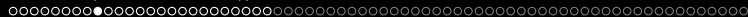
biharmonic / general relativity complex in 3D ($\nabla\nabla$ -Rot_S-Div_T-complex)

($\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain)

biharmonic / general relativity

$$\{0\} \begin{array}{c} \xleftrightarrow{\mathcal{L}_{\{0\}}} \\ \xleftarrow{\pi_{\{0\}}} \end{array} L^2(\Omega) \begin{array}{c} \xleftrightarrow{\nabla\nabla} \\ \xleftarrow{\operatorname{div} \operatorname{Div}_S} \end{array} L^2_S(\Omega) \begin{array}{c} \xleftrightarrow{\operatorname{Rot}_S} \\ \xleftarrow{\operatorname{sym} \operatorname{Rot}_T} \end{array} L^2_T(\Omega) \begin{array}{c} \xleftrightarrow{\operatorname{Div}_T} \\ \xleftarrow{-\operatorname{dev} \nabla} \end{array} L^2(\Omega) \begin{array}{c} \xleftrightarrow{\pi_{RT}} \\ \xleftarrow{\mathcal{L}_{RT}} \end{array} RT$$

complex: $\operatorname{Rot}_S \nabla\nabla = 0$ $\operatorname{Div}_T \operatorname{Rot}_S = 0$



general observations

H_0, H_1, H_2 Hilbert spaces (for simplicity)

$A_0 : D(A_0) \subset H_0 \rightarrow H_1$ lddc (lin, den def, cl)

$A_1 : D(A_1) \subset H_1 \rightarrow H_2$ lddc

$A_0^* : D(A_0^*) \subset H_1 \rightarrow H_0$ lddc (Hilbert space adjoint)

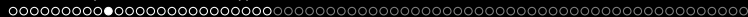
$A_1^* : D(A_1^*) \subset H_2 \rightarrow H_1$ lddc (Hilbert space adjoint)

general complex

$$\dots \begin{array}{c} \cdots \\ \rightleftarrows \\ \cdots \end{array} H_0 \begin{array}{c} A_0 \\ \rightleftarrows \\ A_0^* \end{array} H_1 \begin{array}{c} A_1 \\ \rightleftarrows \\ A_1^* \end{array} H_2 \begin{array}{c} \cdots \\ \rightleftarrows \\ \cdots \end{array} \dots$$

complex: $A_1 A_0 = 0 \quad (\Leftrightarrow \quad A_0^* A_1^* = 0)$

more precisely: $R(A_0) \subset N(A_1) \quad (\Leftrightarrow \quad R(A_1^*) \subset N(A_0^*))$



general observations

$$A_0 : D(A_0) \subset H_0 \rightarrow H_1, \quad A_1 : D(A_1) \subset H_1 \rightarrow H_2 \text{ lddc}$$

$$A_0^* : D(A_0^*) \subset H_1 \rightarrow H_0, \quad A_1^* : D(A_1^*) \subset H_2 \rightarrow H_1 \text{ lddc (Hilbert space adjoints)}$$

general complex ($A_1 A_0 = 0$)

$$\dots \begin{array}{c} \dots \\ \rightleftarrows \\ \dots \end{array} H_0 \begin{array}{c} A_0 \\ \rightleftarrows \\ A_0^* \end{array} H_1 \begin{array}{c} A_1 \\ \rightleftarrows \\ A_1^* \end{array} H_2 \begin{array}{c} \dots \\ \rightleftarrows \\ \dots \end{array} \dots$$

$$Ax = f$$

typical equations/systems

(stat fos)

$$A = A_1$$

(stat sos (sa))

$$A = A_1^* A_1$$

(stat fos (sa/ssa))

$$A = \begin{bmatrix} 0 & \pm A_1^* \\ A_1 & 0 \end{bmatrix}$$

(time-harm fos (sa/ssa))

$$A = \omega - \begin{bmatrix} 0 & \pm A_1^* \\ A_1 & 0 \end{bmatrix}$$

(diff sos)

$$A = \partial_t + A_1^* A_1$$

(wave sos)

$$A = \partial_t^2 + A_1^* A_1$$

(wave fos)

$$A = \partial_t - \begin{bmatrix} 0 & -A_1^* \\ A_1 & 0 \end{bmatrix}$$



general observations

$$Ax = f$$

let's say $A : D(A) \subset H_0 \rightarrow H_1$ linear and H_0, H_1 Hilbert spaces

question: How to solve?



general observations

$$Ax = f$$

$A : D(A) \subset H_0 \rightarrow H_1$ linear

solution theory in the sense of Hadamard

- existence $\Leftrightarrow f \in R(A)$
- uniqueness $\Leftrightarrow A$ inj $\Leftrightarrow N(A) = \{0\} \Leftrightarrow A^{-1}$ exists
- cont dep on $f \Leftrightarrow A^{-1}$ cont

$\Rightarrow x = A^{-1}f \in D(A)$ and cont estimate (Friedrichs/Poincaré type estimate)

$$|x|_{H_0} = |A^{-1}f|_{H_0} \leq c_A |f|_{H_1} = c_A |Ax|_{H_1}$$

\Rightarrow best constant $c_A = |A^{-1}|_{R(A), H_0} \quad |A^{-1}|_{R(A), D(A)} = (c_A^2 + 1)^{1/2}$



general observations

$$A : D(A) \subset H_0 \rightarrow H_1$$

$$A^* : D(A^*) \subset H_1 \rightarrow H_0 \text{ Hilbert space adjoint}$$

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$H_1 = \overline{R(A)} \oplus N(A^*) \quad H_0 = N(A) \oplus \overline{R(A^*)}$$

$$Ax = f$$

solution theory in the sense of Hadamard

- existence $\Leftrightarrow f \in R(A) = N(A^*)^\perp$
- uniqueness $\Leftrightarrow A \text{ inj} \Leftrightarrow N(A) = \{0\} \Leftrightarrow A^{-1} \text{ exists}$
- cont dep on $f \Leftrightarrow A^{-1} \text{ cont} \Leftrightarrow R(A) \text{ cl} \quad (\text{cl range theo})$

fund range cond: $R(A) = \overline{R(A)}$ closed (must hold \rightsquigarrow right setting!)

kernel cond: $N(A) = \{0\}$ (fails in gen \rightsquigarrow proj onto $N(A)^\perp = \overline{R(A^*)}$)



general observations

Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$H_1 = \overline{R(A)} \oplus N(A^*) \quad H_0 = N(A) \oplus \overline{R(A^*)}$$

remarkable observations

- time-dependent problems are simple
in gen $A : D(A) \subset H \rightarrow H$, $A = \partial_t + T$ (gen T skew-sa, or alt l_{ast} $\text{Re } T \geq 0$)

$$N(A) = \{0\} \quad N(A^*) = \{0\} \quad R(A) \text{ (cl)} = N(A^*)^\perp = H$$

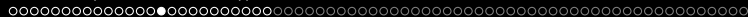
- time-harmonic problems are more complicated
in gen $A : D(A) \subset H \rightarrow H$, $A = -\omega + T$

$$N(A), N(A^*) \text{ (fin dim)} \quad R(A) \text{ (cl, fin co-dim)} = N(A^*)^\perp$$

(Fredholm alternative)

- stat problems are most complicated
in gen $A : D(A) \subset H_0 \rightarrow H_1$

$$\dim N(A) = \dim N(A^*) = \infty \text{ (possibly)} \quad R(A) \text{ (cl, infin co-dim)} = N(A^*)^\perp$$



fa-toolbox for linear (first order) problems/systems

$$Ax = f$$

general theory

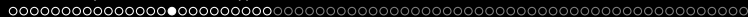
- solution theory
- Friedrichs/Poincaré estimates and constants
- Helmholtz/Hodge/Weyl decompositions
- compact embeddings
- continuous and compact inverse operators
- closed ranges
- variational formulations
- functional a posteriori error estimates
- generalized div-curl-lemma
- ...

idea: solve problem with general and simple linear functional analysis
(\Rightarrow fa-toolbox) ...

literature: probably very well known for ages, but hard to find ...

Friedrichs, Weyl, Hörmander, Fredholm, von Neumann, Riesz, Banach, ... ?

Why not rediscover and extend/modify for our purposes?



1st fundamental observations

$A : D(A) \subset H_0 \rightarrow H_1$ lddc, $A^* : D(A^*) \subset H_1 \rightarrow H_0$ Hilbert space adjoint

(A, A^*) dual pair as $(A^*)^* = \overline{A} = A$

A, A^* may not be inj

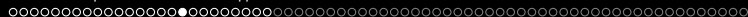
Helmholtz/Hodge/Weyl decompositions (projection theorem)

$$H_1 = N(A^*) \oplus \overline{R(A)} \quad H_0 = N(A) \oplus \overline{R(A^*)}$$

reduced operators restr to $N(A)^\perp$ and $N(A^*)^\perp$

$$\mathcal{A} := A|_{N(A)^\perp} = A|_{\overline{R(A^*)}} \quad \mathcal{A}^* := A^*|_{N(A^*)^\perp} = A^*|_{\overline{R(A)}}$$

$\mathcal{A}, \mathcal{A}^*$ inj $\Rightarrow \mathcal{A}^{-1}, (\mathcal{A}^*)^{-1}$ ex



1st fundamental observations

$A : D(A) \subset H_0 \rightarrow H_1$, $A^* : D(A^*) \subset H_1 \rightarrow H_0$ lddc (A, A^*) dual pair

$$H_1 = N(A^*) \oplus \overline{R(A)} \quad H_0 = N(A) \oplus \overline{R(A^*)}$$

more precisely

$$\mathcal{A} := A|_{\overline{R(A^*)}} : D(\mathcal{A}) \subset \overline{R(A^*)} \rightarrow \overline{R(A)}, \quad D(\mathcal{A}) := D(A) \cap N(A)^\perp = D(A) \cap \overline{R(A^*)}$$

$$\mathcal{A}^* := A^*|_{\overline{R(A)}} : D(\mathcal{A}^*) \subset \overline{R(A)} \rightarrow \overline{R(A^*)}, \quad D(\mathcal{A}^*) := D(A^*) \cap N(A^*)^\perp = D(A^*) \cap \overline{R(A)}$$

$(\mathcal{A}, \mathcal{A}^*)$ dual pair and $\mathcal{A}, \mathcal{A}^*$ inj \Rightarrow

inverse ops exist (and bij)

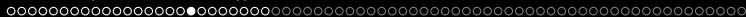
$$\mathcal{A}^{-1} : R(A) \rightarrow D(\mathcal{A}) \quad (\mathcal{A}^*)^{-1} : R(A^*) \rightarrow D(\mathcal{A}^*)$$

refined decompositions

$$D(A) = N(A) \oplus D(\mathcal{A}) \quad D(A^*) = N(A^*) \oplus D(\mathcal{A}^*)$$

\Rightarrow

$$R(A) = R(\mathcal{A}) \quad R(A^*) = R(\mathcal{A}^*)$$



1st fundamental observations

closed range theorem & closed graph theorem \Rightarrow

Lemma (Friedrichs-Poincaré type est/cl range/cont inv)

The following assertions are equivalent:

- (i) $\exists c_A \in (0, \infty) \quad \forall x \in D(\mathcal{A}) \quad |x|_{H_0} \leq c_A |Ax|_{H_1}$
- (i*) $\exists c_{A^*} \in (0, \infty) \quad \forall y \in D(\mathcal{A}^*) \quad |y|_{H_1} \leq c_{A^*} |A^*y|_{H_0}$
- (ii) $R(A) = R(\mathcal{A})$ is closed in H_1 .
- (ii*) $R(A^*) = R(\mathcal{A}^*)$ is closed in H_0 .
- (iii) $\mathcal{A}^{-1} : R(A) \rightarrow D(\mathcal{A})$ is continuous and bijective.
- (iii*) $(\mathcal{A}^*)^{-1} : R(A^*) \rightarrow D(\mathcal{A}^*)$ is continuous and bijective.

In case that one of the latter assertions is true, e.g., (ii), $R(A)$ is closed, we have

$$\begin{aligned} H_0 &= N(A) \oplus R(A^*) & H_1 &= N(A^*) \oplus R(A) \\ D(A) &= N(A) \oplus D(\mathcal{A}) & D(A^*) &= N(A^*) \oplus D(\mathcal{A}^*) \\ D(\mathcal{A}) &= D(A) \cap R(A^*) & D(\mathcal{A}^*) &= D(A^*) \cap R(A) \end{aligned}$$

and $\mathcal{A} : D(\mathcal{A}) \subset R(A^*) \rightarrow R(A)$, $\mathcal{A}^* : D(\mathcal{A}^*) \subset R(A) \rightarrow R(A^*)$.



1st fundamental observations

recall

$$(i) \quad \exists c_A \in (0, \infty) \quad \forall x \in D(\mathcal{A}) \quad |x|_{H_0} \leq c_A |Ax|_{H_1}$$

$$(i^*) \quad \exists c_{A^*} \in (0, \infty) \quad \forall y \in D(\mathcal{A}^*) \quad |y|_{H_1} \leq c_{A^*} |A^*y|_{H_0}$$

'best' consts in (i) and (i*) equal norms of the inv ops and Rayleigh quotients

$$c_A = |\mathcal{A}^{-1}|_{R(A), R(A^*)}$$

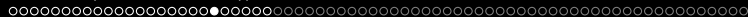
$$c_{A^*} = |(\mathcal{A}^*)^{-1}|_{R(A^*), R(A)}$$

$$\frac{1}{c_A} = \inf_{0 \neq x \in D(\mathcal{A})} \frac{|Ax|_{H_1}}{|x|_{H_0}}$$

$$\frac{1}{c_{A^*}} = \inf_{0 \neq y \in D(\mathcal{A}^*)} \frac{|A^*y|_{H_0}}{|y|_{H_1}}$$

Lemma (Friedrichs-Poincaré type const)

$$c_A = c_{A^*}$$



1st fundamental observations

Lemma (cpt emb/cpt inv)

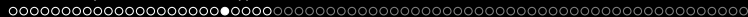
The following assertions are equivalent:

- (i) $D(\mathcal{A}) \hookrightarrow H_0$ is compact.
- (i*) $D(\mathcal{A}^*) \hookrightarrow H_1$ is compact.
- (ii) $\mathcal{A}^{-1} : R(\mathcal{A}) \rightarrow R(\mathcal{A}^*)$ is compact.
- (ii*) $(\mathcal{A}^*)^{-1} : R(\mathcal{A}^*) \rightarrow R(\mathcal{A})$ is compact.

Lemma (Friedrichs-Poincaré type est/cl range/cont inv)

⇓ $D(\mathcal{A}) \hookrightarrow H_0$ compact

- (i) $\exists c_A \in (0, \infty) \quad \forall x \in D(\mathcal{A}) \quad |x|_{H_0} \leq c_A |Ax|_{H_1}$
- (i*) $\exists c_{A^*} \in (0, \infty) \quad \forall y \in D(\mathcal{A}^*) \quad |y|_{H_1} \leq c_{A^*} |A^*y|_{H_0}$
- (ii) $R(\mathcal{A}) = R(\mathcal{A})$ is closed in H_1 .
- (ii*) $R(\mathcal{A}^*) = R(\mathcal{A}^*)$ is closed in H_0 .
- (iii) $\mathcal{A}^{-1} : R(\mathcal{A}) \rightarrow D(\mathcal{A})$ is continuous and bijective.
- (iii*) $(\mathcal{A}^*)^{-1} : R(\mathcal{A}^*) \rightarrow D(\mathcal{A}^*)$ is continuous and bijective.
- (i)-(iii*) equi & the resp Helm deco hold & $|\mathcal{A}^{-1}| = c_A = c_{A^*} = |(\mathcal{A}^*)^{-1}|$



2nd fundamental observations

So far no complex...

$$A_0 : D(A_0) \subset H_0 \rightarrow H_1, \quad A_1 : D(A_1) \subset H_1 \rightarrow H_2 \quad (\text{lddc})$$

$$A_0^* : D(A_0^*) \subset H_1 \rightarrow H_0, \quad A_1^* : D(A_1^*) \subset H_2 \rightarrow H_1 \quad (\text{lddc})$$

general complex ($A_1 A_0 = 0$, i.e., $R(A_0) \subset N(A_1)$ and $R(A_1^*) \subset N(A_0^*)$)

$$\dots \begin{array}{c} \dots \\ \rightleftarrows \\ \dots \end{array} H_0 \begin{array}{c} A_0 \\ \rightleftarrows \\ A_0^* \end{array} H_1 \begin{array}{c} A_1 \\ \rightleftarrows \\ A_1^* \end{array} H_2 \begin{array}{c} \dots \\ \rightleftarrows \\ \dots \end{array} \dots$$

recall Helmholtz deco

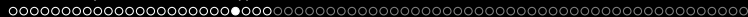
$$H_1 = \overline{R(A_0)} \oplus N(A_0^*)$$

$$\cap \quad \cup \quad \Rightarrow \text{(e.g.) } N(A_1) = \overline{R(A_0)} \oplus \underbrace{(N(A_1) \cap N(A_0^*))}_{=: K_1}$$

$$= N(A_1) \oplus \overline{R(A_1^*)}$$

\Rightarrow refined Helmholtz deco

$$H_1 = \overline{R(A_0)} \oplus K_1 \oplus \overline{R(A_1^*)}$$



2nd fundamental observations

recall

$$D(A_1) = D(\mathcal{A}_1) \cap \overline{R(A_1^*)} \quad R(A_1) = R(\mathcal{A}_1) \quad R(A_1^*) = R(\mathcal{A}_1^*)$$

$$D(A_0^*) = D(\mathcal{A}_0^*) \cap \overline{R(A_0)} \quad R(A_0^*) = R(\mathcal{A}_0^*) \quad R(A_0) = R(\mathcal{A}_0)$$

cohomology group $K_1 = N(A_1) \cap N(A_0^*)$

Lemma (Helmholtz deco I)

$$H_1 = \overline{R(A_0)} \oplus N(A_0^*)$$

$$H_1 = \overline{R(A_1^*)} \oplus N(A_1)$$

$$D(A_0^*) = D(\mathcal{A}_0^*) \oplus N(A_0^*)$$

$$D(A_1) = D(\mathcal{A}_1) \oplus N(A_1)$$

$$N(A_1) = D(\mathcal{A}_0^*) \oplus K_1$$

$$N(A_0^*) = D(\mathcal{A}_1) \oplus K_1$$

$$D(A_1) = \overline{R(A_0)} \oplus (D(A_1) \cap N(A_0^*)) \quad D(A_0^*) = \overline{R(A_1^*)} \oplus (D(A_0^*) \cap N(A_1))$$

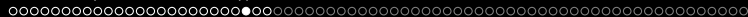
Lemma (Helmholtz deco II)

$$H_1 = \overline{R(A_0)} \oplus K_1 \oplus \overline{R(A_1^*)}$$

$$D(A_1) = \overline{R(A_0)} \oplus K_1 \oplus D(\mathcal{A}_1)$$

$$D(A_0^*) = D(\mathcal{A}_0^*) \oplus K_1 \oplus \overline{R(A_1^*)}$$

$$D(A_1) \cap D(A_0^*) = D(\mathcal{A}_0^*) \oplus K_1 \oplus D(\mathcal{A}_1)$$



2nd fundamental observations

$$K_1 = N(A_1) \cap N(A_0^*) \quad D(A_1) = D(\mathcal{A}_1) \cap \overline{R(A_1^*)} \quad D(A_0^*) = D(\mathcal{A}_0^*) \cap \overline{R(A_0)}$$

Lemma (cpt emb II)

The following assertions are equivalent:

- (i) $D(\mathcal{A}_0) \leftrightarrow H_0$, $D(\mathcal{A}_1) \leftrightarrow H_1$, and $K_1 \leftrightarrow H_1$ are compact.
- (ii) $D(A_1) \cap D(A_0^*) \leftrightarrow H_1$ is compact.

In this case $K_1 < \infty$.

Theorem (fa-toolbox I)

⇓ $D(A_1) \cap D(A_0^*) \leftrightarrow H_1$ compact

- (i) all emb cpt, i.e., $D(\mathcal{A}_0) \leftrightarrow H_0$, $D(\mathcal{A}_1) \leftrightarrow H_1$, $D(\mathcal{A}_0^*) \leftrightarrow H_1$, $D(\mathcal{A}_1^*) \leftrightarrow H_2$ cpt
- (ii) cohomology group K_1 finite dim
- (iii) all ranges closed, i.e., $R(A_0)$, $R(A_0^*)$, $R(A_1)$, $R(A_1^*)$ cl
- (iv) all Friedrichs-Poincaré type est hold
- (v) all Hodge-Helmholtz-Weyl type deco I & II hold with closed ranges



2nd fundamental observations

$$\text{complex} \quad \dots \quad \begin{array}{c} \dots \\ \xrightarrow{A_0} \\ \dots \end{array} \quad H_0 \quad \begin{array}{c} A_0 \\ \xrightarrow{A_1} \\ A_0^* \end{array} \quad H_1 \quad \begin{array}{c} A_1 \\ \xrightarrow{A_2} \\ A_1^* \end{array} \quad H_2 \quad \begin{array}{c} \dots \\ \xrightarrow{\quad} \\ \dots \end{array} \quad \dots$$

Theorem (fa-toolbox I (Friedrichs-Poincaré type est))

$$\Downarrow \quad \boxed{D(A_1) \cap D(A_0^*) \leftrightarrow H_1 \text{ compact}} \quad \Rightarrow \quad \exists \quad |A_i^{-1}| = c_{A_i} = c_{A_i^*} = |(A_i^*)^{-1}| \in (0, \infty)$$

- (i) $\forall x \in D(A_0) \quad |x|_{H_0} \leq c_{A_0} |A_0 x|_{H_1}$
- (i*) $\forall y \in D(A_0^*) \quad |y|_{H_1} \leq c_{A_0} |A_0^* y|_{H_0}$
- (ii) $\forall y \in D(A_1) \quad |y|_{H_1} \leq c_{A_1} |A_1 y|_{H_2}$
- (ii*) $\forall z \in D(A_1^*) \quad |z|_{H_2} \leq c_{A_1} |A_1^* z|_{H_1}$
- (iii) $\forall y \in D(A_1) \cap D(A_0^*) \quad |(1 - \pi_{K_1})y|_{H_1} \leq c_{A_1} |A_1 y|_{H_2} + c_{A_0} |A_0^* y|_{H_0}$

note $\pi_{K_1} y \in K_1$ and $(1 - \pi_{K_1})y \in K_1^\perp$

Remark

enough $R(A_0)$ and $R(A_1)$ cl



2nd fundamental observations

$$\text{complex} \quad \dots \quad \begin{array}{c} \dots \\ \rightleftarrows \\ \dots \end{array} \quad H_0 \quad \begin{array}{c} A_0 \\ \rightleftarrows \\ A_0^* \end{array} \quad H_1 \quad \begin{array}{c} A_1 \\ \rightleftarrows \\ A_1^* \end{array} \quad H_2 \quad \begin{array}{c} \dots \\ \rightleftarrows \\ \dots \end{array} \quad \dots$$

Theorem (fa-toolbox I (Helmholtz deco))

$$\Downarrow \quad \boxed{D(A_1) \cap D(A_0^*) \leftrightarrow H_1 \text{ compact}}$$

$$H_1 = R(A_0) \oplus N(A_0^*)$$

$$H_1 = R(A_1^*) \oplus N(A_1)$$

$$D(A_0^*) = D(\mathcal{A}_0^*) \oplus N(A_0^*)$$

$$D(A_1) = D(\mathcal{A}_1) \oplus N(A_1)$$

$$N(A_1) = D(\mathcal{A}_0^*) \oplus K_1$$

$$N(A_0^*) = D(\mathcal{A}_1) \oplus K_1$$

$$D(A_1) = R(A_0) \oplus (D(A_1) \cap N(A_0^*)) \quad D(A_0^*) = R(A_1^*) \oplus (D(A_0^*) \cap N(A_1))$$

$$H_1 = R(A_0) \oplus K_1 \oplus R(A_1^*)$$

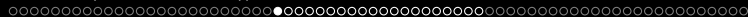
$$D(A_1) = R(A_0) \oplus K_1 \oplus D(\mathcal{A}_1)$$

$$D(A_0^*) = D(\mathcal{A}_0^*) \oplus K_1 \oplus R(A_1^*)$$

$$D(A_1) \cap D(A_0^*) = D(\mathcal{A}_0^*) \oplus K_1 \oplus D(\mathcal{A}_1)$$

Remark

enough $R(A_0)$ and $R(A_1)$ cl



(stat) first order system

(stat) first order system - solution theory

$$\text{complex} \quad \dots \begin{array}{c} \dots \\ \rightleftarrows \\ \dots \end{array} H_0 \begin{array}{c} A_0 \\ \rightleftarrows \\ A_0^* \end{array} H_1 \begin{array}{c} A_1 \\ \rightleftarrows \\ A_1^* \end{array} H_2 \begin{array}{c} \dots \\ \rightleftarrows \\ \dots \end{array} \dots$$

$$A_1 x = f$$

$$\dim N(A_1) = \infty$$

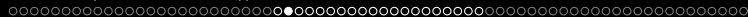
find $x \in D(A_1) \cap D(A_0^*)$ such that the fos

$$\begin{array}{ll} A_1 x = f & (\text{rot } E = F) \\ A_0^* x = g & \text{think of } (-\text{div } E = g) \\ \pi_{K_1} x = k & (\pi_D E = K) \end{array}$$

$$\text{kernel} = \text{cohomology group} = K_1 = N(A_1) \cap N(A_0^*)$$

$$\text{trivially necessary} \quad f \in R(A_1) \quad g \in R(A_0^*) \quad k \in K_1$$

$$\text{apply fa-toolbox}$$



(stat) first order system

(stat) first order system - solution theory

$$\text{complex} \quad \dots \quad \begin{matrix} \dots \\ \rightleftharpoons \\ \dots \end{matrix} \quad H_0 \quad \begin{matrix} A_0 \\ \rightleftharpoons \\ A_0^* \end{matrix} \quad H_1 \quad \begin{matrix} A_1 \\ \rightleftharpoons \\ A_1^* \end{matrix} \quad H_2 \quad \begin{matrix} \dots \\ \rightleftharpoons \\ \dots \end{matrix} \quad \dots$$

$$\text{find } x \in D(A_1) \cap D(A_0^*) \text{ st fos} \quad A_1 x = f \quad A_0^* x = g \quad \pi_{K_1} x = k$$

Theorem (fa-toolbox II (solution theory))

$$\Downarrow \quad \boxed{D(A_1) \cap D(A_0^*) \leftrightarrow H_1 \text{ compact}}$$

$$\text{fos is uniq sol} \quad \Leftrightarrow \quad f \in R(A_1) \quad g \in R(A_0^*) \quad k \in K_1$$

$$x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus K_1 = D(A_1) \cap D(A_0^*)$$

$$\boxed{x_f := \mathcal{A}_1^{-1} f} \in D(\mathcal{A}_1)$$

$$\boxed{x_g := (\mathcal{A}_0^*)^{-1} g} \in D(\mathcal{A}_0^*)$$

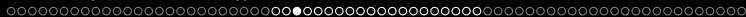
$$\text{dep cont on data} \quad |x|_{H_1} \leq |x_f|_{H_1} + |x_g|_{H_1} + |k|_{H_1} \leq c_{A_1} |f|_{H_2} + c_{A_0} |g|_{H_0} + |k|_{H_1}$$

moreover

$$\pi_{R(A_1^*)} x = x_f \quad \pi_{R(A_0)} x = x_g \quad \pi_{K_1} x = k \quad |x|_{H_1}^2 = |x_f|_{H_1}^2 + |x_g|_{H_1}^2 + |k|_{H_1}^2$$

Remark

enough $R(A_0)$ and $R(A_1)$ cl



(stat) first order system - variational formulations

$$x := x_f + x_g + k \in D(\mathcal{A}_1) \oplus D(\mathcal{A}_0^*) \oplus K_1 = D(A_1) \cap D(A_0^*)$$

$$x_f := \mathcal{A}_1^{-1} f \in D(\mathcal{A}_1) = D(A_1) \cap R(A_1^*) = D(A_1) \cap N(A_0^*) \cap K_1^\perp$$

$$x_g := (\mathcal{A}_0^*)^{-1} g \in D(\mathcal{A}_0^*) = D(A_0^*) \cap R(A_0) = D(A_0^*) \cap N(A_1) \cap K_1^\perp$$

$$A_1 x = f$$

$$A_1 x_f = f$$

$$A_1 x_g = 0$$

$$A_1 k = 0$$

$$A_0^* x = g$$

$$A_0^* x_f = 0$$

$$A_0^* x_g = g$$

$$A_0^* k = 0$$

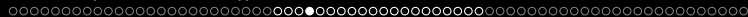
$$\pi_{K_1} x = k$$

$$\pi_{K_1} x_f = 0$$

$$\pi_{K_1} x_g = 0$$

$$\pi_{K_1} k = k$$

- option I: find x_f and x_g separately $\Rightarrow x = x_f + x_g + k$
- option II: find x directly



(stat) first order system

(stat) first order system - variational formulations I

finding

$$x_f := \mathcal{A}_1^{-1} f \in D(\mathcal{A}_1) = D(A_1) \cap \underbrace{R(A_1^*)}_{=R(\mathcal{A}_1^*)} = D(A_1) \cap N(A_0^*) \cap K_1^\perp$$

$$A_1 x_f = f$$

$$A_0^* x_f = 0$$

$$\pi_{K_1} x_f = 0$$

at least two options

- option Ia: multiply $A_1 x_f = f$ by $A_1 \xi \Rightarrow$

$$\forall \xi \in D(\mathcal{A}_1) \quad \langle A_1 x_f, A_1 \xi \rangle_{H_2} = \langle f, A_1 \xi \rangle_{H_2}$$

weak form of $A_1^* A_1 x_f = A_1^* f$

- option Ib: repr $x_f = A_1^* y_f$ with potential $y_f = (A_1^*)^{-1} x_f \in D(\mathcal{A}_1^*)$
and mult by x_f by $A_1^* \phi \Rightarrow$

$$\forall \phi \in D(\mathcal{A}_1^*) \quad \langle A_1^* y_f, A_1^* \phi \rangle_{H_1} = \langle x_f, A_1^* \phi \rangle_{H_1} = \langle A_1 x_f, \phi \rangle_{H_2} = \langle f, \phi \rangle_{H_2}$$

weak form of $A_1 x_f = f$ and $A_1 A_1^* y_f = f$ analogously for x_g

(stat) first order system - variational formulations I

Theorem

Let $D(\mathcal{A}_1) \cap D(\mathcal{A}_0^*) \hookrightarrow H_1$ be compact and let $f \in R(\mathcal{A}_1)$ and $g \in R(\mathcal{A}_0^*)$.
The part sol x_f and x_g can be found by the following 4 var form:

$$(i) \quad \exists^1 \tilde{x}_f \in D(\mathcal{A}_1) \quad st \quad \forall \xi \in D(\mathcal{A}_1) \quad \langle \mathcal{A}_1 \tilde{x}_f, \mathcal{A}_1 \xi \rangle_{H_2} = \langle f, \mathcal{A}_1 \xi \rangle_{H_2}$$

$$which \text{ even holds for all } \xi \in D(\mathcal{A}_1). \quad \Rightarrow \quad \boxed{\tilde{x}_f = x_f}$$

$$(i') \quad \exists^1 y_f \in D(\mathcal{A}_1^*) \quad st \quad \forall \phi \in D(\mathcal{A}_1^*) \quad \langle \mathcal{A}_1^* y_f, \mathcal{A}_1^* \phi \rangle_{H_1} = \langle f, \phi \rangle_{H_2}$$

$$which \text{ even holds for all } \phi \in D(\mathcal{A}_1^*). \quad \Rightarrow \quad \boxed{\mathcal{A}_1^* y_f = x_f}$$

$$(ii) \quad \exists^1 \tilde{x}_g \in D(\mathcal{A}_0^*) \quad st \quad \forall \zeta \in D(\mathcal{A}_0^*) \quad \langle \mathcal{A}_0^* \tilde{x}_g, \mathcal{A}_0^* \zeta \rangle_{H_0} = \langle g, \mathcal{A}_0^* \zeta \rangle_{H_0}$$

$$which \text{ even holds for all } \zeta \in D(\mathcal{A}_0^*). \quad \Rightarrow \quad \boxed{\tilde{x}_g = x_g}$$

$$(ii') \quad \exists^1 z_g \in D(\mathcal{A}_0) \quad st \quad \forall \varphi \in D(\mathcal{A}_0) \quad \langle \mathcal{A}_0 z_g, \mathcal{A}_0 \varphi \rangle_{H_1} = \langle g, \varphi \rangle_{H_0}$$

$$which \text{ even holds for all } \varphi \in D(\mathcal{A}_0). \quad \Rightarrow \quad \boxed{\mathcal{A}_0 z_g = x_g}$$



(stat) first order system

(stat) first order system - variational formulations I

$$\text{e.g. } \exists^1 \tilde{x}_f \in D(\mathcal{A}_1) \quad \text{st} \quad \forall \xi \in D(\mathcal{A}_1) \quad \langle \mathcal{A}_1 \tilde{x}_f, \mathcal{A}_1 \xi \rangle_{H_2} = \langle f, \mathcal{A}_1 \xi \rangle_{H_2} \quad \Rightarrow \quad \boxed{\tilde{x}_f = x_f}$$

Helmholtz deco \Rightarrow

$$\begin{aligned} \tilde{x}_f \in D(\mathcal{A}_1) &= D(\mathcal{A}_1) \cap R(\mathcal{A}_1^*) = D(\mathcal{A}_1) \cap N(\mathcal{A}_1)^\perp = D(\mathcal{A}_1) \cap (R(\mathcal{A}_0) \oplus K_1)^\perp \\ &= D(\mathcal{A}_1) \cap R(\mathcal{A}_0)^\perp \cap K_1^\perp \end{aligned}$$

 \Rightarrow saddle point formulations/double (multiple) saddle point formulations

Theorem

Let $D(\mathcal{A}_1) \cap D(\mathcal{A}_0^*) \leftrightarrow H_1$ be compact and let $f \in R(\mathcal{A}_1)$ and $g \in R(\mathcal{A}_0^*)$.
The part sol x_f and x_g can be found by the following 4 var form:

$$(i) \quad \exists^1 (\tilde{x}_f, u, h) \in D(\mathcal{A}_1) \times D(\mathcal{A}_0) \times K_1 \quad \text{st} \quad \forall (\xi, \varphi, \kappa) \in D(\mathcal{A}_1) \times D(\mathcal{A}_0) \times K_1$$

$$\langle \mathcal{A}_1 \tilde{x}_f, \mathcal{A}_1 \xi \rangle_{H_2} + \langle \mathcal{A}_0 u, \xi \rangle_{H_1} + \langle h, \xi \rangle_{H_1} = \langle f, \mathcal{A}_1 \xi \rangle_{H_2}$$

$$\langle \tilde{x}_f, \mathcal{A}_0 \varphi \rangle_{H_1} = 0$$

$$\langle \tilde{x}_f, \kappa \rangle_{H_1} = 0$$

$$\Rightarrow \quad u = 0 \quad h = 0 \quad \boxed{\tilde{x}_f = x_f}$$

(i') analogously for y_f (ii) analogously for \tilde{x}_g (ii') analogously for z_g



(stat) first order system

(stat) first order system - variational formulations I

latter tripple saddle point formulation

$$\begin{aligned} \exists^1 (\tilde{x}_f, u, h) \in D(A_1) \times D(A_0) \times K_1 \quad \text{st} \quad \forall (\xi, \varphi, \kappa) \in D(A_1) \times D(A_0) \times K_1 \\ \langle A_1 \tilde{x}_f, A_1 \xi \rangle_{H_2} + \langle A_0 u, \xi \rangle_{H_1} + \langle h, \xi \rangle_{H_1} = \langle f, A_1 \xi \rangle_{H_2} \\ \langle \tilde{x}_f, A_0 \varphi \rangle_{H_1} = 0 \\ \langle \tilde{x}_f, \kappa \rangle_{H_1} = 0 \end{aligned}$$

is weak formulation of

$$A_1^* A_1 \tilde{x}_f + A_0 u + h = A_1^* f \quad A_0^* \tilde{x}_f = 0 \quad \pi_{K_1} \tilde{x}_f = 0$$

i.e., in formal matrix notation

$$\begin{bmatrix} A_1^* A_1 & A_0 & \iota_{K_1} \\ A_0^* & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_f \\ u \\ h \end{bmatrix} = \begin{bmatrix} A_1^* f \\ 0 \\ 0 \end{bmatrix}$$

Note $u = 0, \quad h = 0,$

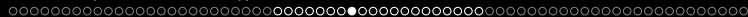
$$\boxed{\tilde{x}_f = x_f}$$

potential y_f

$$\begin{bmatrix} A_1 A_1^* & A_2^* & \iota_{K_2} \\ A_2 & 0 & 0 \\ \pi_{K_2} = \iota_{K_2}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} y_f \\ v \\ h_2 \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}$$

Note $v = 0, \quad h_2 = 0,$

$$\boxed{A_1^* y_f = x_f}$$



(stat) first order system

(stat) first order system - variational formulations II

$$\begin{bmatrix} A_1^* A_1 & A_0 & \iota_{K_1} \\ A_0^* & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_f \\ u \\ h \end{bmatrix} = \begin{bmatrix} A_1^* f \\ 0 \\ 0 \end{bmatrix}$$

Note $u = 0, \quad h = 0, \quad \boxed{\tilde{x}_f = x_f}$

SAME formulation can be used to compute $x = x_f + x_g + k$ directly!

$$\begin{bmatrix} A_1^* A_1 & A_0 & \iota_{K_1} \\ A_0^* & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ u \\ h \end{bmatrix} = \begin{bmatrix} A_1^* f \\ g \\ k \end{bmatrix}$$

$\Rightarrow u = 0, \quad h = 0, \quad \boxed{\tilde{x} = x}$

Remark

special case $A_0 = \overset{\circ}{\nabla}$ $A_0^* = -\text{div}$ or $A_0 = \nabla$ $A_0^* = -\overset{\circ}{\text{div}}$
 $A_1 = \overset{\circ}{\text{rot}}$ $A_1^* = \text{rot}$ $A_0 = \text{rot}$ $A_0^* = \overset{\circ}{\text{rot}}$

var form recently proposed by

Alonso Rodriguez, A., Bertolazzi E., and Valli A.: *The curl-div system: theory and finite element approximation, talk/preprint, 2018*



(stat) first order system - variational formulations II

Theorem

Let $D(A_1) \cap D(A_0^*) \leftrightarrow H_1$ be compact and let $f \in R(A_1)$ and $g \in R(A_0^*)$.
 x can be found by the following 2 double saddle point var form:

$$(i) \exists^1 (\tilde{x}, u, h_1) \in D(A_1) \times D(A_0) \times K_1 \quad \text{st} \quad \forall (\xi, \varphi, \kappa) \in D(A_1) \times D(A_0) \times K_1$$

$$\langle A_1 \tilde{x}, A_1 \xi \rangle_{H_2} + \langle A_0 u, \xi \rangle_{H_1} + \langle h_1, \xi \rangle_{H_1} = \langle f, A_1 \xi \rangle_{H_2}$$

$$\langle \tilde{x}, A_0 \varphi \rangle_{H_1} = \langle g, \varphi \rangle_{H_0}$$

$$\langle \tilde{x}, \kappa \rangle_{H_1} = \langle k, \kappa \rangle_{H_1}$$

$$\Rightarrow u = 0, \quad h_1 = 0, \quad \boxed{\tilde{x} = x}$$

$$(ii) \exists^1 (\hat{x}, v, h_2) \in D(A_0^*) \times D(A_1^*) \times K_1 \quad \text{st} \quad \forall (\zeta, \phi, \kappa) \in D(A_0^*) \times D(A_1^*) \times K_1$$

$$\langle A_0^* \hat{x}, A_0^* \zeta \rangle_{H_0} + \langle A_1^* v, \zeta \rangle_{H_1} + \langle h_2, \zeta \rangle_{H_1} = \langle g, A_0^* \zeta \rangle_{H_0}$$

$$\langle \hat{x}, A_1^* \phi \rangle_{H_1} = \langle f, \phi \rangle_{H_2}$$

$$\langle \hat{x}, \kappa \rangle_{H_1} = \langle k, \kappa \rangle_{H_1}$$

$$\Rightarrow v = 0, \quad h_2 = 0, \quad \boxed{\hat{x} = x}$$



(stat) first order system

(stat) first order system - variational formulations II

form matrix not

$$\begin{bmatrix} A_1^* A_1 & A_0 & \iota_{K_1} \\ A_0^* & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ u \\ h_1 \end{bmatrix} = \begin{bmatrix} A_1^* f \\ g \\ k \end{bmatrix}$$

$$\Rightarrow u = 0, \quad h_1 = 0, \quad \boxed{\tilde{x} = x}$$

$$\begin{bmatrix} A_0 A_0^* & A_1^* & \iota_{K_1} \\ A_1 & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ v \\ h_2 \end{bmatrix} = \begin{bmatrix} A_0 g \\ f \\ k \end{bmatrix}$$

$$\Rightarrow v = 0, \quad h_2 = 0, \quad \boxed{\hat{x} = x}$$



(stat) first order system - variational formulations II

$$\begin{aligned}
 D(\mathcal{A}_0) &= D(A_0) \cap R(A_0^*) = D(A_0) \cap N(A_0)^\perp = D(A_0) \cap (R(\mathcal{A}_{-1}) \oplus K_0)^\perp \\
 &= D(A_0) \cap R(\mathcal{A}_{-1})^\perp \cap K_0^\perp
 \end{aligned}$$

$$\begin{aligned}
 D(\mathcal{A}_1^*) &= D(A_1^*) \cap R(A_1) = D(A_1^*) \cap N(A_1^*)^\perp = D(A_1^*) \cap (R(\mathcal{A}_2^*) \oplus K_2)^\perp \\
 &= D(A_1^*) \cap R(\mathcal{A}_2^*)^\perp \cap K_2^\perp
 \end{aligned}$$



(stat) first order system - variational formulations II

$$D(\mathcal{A}_0) = D(A_0) \cap R(\mathcal{A}_{-1})^\perp \cap K_0^\perp$$

$$D(\mathcal{A}_1^*) = D(A_1^*) \cap R(\mathcal{A}_2^*)^\perp \cap K_2^\perp$$

Theorem

Let $D(A_1) \cap D(A_0^*) \leftrightarrow H_1$ be compact and let $f \in R(A_1)$ and $g \in R(A_0^*)$.
 x can be found by the following quadruple saddle point var form:

$$\exists^1 (\tilde{x}, u, y, h_1, h_0) \in D(A_1) \times D(A_0) \times D(\mathcal{A}_{-1}) \times K_1 \times K_0 \quad \text{st}$$

$$\forall (\xi, \varphi, \vartheta, \kappa, \lambda) \in D(A_1) \times D(A_0) \times D(A_{-1}) \times K_1 \times K_0$$

$$\langle A_1 \tilde{x}, A_1 \xi \rangle_{H_2} + \langle A_0 u, \xi \rangle_{H_1} + \langle h_1, \xi \rangle_{H_1} = \langle f, A_1 \xi \rangle_{H_2}$$

$$\langle \tilde{x}, A_0 \varphi \rangle_{H_1} + \langle A_{-1} y, \varphi \rangle_{H_0} + \langle h_0, \varphi \rangle_{H_0} = \langle g, \varphi \rangle_{H_0}$$

$$\langle u, A_{-1} \vartheta \rangle_{H_0} = 0$$

$$\langle \tilde{x}, \kappa \rangle_{H_1} = \langle k, \kappa \rangle_{H_1}$$

$$\langle u, \lambda \rangle_{H_0} = 0$$

$$\Rightarrow \quad u = 0, \quad y = 0, \quad h_1 = 0, \quad h_0 = 0, \quad \boxed{\tilde{x} = x}$$



(stat) first order system

(stat) first order system - variational formulations II

$$D(\mathcal{A}_0) = D(A_0) \cap R(\mathcal{A}_{-1})^\perp \cap K_0^\perp$$

$$D(\mathcal{A}_1^*) = D(A_1^*) \cap R(\mathcal{A}_2^*)^\perp \cap K_2^\perp$$

Theorem

Let $D(A_1) \cap D(A_0^*) \leftrightarrow H_1$ be compact and let $f \in R(A_1)$ and $g \in R(A_0^*)$.
 x can be found by the following quadruple saddle point var form:

$$\exists^1 (\hat{x}, v, z, h_1, h_2) \in D(A_0^*) \times D(A_1^*) \times D(\mathcal{A}_2^*) \times K_1 \times K_2 \quad \text{st}$$

$$\forall (\zeta, \phi, \theta, \kappa, \lambda) \in D(A_0^*) \times D(A_1^*) \times D(A_2^*) \times K_1 \times K_2$$

$$\langle A_0^* \hat{x}, A_0^* \zeta \rangle_{H_0} + \langle A_1^* v, \zeta \rangle_{H_1} + \langle h_1, \zeta \rangle_{H_1} = \langle g, A_0^* \zeta \rangle_{H_0}$$

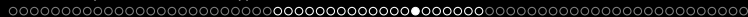
$$\langle \hat{x}, A_1^* \phi \rangle_{H_1} + \langle A_2^* z, \phi \rangle_{H_2} + \langle h_2, \phi \rangle_{H_2} = \langle f, \phi \rangle_{H_2}$$

$$\langle v, A_2^* \theta \rangle_{H_2} = 0$$

$$\langle \hat{x}, \kappa \rangle_{H_1} = \langle k, \kappa \rangle_{H_1}$$

$$\langle v, \lambda \rangle_{H_2} = 0$$

$$\Rightarrow \quad v = 0, \quad z = 0, \quad h_1 = 0, \quad h_2 = 0, \quad \boxed{\hat{x} = x}$$



(stat) first order system

(stat) first order system - variational formulations II

form matrix not

$$\begin{bmatrix} A_1^* A_1 & A_0 & 0 & \iota_{K_1} & 0 \\ A_0^* & 0 & A_{-1} & 0 & \iota_{K_0} \\ 0 & A_{-1}^* & 0 & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 & 0 & 0 \\ 0 & \pi_{K_0} = \iota_{K_0}^* & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ u \\ y \\ h_1 \\ h_0 \end{bmatrix} = \begin{bmatrix} A_1^* f \\ g \\ 0 \\ k \\ 0 \end{bmatrix}$$

note $u = 0, \quad y = 0, \quad h_1 = 0, \quad h_0 = 0,$ $\tilde{x} = x$

$$\begin{bmatrix} A_0 A_0^* & A_1^* & 0 & \iota_{K_1} & 0 \\ A_1 & 0 & A_2^* & 0 & \iota_{K_2} \\ 0 & A_2 & 0 & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 & 0 & 0 \\ 0 & \pi_{K_2} = \iota_{K_2}^* & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ v \\ z \\ h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} A_0 g \\ f \\ 0 \\ k \\ 0 \end{bmatrix}$$

note $v = 0, \quad z = 0, \quad h_1 = 0, \quad h_2 = 0,$ $\hat{x} = x$

typical situation in 3D:

- K_{-1}, K_0, K_3, K_4 trivial only K_1, K_2 non-trivial (Dirichlet/Neumann fields)
- $A_{-2}^* = 0 \Rightarrow N(A_{-2}^*) = H_{-1}$
- $A_4 = 0 \Rightarrow N(A_4) = H_4$
- $N(A_3), N(A_{-1}^*)$ finite co-dim



(stat) first order system - variational formulations II

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recall

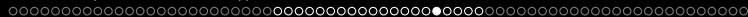
$$\begin{aligned}
 D(\mathcal{A}_i) &= D(A_i) \cap R(A_{i-1})^\perp \cap K_i^\perp & D(\mathcal{A}_i^*) &= D(A_i^*) \cap R(A_{i+1}^*)^\perp \cap K_{i+1}^\perp \\
 &= D(A_i) \cap N(A_{i-1}^*) \cap K_i^\perp & &= D(A_i^*) \cap N(A_{i+1}) \cap K_{i+1}^\perp
 \end{aligned}$$

always in 3D

$$\begin{aligned}
 D(\mathcal{A}_{-1}) &= D(A_{-1}) & D(\mathcal{A}_3^*) &= D(A_3^*) \\
 D(\mathcal{A}_0) &= D(A_0) \cap N(A_{-1}^*) & D(\mathcal{A}_2^*) &= D(A_2^*) \cap N(A_3)
 \end{aligned}$$

often in 3D

$$\begin{aligned}
 D(\mathcal{A}_0) &= D(A_0) & D(\mathcal{A}_2^*) &= D(A_2^*)
 \end{aligned}$$



(stat) first order system

(stat) first order system - variational formulations II

always in 3D

$$D(\mathcal{A}_{-1}) = D(A_{-1})$$

$$D(\mathcal{A}_3^*) = D(A_3^*)$$

$$D(\mathcal{A}_0) = D(A_0) \cap N(A_{-1}^*)$$

$$D(\mathcal{A}_2^*) = D(A_2^*) \cap N(A_3)$$

often in 3D

$$D(\mathcal{A}_0) = D(A_0)$$

$$D(\mathcal{A}_2^*) = D(A_2^*)$$

always in 3D: test spaces (K_0 trivial)

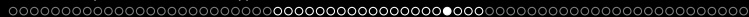
$$D(A_1) \times D(A_0) \times D(\mathcal{A}_{-1}) \times K_1 = D(A_1) \times D(A_0) \times D(A_{-1}) \times K_1 \quad \text{OK}$$

$$\begin{bmatrix} A_1^* A_1 & A_0 & 0 & \iota_{K_1} \\ A_0^* & 0 & A_{-1} & 0 \\ 0 & A_{-1}^* & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ u \\ y \\ h_1 \end{bmatrix} = \begin{bmatrix} A_1^* f \\ g \\ 0 \\ k \end{bmatrix}$$

often in 3D: test spaces

$$D(A_0^*) \times D(A_1^*) \times D(\mathcal{A}_2^*) \times K_1 \times K_2 = D(A_0^*) \times D(A_1^*) \times D(A_2^*) \times K_1 \times K_2 \quad \text{OK}$$

$$\begin{bmatrix} A_0 A_0^* & A_1^* & 0 & \iota_{K_1} & 0 \\ A_1 & 0 & A_2^* & 0 & \iota_{K_2} \\ 0 & A_2 & 0 & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 & 0 & 0 \\ 0 & \pi_{K_2} = \iota_{K_2}^* & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ v \\ z \\ h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} A_0 g \\ f \\ 0 \\ k \\ 0 \end{bmatrix}$$



(stat) first order system

(stat) first order system - variational formulations II

always in 3D

$$D(\mathcal{A}_{-1}) = D(A_{-1})$$

$$D(\mathcal{A}_3^*) = D(A_3^*)$$

always in 3D: test spaces (K_0 trivial)

$$D(A_1) \times D(A_0) \times D(\mathcal{A}_{-1}) \times K_1 = D(A_1) \times D(A_0) \times D(A_{-1}) \times K_1 \quad \text{OK}$$

$$\begin{bmatrix} A_1^* A_1 & A_0 & 0 & \iota_{K_1} \\ A_0^* & 0 & A_{-1} & 0 \\ 0 & A_{-1}^* & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ u \\ y \\ h_1 \end{bmatrix} = \begin{bmatrix} A_1^* f \\ g \\ 0 \\ k \end{bmatrix}$$

always in 3D: test spaces (K_3 trivial)

$$D(A_0^*) \times D(A_1^*) \times D(A_2^*) \times D(\mathcal{A}_3^*) \times K_1 \times K_2 = D(A_0^*) \times D(A_1^*) \times D(A_2^*) \times D(A_3^*) \times K_1 \times K_2$$

$$\begin{bmatrix} A_0 A_0^* & A_1^* & 0 & 0 & \iota_{K_1} & 0 \\ A_1 & 0 & A_2^* & 0 & 0 & \iota_{K_2} \\ 0 & A_2 & 0 & A_3^* & 0 & 0 \\ 0 & 0 & A_3 & 0 & 0 & 0 \\ \pi_{K_1} = \iota_{K_1}^* & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi_{K_2} = \iota_{K_2}^* & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ v \\ z \\ w \\ h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} A_0 g \\ f \\ 0 \\ 0 \\ k \\ 0 \end{bmatrix}$$



(stat) first order system

(stat) first order system - a posteriori error estimates

problem: $\boxed{\text{find } x \in D(A_1) \cap D(A_0^*) \text{ st } A_1 x = f \quad A_0^* x = g \quad \pi_{K_1} x = k}$

'very' non-conforming 'approximation' of x : $\boxed{\tilde{x} \in H_1}$

def., dcmp. err. $\boxed{e = x - \tilde{x}} = \pi_{R(A_0)} e + \pi_{K_1} e + \pi_{R(A_1^*)} e \in H_1 = R(A_0) \oplus K_1 \oplus R(A_1^*)$

Theorem (sharp upper bounds)

Let $\tilde{x} \in H_1$ and $e = x - \tilde{x}$. Then

$$|e|_{H_1}^2 = |\pi_{R(A_0)} e|_{H_1}^2 + |\pi_{K_1} e|_{H_1}^2 + |\pi_{R(A_1^*)} e|_{H_1}^2$$

$$|\pi_{R(A_0)} e|_{H_1} = \min_{\phi \in D(A_0^*)} (c_{A_0} |A_0^* \phi - g|_{H_0} + |\phi - \tilde{x}|_{H_1}) \quad \boxed{\text{reg } (A_0 A_0^* + 1)\text{-prbl in } D(A_0^*)}$$

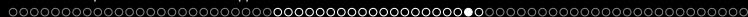
$$|\pi_{R(A_1^*)} e|_{H_1} = \min_{\varphi \in D(A_1)} (c_{A_1} |A_1 \varphi - f|_{H_2} + |\varphi - \tilde{x}|_{H_1}) \quad \boxed{\text{reg } (A_1^* A_1 + 1)\text{-prbl in } D(A_1)}$$

$$|\pi_{K_1} e|_{H_1} = |\pi_{K_1} \tilde{x} - k|_{H_1} = \min_{\substack{\xi \in D(A_0) \\ \zeta \in D(A_1^*)}} |A_0 \xi + A_1^* \zeta + \tilde{x} - k|_{H_1} \quad \boxed{\text{cpld } (A_0^* A_0) \text{-}(A_1 A_1^*)\text{-sys in } D(A_0) \text{-} D(A_1^*)}$$

Remark

Even $\pi_{K_1} e = k - \pi_{K_1} \tilde{x}$ and the minima are attained at

$$\hat{\phi} = \pi_{R(A_0)} e + \tilde{x}, \quad \hat{\varphi} = \pi_{R(A_1^*)} e + \tilde{x}, \quad A_0 \hat{\xi} + A_1^* \hat{\zeta} = (\pi_{K_1} - 1) \tilde{x}.$$



(stat) first order system

(stat) first order system - a posteriori error estimates

problem: $\boxed{\text{find } x \in D(A_1) \cap D(A_0^*) \text{ st } A_1 x = f \quad A_0^* x = g \quad \pi_{K_1} x = k}$

'very' non-conforming 'approximation' of x : $\boxed{\tilde{x} \in H_1}$

def., dcmp. err. $\boxed{e = x - \tilde{x}} = \pi_{R(A_0)} e + \pi_{K_1} e + \pi_{R(A_1^*)} e \in H_1 = R(A_0) \oplus K_1 \oplus R(A_1^*)$

Theorem (sharp lower bounds)

Let $\tilde{x} \in H_1$ and $e = x - \tilde{x}$. Then

$$|e|_{H_1}^2 = |\pi_{R(A_0)} e|_{H_1}^2 + |\pi_{K_1} e|_{H_1}^2 + |\pi_{R(A_1^*)} e|_{H_1}^2$$

$$|\pi_{R(A_0)} e|_{H_1}^2 = \max_{\phi \in D(A_0)} (2\langle g, \phi \rangle_{H_0} - \langle 2\tilde{x} + A_0 \phi, A_0 \phi \rangle_{H_1})$$

 $(A_0^* A_0)\text{-prbl in } D(A_0)$

$$|\pi_{R(A_1^*)} e|_{H_1}^2 = \max_{\varphi \in D(A_1^*)} (2\langle f, \varphi \rangle_{H_2} - \langle 2\tilde{x} + A_1^* \varphi, A_1^* \varphi \rangle_{H_1})$$

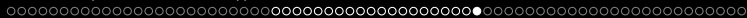
 $(A_1 A_0^*)\text{-prbl in } D(A_1^*)$

$$|\pi_{K_1} e|_{H_1}^2 = \max_{\psi \in K_1} (2\langle k - \tilde{x}, \psi \rangle_{H_1} - \langle \psi, \psi \rangle_{H_1})$$

$$\pi_{K_1} e = k - \pi_{K_1} \tilde{x}$$

Remark

The maxima are attained at $\phi \in D(A_0)$ with $A_0 \phi = \pi_{A_0} e$
and $\varphi \in D(A_1^*)$ with $A_1^* \varphi = \pi_{R(A_1^*)} e$ and $\psi = \pi_{K_1} e$



A_0^* - A_1 -lemma (generalized global div-curl-lemma)

Lemma (A_0^* - A_1 -lemma)

Let $D(A_1) \cap D(A_0^*) \hookrightarrow H_1$ be compact, and

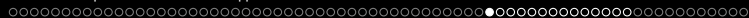
(i) (x_n) bounded in $D(A_1)$,

(ii) (y_n) bounded in $D(A_0^*)$.

$\Rightarrow \exists x \in D(A_1)$, $y \in D(A_0^*)$ and subsequences st

$x_n \rightharpoonup x$ in $D(A_1)$ and $y_n \rightharpoonup y$ in $D(A_0^*)$ as well as

$$\langle x_n, y_n \rangle_{H_1} \rightarrow \langle x, y \rangle_{H_1}.$$



applications: fos & sos (first and second order systems)

classical de Rham complex in 3D (∇ -rot-div-complex)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot{\cup} \Gamma_n}$

(electro-magneto dynamics, Maxwell's equations)

$$\{0\} \begin{array}{c} \xrightarrow{\iota_{\{0\}}} \\ \xleftarrow{\pi_{\{0\}}} \end{array} L^2 \begin{array}{c} \xrightarrow{\nabla} \\ \xleftarrow{-\operatorname{div}} \end{array} L^2 \begin{array}{c} \xrightarrow{\operatorname{rot}} \\ \xleftarrow{\operatorname{rot}} \end{array} L^2 \begin{array}{c} \xrightarrow{\operatorname{div}} \\ \xleftarrow{-\nabla} \end{array} L^2 \begin{array}{c} \xrightarrow{\pi_{\mathbb{R}}} \\ \xleftarrow{\iota_{\mathbb{R}}} \end{array} \mathbb{R}$$

mixed boundary conditions and inhomogeneous and anisotropic media

$$\{0\} \text{ or } \mathbb{R} \begin{array}{c} \xrightarrow{\iota} \\ \xleftarrow{\pi} \end{array} L^2 \begin{array}{c} \xrightarrow{\nabla_{\Gamma_t}} \\ \xleftarrow{-\operatorname{div}_{\Gamma_n} \varepsilon} \end{array} L^2_{\varepsilon} \begin{array}{c} \xrightarrow{\operatorname{rot}_{\Gamma_t}} \\ \xleftarrow{\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}} \end{array} L^2 \begin{array}{c} \xrightarrow{\operatorname{div}_{\Gamma_t}} \\ \xleftarrow{-\nabla_{\Gamma_n}} \end{array} L^2 \begin{array}{c} \xrightarrow{\pi} \\ \xleftarrow{\iota} \end{array} \mathbb{R} \text{ or } \{0\}$$



applications: fos & sos (first and second order systems)

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$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot{\cup} \Gamma_n}$

(electro-magneto dynamics, Maxwell's equations with mixed boundary conditions)

$$\{0\} \text{ or } \mathbb{R} \xleftrightarrow[\pi]{\ell} L^2 \xleftrightarrow[-\operatorname{div}_{\Gamma_n} \varepsilon]{\nabla_{\Gamma_t}} L^2_{\varepsilon} \xleftrightarrow[\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}]{\operatorname{rot}_{\Gamma_t}} L^2 \xleftrightarrow[-\nabla_{\Gamma_n}]{\operatorname{div}_{\Gamma_t}} L^2 \xleftrightarrow[\ell]{\pi} \mathbb{R} \text{ or } \{0\}$$

related fos

$$\nabla_{\Gamma_t} u = A \quad \text{in } \Omega \quad | \quad \operatorname{rot}_{\Gamma_t} E = J \quad \text{in } \Omega \quad | \quad \operatorname{div}_{\Gamma_t} H = k \quad \text{in } \Omega \quad | \quad \pi v = b \quad \text{in } \Omega$$

$$\pi u = a \quad \text{in } \Omega \quad | \quad -\operatorname{div}_{\Gamma_n} \varepsilon E = j \quad \text{in } \Omega \quad | \quad \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} H = K \quad \text{in } \Omega \quad | \quad -\nabla_{\Gamma_n} v = B \quad \text{in } \Omega$$

related sos

$$-\operatorname{div}_{\Gamma_n} \varepsilon \nabla_{\Gamma_t} u = j \quad \text{in } \Omega \quad | \quad \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} \operatorname{rot}_{\Gamma_t} E = K \quad \text{in } \Omega \quad | \quad -\nabla_{\Gamma_n} \operatorname{div}_{\Gamma_t} H = B \quad \text{in } \Omega$$

$$\pi u = a \quad \text{in } \Omega \quad | \quad -\operatorname{div}_{\Gamma_n} \varepsilon E = j \quad \text{in } \Omega \quad | \quad \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} H = K \quad \text{in } \Omega$$

corresponding compact embeddings:

$$D(\nabla_{\Gamma_t}) \cap D(\pi) = D(\nabla_{\Gamma_t}) = H_{\Gamma_t}^1 \hookrightarrow L^2 \quad (\text{Rellich's selection theorem})$$

$$D(\operatorname{rot}_{\Gamma_t}) \cap D(-\operatorname{div}_{\Gamma_n} \varepsilon) = R_{\Gamma_t} \cap \varepsilon^{-1} D_{\Gamma_n} \hookrightarrow L^2_{\varepsilon} \quad (\text{Weck's selection theorem, '74})$$

$$D(\operatorname{div}_{\Gamma_t}) \cap D(\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}) = D_{\Gamma_t} \cap R_{\Gamma_n} \hookrightarrow L^2 \quad (\text{Weck's selection theorem, '74})$$

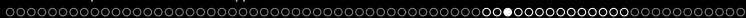
$$D(\nabla_{\Gamma_n}) \cap D(\pi) = D(\nabla_{\Gamma_n}) = H_{\Gamma_n}^1 \hookrightarrow L^2 \quad (\text{Rellich's selection theorem})$$

Weck's selection theorem for weak Lip. dom. and mixed bc: Bauer/Py/Schomburg ('16)

Weck's selection theorem (Weck '74, (Habil. '72) stimulated by Rolf Leis)

(Weber '80, Picard '84, Costabel '90, Witsch '93, Jochmann '97, Kuhn '99, Picard/Weck/Witsch '01,

Py '96, '03, '06, '07, '08)



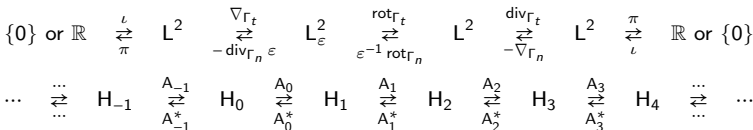
applications: fos & sos (first and second order systems)

classical de Rham complex in 3D (∇ -rot-div-complex)

$$\begin{aligned}
 \operatorname{rot} E &= F && \text{in } \Omega \\
 -\operatorname{div} \varepsilon E &= g && \text{in } \Omega \\
 \nu \times E &= 0 && \text{at } \Gamma_t \\
 \nu \cdot \varepsilon E &= 0 && \text{at } \Gamma_n
 \end{aligned}$$

non-trivial kernel $\mathcal{H}_{D,\varepsilon} = \{H \in L^2 : \operatorname{rot} H = 0, \operatorname{div} \varepsilon H = 0, \nu \times H|_{\Gamma_t} = 0, \nu \cdot \varepsilon H|_{\Gamma_n} = 0\}$
 additional condition on Dirichlet/Neumann fields for uniqueness

$$\pi_D E = K \in \mathcal{H}_{D,\varepsilon}$$



$$\begin{array}{llll}
 \text{find } E \in R_{\Gamma_t}(\Omega) \cap \varepsilon^{-1} D_{\Gamma_n}(\Omega) & \text{st} & \text{(fos)} & \text{find } x \in D(A_1) \cap D(A_0^*) & \text{st} \\
 \operatorname{rot}_{\Gamma_t} E = F & & & A_1 x = f & \\
 -\operatorname{div}_{\Gamma_n} \varepsilon E = g & & \text{translation} & A_0^* x = g & \\
 \pi_{D/N} E = K & & & \pi_{K_1} x = k &
 \end{array}$$

classical de Rham complex in 3D (∇ -rot-div-complex)

$c_{A_0} = c_{fp}$ (Friedrichs/Poincaré constant) and $c_{A_1} = c_m$ (Maxwell constant)

Lemma/Theorem \Downarrow $D(A_1) \cap D(A_0^*) \Leftrightarrow L_\varepsilon^2(\Omega)$ compact

(i) all Friedrichs-Poincaré type est hold

$$\forall \varphi \in D(\mathcal{A}_0) \quad |\varphi|_{H_0} \leq c_{A_0} |A_0 \varphi|_{H_1} \quad \Leftrightarrow \quad \forall \varphi \in H_{\Gamma_t}^1 \quad |\varphi|_{L^2} \leq c_{fp} |\nabla \varphi|_{L_\varepsilon^2}$$

$$\forall \phi \in D(\mathcal{A}_0^*) \quad |\phi|_{H_1} \leq c_{A_0} |A_0^* \phi|_{H_0} \quad \Leftrightarrow \quad \forall \Phi \in \varepsilon^{-1} D_{\Gamma_n} \cap \nabla H_{\Gamma_t}^1 \quad |\Phi|_{L_\varepsilon^2} \leq c_{fp} |\operatorname{div} \varepsilon \Phi|_{L^2}$$

$$\forall \varphi \in D(\mathcal{A}_1) \quad |\varphi|_{H_1} \leq c_{A_1} |A_1 \varphi|_{H_2} \quad \Leftrightarrow \quad \forall \Phi \in R_{\Gamma_t} \cap \varepsilon^{-1} \operatorname{rot} R_{\Gamma_n} \quad |\Phi|_{L_\varepsilon^2} \leq c_m |\operatorname{rot} \Phi|_{L^2}$$

$$\forall \psi \in D(\mathcal{A}_1^*) \quad |\psi|_{H_2} \leq c_{A_1} |A_1^* \psi|_{H_1} \quad \Leftrightarrow \quad \forall \Psi \in R_{\Gamma_n} \cap \operatorname{rot} R_{\Gamma_t} \quad |\Psi|_{L^2} \leq c_m |\operatorname{rot} \Psi|_{L_\varepsilon^2}$$

(ii) all ranges $R(A_0) = \nabla H_{\Gamma_t}^1$, $R(A_1) = \operatorname{rot} R_{\Gamma_t}$, $R(A_0^*) = \operatorname{div} D_{\Gamma_n}$ are cl in L^2

(iii) the inverse ops $(\widetilde{\nabla}_{\Gamma_t})^{-1}$, $(\widetilde{\operatorname{div}}_{\Gamma_n} \varepsilon)^{-1}$, $(\widetilde{\operatorname{rot}}_{\Gamma_t})^{-1}$, $(\widetilde{\varepsilon^{-1} \operatorname{rot}}_{\Gamma_n})^{-1}$ are cont, even cpt

(iv) all Helmholtz decomposition hold, e.g.,

$$H_1 = R(A_0) \oplus K_1 \oplus R(A_1^*) \quad \Leftrightarrow \quad L_\varepsilon^2 = \nabla H_{\Gamma_t}^1 \oplus_{L_\varepsilon^2} \mathcal{H}_{D,\varepsilon} \oplus_{L_\varepsilon^2} \varepsilon^{-1} \operatorname{rot} R_{\Gamma_n}$$

(v) solution theory

(vi) variational formulations

(vii) functional a posteriori error estimates

(viii) div-curl-lemma

(ix) ...



applications: fos & sos (first and second order systems)

classical de Rham complex in 3D (∇ -rot-div-complex)

$$\{0\} \text{ or } \mathbb{R} \xrightleftharpoons[\pi]{} L^2 \xrightleftharpoons[-\operatorname{div}_{\Gamma_n} \varepsilon]{\nabla_{\Gamma_t}} L^2_\varepsilon \xrightleftharpoons[\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}]{\operatorname{rot}_{\Gamma_t}} L^2 \xrightleftharpoons[-\nabla_{\Gamma_n}]{\operatorname{div}_{\Gamma_t}} L^2 \xrightleftharpoons[\iota]{\pi} \mathbb{R} \text{ or } \{0\}$$

variational formulations

$$\text{var space } (\tilde{E}, u, r, H) \in \boxed{R_{\Gamma_t} \times H_{\Gamma_t}^1 \times \{0\}/\mathbb{R} \times \mathcal{H}_{D,\varepsilon}}$$

$$\begin{bmatrix} \mu \operatorname{rot}_{\Gamma_n} \operatorname{rot}_{\Gamma_t} & \operatorname{grad}_{\Gamma_t} & 0 & \iota_{\mathcal{H}_{D,\varepsilon}} \\ -\operatorname{div}_{\Gamma_n} \varepsilon & 0 & \iota_{\{0\}/\mathbb{R}} & 0 \\ 0 & \pi_{\{0\}/\mathbb{R}} & 0 & 0 \\ \pi_{\mathcal{H}_{D,\varepsilon}} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{E} \\ u \\ r \\ H \end{bmatrix} = \begin{bmatrix} \mu \operatorname{rot}_{\Gamma_n} F \\ g \\ 0 \\ K \end{bmatrix}$$

$$\text{var space } (\hat{E}, U, v, r, H, \tilde{H}) \in \boxed{\varepsilon^{-1} D_{\Gamma_n} \times R_{\Gamma_n} \times H_{\Gamma_n}^1 \times \{0\}/\mathbb{R} \times \mathcal{H}_{D,\varepsilon} \times \mathcal{H}_N}$$

$$\begin{bmatrix} -\operatorname{grad}_{\Gamma_t} \operatorname{div}_{\Gamma_n} \varepsilon & \mu \operatorname{rot}_{\Gamma_n} & 0 & 0 & \iota_{\mathcal{H}_{D,\varepsilon}} & 0 \\ \operatorname{rot}_{\Gamma_t} & 0 & -\operatorname{grad}_{\Gamma_n} & 0 & 0 & \iota_{\mathcal{H}_N} \\ 0 & \operatorname{div}_{\Gamma_t} & 0 & \iota_{\{0\}/\mathbb{R}} & 0 & 0 \\ 0 & 0 & \pi_{\{0\}/\mathbb{R}} & 0 & 0 & 0 \\ \pi_{\mathcal{H}_{D,\varepsilon}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi_{\mathcal{H}_N} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{E} \\ U \\ v \\ r \\ H \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} \operatorname{grad}_{\Gamma_t} g \\ F \\ 0 \\ 0 \\ K \\ 0 \end{bmatrix}$$

$$\text{note } u = v = 0, \quad r = 0, \quad U = H = \tilde{H} = 0, \quad \boxed{\tilde{E} = \hat{E} = E}$$



classical de Rham complex in 3D (∇ -rot-div-complex)

Theorem (sharp upper bounds)

Let $\tilde{E} \in L^2_\varepsilon$ (very non-conforming approximation of E !) and $e := E - \tilde{E}$. Then

$$|e|_{L^2_\varepsilon}^2 = |\pi_R(\nabla_{\Gamma_t})e|_{L^2_\varepsilon}^2 + |\pi_{R(\varepsilon^{-1}\text{rot}_{\Gamma_n})}e|_{L^2_\varepsilon}^2 + |\pi_{\mathcal{H}_{D,\varepsilon}}e|_{L^2_\varepsilon}^2$$

$$= \min_{\Phi \in \varepsilon^{-1}D_{\Gamma_n}} (c_f |\text{div } \varepsilon \Phi + g|_{L^2} + |\Phi - \tilde{E}|_{L^2_\varepsilon})^2$$

reg $(-\nabla_{\Gamma_t} \text{div}_{\Gamma_n} + 1)$ -prbl in D_{Γ_n}

$$+ \min_{\Phi \in R_{\Gamma_t}} (c_m |\text{rot } \Phi - F|_{L^2} + |\Phi - \tilde{E}|_{L^2_\varepsilon})^2$$

reg $(\text{rot}_{\Gamma_n} \text{rot}_{\Gamma_t} + 1)$ -prbl in R_{Γ_t}

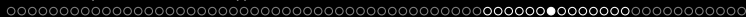
$$+ \min_{\phi \in H^1_{\Gamma_t}, \Psi \in R_{\Gamma_n}} |\nabla \phi + \varepsilon^{-1} \text{rot } \Psi + \tilde{E} - K|_{L^2_\varepsilon}^2$$

cpld $(-\text{div}_{\Gamma_n} \nabla_{\Gamma_t})$ - $(\text{rot}_{\Gamma_t} \text{rot}_{\Gamma_n})$ -sys in $H^1_{\Gamma_t}$ - R_{Γ_n}

Remark

- $(\text{rot}_{\Gamma_t} \text{rot}_{\Gamma_n})$ -prbl needs saddle point formulation
- Ω top trv $\Rightarrow \pi_D = 0$ and $R_{\Gamma_t,0} = \nabla H^1_{\Gamma_t}$ and $D_{\Gamma_n,0} = \text{rot } R_{\Gamma_n}$

$$\bullet \quad \Omega \text{ convex and } \varepsilon = \mu = 1 \text{ and } \Gamma_t = \Gamma \text{ or } \Gamma_n = \Gamma \Rightarrow c_f \leq c_m \leq c_p \leq \frac{\text{diam } \Omega}{\pi}$$



classical de Rham complex in 3D (∇ -rot-div-complex)

Theorem (sharp lower bounds)

Let $\tilde{E} \in L_\varepsilon^2$ (very non-conforming approximation of E !) and $e := E - \tilde{E}$. Then

$$|e|_{L_\varepsilon^2}^2 = |\pi_{R(\nabla_{\Gamma_t})} e|_{L_\varepsilon^2}^2 + |\pi_{R(\varepsilon^{-1} \text{rot}_{\Gamma_n})} e|_{L_\varepsilon^2}^2 + |\pi_{\mathcal{H}_{D,\varepsilon}} e|_{L_\varepsilon^2}^2$$

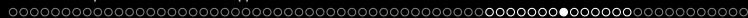
$$= \max_{\varphi \in \mathcal{H}_{\Gamma_t}^1} (2\langle g, \varphi \rangle_{L^2} - \langle 2\tilde{E} + \text{grad } \varphi, \varepsilon \text{ grad } \varphi \rangle_{L^2})$$

$\text{reg}(-\nabla_{\Gamma_t} \text{div}_{\Gamma_n} + 1)$ -prbl in D_{Γ_n}

$$+ \max_{\Psi \in R_{\Gamma_n}} (2\langle F, \Psi \rangle_{L^2} - \langle 2\tilde{E} + \mu \text{rot } \Psi, \text{rot } \Psi \rangle_{L^2})$$

$\text{reg}(\text{rot}_{\Gamma_n} \text{rot}_{\Gamma_t} + 1)$ -prbl in R_{Γ_t}

$$+ \max_{\Psi \in \mathcal{H}_{D,\varepsilon}} \langle 2(K - \tilde{E}) - \Psi, \Psi \rangle_{L_\varepsilon^2}$$



div-curl-lemma

Lemma (div-curl-lemma (global version))

Assumptions:

- (i) (E_n) bounded in $L^2(\Omega)$
- (i') (H_n) bounded in $L^2(\Omega)$
- (ii) $(\operatorname{rot} E_n)$ bounded in $L^2(\Omega)$
- (ii') $(\operatorname{div} \varepsilon H_n)$ bounded in $L^2(\Omega)$
- (iii) $\nu \times E_n = 0$ on Γ_t , i.e., $E_n \in R_{\Gamma_t}(\Omega)$
- (iii') $\nu \cdot \varepsilon H_n = 0$ on Γ_n , i.e., $H_n \in \varepsilon^{-1} D_{\Gamma_n}(\Omega)$

$\Rightarrow \exists E, H$ and subsequences st

$E_n \rightarrow E, \operatorname{rot} E_n \rightarrow \operatorname{rot} E$ and $H_n \rightarrow H, \operatorname{div} H_n \rightarrow \operatorname{div} H$ in $L^2(\Omega)$ and

$$\langle E_n, H_n \rangle_{L^2_\varepsilon(\Omega)} \rightarrow \langle E, H \rangle_{L^2_\varepsilon(\Omega)}$$



applications: fos & sos (first and second order systems)

de Rham complex in ND or on Riemannian manifolds (d-complex)

$\Omega \subset \mathbb{R}^N$ bd w. Lip. dom. or Ω Riemannian manifold with cpt cl. and Lip. boundary Γ
(generalized Maxwell equations)

$$\{0\} \begin{array}{c} \hookrightarrow \\ \xleftrightarrow{\pi_{\{0\}}} \end{array} L^{2,0} \begin{array}{c} \xrightarrow{d} \\ \xleftrightarrow{-\delta} \end{array} L^{2,1} \begin{array}{c} \xrightarrow{d} \\ \xleftrightarrow{-\delta} \end{array} \dots L^{2,q} \begin{array}{c} \xrightarrow{d} \\ \xleftrightarrow{-\delta} \end{array} L^{2,q+1} \dots L^{2,N-1} \begin{array}{c} \xrightarrow{d} \\ \xleftrightarrow{-\delta} \end{array} L^{2,N} \begin{array}{c} \xrightarrow{\pi_{\mathbb{R}}} \\ \xleftrightarrow{\iota_{\mathbb{R}}} \end{array} \mathbb{R}$$



applications: fos & sos (first and second order systems)

de Rham complex in ND or on Riemannian manifolds (d-complex)

$\Omega \subset \mathbb{R}^N$ bd w. Lip. dom. or Ω Riemannian manifold with cpt cl. and Lip. boundary Γ
(generalized Maxwell equations)

$$\{0\} \text{ or } \mathbb{R} \xrightarrow{\begin{smallmatrix} L \\ \rightleftharpoons \\ \pi \end{smallmatrix}} L^{2,0} \begin{array}{c} d_{\Gamma_t}^0 \\ \rightleftharpoons \\ -\delta_{\Gamma_n}^1 \end{array} L^{2,1} \begin{array}{c} d_{\Gamma_t}^1 \\ \rightleftharpoons \\ -\delta_{\Gamma_n}^2 \end{array} \dots L^{2,q} \begin{array}{c} d_{\Gamma_t}^q \\ \rightleftharpoons \\ -\delta_{\Gamma_n}^{q+1} \end{array} L^{2,q+1} \dots L^{2,N-1} \begin{array}{c} d_{\Gamma_t}^{N-1} \\ \rightleftharpoons \\ -\delta_{\Gamma_n}^N \end{array} L^{2,N} \xrightarrow{\begin{smallmatrix} \pi \\ \rightleftharpoons \\ L \end{smallmatrix}} \mathbb{R} \text{ or } \{0\}$$

related fos

$$\begin{aligned} d_{\Gamma_t}^q E &= F && \text{in } \Omega \\ -\delta_{\Gamma_n}^q E &= G && \text{in } \Omega \end{aligned}$$

related sos

$$\begin{aligned} -\delta_{\Gamma_n}^{q+1} d_{\Gamma_t}^q E &= F && \text{in } \Omega \\ -\delta_{\Gamma_n}^q E &= G && \text{in } \Omega \end{aligned}$$

includes: EMS rot / div, Laplacian, rot rot, and more...

corresponding compact embeddings:

$$D(d_{\Gamma_t}^q) \cap D(\delta_{\Gamma_n}^q) \hookrightarrow L^{2,q} \quad (\text{Weck's selection theorems, '74})$$

Weck's selection theorem for Lip. manifolds and mixed bc: Bauer/Py/Schomburg ('17)

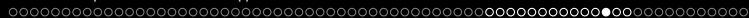


applications: fos & sos (first and second order systems)

elasticity complex in 3D (sym ∇ -Rot Rot $_{\mathbb{S}}^T$ -Div $_{\mathbb{S}}$ -complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\begin{array}{ccccccc}
 \{0\} & \begin{array}{c} \iota_{\{0\}} \\ \rightleftharpoons \\ \pi_{\{0\}} \end{array} & L^2 & \begin{array}{c} \text{sym } \nabla \\ \rightleftharpoons \\ -\text{Div}_{\mathbb{S}} \end{array} & L^2_{\mathbb{S}} & \begin{array}{c} \text{Rot Rot}_{\mathbb{S}}^T \\ \rightleftharpoons \\ \text{Rot Rot}_{\mathbb{S}}^T \end{array} & L^2_{\mathbb{S}} & \begin{array}{c} \text{Div}_{\mathbb{S}} \\ \rightleftharpoons \\ -\text{sym } \nabla \end{array} & L^2 & \begin{array}{c} \pi_{\text{RM}} \\ \rightleftharpoons \\ \iota_{\text{RM}} \end{array} & \text{RM}
 \end{array}$$



elasticity complex in 3D (sym ∇ -Rot Rot $_S^T$ -Div $_S$ -complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \begin{array}{c} \overset{\mathcal{L}\{0\}}{\rightleftarrows} \\ \underset{\pi\{0\}}{\rightleftarrows} \end{array} L^2 \begin{array}{c} \overset{\text{sym } \nabla}{\rightleftarrows} \\ \underset{-\text{Div}_S}{\rightleftarrows} \end{array} L_S^2 \begin{array}{c} \overset{\text{Rot Rot}_S^T}{\rightleftarrows} \\ \underset{\text{Rot Rot}_S^T}{\rightleftarrows} \end{array} L_S^2 \begin{array}{c} \overset{\text{Div}_S}{\rightleftarrows} \\ \underset{-\text{sym } \nabla}{\rightleftarrows} \end{array} L^2 \begin{array}{c} \overset{\pi_{RM}}{\rightleftarrows} \\ \underset{\mathcal{L}_{RM}}{\rightleftarrows} \end{array} \text{RM}$$

related fos (Rot Rot $_{S,\Gamma}^T$, Rot Rot $_S^T$ first order operators!)

$$\begin{array}{l|l|l|l} \text{sym } \nabla_\Gamma v = M & \text{in } \Omega & | & \text{Rot Rot}_{S,\Gamma}^T M = F & \text{in } \Omega & | & \text{Div}_{S,\Gamma} N = g & \text{in } \Omega & | & \pi v = r & \text{in } \Omega \\ \pi v = 0 & \text{in } \Omega & | & -\text{Div}_S M = f & \text{in } \Omega & | & \text{Rot Rot}_S^T N = G & \text{in } \Omega & | & -\text{sym } \nabla v = M & \text{in } \Omega \end{array}$$

related sos (Rot Rot $_S^T$ Rot Rot $_{S,\Gamma}^T$ second order operator!)

$$\begin{array}{l|l|l|l} -\text{Div}_S \text{sym } \nabla_\Gamma v = f & \text{in } \Omega & | & \text{Rot Rot}_S^T \text{Rot Rot}_{S,\Gamma}^T M = G & \text{in } \Omega & | & -\text{sym } \nabla \text{Div}_{S,\Gamma} N = M & \text{in } \Omega \\ \pi v = 0 & \text{in } \Omega & | & -\text{Div}_S M = f & \text{in } \Omega & | & \text{Rot Rot}_S^T N = G & \text{in } \Omega \end{array}$$

corresponding compact embeddings:

$$\begin{array}{ll} D(\text{sym } \nabla_\Gamma) \cap D(\pi) = D(\nabla_\Gamma) = H_\Gamma^1 \hookrightarrow L^2 & \text{(Rellich's selection theorem and Korn ineq.)} \\ D(\text{Rot Rot}_{S,\Gamma}^T) \cap D(\text{Div}_S) \hookrightarrow L_S^2 & \text{(new selection theorem)} \\ D(\text{Div}_{S,\Gamma}) \cap D(\text{Rot Rot}_S^T) \hookrightarrow L_S^2 & \text{(new selection theorem)} \\ D(\pi) \cap D(\text{sym } \nabla) = D(\nabla) = H^1 \hookrightarrow L^2 & \text{(Rellich's selection theorem and Korn ineq.)} \end{array}$$

two new selection theorems for strong Lip. dom.: Py/Schomburg/Zulehner ('18)



applications: fos & sos (first and second order systems)

biharmonic / general relativity complex in 3D ($\nabla\nabla$ -Rot_S-Div_T-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\begin{array}{ccccccc}
 \{0\} & \begin{array}{c} \overset{\iota_{\{0\}}}{\rightleftarrows} \\ \underset{\pi_{\{0\}}}{\rightleftarrows} \end{array} & L^2 & \begin{array}{c} \overset{\nabla\nabla}{\rightleftarrows} \\ \underset{\operatorname{div} \operatorname{Div}_S}{\rightleftarrows} \end{array} & L^2_S & \begin{array}{c} \overset{\operatorname{Rot}_S}{\rightleftarrows} \\ \underset{\operatorname{sym} \operatorname{Rot}_T}{\rightleftarrows} \end{array} & L^2_T & \begin{array}{c} \overset{\operatorname{Div}_T}{\rightleftarrows} \\ \underset{-\operatorname{dev} \nabla}{\rightleftarrows} \end{array} & L^2 & \begin{array}{c} \overset{\pi_{RT}}{\rightleftarrows} \\ \underset{\iota_{RT}}{\rightleftarrows} \end{array} & RT
 \end{array}$$



applications: fos & sos (first and second order systems)

biharmonic / general relativity complex in 3D ($\nabla\nabla$ -Rot $_{\mathbb{S}}$ -Div $_{\mathbb{T}}$ -complex) $\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \begin{array}{c} \xrightarrow{\iota_{\{0\}}} \\ \xrightarrow{\pi_{\{0\}}} \end{array} L^2 \begin{array}{c} \xrightarrow{\nabla\nabla} \\ \xrightarrow{\text{div Div}_{\mathbb{S}}} \end{array} L^2_{\mathbb{S}} \begin{array}{c} \xrightarrow{\text{Rot}_{\mathbb{S}}} \\ \xrightarrow{\text{sym Rot}_{\mathbb{T}}} \end{array} L^2_{\mathbb{T}} \begin{array}{c} \xrightarrow{\text{Div}_{\mathbb{T}}} \\ \xrightarrow{-\text{dev } \nabla} \end{array} L^2 \begin{array}{c} \xrightarrow{\pi_{\text{RT}}} \\ \xrightarrow{\iota_{\text{RT}}} \end{array} \text{RT}$$

related fos ($\nabla\nabla_{\Gamma}$, $\text{div Div}_{\mathbb{S}}$ first order operators!)

$$\begin{array}{l} \nabla\nabla_{\Gamma} u = M \quad \text{in } \Omega \quad | \quad \text{Rot}_{\mathbb{S},\Gamma} M = F \quad \text{in } \Omega \quad | \quad \text{Div}_{\mathbb{T},\Gamma} N = g \quad \text{in } \Omega \quad | \quad \pi v = r \quad \text{in } \Omega \\ \pi u = 0 \quad \text{in } \Omega \quad | \quad \text{div Div}_{\mathbb{S}} M = f \quad \text{in } \Omega \quad | \quad \text{sym Rot}_{\mathbb{T}} N = G \quad \text{in } \Omega \quad | \quad -\text{dev } \nabla v = T \quad \text{in } \Omega \end{array}$$

related sos ($\text{div Div}_{\mathbb{S}} \nabla\nabla_{\Gamma} = \Delta_{\Gamma}^2$ second order operator!)

$$\begin{array}{l} \text{div Div}_{\mathbb{S}} \nabla\nabla_{\Gamma} u = \Delta_{\Gamma}^2 u = f \quad \text{in } \Omega \quad | \quad \text{sym Rot}_{\mathbb{T}} \text{Rot}_{\mathbb{S},\Gamma} M = G \quad \text{in } \Omega \quad | \quad -\text{dev } \nabla \text{Div}_{\mathbb{T},\Gamma} N = T \quad \text{in } \Omega \\ \pi u = 0 \quad \text{in } \Omega \quad | \quad \text{div Div}_{\mathbb{S}} M = f \quad \text{in } \Omega \quad | \quad \text{sym Rot}_{\mathbb{T}} N = G \quad \text{in } \Omega \end{array}$$

corresponding compact embeddings:

$$D(\nabla\nabla_{\Gamma}) \cap D(\pi) = D(\nabla\nabla_{\Gamma}) = H_{\Gamma}^2 \hookrightarrow L^2 \quad (\text{Rellich's selection theorem})$$

$$D(\text{Rot}_{\mathbb{S},\Gamma}) \cap D(\text{div Div}_{\mathbb{S}}) \hookrightarrow L^2_{\mathbb{S}} \quad (\text{new selection theorem})$$

$$D(\text{Div}_{\mathbb{T},\Gamma}) \cap D(\text{sym Rot}_{\mathbb{T}}) \hookrightarrow L^2_{\mathbb{T}} \quad (\text{new selection theorem})$$

$$D(\pi) \cap D(\text{dev } \nabla) = D(\text{dev } \nabla) = D(\nabla) = H^1 \hookrightarrow L^2 \quad (\text{Rellich's selection theorem and Korn type ineq.})$$

two new selection theorems for strong Lip. dom. and Korn Type ineq.: Py/Zulehner ('16)



literature (fa-toolbox, complexes, a posteriori error estimates, ...)

results of this talk:

- Py: *Solution Theory and Functional A Posteriori Error Estimates for General First Order Systems with Applications to Electro-Magneto-Statics*, (NFAO) Numerical Functional Analysis and Optimization, 2018

(paper contains main results of this talk)



literature (a posteriori error estimates of functional type)

- Repin, S.: *A posteriori error estimates for variational problems with uniformly convex functionals*,
(MC) Mathematics of Computation, 2000
- Neittaanmäki, P., Repin, S.: *Reliable methods for computer simulation, error control and a posteriori estimates*,
Elsevier, 2004
- Repin, S.: *A posteriori estimates for partial differential equations*,
Radon Series on Applied Mathematics, De Gruyter, 2008
- Py, Repin, S.: *Functional A Posteriori Error Estimates for Elliptic Problems in Exterior Domains*,
(PMA) Problemy Matematicheskogo Analiza/ (JMS) Journal of Mathematical Sciences (Springer New York), 2009
- Py, Repin, S.: *Two-sided a posteriori error bounds for electro-magneto static problems*,
Zapiski POMI/ (JMS) Journal of Mathematical Sciences (Springer New York), 2009
- Mali, O., Neittaanmäki, P., Repin, S.: *Accuracy verification methods, theory and algorithms*,
Springer, 2014



literature (a posteriori error equations)

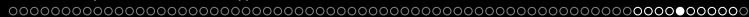
- Anjam, I., Py: *Functional a posteriori error control for conforming mixed approximations of coercive problems with lower order terms*, (CMAM) Computational Methods in Applied Mathematics, 2016
- Anjam, I., Py: *An Elementary Method of Deriving A Posteriori Error Equalities and Estimates for Linear Partial Differential Equations*, (CMAM) Computational Methods in Applied Mathematics, 2018



literature (complexes, Friedrichs type constants, Maxwell constants)

results of this talk:

- Py: *On Constants in Maxwell Inequalities for Bounded and Convex Domains*, Zapiski POMI/ (JMS) Journal of Mathematical Sciences (Springer New York), 2015
- Py: *On Maxwell's and Poincare's Constants*, (DCDS) Discrete and Continuous Dynamical Systems - Series S, 2015
- Py: *On the Maxwell Constants in 3D*, (M2AS) Mathematical Methods in the Applied Sciences, 2017
- Py: *On the Maxwell and Friedrichs/Poincare Constants in ND*, submitted, 2017

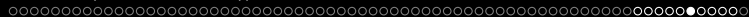


literature (complexes, Friedrichs type constants, compact embeddings)

- Weck, N.: *Maxwell's boundary value problems on Riemannian manifolds with nonsmooth boundaries*,
(JMA2) Journal of Mathematical Analysis and Applications, 1974 (1972)
- Picard, R.: *An elementary proof for a compact imbedding result in generalized electromagnetic theory*,
(MZ) Mathematische Zeitschrift, 1984
- Witsch, K.-J.: *A remark on a compactness result in electromagnetic theory*,
(M2AS) Mathematical Methods in the Applied Sciences, 1993

results of this talk:

- Bauer, S., Py, Schomburg, M.: *The Maxwell Compactness Property in Bounded Weak Lipschitz Domains with Mixed Boundary Conditions*,
(SIMA) SIAM Journal on Mathematical Analysis, 2016
- Zulehner, W., Py: *On Closed and Exact Grad grad- and div Div-Complexes, Corresponding Compact Embeddings for Tensor Rotations, and a Related Decomposition Result for Biharmonic Problems in 3D*,
submitted, 2016
- Py, Schomburg, M., Zulehner, W.: *Compact Embeddings, Friedrichs/Poincaré Type Estimates, Helmholtz Type Decompositions, and a General Toolbox for the Elasticity Complex in 3D*,
in preparation, 2018



literature (div-curl-lemma)

original papers (local div-curl-lemma):

- Murat, F.: *Compacité par compensation*,
Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 1978
- Tartar, L.: *Compensated compactness and applications to partial differential equations*,
Nonlinear analysis and mechanics, Heriot-Watt symposium, 1979



literature (div-curl-lemma)

recent papers (global div-curl-lemma, H^1 -detour):

- Gloria, A., Neukamm, S., Otto, F.: *Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics*, (IM) Invent. Math., 2015
- Kozono, H., Yanagisawa, T.: *Global compensated compactness theorem for general differential operators of first order*, (ARMA) Arch. Ration. Mech. Anal., 2013
- Schweizer, B.: *On Friedrichs inequality, Helmholtz decomposition, vector potentials, and the div-curl lemma*, accepted preprint, 2018

recent papers (global div-curl-lemma, general results/this talk):

- Waurick, M.: *A Functional Analytic Perspective to the div-curl Lemma*, (JOP) J. Operator Theory, 2018
- Py: *A Global div-curl-Lemma for Mixed Boundary Conditions in Weak Lipschitz Domains and a Corresponding Generalized A_0^* - A_1 -Lemma in Hilbert Spaces*, (ANA) Analysis (Munich), 2018



literature (complexes, optimization, and a posteriori error estimates for degenerated magneto statics)

- Py, Yousept, I.: *A Posteriori Error Analysis for the Optimal Control of Magneto-Static Fields*,
(M2NA) ESAIM: Mathematical Modelling and Numerical Analysis, 2017



literature (full time-dependent Maxwell equations)

- Py, Picard, R.: *A Note on the Justification of the Eddy Current Model in Electrodynamics*,
(M2AS) Mathematical Methods in the Applied Sciences, 2017



literature (Maxwell's equations and more...)

upcoming books:

- Langer, U., Py, Repin, S. (Eds): *Maxwell's equations. Analysis and numerics*, Radon Series on Applied Mathematics, De Gruyter, 2018
- Py: *Maxwell's Equations: Hilbert Space Methods for the Theory of Electromagnetism*, Radon Series on Applied Mathematics, De Gruyter, 2020

(last book: contains all results of this talk and more...)

... the world is full of complexes ... ;)

⇒ relaxing at

AANMPDE 11

11th Workshop on Analysis and Advanced Numerical Methods
for Partial Differential Equations (not only) for Junior Scientists

<http://www.mit.jyu.fi/scoma/AANMPDE11>

August 6–10 2018, Särkisaari, Finland

organizers: Ulrich Langer, Dirk Pauly, Sergey Repin

